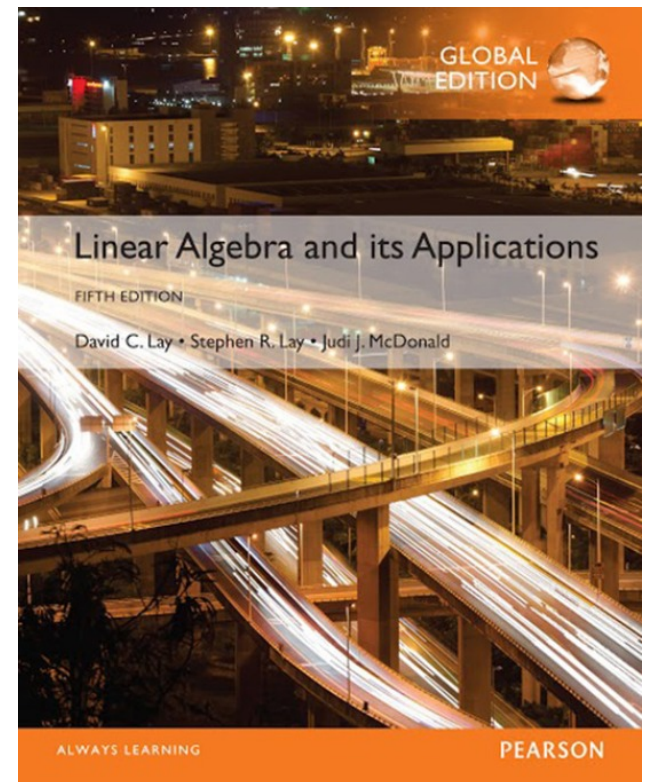


# 5

## Eigenvalues and Eigenvectors

### 5.6

### DISCRETE DYNAMICAL SYSTEMS



# GRAPHICAL DESCRIPTION OF SOLUTIONS

- When  $A$  is  $2 \times 2$ , algebraic calculations can be supplemented by a geometric description of a system's evolution.
- We can view the equation  $x_{k+1} = Ax_k$  as a description of what happens to an initial point  $x_0$  in  $\mathbb{R}^2$  as it is transformed repeatedly by the mapping  $x \mapsto Ax$
- The graph of  $x_0, x_1, \dots$  is called a **trajectory** of the dynamical system.

# GRAPHICAL DESCRIPTION OF SOLUTIONS

- **Example 2** Plot several trajectories of the dynamical system  $x_{k+1} = Ax_k$  when

$$A = \begin{bmatrix} .80 & 0 \\ 0 & .64 \end{bmatrix}$$

- **Solution** The eigenvalues of  $A$  are .8 and .64, with eigenvectors  $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . If  $x_0 = c_1 v_1 + c_2 v_2$ , then

$$x_k = c_1 (.8)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 (.64)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Of course,  $x_k$  tends to 0 because  $(.8)^k$  and  $(.64)^k$  both approach 0 as  $k \rightarrow \infty$ . But the way  $x_k$  goes toward 0 is interesting. See Fig. 1 on the next slide.

# GRAPHICAL DESCRIPTION OF SOLUTIONS

Figure 1 shows the first few terms of several trajectories that begin at points on the boundary of the box with corners at  $(\pm 3, \pm 3)$ . The points on each trajectory are connected by a thin curve, to make the trajectory easier to see.

