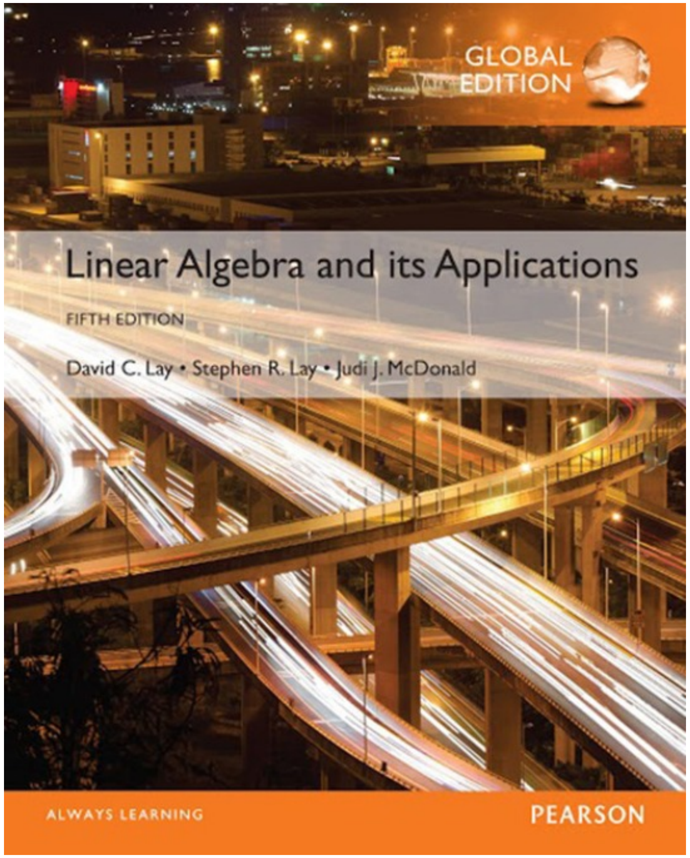
4 Vector Spaces

4.6

RANK



PEARSON

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THE ROW SPACE

- If A is an $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$.
- The set of all linear combinations of the row vectors is called the **row space** of A and is denoted by Row A.
- Each row has n entries, so Row A is a subspace of $\mathbb{R}^n$.
- Since the rows of A are identified with the columns of A^T, we could also write Col A^T in place of Row A.

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THE ROW SPACE

- **Theorem 13:** If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

- **Proof:** If B is obtained from A by row operations, the rows of B are linear combinations of the rows of A.

- It follows that any linear combination of the rows of B is automatically a linear combination of the rows of A.

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THE ROW SPACE

- Thus the row space of B is contained in the row space of A.

- Since row operations are reversible, the same argument shows that the row space of A is a subset of the row space of B.

- So the two row spaces are the same.

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THE ROW SPACE

- If B is in echelon form, its nonzero rows are linearly independent because no nonzero row is a linear combination of the nonzero rows below it. (Apply Theorem 4 to the nonzero rows of B in reverse order, with the first row last).

- Thus the nonzero rows of B form a basis of the (common) row space of B and A.

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THE ROW SPACE

- **Example 2:** Find bases for the row space, the column space, and the null space of the matrix

$$\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

- **Solution:** To find bases for the row space and the column space, row reduce A to an echelon form:

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THE ROW SPACE

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- By Theorem 13, the first three rows of B form a basis for the row space of A (as well as for the row space of B).

- Thus

  Basis for Row A: $\{(1,3,-5,1,5),(0,1,-2,2,-7),(0,0,0,-4,20)\}$

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THE ROW SPACE

- For the column space, observe from B that the pivots are in columns 1, 2, and 4.
- Hence columns 1, 2, and 4 of A (not B) form a basis for Col A:

$$\text{Basis for Col } A : \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

- Notice that any echelon form of A provides (in its nonzero rows) a basis for Row A and also identifies the pivot columns of A for Col A.

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THE ROW SPACE

- However, for Nul A, we need the *reduced echelon form*.

