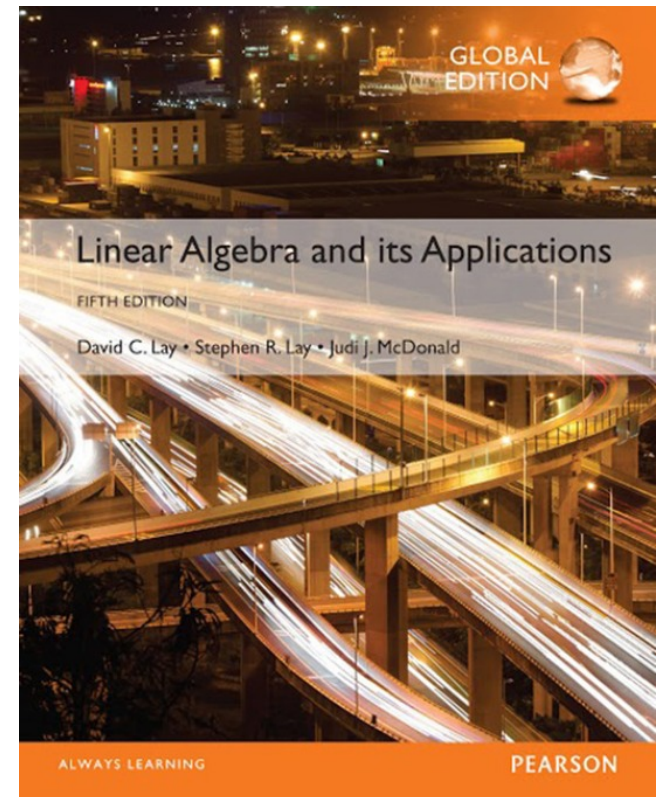


4

Vector Spaces

4.4

COORDINATE SYSTEMS



THE UNIQUE REPRESENTATION THEOREM

- **Theorem 7:** Let $B = \{b_1, \dots, b_n\}$ be a basis for vector space V . Then for each \mathbf{x} in V , there exists a unique set of scalars c_1, \dots, c_n such that

$$\mathbf{x} = c_1 b_1 + \dots + c_n b_n \quad (1)$$

- **Proof:** Since B spans V , there exist scalars such that (1) holds.
- Suppose \mathbf{x} also has the representation

$$\mathbf{x} = d_1 b_1 + \dots + d_n b_n$$

for scalars d_1, \dots, d_n .

THE UNIQUE REPRESENTATION THEOREM

- Then, subtracting, we have

$$0 = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \dots + (c_n - d_n)\mathbf{b}_n \quad (2)$$

- Since B is linearly independent, the weights in (2) must all be zero. That is, $c_j = d_j$ for $1 \leq j \leq n$.
- **Definition:** Suppose $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V and \mathbf{x} is in V . **The coordinates of \mathbf{x} relative to the basis B (or the B -coordinate of \mathbf{x})** are the weights c_1, \dots, c_n such that $\mathbf{x} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n$

THE UNIQUE REPRESENTATION THEOREM

- If c_1, \dots, c_n are the **B**-coordinates of \mathbf{x} , then the vector in \mathbb{R}^n

$$[\mathbf{x}]_{\mathbf{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **coordinate vector of \mathbf{x} (relative to \mathbf{B})**, or the **B-coordinate vector of \mathbf{x}** .

- The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathbf{B}}$ is the **coordinate mapping (determined by \mathbf{B})**.

COORDINATES IN \mathbb{R}^n

- When a basis B for \mathbb{R}^n is fixed, the B -coordinate vector of a specified \mathbf{x} is easily found, as in the example below.

- **Example 1:** Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and

$B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_B$ of \mathbf{x} relative to B .

- **Solution:** The B -coordinate c_1, c_2 of \mathbf{x} satisfy

$$c_1 \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\mathbf{b}_1} + c_2 \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\mathbf{b}_2} = \underbrace{\begin{bmatrix} 4 \\ 5 \end{bmatrix}}_{\mathbf{x}}$$

COORDINATES IN \mathbb{R}^n

or

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad (3)$$

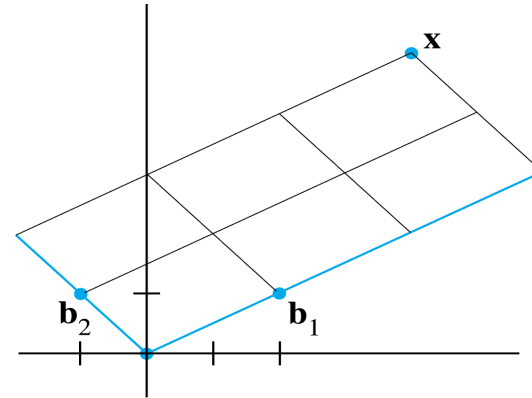
b_1 b_2 x

- This equation can be solved by row operations on an augmented matrix or by using the inverse of the matrix on the left.
- In any case, the solution is $c_1 = 3, c_2 = 2$.
- Thus $x = 3b_1 + 2b_2$ and

$$[x]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

COORDINATES IN \mathbb{R}^n

- See the following figure.



The \mathcal{B} -coordinate vector of \mathbf{x} is $(3, 2)$.

- The matrix in (3) changes the \mathcal{B} -coordinates of a vector \mathbf{x} into the standard coordinates for \mathbf{x} .
- An analogous change of coordinates can be carried out in \mathbb{R}^n for a basis $\mathcal{B} = \{b_1, \dots, b_n\}$.
- Let $P_{\mathcal{B}} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$

COORDINATES IN \mathbb{R}^n

- Then the vector equation

$$\mathbf{x} = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \dots + c_n \mathbf{b}_n$$

is equivalent to

(4)
$$\mathbf{x} = P_B [\mathbf{x}]_B$$

- P_B is called the **change-of-coordinates matrix** from B to the standard basis in \mathbb{R}^n .
- Left-multiplication by P_B transforms the coordinate vector $[\mathbf{x}]_B$ into \mathbf{x} .
- Since the columns of P_B form a basis for \mathbb{R}^n , P_B is invertible (by the Invertible Matrix Theorem).