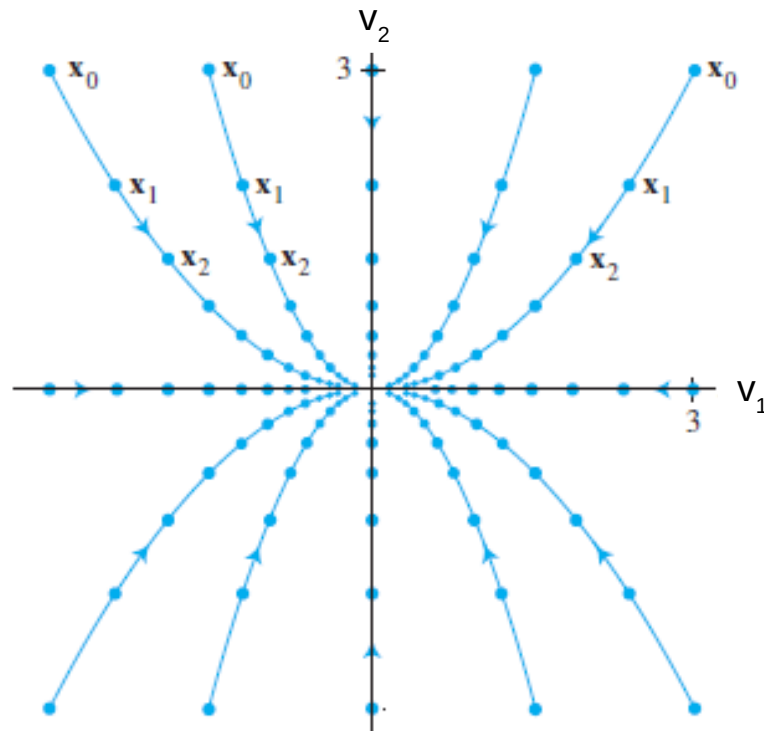


FIGURE 1 The origin as an attractor.

# GRAPHICAL DESCRIPTION OF SOLUTIONS

- In example 2, the origin is called an **attractor** of the dynamical system because all trajectories tend toward 0.
- In the next example, both eigenvalues of  $A$  are larger than 1 in magnitude, and 0 is called a **repeller** of the dynamical system.
- **Example 3** Plot several typical solutions of the equation  $\mathbf{x}_{k+1} = A\mathbf{x}_k$ , where

$$A = \begin{bmatrix} 1.44 & 0 \\ 0 & 1.2 \end{bmatrix}$$

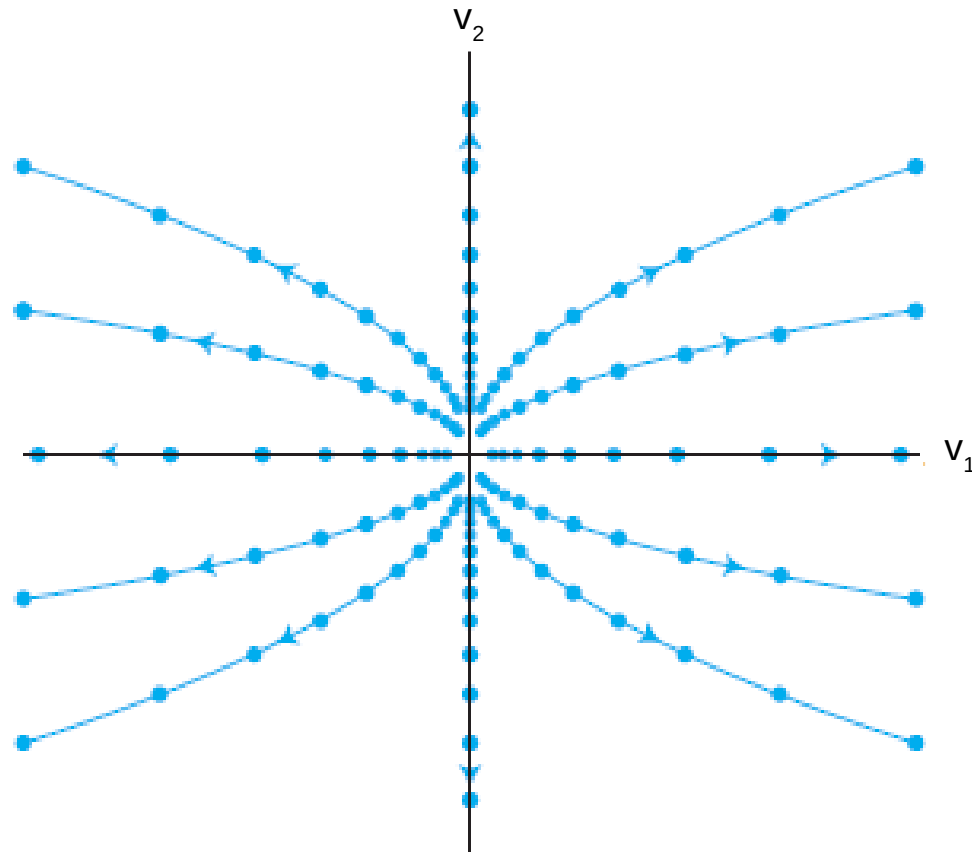
# GRAPHICAL DESCRIPTION OF SOLUTIONS

- **Solution** The eigenvalues of  $A$  are 1.44 and 1.2. If  $x_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ , then

$$x_k = c_1(1.44)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(1.2)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Both terms grow in size, but the first term grows faster.
- So the direction of greatest repulsion is the line through 0 and the eigenvector for the eigenvalue of larger magnitude.
- Fig. 2 on the next slide shows several trajectories that begin at points quite close to 0.

# GRAPHICAL DESCRIPTION OF SOLUTIONS



**FIGURE 2** The origin as a repeller.