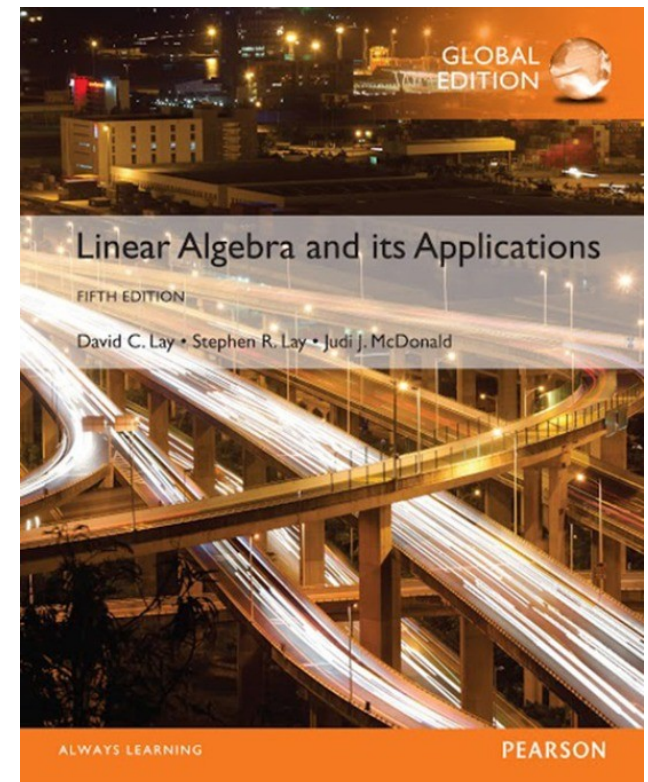


# 3

## Determinants

### 3.1

## INTRODUCTION TO DETERMINANTS



# INTRODUCTION TO DETERMINANTS

- **Definition:** For  $n \geq 2$ , the **determinant** of an  $n \times n$  matrix  $A = [a_{ij}]$  is the sum of  $n$  terms of the form  $\pm a_{1j} \det A_{1j}$ , with plus and minus signs alternating, where the entries  $a_{11}, a_{12}, \dots, a_{1n}$  are from the first row of  $A$ . In symbols,

$$\begin{aligned}\det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \dots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

# INTRODUCTION TO DETERMINANTS

- **Example 1** Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

- **Solution** Compute  $\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$ :

$$\det A = 1 \cdot \det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$= 1(0 - 2) - 5(0 - 0) + 0(-4 - 0) = -2$$

# INTRODUCTION TO DETERMINANTS

- Another common notation for the determinant of a matrix uses a pair of vertical lines in place of brackets.
- Thus the calculation in Example 1 can be written as

$$\det A = 1 \cdot \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = \dots = -2$$

- To state the next theorem, it is convenient to write the definition of  $\det A$  in a slightly different form. Given  $A = [a_{ij}]$ , the  **$(i, j)$ -cofactor** of  $A$  is the number  $C_{ij}$  given by

$$C_{ij} = (-1)^{i+j} \det A_{ij} \quad (4)$$

# INTRODUCTION TO DETERMINANTS

■ Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

- This formula is called a **cofactor expansion across the first row of  $A$** .

- **Theorem 1:** The determinant of an  $n \times n$  matrix  $A$  can be computed by a cofactor across any row or down any column. The expansion across the  $i$ th row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

- The cofactor expansion down the  $j$ th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

# INTRODUCTION TO DETERMINANTS

- **Example 2** Use a cofactor expansion across the third row to compute  $\det A$ , where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

- **Solution** Compute

$$\begin{aligned} \det A &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= (-1)^{3+1}a_{31}\det A_{31} + (-1)^{3+2}a_{32}\det A_{32} + (-1)^{3+3}a_{33}\det A_{33} \\ &= 0 \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} \\ &= 0 + 2(-1) + 0 = -2 \end{aligned}$$

# INTRODUCTION TO DETERMINANTS

- **Theorem 2:** If  $A$  is a triangular matrix, then  $\det A$  is the product of the entries on the main diagonal of  $A$ .

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ * & a_{22} & 0 & 0 \\ * & * & a_{33} & 0 \\ * & * & * & \dots & a_{nn} \end{bmatrix}$$

Lower triangular matrix