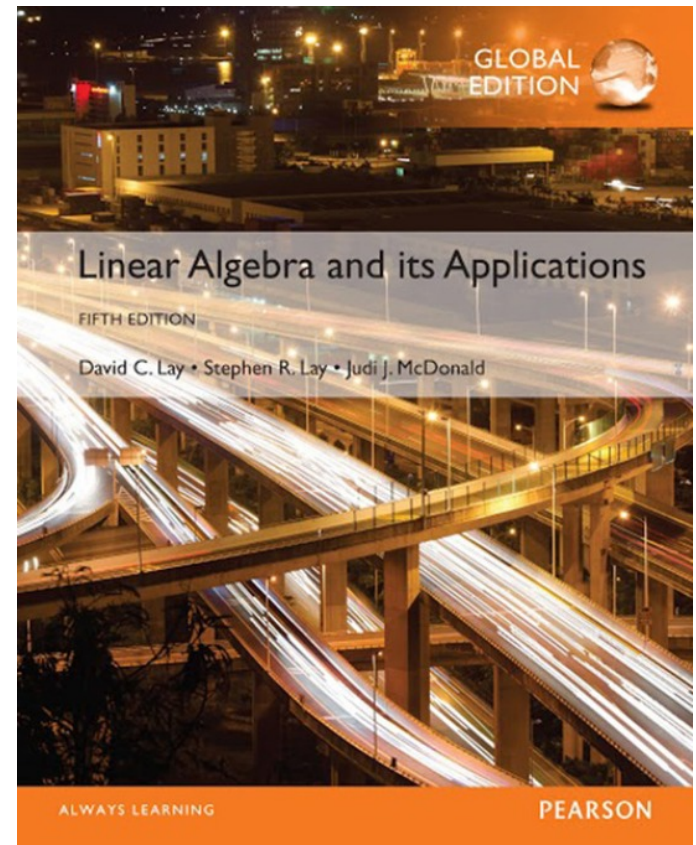


1

Linear Equations in Linear Algebra

1.4

THE MATRIX EQUATION $Ax = b$



MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **Definition:** If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if \mathbf{x} is in \mathbb{R}^n , then the **product of A and \mathbf{x}** , denoted by $A\mathbf{x}$, is the **linear combination of the columns of A using the corresponding entries in \mathbf{x} as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n.$$

- Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in \mathbf{x} .

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **Example 2:** For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in \mathbb{R}^m , write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.
- **Solution:** Place $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the columns of a matrix A and place the weights 3, -5 , and 7 into a vector \mathbf{x} .
- That is,

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\mathbf{x}.$$

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- Now, write the system of linear equations as a vector equation involving a linear combination of vectors.
- For example, the following system

$$x_1 + 2x_2 - x_3 = 4 \quad (1)$$

$$-5x_2 + 3x_3 = 1$$

is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \quad (2)$$

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- As in the example, the linear combination on the left side is a matrix times a vector, so that (2) becomes

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \quad (3)$$

- Equation (3) has the form $A\mathbf{x} = \mathbf{b}$. Such an equation is called a **matrix equation**, to distinguish it from a vector equation such as shown in (2).

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

THEOREM 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if \mathbf{b} is in \mathbb{R}^m , then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}]$$

EXISTENCE OF SOLUTIONS

- The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if \mathbf{b} is a linear combination of the columns of A .

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A , either they are all true statements or they are all false.

- a. For each \mathbf{b} in \mathbb{R}^m , the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
- b. Each \mathbf{b} in \mathbb{R}^m is a linear combination of the columns of A .
- c. The columns of A span \mathbb{R}^m .
- d. A has a pivot position in every row.

COMPUTATION OF $A\mathbf{x}$

- **Example 4:** Compute $A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$
and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.
- **Solution:** From the definition,

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

COMPUTATION OF $A\mathbf{x}$

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} \quad (1)$$

$$= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}.$$

- The first entry in the product $A\mathbf{x}$ is a sum of products (sometimes called a *dot product*), using the first row of A and the entries in \mathbf{x} .

COMPUTATION OF $A\mathbf{x}$

- That is,
$$\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \end{bmatrix}.$$

- Similarly, the second entry in $A\mathbf{x}$ can be calculated by multiplying the entries in the second row of A by the corresponding entries in \mathbf{x} and then summing the resulting products.

$$\begin{bmatrix} -1 & 5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + 5x_2 - 3x_3 \end{bmatrix}$$

ROW-VECTOR RULE FOR COMPUTING $A\mathbf{x}$

- Likewise, the third entry in $A\mathbf{x}$ can be calculated from the third row of A and the entries in \mathbf{x} .
- If the product $A\mathbf{x}$ is defined, then the i th entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector \mathbf{x} .
- The matrix with 1's on the diagonal and 0's elsewhere is called an **identity matrix** and is denoted by I .

- For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix.

PROPERTIES OF THE MATRIX-VECTOR PRODUCT $A\mathbf{x}$

- **Theorem 5:** If A is an $m \times n$ matrix, \mathbf{u} and \mathbf{v} are vectors in \mathbb{R}^n , and c is a scalar, then

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v};$$

- a. $A(c\mathbf{u}) = c(A\mathbf{u})$.

- **Proof:** For simplicity, take $n = 3$, $A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$, and \mathbf{u}, \mathbf{v} in \mathbb{R}^3 .
- For $i = 1, 2, 3$, let u_i and v_i be the i th entries in \mathbf{u} and \mathbf{v} , respectively.

PROPERTIES OF THE MATRIX-VECTOR PRODUCT $A\mathbf{x}$

THEOREM 5

If A is an $m \times n$ matrix, \mathbf{u} and \mathbf{v} are vectors in \mathbb{R}^n , and c is a scalar, then

- a.* $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;
- b.* $A(c\mathbf{u}) = c(A\mathbf{u})$.

- **Proof:** For simplicity, take $n = 3$, $A = [a_1 \ a_2 \ a_3]$, and \mathbf{u}, \mathbf{v} in \mathbb{R}^3 .
- For $i = 1, 2, 3$, let u_i and v_i be the i th entries in \mathbf{u} and \mathbf{v} , respectively.

PROPERTIES OF THE MATRIX-VECTOR PRODUCT $A\mathbf{x}$

- To prove statement (a), compute $A(\mathbf{u} + \mathbf{v})$ as a linear combination of the columns of A using the entries in $\mathbf{u} + \mathbf{v}$ as weights.

$$\begin{aligned} A(\mathbf{u} + \mathbf{v}) &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \\ &= (u_1 + v_1)\mathbf{a}_1 + (u_2 + v_2)\mathbf{a}_2 + (u_3 + v_3)\mathbf{a}_3 \\ &= (u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3) \\ &= A\mathbf{u} + A\mathbf{v} \end{aligned}$$

Entries in $\mathbf{u} + \mathbf{v}$

Columns of A

PROPERTIES OF THE MATRIX-VECTOR PRODUCT $A\mathbf{x}$

- To prove statement (b), compute $A(c\mathbf{u})$ as a linear combination of the columns of A using the entries in $c\mathbf{u}$ as weights.

$$\begin{aligned} A(c\mathbf{u}) &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = (cu_1)\mathbf{a}_1 + (cu_2)\mathbf{a}_2 + (cu_3)\mathbf{a}_3 \\ &= c(u_1\mathbf{a}_1) + c(u_2\mathbf{a}_2) + c(u_3\mathbf{a}_3) \\ &= c(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) \\ &= c(A\mathbf{u}) \end{aligned}$$