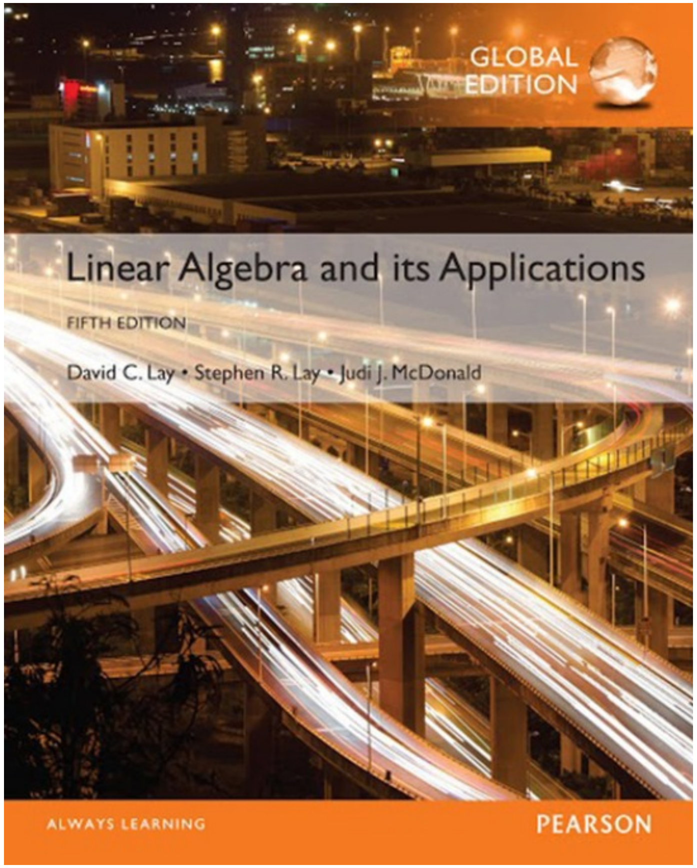
2 Matrix Algebra

2.3 CHARACTERIZATIONS OF INVERTIBLE MATRICES



THE INVERTIBLE MATRIX THEOREM

- **Theorem 8**: Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.
  a. A is an invertible matrix.
  b. A is row equivalent to the $n \times n$ identity matrix.
  c. A has n pivot positions.
  d. The equation $A\mathrm{x} = 0$ has only the trivial solution.
  e. The columns of A form a linearly independent set.

THE INVERTIBLE MATRIX THEOREM

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

THE INVERTIBLE MATRIX THEOREM

- First, we need some notation.
- If the truth of statement (a) always implies that statement (j) is true, we say that (a) *implies* (j) and write (a) $\Rightarrow$ (j) .
- The proof will establish the “circle” of implications as shown in the following figure.

$$\begin{tikzcd} & (a) \arrow[dr, Rightarrow] & \\ (b) \arrow[ur, Rightarrow] & & (j) \arrow[d, Rightarrow] \\ (c) \arrow[u, Rightarrow] & & (d) \arrow[ll, Rightarrow] \end{tikzcd}$$

- If any one of these five statements is true, then so are the others.

THE INVERTIBLE MATRIX THEOREM

- Finally, the proof will link the remaining statements of the theorem to the statements in this circle.
- **Proof:** If statement (a) is true, then A^{-1} works for C in (j), so $(a) \Rightarrow (j)$.
- Next, $(j) \Rightarrow (d)$.
- Also, $(d) \Rightarrow (c)$.
- If A is square and has n pivot positions, then the pivots must lie on the main diagonal, in which case the reduced echelon form of A is I_n.
- Thus $(c) \Rightarrow (b)$.
- Also, $(b) \Rightarrow (a)$.

THE INVERTIBLE MATRIX THEOREM

$$
\begin{tikzcd}
 & \text{(k)} \arrow[dr, Rightarrow] & \\
\text{(a)} \arrow[ur, Rightarrow] & & \text{(g)} \arrow[ll, Rightarrow]
\end{tikzcd}
$$

$$\text{(g)} \Leftrightarrow \text{(h)} \Leftrightarrow \text{(i)}$$

$$\text{(d)} \Leftrightarrow \text{(e)} \Leftrightarrow \text{(f)}$$

$$\text{(a)} \Leftrightarrow \text{(l)}$$

THE INVERTIBLE MATRIX THEOREM

- This completes the circle in the previous figure.
- Next, $(a) \Rightarrow (k)$ because A^{-1} works for D.
- Also, $(k) \Rightarrow (g)$ and $(g) \Rightarrow (a)$.
- So (k) and (g) are linked to the circle.
- Further, (g), (h), and (i) are equivalent for any matrix.
- Thus, (h) and (i) are linked through (g) to the circle.
- Since (d) is linked to the circle, so are (e) and (f), because (d), (e), and (f) are all equivalent for *any* matrix A.
- Finally, $(a) \Rightarrow (l)$ and $(l) \Rightarrow (a)$.
- This completes the proof.

