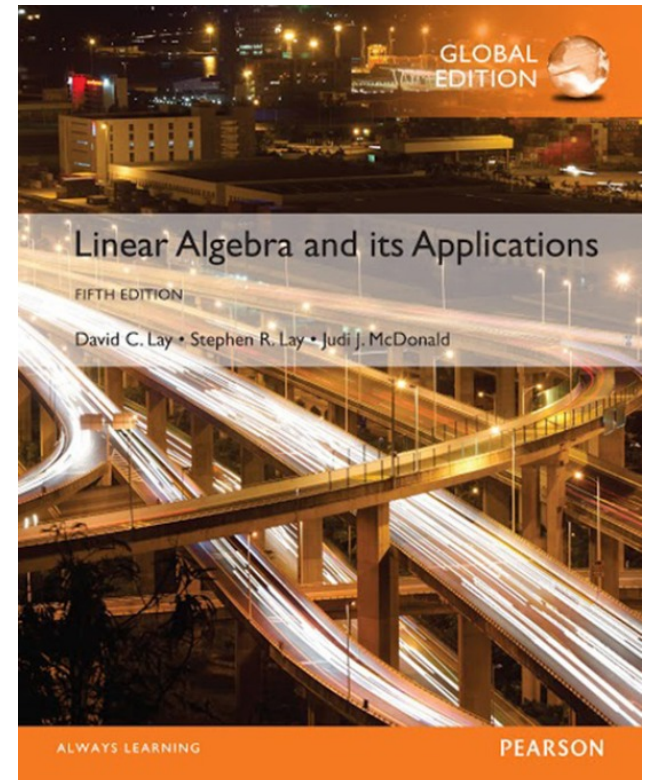


5

Eigenvalues and Eigenvectors

5.4

EIGENVECTORS AND LINEAR TRANSFORMATIONS



THE MATRIX OF A LINEAR TRANSFORMATION

- Given any x in V , the coordinate vector $[x]_{\beta}$ is in \mathbb{R}^n and the coordinate vector of its image, $[T(x)]_{\mathcal{C}}$ is in \mathbb{R}^m , as shown in Fig. 1 below.

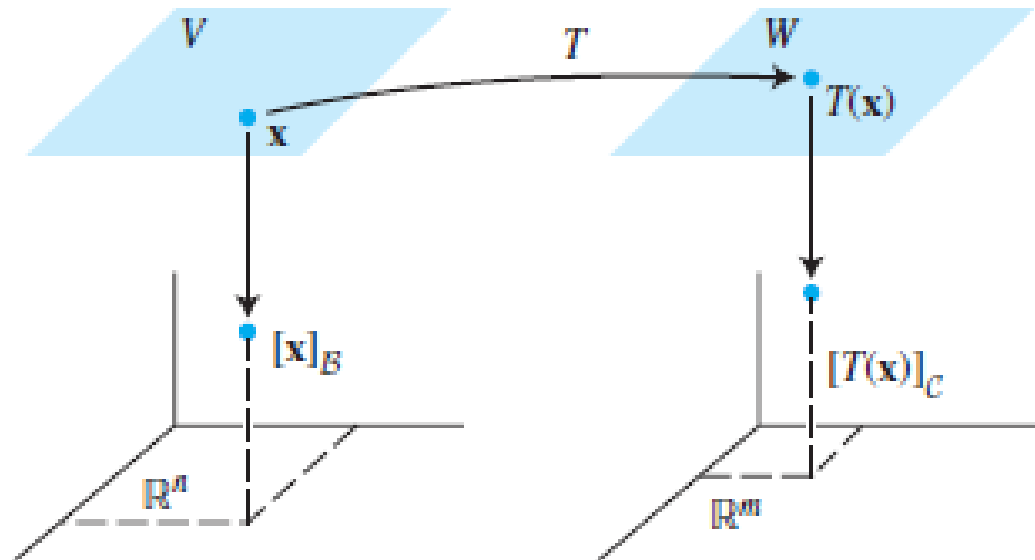


FIGURE 1 A linear transformation from V to W .

THE MATRIX OF A LINEAR TRANSFORMATION

- The connection between $[x]_{\beta}$ and $[T(x)]_{\mathcal{C}}$ is easy to find. Let $\{b_1, \dots, b_n\}$ be the basis β for V . If $x = r_1 b_1 + \dots + r_n b_n$, then,

$$[x]_{\beta} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

- And

$$T(x) = T(r_1 b_1 + \dots + r_n b_n) = r_1 T(b_1) + \dots + r_n T(b_n) \quad (1)$$

- because T is linear.

