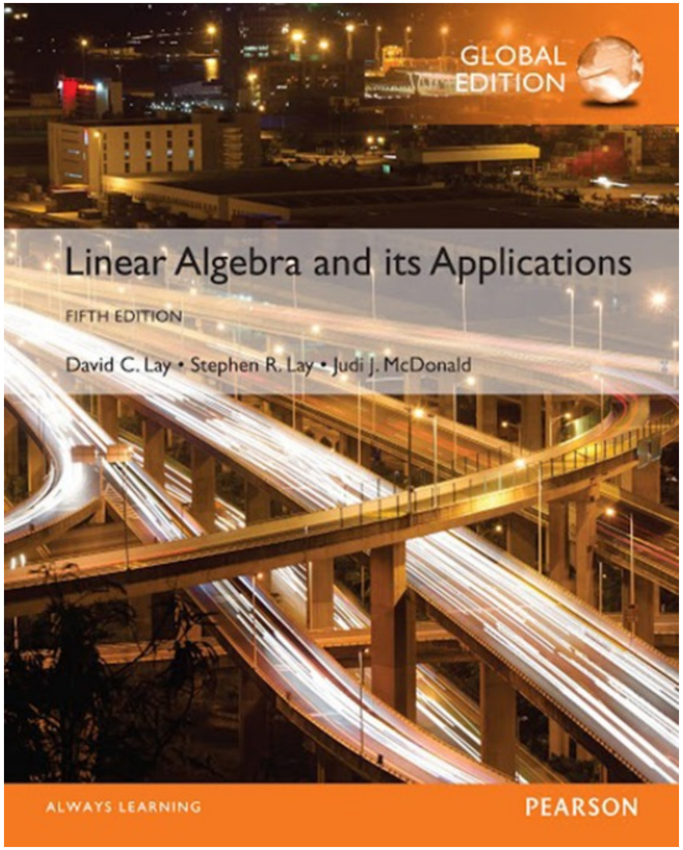
1 Linear Equations in Linear Algebra

1.9

THE MATRIX OF A LINEAR TRANSFORMATION



THE MATRIX OF A LINEAR TRANSFORMATION

- **Theorem 10**: Let T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(x) = Ax \text{ for all x in } \mathbb{R}^n$$

- In fact, A is the m $\times$ n matrix whose j^{th} column is the vector $T(e_j)$, where e_j is the j^{th} column of the identity matrix in $\mathbb{R}^n$

$$A = [T(e_1) \ldots T(e_1)] \qquad (3)$$

THE MATRIX OF A LINEAR TRANSFORMATION

- **Proof**: Write x= $I_n x = [e_1 \ \ldots e_2]x = x_1 e_1 + \ \ldots + x_n e_n$, and use the linearity of T to compute

$$T(x) = T(x_1 e_1 + \cdots + x_n e_n) = x_1 T(e_1) + \cdots + x_n T(e_n)$$

$$= [T(e_1) \quad \ldots \quad T(e_1)] \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = Ax$$

THE MATRIX OF A LINEAR TRANSFORMATION

- The matrix A in (3) is called the **standard matrix for the linear transformation** T.
- We know now that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be viewed as a matrix transformation, and vice versa. The term *linear transformation* focuses on a property of a mapping, while *matrix transformation* describes how such a mapping is implemented, as the example on the next slide illustrates.

THE MATRIX OF A LINEAR TRANSFORMATION

- **Example 2**: Find the standard matrix A for the dilation transformation T(x)= 3x, for x in $\mathbb{R}^2$.
- **Solution**: Write

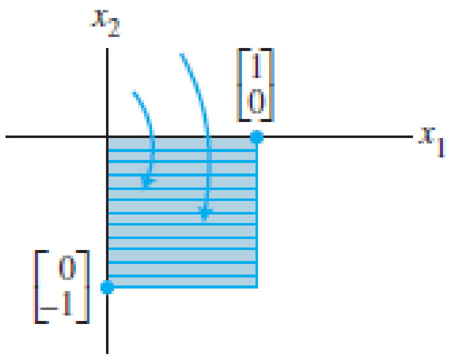
$$T(e_1)= e_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \; and \; T(e_1)= e_1= \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\mathrm{A} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

