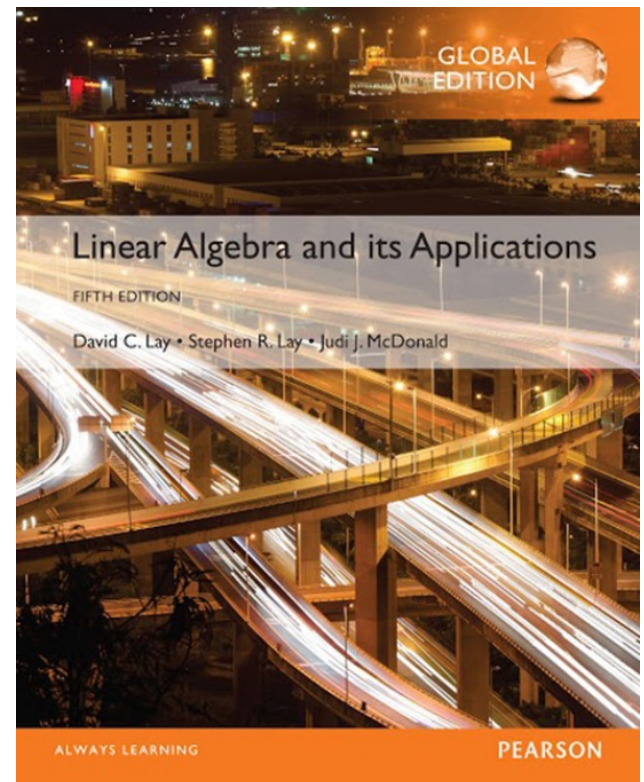


# 5

# Eigenvalues and Eigenvectors

## 5.5

## COMPLEX EIGENVALUES



# COMPLEX EIGENVALUES

- The matrix eigenvalue-eigenvector theory already developed for  $\mathbb{R}^n$  applies equally well to  $\mathbb{C}^n$ .
- So a complex scalar  $\lambda$  satisfied  $\det(A - \lambda I) = 0$  if and only if there is a nonzero vector  $\mathbf{x}$  in  $\mathbb{C}^n$  such that  $A\mathbf{x} = \lambda\mathbf{x}$ .
- We call  $\lambda$  a **(complex) eigenvalue** and  $\mathbf{x}$  a **(complex) eigenvector** corresponding to  $\lambda$ .

# COMPLEX EIGENVALUES

- **Example 1** If  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ , then the linear transformation  $x \mapsto Ax$  on  $\mathbb{R}^2$  rotates the plane counterclockwise through a quarter-turn.
- The action of  $A$  is periodic, since after four quarter-turns, a vector is back where it started.
- Obviously, no nonzero vector is mapped into a multiple of itself, so  $A$  has no eigenvectors in  $\mathbb{R}^2$  and hence no real eigenvalues.
- In fact, the characteristic equation of  $A$  is

$$\lambda^2 + 1 = 0$$

# COMPLEX EIGENVALUES

- The only roots are complex:  $\lambda = i$  and  $\lambda = -i$ . However, if we permit  $A$  to act on  $\mathbb{C}^2$ , then

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix}$$
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$$

- Thus  $i$  and  $-i$  are eigenvalues, with  $\begin{bmatrix} 1 \\ -i \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ i \end{bmatrix}$  as corresponding eigenvectors.

# REAL AND IMAGINARY PARTS OF VECTORS

- The complex conjugate of a complex vector  $x$  in  $\mathbb{C}^n$  is the vector  $\bar{x}$  in  $\mathbb{C}^n$  whose entries are the complex conjugates of the entries in  $x$ .
- The **real** and **imaginary parts** of a complex vector  $x$  are the vectors  $\text{Re } x$  and  $\text{Im } x$  in  $\mathbb{R}^n$  formed from the real and imaginary parts of the entries of  $x$ .