THE INVERTIBLE MATRIX THEOREM

- Theorem 8 could also be written as "The equation $A\mathbf{x} = \mathbf{b}$ has a *unique* solution for each **b** in $\mathbb{R}^n$."
- This statement implies (b) and hence implies that A is invertible.
- The following fact follows from Theorem 8.

  Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.
- The Invertible Matrix Theorem divides the set of all $n \times n$ matrices into two disjoint classes: the invertible (nonsingular) matrices, and the noninvertible (singular) matrices.

THE INVERTIBLE MATRIX THEOREM

- Each statement in the theorem describes a property of every $n \times n$ invertible matrix.

- The *negation* of a statement in the theorem describes a property of every $n \times n$ singular matrix.

- For instance, an $n \times n$ singular matrix is *not* row equivalent to I_n, does *not* have n pivot position, and has linearly *dependent* columns.

THE INVERTIBLE MATRIX THEOREM

- **Example 1:** Use the Invertible Matrix Theorem to decide if A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

- **Solution:**

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

THE INVERTIBLE MATRIX THEOREM

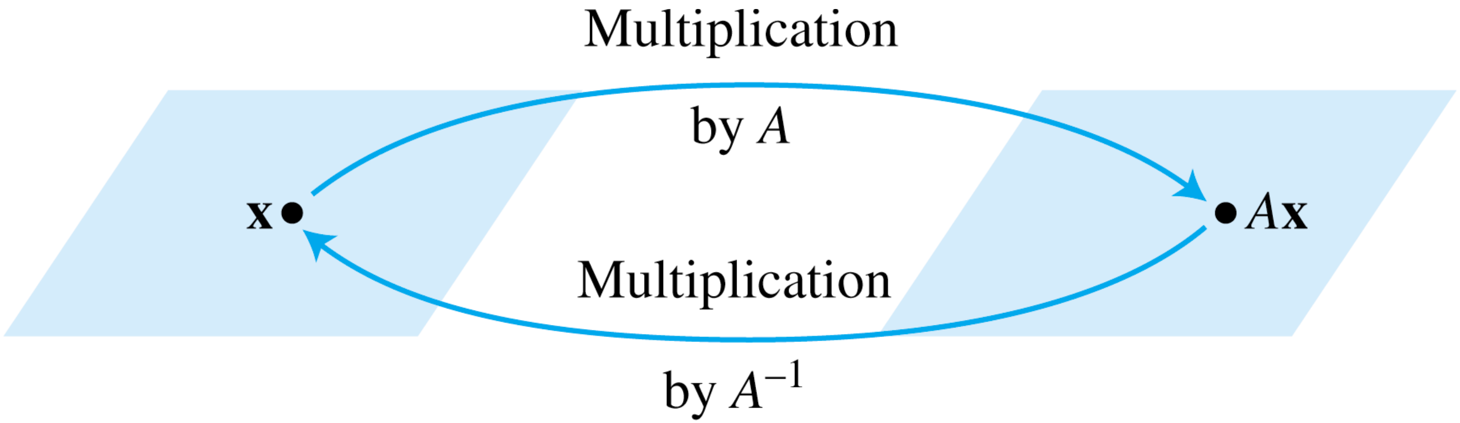
- So A has three pivot positions and hence is invertible, by the Invertible Matrix Theorem, statement (c).

- The Invertible Matrix Theorem *applies only to square matrices*.

- For example, if the columns of a 4×3 matrix are linearly independent, we cannot use the Invertible Matrix Theorem to conclude anything about the existence or nonexistence of solutions of equation of the form $A\mathbf{x} = \mathbf{b}$.

INVERTIBLE LINEAR TRANSFORMATIONS

- Matrix multiplication corresponds to composition of linear transformations.
- When a matrix A is invertible, the equation $A^{-1}A\mathbf{x} = \mathbf{x}$ can be viewed as a statement about linear transformations. See the following figure.



A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

INVERTIBLE LINEAR TRANSFORMATIONS

- A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathrm{x})) = \mathrm{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \qquad (1)$$
$$T(S(\mathrm{x})) = \mathrm{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \qquad (2)$$

- **Theorem 9**: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathrm{x}) = A^{-1}\mathrm{x}$ is the unique function satisfying equation (1) and (2).

INVERTIBLE LINEAR TRANSFORMATIONS

- **Proof:** Suppose that T is invertible.
- Then (2) shows that T is onto $\mathbb{R}^n$, for if $\mathbf{b}$ is in $\mathbb{R}^n$ and $\mathrm{x} = S(\mathrm{b})$, then $T(\mathrm{x}) = T(S(\mathrm{b})) = \mathrm{b}$, so each $\mathbf{b}$ is in the range of T.
- Thus A is invertible, by the Invertible Matrix Theorem, statement (i).
- Conversely, suppose that A is invertible, and let $S(\mathrm{x}) = A^{-1}\mathrm{x}$. Then, S is a linear transformation, and S satisfies (1) and (2).
- For instance, $S(T(\mathrm{x})) = S(A\mathrm{x}) = A^{-1}(A\mathrm{x}) = \mathrm{x}$.
- Thus, T is invertible.