THE MATRIX OF A LINEAR TRANSFORMATION

- Now, since the coordinate mapping from W to \mathbb{R}^m is linear, equation (1) leads to

$$[T(x)]_C = r_1[T(b_1)]_C + \cdots + r_n[T(b_n)]_C \quad (2)$$

- Since C -coordinate vectors are in \mathbb{R}^m , the vector equation (2) can be written as a matrix equation, namely,

$$[T(x)]_C = M[x]_\beta \quad (3)$$

- where

$$M = [[T(b_1)]_C \ [T(b_2)]_C \ \cdots \ [T(b_n)]_C] \quad (4)$$

THE MATRIX OF A LINEAR TRANSFORMATION

- The matrix M is a matrix representation of T , called the matrix for T relative to the bases β and C . See Fig. 2 below:

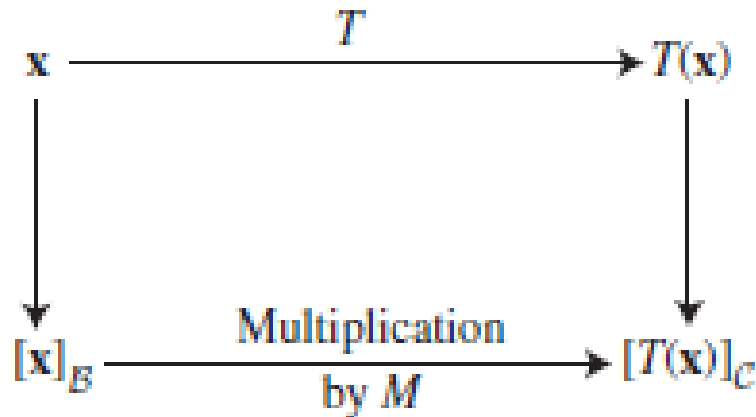


FIGURE 2

THE MATRIX OF A LINEAR TRANSFORMATION

- **Example 1** Suppose $\beta = \{b_1, b_2\}$ is a basis for V and $C = \{c_1, c_2, c_3\}$ is a basis for W . Let $T: V \rightarrow W$ be a linear transformation with the property that

$$T(b_1) = 3c_1 - 2c_2 + 5c_3 \quad \text{and} \quad T(b_2) = 4c_1 + 7c_2 - c_3$$

- Find the matrix M for T relative to β and C .

THE MATRIX OF A LINEAR TRANSFORMATION

- **Solution** The C -coordinate vectors of the images of b_1 and b_2 are

$$[T(b_1)]_C = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \text{ and } [T(b_2)]_C = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$$

- Hence

$$M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$$

- If β and C are bases for the same space V and if T is the identity transformation $T(x) = x$ for x in V , then matrix M in (4) is just a change-of-coordinates matrix.

LINEAR TRANSFORMATIONS FROM V INTO V

- In the common case where W is the same V and the basis C is the same as β , then the matrix M in (4) is called the **matrix for T relative to β** , or simply the **β -matrix for T** , and is denoted by $[T]_{\beta}$.
- See Fig. 3 below

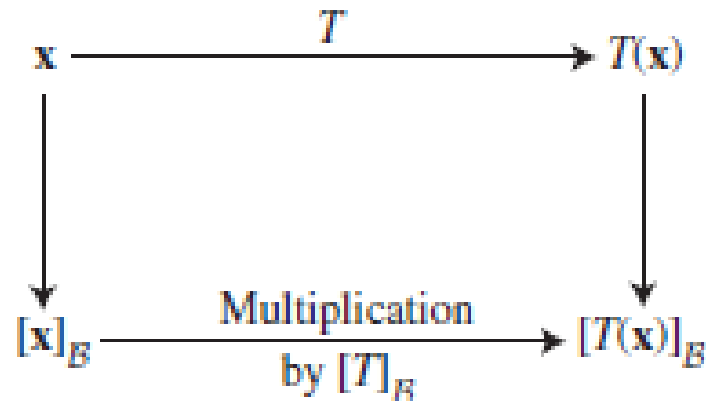


FIGURE 3

LINEAR TRANSFORMATIONS ON \mathbb{R}^n

- **Theorem 8:** Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If β is the basis for \mathbb{R}^n formed from the columns of P , then D is the β -matrix for the transformation $x \mapsto Ax$.
- **Proof** Denote the columns of P by b_1, \dots, b_n , so that $\beta = \{b_1, \dots, b_n\}$ and $P = [b_1 \dots b_n]$. In this case, P is the change-of-coordinates matrix P_β discussed in Section 4.4, where

$$P[x]_\beta = x \quad \text{and} \quad [x]_\beta = P^{-1}x$$

LINEAR TRANSFORMATIONS ON \mathbb{R}^n

- If $T(x) = Ax$ for x in \mathbb{R}^n , then

$$\begin{aligned}[T]_{\beta} &= [[T(b_1)]_{\beta} \quad \dots \quad [T(b_n)]_{\beta}] && \text{Definition of } [T]_{\beta} \\ &= [[Ab_1]_{\beta} \quad \dots \quad [Ab_n]_{\beta}] && \text{Since } T(x) = Ax \\ &= [P^{-1}Ab_1 \quad \dots \quad P^{-1}Ab_n] && \text{Change of coordinates} \\ &= P^{-1}A[b_1 \quad \dots \quad b_n] && \text{Matrix multiplication} \\ &= P^{-1}AP\end{aligned}$$

- Since $A = PDP^{-1}$, we have $[T]_{\beta} = P^{-1}AP = D$.

LINEAR TRANSFORMATIONS ON \mathbb{R}^n

- **Example 3** Define $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a basis β for \mathbb{R}^2 with the property that the β -matrix for T is a diagonal matrix.
- **Solution** From Example 2 in Section 5.3 we know that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

- The columns of P , call them \mathbf{b}_1 and \mathbf{b}_2 , are eigenvectors of A . By Theorem 8, D is the β -matrix for T when $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$. The mappings $\mathbf{x} \mapsto A\mathbf{x}$ and $\mathbf{u} \mapsto D\mathbf{u}$ describe the same linear transformation, relative to different bases.