# REAL AND IMAGINARY PARTS OF VECTORS

■ **Example 4** If  $x = \begin{bmatrix} 3 - i \\ i \\ 2 + 5i \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + i \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$ , then

$$\operatorname{Re} x = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \quad \operatorname{Im} x = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}, \quad \text{and } \bar{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} - i \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 + i \\ -i \\ 2 - 5i \end{bmatrix}$$

# EIGENVALUES AND EIGENVECTORS OF A REAL MATRIX THAT ACTS ON

- **Theorem 9:** Let  $A$  be a real  $2 \times 2$  matrix with a complex eigenvalue  $\lambda = a - bi$  ( $b \neq 0$ ) and an associated eigenvector  $\mathbf{v}$  in  $\mathbb{C}^2$ . Then

$$A = PCP^{-1}, \text{ where } P = [\operatorname{Re} \mathbf{v} \quad \operatorname{Im} \mathbf{v}] \text{ and } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

# PROOF

Let  $\boldsymbol{v}$  be a vector in  $\mathbb{C}^n$ , and  $A$  be a real  $n \times n$  matrix. First, we show that  $\operatorname{Re}(A\boldsymbol{v}) = A \operatorname{Re}(\boldsymbol{v})$  and  $\operatorname{Im}(A\boldsymbol{v}) = A \operatorname{Im}(\boldsymbol{v})$ .

- ▶ We have:  $\boldsymbol{v} = \operatorname{Re}(\boldsymbol{v}) + \operatorname{Im}(\boldsymbol{v}) i$ .
- ▶ So  $A\boldsymbol{v} = A \operatorname{Re}(\boldsymbol{v}) + A \operatorname{Im}(\boldsymbol{v}) i$ .
- ▶ Since  $A$  is real, so are  $A \operatorname{Re}(\boldsymbol{v})$  and  $A \operatorname{Im}(\boldsymbol{v})$ .
- ▶ Thus  $A \operatorname{Re}(\boldsymbol{v})$  is the real part of  $A\boldsymbol{v}$  and  $A \operatorname{Im}(\boldsymbol{v})$  is the imaginary part of  $A\boldsymbol{v}$ .



## PROOF

Let  $A$  be a real  $n \times n$  matrix, and  $v$  be a complex eigenvector of it ( $Re(v), Im(v) \neq 0$ ). Second, we show that  $Re(v)$  and  $Im(v)$  are linearly independent.

- ▶ Let  $\bar{v}$  be the complex conjugate of  $v$ . It is easy to see that  $v$  and  $\bar{v}$  are linearly independent!
- ▶ So, the following has only trivial solution:

$$c_1(Re(v) + Im(v) i) + c_2(Re(\bar{v}) + Im(\bar{v}) i) = 0 \quad (1)$$

- ▶ Eq. 1 can be simplified as follows:

$$(c_1 + c_2)Re(v) + i(c_1 - c_2)Im(v) = 0 \quad (2)$$

But the solution to Eq. 2 is equivalent to the solution of the following:  $k_1 Re(v) + k_2 Im(v) = 0$ .

- ▶ Therefore,  $Re(v)$  and  $Im(v)$  are linearly independent.

## PROOF

- ▶ If  $\lambda = a - b i$ , then  $A\mathbf{v} = \lambda\mathbf{v} = (a - b i)(\operatorname{Re}(\mathbf{v}) + \operatorname{Im}(\mathbf{v}) i)$ .
- ▶ This gives:  
$$A\mathbf{v} = (a \operatorname{Re}(\mathbf{v}) + b \operatorname{Im}(\mathbf{v})) + (a \operatorname{Im}(\mathbf{v}) - b \operatorname{Re}(\mathbf{v})) i = \operatorname{Re}(A\mathbf{v}) + \operatorname{Im}(A\mathbf{v}) i.$$
- ▶ By the previous slide, we have:
  - ▶  $A \operatorname{Re}(\mathbf{v}) = \operatorname{Re}(A\mathbf{v}) = a \operatorname{Re}(\mathbf{v}) + b \operatorname{Im}(\mathbf{v})$
  - ▶  $A \operatorname{Im}(\mathbf{v}) = \operatorname{Im}(A\mathbf{v}) = -b \operatorname{Re}(\mathbf{v}) + a \operatorname{Im}(\mathbf{v})$
- ▶ Let  $P = [\operatorname{Re}(\mathbf{v}) \quad \operatorname{Im}(\mathbf{v})]$ . We have:  $A \operatorname{Re}(\mathbf{v}) = P \begin{pmatrix} a \\ b \end{pmatrix}$  and  
$$A \operatorname{Im}(\mathbf{v}) = P \begin{pmatrix} -b \\ a \end{pmatrix}$$
- ▶ Therefore,  $AP = A[\operatorname{Re}(\mathbf{v}) \quad \operatorname{Im}(\mathbf{v})] = \left[ P \begin{pmatrix} a \\ b \end{pmatrix} \quad P \begin{pmatrix} -b \\ a \end{pmatrix} \right] =$   
$$P \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = P C$$