GEOMETRIC LINEAR TRANSFORMATIONS OF $\mathbb{R}^2$

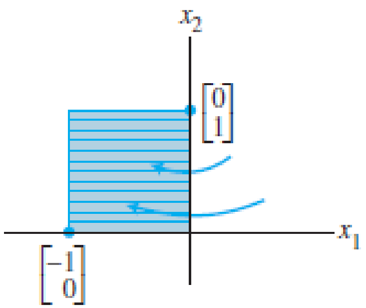
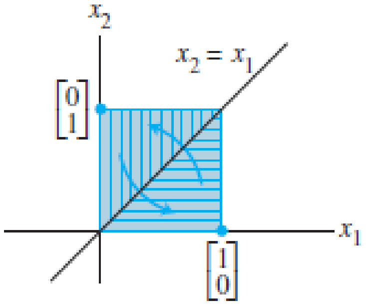
- Tables 1-4 illustrate other common geometric linear transformations of the plane.

TABLE 1 Reflections

Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$

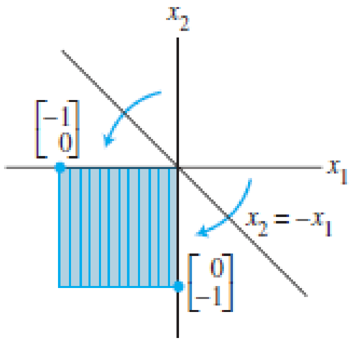
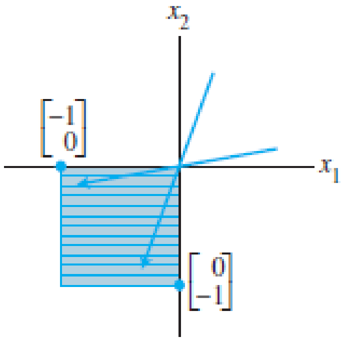
EXISTENCE AND UNIQUENESS QUESTIONS

- Table 1 continued:

Reflection through the x_2-axis		$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

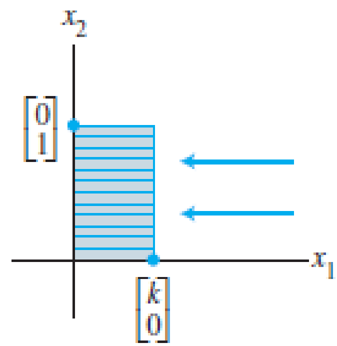
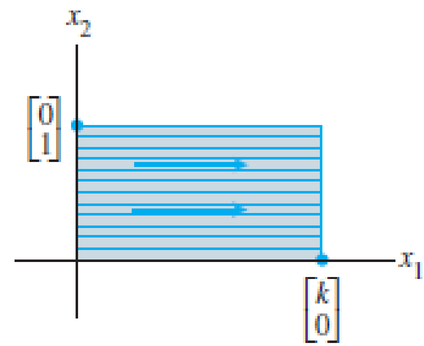
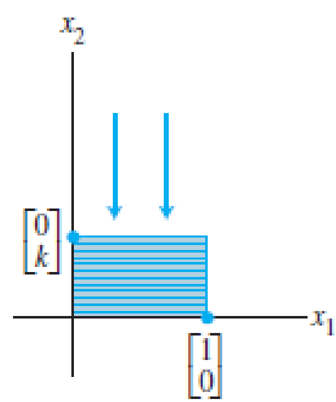
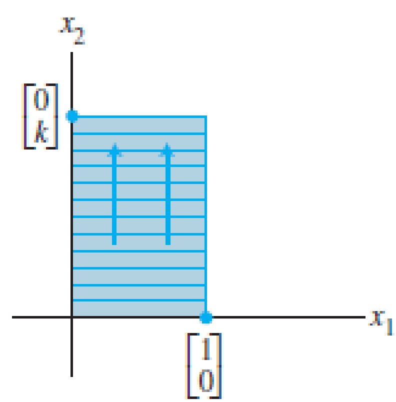
EXISTENCE AND UNIQUENESS QUESTIONS

- Table 1 continued:

Reflection through the line $x_2 = -x_1$		$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$
Reflection through the origin		$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

