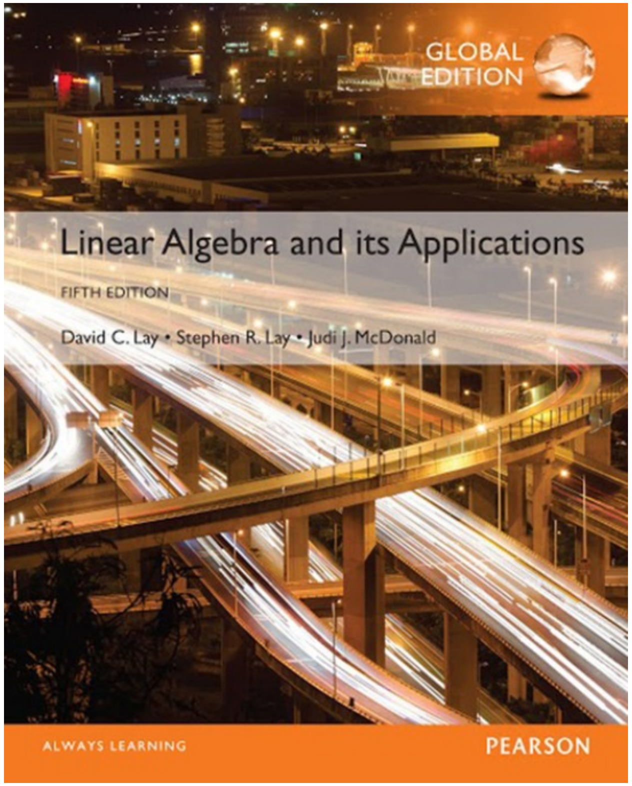
4 Determinants

4.7

CHANGE OF BASIS



CHANGE OF BASIS

- **Example 1** Consider two bases $\beta = \{\mathrm{b}_1, \mathrm{b}_2\}$ and $C = \{\mathrm{c}_1, \mathrm{c}_2\}$ for a vector space V, such that

$$b_1 = 4c_1 + c_2 \quad \text{and} \quad b_2 = -6c_1 + c_2 \tag{1}$$

- Suppose

$$x = 3b_1 + b_2 \tag{2}$$

- That is, suppose $[x]_\beta = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[x]_C$.

CHANGE OF BASIS

- **Solution** Apply the coordinate mapping determined by C to $\mathbf{x}$ in (2). Since the coordinate mapping is a linear transformation,

$$[x]_C = [3\mathrm{b}_1 + \mathrm{b}_2]_C$$
$$= 3[\mathrm{b}_1]_C + [\mathrm{b}_2]_C$$

- We can write the vector equation as a matrix equation, using the vectors in the linear combination as the columns of a matrix:

$$[x]_C = [\mathrm{b}_1 \quad \mathrm{b}_2]_C \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad (3)$$

CHANGE OF BASIS

- This formula gives $[x]_C$, once we know the columns of the matrix. From (1),

$$[\mathrm{b}_1]_C = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{and } [\mathrm{b}_2]_C = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

- Thus, (3) provides the solution:

$$[x]_C = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

- The *C*-coordinates of **x** match those of the **x** in Fig. 1, as seen on the next slide.

