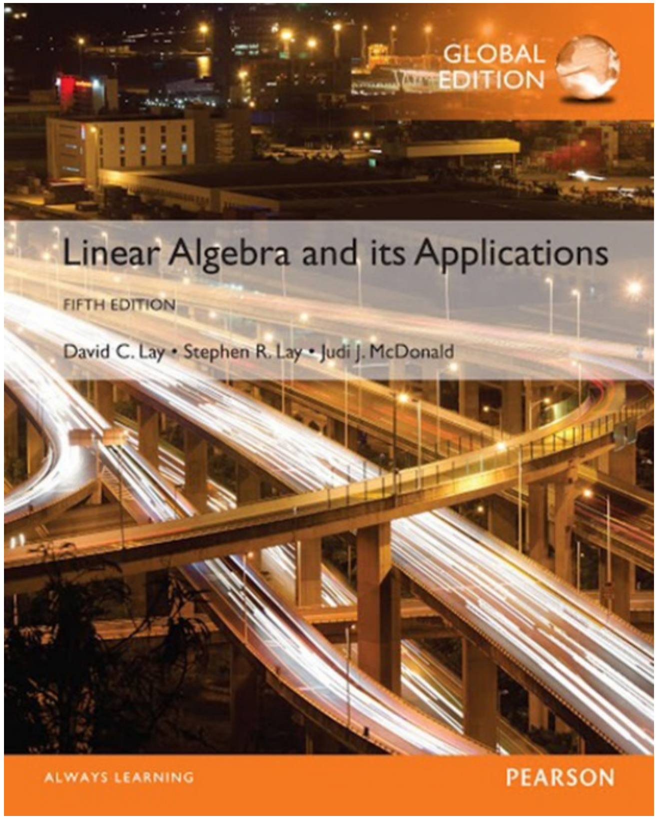
4 Vector Spaces

4.2

NULL SPACES, COLUMN SPACES, AND LINEAR TRANSFORMATIONS



NULL SPACE OF A MATRIX

- **Definition:** The **null space** of an $m \times n$ matrix A, written as Nul$A$, is the set of all solutions of the homogeneous equation $A\mathrm{x} = 0$. In set notation,

$$\text{Nul } A = \{\mathrm{x} : \mathrm{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathrm{x} = 0\}.$$

- **Theorem 2:** The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathrm{x} = 0$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.
- **Proof:** Nul$A$ is a subset of $\mathbb{R}^n$ because A has n columns.
- We need to show that Nul$A$ satisfies the three properties of a subspace.

NULL SPACE OF A MATRIX

- **0** is in Null A.
- Next, let **u** and **v** represent any two vectors in Nul A.
- Then

$$A\mathrm{u} = 0 \text{ and } A\mathrm{v} = 0$$

- To show that $\mathrm{u} + \mathrm{v}$ is in Nul A, we must show that $A(\mathrm{u} + \mathrm{v}) = 0$.
- Using a property of matrix multiplication, compute

$$A(\mathrm{u} + \mathrm{v}) = A\mathrm{u} + A\mathrm{v} = 0 + 0 = 0$$

- Thus $\mathrm{u} + \mathrm{v}$ is in Nul A, and Nul A is closed under vector addition.

NULL SPACE OF A MATRIX

- Finally, if c is any scalar, then
$$A(c\mathrm{u}) = c(A\mathrm{u}) = c(0) = 0$$
which shows that $c\mathbf{u}$ is in Nul$A$.
- Thus Nul$A$ is a subspace of $\mathbb{R}^n$.

- **An Explicit Description of Nul*A***
- There is no obvious relation between vectors in Nul$A$ and the entries in A.
- We say that Nul$A$ is defined *implicitly*, because it is defined by a condition that must be checked.

NULL SPACE OF A MATRIX

- No explicit list or description of the elements in Nul A is given.
- *Solving* the equation $A\mathbf{x} = \mathbf{0}$ amounts to producing an *explicit* description of Nul A.

- **Example 3:** Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

NULL SPACE OF A MATRIX

- **Solution:** The first step is to find the general solution of $A\mathrm{x} = 0$ in terms of free variables.

- Row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to *reduce* echelon form in order to write the basic variables in terms of the free variables:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} x_1 - 2x_2 - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

NULL SPACE OF A MATRIX

- The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$, with x_2, x_4, and x_5 free.
- Next, decompose the vector giving the general solution into a linear combination of *vectors where the weights are the free variables*. That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{w}}$$

NULL SPACE OF A MATRIX

$$= x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w} \qquad (3)$$

- Every linear combination of **u**, **v**, and **w** is an element of Nul A.
- Thus {**u**, **v**, **w**} is a spanning set for Nul A.