- Further row operations on B yield

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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THE ROW SPACE

- The equation $A\mathrm{x} = 0$ is equivalent to $C\mathrm{x} = 0$, that is,

$$\begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 - 2x_3 + 3x_5 &= 0 \\ x_4 - 5x_5 &= 0 \end{aligned}$$

- So $x_1 = -x_3 - x_5$, $x_2 = 2x_3 - 3x_5$, $x_4 = 5x_5$, with x_3 and x_5 free variables.

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THE ROW SPACE

- The calculations show that

$$\text{Basis for Nul } A: \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

- Observe that, unlike the basis for Col A, the bases for Row A and Nul A have no simple connection with the entries in A itself.

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THE RANK THEOREM

- **Definition:** The **rank** of A is the dimension of the column space of A.
- Since Row A is the same as Col A^T, the dimension of the row space of A is the rank of A^T.
- The dimension of the null space is sometimes called the **nullity** of A.
- **Theorem 14:** The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n$$

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THE RANK THEOREM

- **Proof:** By Theorem 6, rank A is the number of pivot columns in A.
- Equivalently, rank A is the number of pivot positions in an echelon form B of A.
- Since B has a nonzero row for each pivot, and since these rows form a basis for the row space of A, the rank of A is also the dimension of the row space.
- The dimension of Nul A equals the number of free variables in the equation $A\mathbf{x} = 0$.
- Expressed another way, the dimension of Nul A is the number of columns of A that are *not* pivot columns.

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THE RANK THEOREM

- (It is the number of these columns, not the columns themselves, that is related to Nul A).

- Obviously,

$$\left\{\begin{array}{c}\text{number of}\\ \text{pivot columns}\end{array}\right\}+\left\{\begin{array}{c}\text{number of}\\ \text{nonpivot columns}\end{array}\right\}=\left\{\begin{array}{c}\text{number of}\\ \text{columns}\end{array}\right\}$$

- This proves the theorem.

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THE RANK THEOREM

- **Example 3:**
  a. If A is a 7×9 matrix with a two-dimensional null space, what is the rank of A?
  b. Could a 6×9 matrix have a two-dimensional null space?
- **Solution:**
  a. Since A has 9 columns, $(\text{rank } A) + 2 = 9$, and hence rank $A = 7$.
  b. No. If a 6×9 matrix, call it B, has a two-dimensional null space, it would have to have rank 7, by the Rank Theorem.

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THE INVERTIBLE MATRIX THEOREM (CONTINUED)

- But the columns of B are vectors in $\mathbb{R}^6$, and so the dimension of Col B cannot exceed 6; that is, rank B cannot exceed 6.

- **Theorem:** Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

  m. The columns of A form a basis of $\mathbb{R}^n$.

  n. Col $A = \mathbb{R}^n$

  o. Dim Col $A = n$

  p. rank $A = n$

RANK AND THE INVERTIBLE MATRIX THEOREM

q. $\text{Nul } A = \{0\}$

r. $\text{Dim Nul } A = 0$

- **Proof:** Statement (m) is logically equivalent to statements (e) and (h) regarding linear independence and spanning.

- The other five statements are linked to the earlier ones of the theorem by the following chain of almost trivial implications:

$$(g) \Rightarrow (n) \Rightarrow (o) \Rightarrow (p) \Rightarrow (r) \Rightarrow (q) \Rightarrow (d)$$

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RANK AND THE INVERTIBLE MATRIX THEOREM

- Statement (g), which says that the equation $A\mathrm{x} = \mathrm{b}$ has at least one solution for each **b** in $\mathbb{R}^n$, implies (n), because Col A is precisely the set of all **b** such that the equation $A\mathrm{x} = \mathrm{b}$ is consistent.

- The implications $(\mathrm{n}) \Rightarrow (\mathrm{o}) \Rightarrow (\mathrm{p})$ follow from the definitions of dimension and rank.

- If the rank of A is n, the number of columns of A, then dim Nul $A = 0$, by the Rank Theorem, and so Nul $A = \{0\}$.

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RANK AND THE INVERTIBLE MATRIX THEOREM

- Thus (p) $\Rightarrow$ (r) $\Rightarrow$ (q).
- Also, (q) implies that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, which is statement (d).
- Since statements (d) and (g) are already known to be equivalent to the statement that A is invertible, the proof is complete.

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