CHANGE OF BASIS

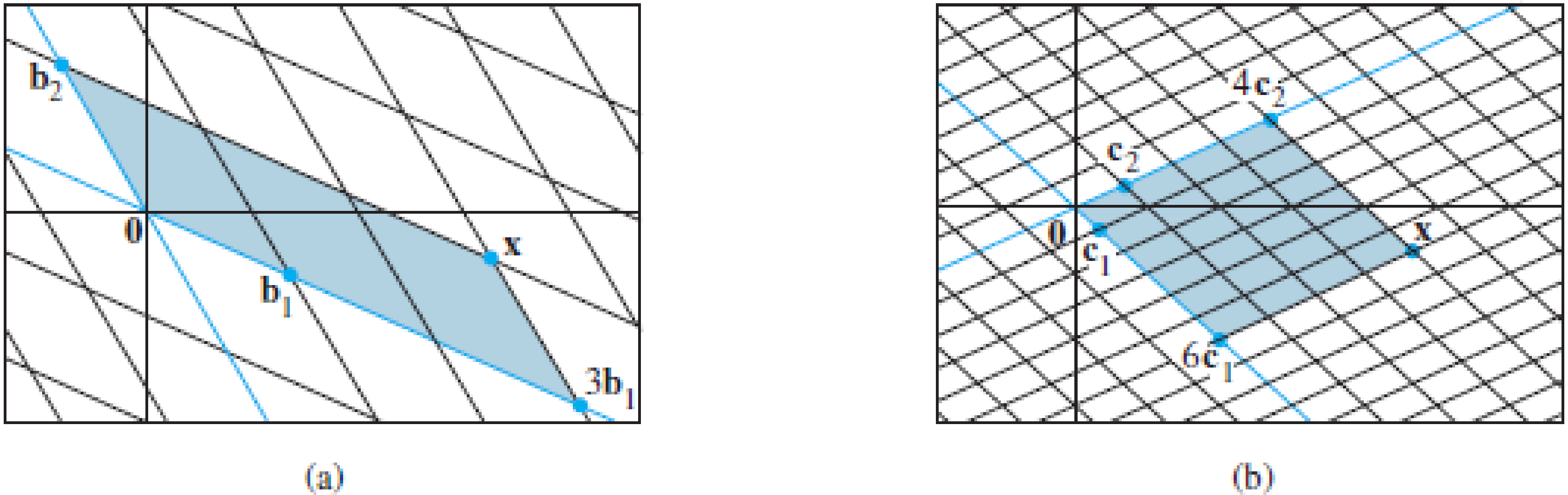


FIGURE 1 Two coordinate systems for the same vector space.

CHANGE OF BASIS

- **Theorem 15**: Let $\beta = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$ for a vector space V. Then there is a unique n × n matrix $\underset{C\leftarrow\beta}{P}$ such that

$$[x]_C = \underset{C\leftarrow\beta}{P}[x]_\beta \qquad (4)$$

- The columns of $\underset{C\leftarrow\beta}{P}$ are the C-coordinate vectors of the vectors in the basis β. That is,

$$\underset{C\leftarrow\beta}{P} = [\,[b_1]_C [b_2]_C \quad \cdots \quad [b_n]_C] \qquad (5)$$

CHANGE OF BASIS

- The matrix $\underset{c\leftarrow\beta}{P}$ in Theorem 15 is called the **change-of-coordinates matrix from β to C**. Multiplication by $\underset{c\leftarrow\beta}{P}$ converts β-coordinates into C-coordinates.
- Figure 2 below illustrates the change-of-coordinates equation (4).