1. The spanning set produced by the method in Example (3) is automatically linearly independent because the free variables are the weights on the spanning vectors.
2. When Nul A contains nonzero vectors, the number of vectors in the spanning set for Nul A equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

COLUMN SPACE OF A MATRIX

- **Definition:** The column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$, then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

- **Theorem 3:** The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.
- A typical vector in Col A can be written as $A\mathbf{x}$ for some $\mathbf{x}$ because the notation $A\mathbf{x}$ stands for a linear combination of the columns of A. That is,

$$\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}.$$

COLUMN SPACE OF A MATRIX

- The notation $A\mathbf{x}$ for vectors in Col A also shows that Col A is the *range* of the linear transformation $\mathrm{x} \mapsto A\mathrm{x}$.
- The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathrm{x} = \mathrm{b}$ has a solution for each **b** in $\mathbb{R}^m$.
- **Example 7:** Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $\mathrm{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathrm{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

COLUMN SPACE OF A MATRIX

a. Determine if **u** is in Nul A. Could **u** be in Col A?
b. Determine if **v** is in Col A. Could **v** be in Nul A?

- **Solution:**

a. An explicit description of Nul A is not needed here. Simply compute the product $A\mathbf{u}$.

$$A\mathrm{u} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

COLUMN SPACE OF A MATRIX

- $\mathbf{u}$ is *not* a solution of $A\mathrm{x} = 0$, so $\mathbf{u}$ is not in Nul$A$.
- Also, with four entries, $\mathbf{u}$ could not possibly be in Col A, since Col A is a subspace of $\mathbb{R}^3$.

b. Reduce $\begin{bmatrix} A & \mathrm{v} \end{bmatrix}$ to an echelon form.

$$\begin{bmatrix} A & \mathrm{v} \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

- The equation $A\mathrm{x} = \mathrm{v}$ is consistent, so $\mathbf{v}$ is in Col A.

  - With only three entries, **v** could not possibly be in Nul$A$, since Nul$A$ is a subspace of $\mathbb{R}^4$.
- Subspaces of vector spaces other than $\mathbb{R}^n$ are often described in terms of a linear transformation instead of a matrix.
- **Definition:** A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector **x** in V a unique vector $T(\mathbf{x})$ in W, such that

   i. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all **u**, **v** in V, and

   ii. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all **u** in V and all scalars c.

KERNEL AND RANGE OF A LINEAR TRANSFORMATION

- The **kernel** (or **null space**) of such a T is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = 0$ (the zero vector in W).
- The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.
- The kernel of T is a subspace of V.
- The range of T is a subspace of W.

CONTRAST BETWEEN NUL *A* AND COL *A* FOR AN $m \times n$ MATRIX *A*

Nul$A$	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	1. Col A is a subspace of $\mathbb{R}^m$.
2. Nul$A$ is implicitly defined; *i.e.*, you are given only a condition $(A\mathrm{x} = 0)$ that vectors in Nul$A$ must satisfy.	2. Col A is explicitly defined; *i.e.*, you are told how to build vectors in Col A.

CONTRAST BETWEEN NUL *A* AND COL *A* FOR AN $m \times n$ MATRIX *A*

3. It takes time to find vectors in Nul A. Row operations on $\begin{bmatrix} A & 0 \end{bmatrix}$ are required.	3. It is easy to find vectors in Col A. The columns of a are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	4. There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.

CONTRAST BETWEEN NUL A AND COL A FOR AN $m \times n$ MATRIX A

5. A typical vector **v** in Nul A has the property that $A\mathrm{v} = 0$.	5. A typical vector **v** in Col A has the property that the equation $A\mathrm{x} = \mathrm{v}$ is consistent.
6. Given a specific vector **v**, it is easy to tell if **v** is in Nul A. Just compare $A\mathbf{v}$.	6. Given a specific vector **v**, it may take time to tell if v is in Col A. Row operations on $\begin{bmatrix} A & \mathrm{v} \end{bmatrix}$ are required.

CONTRAST BETWEEN NUL A AND COL A FOR AN $m \times n$ MATRIX A

7. Nul $A = \{0\}$ if and only if the equation $A\mathbf{x} = 0$ has only the trivial solution.	7. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{0\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	8. Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