COORDINATES IN \mathbb{R}^n

- Left-multiplication by P_B^{-1} converts \mathbf{x} into its B-coordinate vector:

$$P_B^{-1} \mathbf{x} = [\mathbf{x}]_B$$

- The correspondence $\mathbf{x} \mapsto [\mathbf{x}]_B$, produced by P_B^{-1} , is the coordinate mapping.
- Since P_B^{-1} is an invertible matrix, the coordinate mapping is a one-to-one linear transformation from \mathbb{R}^n onto \mathbb{R}^n , by the Invertible Matrix Theorem.

THE COORDINATE MAPPING

- **Theorem 8:** Let $B = \{b_1, \dots, b_n\}$ be a basis for a vector space V . Then the coordinate mapping $x \mapsto [x]_B$ is a one-to-one linear transformation from V onto \mathbb{R}^n .

- **Proof:** Take two typical vectors in V , say,

$$u = c_1 b_1 + \dots + c_n b_n$$
$$w = d_1 b_1 + \dots + d_n b_n$$

- Then, using vector operations,
 $u + w = (c_1 + d_1)b_1 + \dots + (c_n + d_n)b_n$

THE COORDINATE MAPPING

- It follows that

$$[\mathbf{u} + \mathbf{w}]_B = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\mathbf{u}]_B + [\mathbf{w}]_B$$

- So the coordinate mapping preserves addition.
- If r is any scalar, then

$$r\mathbf{u} = r(c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n) = (rc_1)\mathbf{b}_1 + \dots + (rc_n)\mathbf{b}_n$$

THE COORDINATE MAPPING

- So

$$[r\mathbf{u}]_B = \begin{bmatrix} r c_1 \\ \vdots \\ r c_n \end{bmatrix} = r \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = r [\mathbf{u}]_B$$

- Thus the coordinate mapping also preserves scalar multiplication and hence is a linear transformation.
- The linearity of the coordinate mapping extends to linear combinations.
- If $\mathbf{u}_1, \dots, \mathbf{u}_p$ are in V and if c_1, \dots, c_p are scalars, then
$$[c_1 \mathbf{u}_1 + \dots + c_p \mathbf{u}_p]_B = c_1 [\mathbf{u}_1]_B + \dots + c_p [\mathbf{u}_p]_B \quad (5)$$

THE COORDINATE MAPPING

- In words, (5) says that the B-coordinate vector of a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_p$ is the *same* linear combination of their coordinate vectors.
- The coordinate mapping in Theorem 8 is an important example of an *isomorphism* from V onto \mathbb{R}^n .
- In general, a one-to-one linear transformation from a vector space V onto a vector space W is called an **isomorphism** from V onto W .
- The notation and terminology for V and W may differ, but the two spaces are indistinguishable as vector spaces.

THE COORDINATE MAPPING

- *Every vector space calculation in V is accurately reproduced in W , and vice versa.*
- In particular, any real vector space with a basis of n vectors is indistinguishable from \mathbb{R}^n .

- **Example 7:** Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$,

and $B = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then B is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if \mathbf{x} is in H , and if it is, find the coordinate vector of \mathbf{x} relative to B .

THE COORDINATE MAPPING

- **Solution:** If \mathbf{x} is in H , then the following vector equation is consistent:

$$c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

- The scalars c_1 and c_2 , if they exist, are the B-coordinates of \mathbf{x} .

THE COORDINATE MAPPING

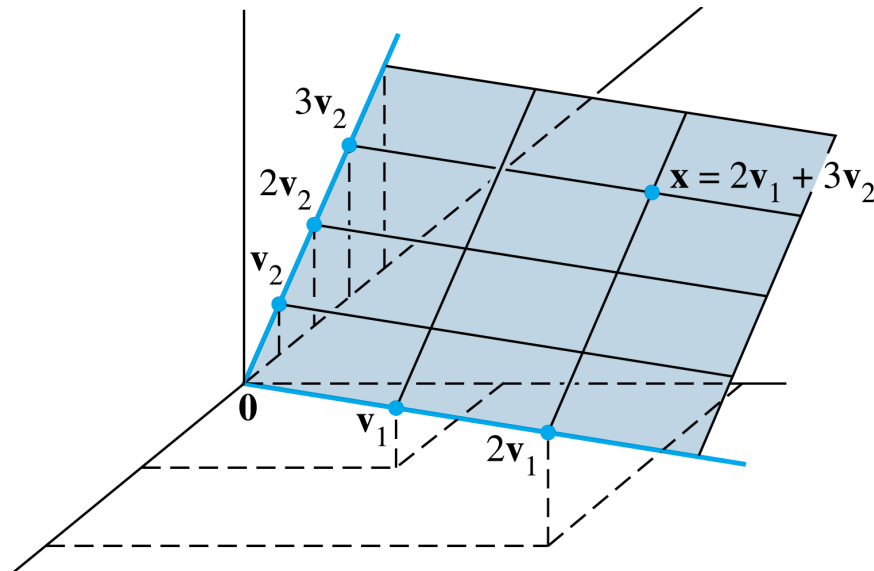
- Using row operations, we obtain

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

- Thus $c_1 = 2$, $c_2 = 3$ and $[\mathbf{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

THE COORDINATE MAPPING

- The coordinate system on H determined by B is shown in the following figure.



A coordinate system on a plane H in \mathbb{R}^3 .