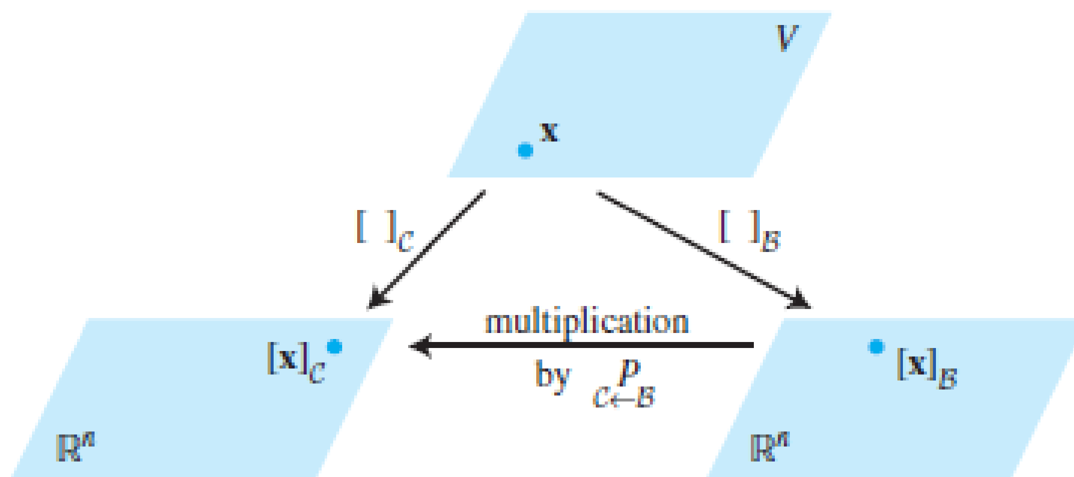


FIGURE 2 Two coordinate systems for V.

CHANGE OF BASIS

- The columns of $\underset{C \leftarrow \beta}{P}$ are linearly independent because they are the coordinate vectors of the linearly independent set β.

- Since $\underset{C \leftarrow \beta}{P}$ is square, it must be invertible, by the Invertible Matrix Theorem. Left-multiplying both sides of equation (4) by $(\underset{C \leftarrow \beta}{P})^{-1}$ yields

$$(\underset{C \leftarrow \beta}{P})^{-1} [x]_C = [x]_\beta$$

- Thus $(\underset{C \leftarrow \beta}{P})^{-1}$ is the matrix that converts C-coordinates into β–coordinates. That is,

$$(\underset{C \leftarrow \beta}{P})^{-1} = \underset{\beta \leftarrow C}{P} \qquad (6)$$

CHANGE OF BASIS IN $\mathbb{R}^n$

- If $\beta = \{b_1, \ldots, b_n\}$ and $\mathcal{E}$ is the standard basis $\{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$, then $[b_1]_{\mathcal{E}} = b_1$, and likewise for the other vectors in β. In this case, $\underset{\mathcal{E} \leftarrow \beta}{P}$ is thesame as the change-of-coordinates matrix P_β introduced in Section 4.4, namely,

$$P_\beta = [\, b_1 \quad b_2 \ldots b_n]$$

- To change coordinates between two nonstandard bases in $\mathbb{R}^n$,we need Theorem 15. The theorem shows that to solve the change-of-basis problem, we need the coordinate vectors of the old basis relative to the new basis.

CHANGE OF BASIS IN $\mathbb{R}^n$

- **Example 2** Let $\mathrm{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathrm{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathrm{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathrm{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ and consider the bases for $\mathbb{R}^n$ given by $\beta = \{\mathrm{b}_1, \mathrm{b}_2\}$ and $C = \{\mathrm{c}_1, \mathrm{c}_2\}$. Find the change-of-coordinates matrix from β to C.

- **Solution** The matrix $\beta \overset{P}{\leftarrow} C$ involves the C-coordinate vectors of b_1 and b_2. Let $[\mathrm{b}_1]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\mathrm{b}_2]_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then, by definition,

$$[c_1 \quad c_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathrm{b}_1 \quad \text{and} \quad [c_1 \quad c_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathrm{b}_2$$

CHANGE OF BASIS IN $\mathbb{R}^n$

- To solve both systems simultaneously, augment the coefficient matrix with b_1 and b_2, and row reduce:

$$[c_1 \;\; c_2 \;\vdots\; b_1 \;\; b_2] = \begin{bmatrix} 1 & 3 & \vdots & -9 & -5 \\ -4 & -5 & \vdots & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \vdots & 6 & 4 \\ 0 & 1 & \vdots & -5 & -3 \end{bmatrix} \qquad (7)$$

- Thus

$$[b_1]_C = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \text{and} \;\; [b_2]_C = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

- The desired change-of-coordinates matrix is therefore

$$\underset{C\leftarrow \beta}{P} = [[b_1]_c \;\; [b_2]_c] = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$