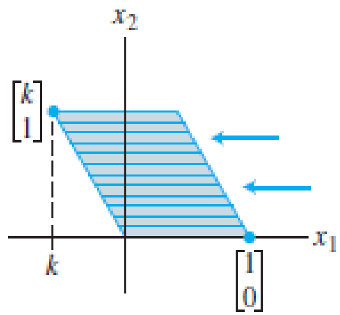
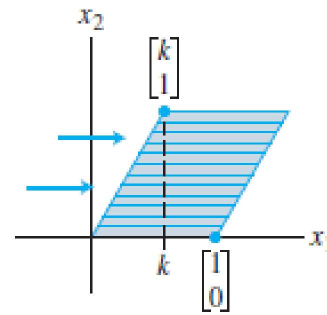
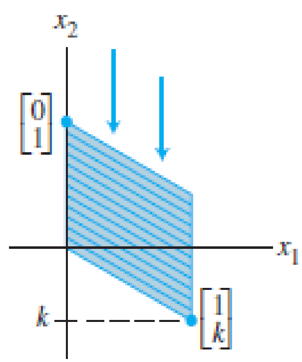
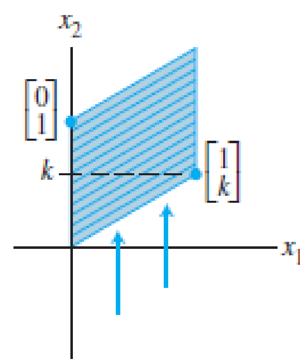
EXISTENCE AND UNIQUENESS QUESTIONS

TABLE 2 Contractions and Expansions

Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$

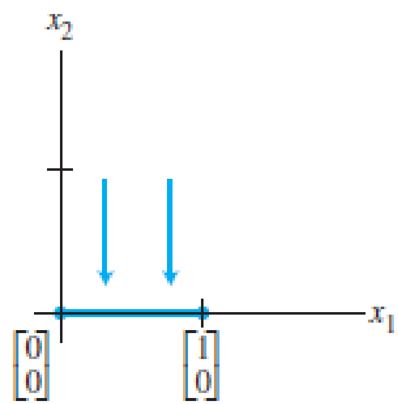
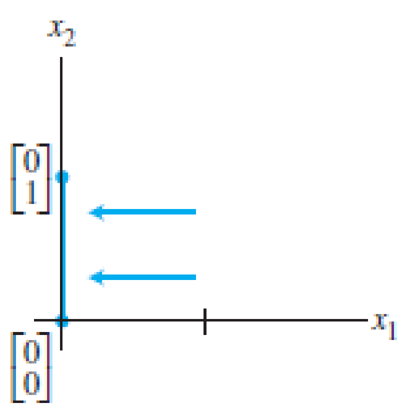
EXISTENCE AND UNIQUENESS QUESTIONS

TABLE 3 **Shears**

Transformation	Image of the Unit Square		Standard Matrix
Horizontal shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

EXISTENCE AND UNIQUENESS QUESTIONS

TABLE 4 Projections

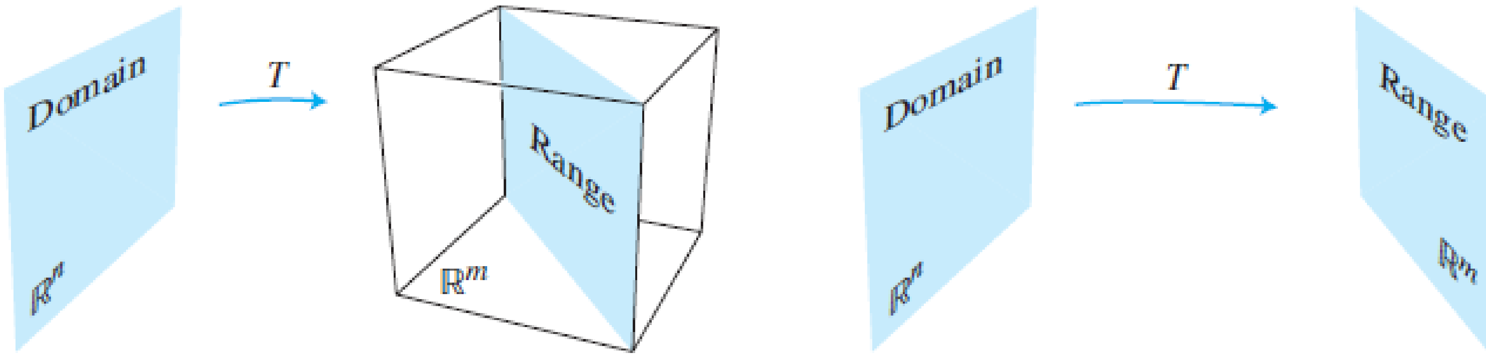
Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

