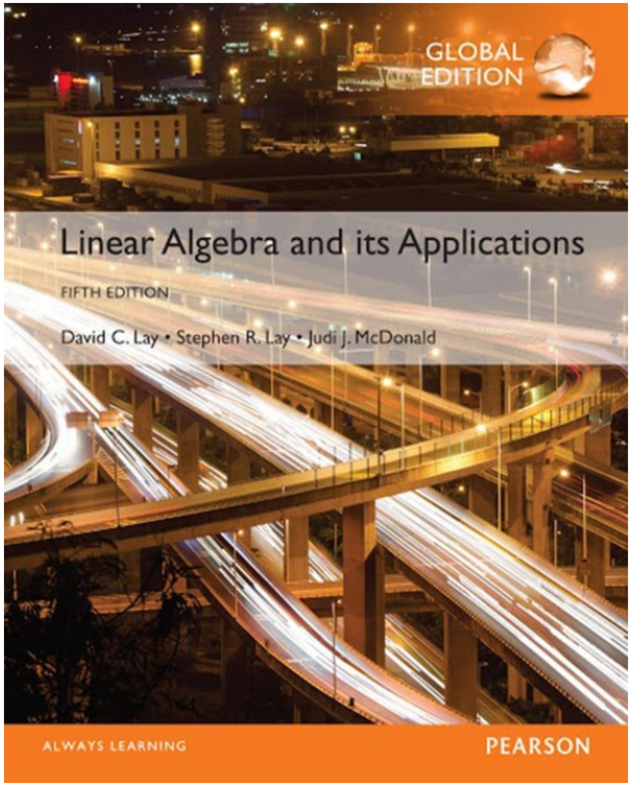
2 Matrix Algebra

2.9

DIMENSION AND RANK



COORDINATE SYSTEMS

- Suppose $\beta = \{b_1, \ldots, bp\}$ is a basis for H, and suppose a vector **x** in H can be generated in two ways, say,

  $x = c_1 b_1 + \cdots + c_p b_p$ and $x = d_1 b_1 + \cdots + d_p b_p$ (1)

- Then, subtracting gives

  $$0 = x - x = (c_1 - d_1)b_1 + \cdots + (c_p - dp)b_p \qquad (2)$$

- Since β is linearly independent, the weights in (2) must all be zero. That is, $c_j = d_j$ for $1 \le j \le p$, which shows that the two representations in (1) are actually the same.

COORDINATE SYSTEMS

- **Definition**: Suppose the set $\beta = \{b_1, \ldots, bp\}$ is a basis for a subspace H. For each x in H, the **coordinates of x relative to the basis**β are the weights $c_1, \ldots, c_p$ such that $x = c_1 b_1 + \cdots + c_p b_p$, and the vector in $\mathbb{R}^p$

$$[x]_\beta = \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_p \end{bmatrix}$$

- is called the **coordinate vector of x (relative to β)** or the **β-coordinate vector of x**.

COORDINATE SYSTEMS

- **Example 1** Let $v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $x = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, and $\beta = \{v_1, v_2\}$. Then β is a basis for H = Span $\{v_1, v_2\}$ because v_1 and v_2 are linearly independent. Determine if x is in *H*, and if it is, find the coordinate vector of x relative to β.
- **Solution** If x is in *H*, then the following vector equation is consistent:

$$c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

COORDINATE SYSTEMS

- The scalars c_1 and c_2, if they exist, are the β-coordinates of $\mathbf{x}$. Row operations show that

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

- Thus $c_1 = 2$, $c_2 = 3$ and $[x]_\beta = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The basis β determines a "coordinate system" on H, which can be visualized by the grid shown in Fig. 1 on the next slide.

COORDINATE SYSTEMS

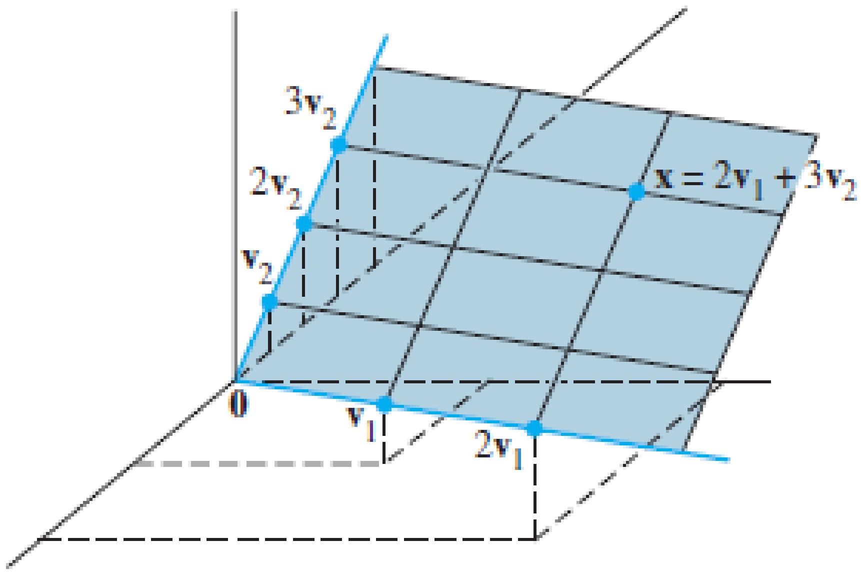


FIGURE 1 A coordinate system on a plane H in $\mathbb{R}^3$.

THE DIMESION OF A SUBSPACE

- **Definition**: The **dimension** of a nonzero subspace H, denoted by dim H, is the number of vectors in any basis for H. The dimension of the zero subspace $\{0\}$ is defined to be zero.

- **Definition:** The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A.

THE DIMESION OF A SUBSPACE

- **Example 3** Determine the rank of the matrix

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

- **Solution** Reduce A to echelon form:

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

- The matrix A has 3 pivot columns, so rank $A = 3$.

THE DIMESION OF A SUBSPACE

- **Theorem 14** If a matrix A has n columns, then rank A + dim Nul$A = n$.

- **Theorem 15** Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

RANK AND THE INVERTIBLE MATRIX THEOREM

- **The Invertible Theorem (continued)** Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. Col $A = \mathbb{R}^n$

o. dim Col $A = n$

p. rank $A = n$

q. Nul$A = \{0\}$

r. dim Nul$A = 0$

RANK AND THE INVERTIBLE MATRIX THEOREM

- **Proof** Statement (m) is logically equivalent to statements (e) and (h) regarding linear independence and spanning. The other five statements are linked to the earlier ones of the theorem by the following chain of almost trivial implications:

$$(g) \Longrightarrow (n) \Rightarrow (o) \Rightarrow (p) \Rightarrow (r) \Rightarrow (q) \Rightarrow (d)$$

- Statement (g), which says that the equation Ax = b has at least one solution for each **b** in $\mathbb{R}^n$, implies statement (n), because Col A is precisely the set of all **b** such that the equation Ax = b is consistent.

RANK AND THE INVERTIBLE MATRIX THEOREM

- The implications $(n) \Rightarrow (o) \Rightarrow (p)$ follow from the definitions of *dimension* and *rank*.
- If the rank of A is n, the number of columns of A, then dim Nul$A$ = 0, by the Rank Theorem, and so Nul$A$ $\{0\}$. Thus $(p) \Rightarrow (r) \Rightarrow (q)$.
- Also, statement (q) implies that the equation Ax = 0 has only the trivial solution, which is statement (d).
- Since statements (d) and (g) are already known to be equivalent to the statement that A is invertible, the proof is complete.