EXISTENCE AND UNIQUENESS QUESTIONS

- **Definition**: A mapping T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^n$ if each **b** in $\mathbb{R}^m$ is the image of *at least one* **x** in $\mathbb{R}^n$.

- Equivalently, T is onto $\mathbb{R}^m$ when the range of *T* is all of the codomain $\mathbb{R}^m$. That is, *T* maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if, for each **b** in the codomain $\mathbb{R}^m$, there exists at least one solution of T(x)= b . "Does T map $\mathbb{R}^n$ onto $\mathbb{R}^m$?" is an existence question. The mapping *T* is not onto when there is some **b** in $\mathbb{R}^m$ for which the equation T(x)=b has no solution. See the figure on the next slide.

EXISTENCE AND UNIQUENESS QUESTIONS



T is *not* onto $\mathbb{R}^m$ T *is* onto $\mathbb{R}^m$

FIGURE 3 Is the range of T all of $\mathbb{R}^m$?

- **Definition**: A mapping T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** if each **b** in $\mathbb{R}^m$ is the image of *at most one* **x** in $\mathbb{R}^n$.

EXISTENCE AND UNIQUENESS QUESTIONS

- **Example 4**: Let T be the linear transformation whose standard matrix is

$$A=\begin{bmatrix}1 & -4 & 8 & 1\\ 0 & 2 & -1 & 3\\ 0 & 0 & 0 & 5\end{bmatrix}$$

- Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

EXISTENCE AND UNIQUENESS QUESTIONS

- **Solution**: Since A happens to be in echelon form, we can see at once that A has a pivot position in each row. By Theorem 4 in Section 1.4, for each **b** in $\mathbb{R}^3$, the equation Ax=b is consistent. In other words, the linear transformation T maps $\mathbb{R}^4$(its domain) onto $\mathbb{R}^3$.
- However, since the equation Ax=b has a free variable (because there are four variables and only three basic variables), each **b** is the image of more than one **x**. This is, T is *not* one-to-one.

EXISTENCE AND UNIQUENESS QUESTIONS

- **Theorem 11**: Let T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation T(x)=0 has only the trivial solution.
- **Proof**: Since *T* is linear, T(0) =0. If *T* is one-to-one, then the equation T(x)=0 has at most one solution and hence only the trivial solution.
- If *T* is not one-to-one, then there is a b that is the image of at least two different vectors in $\mathbb{R}^n$--say, **u** and **v**. That is T(u)=*b* and T(v)=*b*. But then, since *T* is linear,

$$T(u - v) = T(u) - T(v) = b - b = 0$$

EXISTENCE AND UNIQUENESS QUESTIONS

- The vector **u** – **v** is not zero, since u ≠ v. Hence the equation T(x)=0 has more than one solution. So, either the two conditions in the theorem are both true or they are both false.
- **Theorem 12**: Let T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

a) *T* maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of *A* span $\mathbb{R}^m$;

b) *T* is one-to-one if and only if the columns of *A* are linearly independent.

EXISTENCE AND UNIQUENESS QUESTIONS

- **Proof:**

a) By Theorem 4 in Section 1.4, the columns of A span $\mathbb{R}^m$ if and only if for each b in $\mathbb{R}^m$ the equation Ax=b is consistent—in other words, if and only if for every **b**, the equation T(x)=b has at least one solution. This is true if and only if T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

b) The equations T(x)=0 and Ax=0 are the same except for notation. So, by Theorem 11, T is one-to-one if and only if Ax=0 has only the trivial solution. This happens if and only if the columns of A are linearly independent.