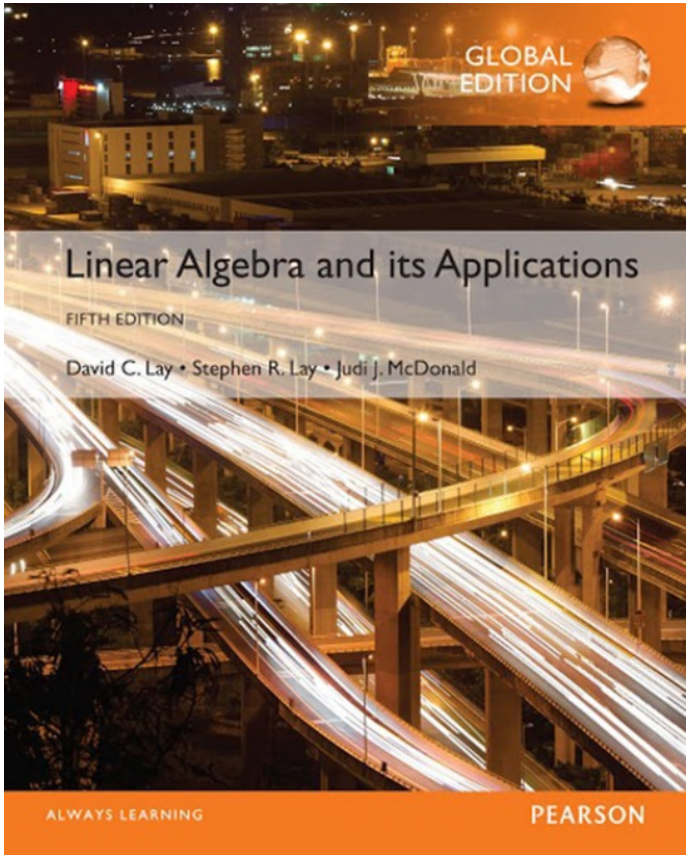
4 Vector Spaces

4.3

LINEARLY INDEPENDENT SETS; BASES



LINEAR INDEPENDENT SETS; BASES

- An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation

$$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \ldots + c_p\mathrm{v}_p = 0 \qquad (1)$$

has *only* the trivial solution, $c_1 = 0, \ldots, c_p = 0$.

- The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if (1) has a nontrivial solution, *i.e.*, if there are some weights, $c_1, \ldots, c_p$, *not all zero*, such that (1) holds.
- In such a case, (1) is called a **linear dependence relation** among $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

LINEAR INDEPENDENT SETS; BASES

- **Theorem 4:** An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathrm{v}_1 \neq 0$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathrm{v}_1, \ldots, \mathrm{v}_{j-1}$.

- **Definition:** Let H be a subspace of a vector space V. An indexed set of vectors $\mathsf{B} = \{\mathrm{b}_1, \ldots, \mathrm{b}_p\}$ in V is a basis for H if
  (i) B is a linearly independent set, and
  (ii) The subspace spanned by B coincides with H; that is, $H = \mathrm{Span}\{\mathrm{b}_1, \ldots, \mathrm{b}_p\}$

LINEAR INDEPENDENT SETS; BASES

- The definition of a basis applies to the case when $H = V$, because any vector space is a subspace of itself.

- Thus a basis of V is a linearly independent set that spans V.

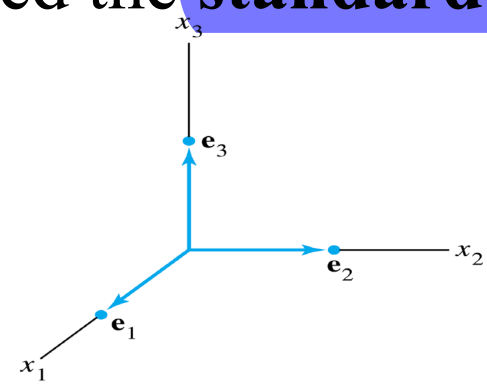
- When $H \neq V$, condition (ii) includes the requirement that each of the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_p$ must belong to H, because Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ contains $\mathbf{b}_1, \ldots, \mathbf{b}_p$.

STANDARD BASIS

- Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the columns of the $n \times n$ matrix, I_n.
- That is,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called the **standard basis for** $\mathbb{R}^n$. See the following figure.



The standard basis for $\mathbb{R}^3$.

THE SPANNING SET THEOREM

- **Theorem 5:** Let $S = \{\mathrm{v}_1,\ldots,\mathrm{v}_p\}$ be a set in V, and let $H = \mathrm{Span}\{\mathrm{v}_1,\ldots,\mathrm{v}_p\}$.
  a. If one of the vectors in S—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.
  b. If $H \neq \{0\}$, some subset of S is a basis for H.
- **Proof:**
  a. By rearranging the list of vectors in S, if necessary, we may suppose that $\mathbf{v}_p$ is a linear combination of $\mathrm{v}_1,\ldots,\mathrm{v}_{p-1}$—say,

THE SPANNING SET THEOREM

$$\mathbf{v}_p = a_1\mathbf{v}_1 + \ldots + a_{p-1}\mathbf{v}_{p-1} \qquad (3)$$

- Given any **x** in *H*, we may write
  $$\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_{p-1}\mathbf{v}_{p-1} + c_p\mathbf{v}_p \qquad (4)$$
  for suitable scalars $c_1, \ldots, c_p$.

- Substituting the expression for $\mathbf{v}_p$ from (3) into (4), it is easy to see that **x** is a linear combination of $\mathbf{v}_1, \ldots \mathbf{v}_{p-1}$.

- Thus $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$ spans *H*, because **x** was an arbitrary element of *H*.

THE SPANNING SET THEOREM

b. If the original spanning set S is linearly independent, then it is already a basis for H.
   - Otherwise, one of the vectors in S depends on the others and can be deleted, by part (a).
   - So long as there are two or more vectors in the spanning set, we can repeat this process until the spanning set is linearly independent and hence is a basis for H.
   - If the spanning set is eventually reduced to one vector, that vector will be nonzero (and hence linearly independent) because $H \neq \{0\}$.

THE SPANNING SET THEOREM

- **Example 7:** Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$

  and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
  Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, and show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Then find a basis for the subspace H.

- **Solution:** Every vector in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ belongs to H because

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_3$$

THE SPANNING SET THEOREM

- Now let **x** be any vector in H—say,

$$\mathrm{x} = c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3\mathrm{v}_3\,.$$

- Since $\mathrm{v}_3 = 5\mathrm{v}_1 + 3\mathrm{v}_2$, we may substitute

$$\mathrm{x} = c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3(5\mathrm{v}_1 + 3\mathrm{v}_2)$$

$$= (c_1 + 5c_3)\mathrm{v}_1 + (c_2 + 3c_3)\mathrm{v}_2$$

- Thus **x** is in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$, so every vector in H already belongs to Span $\{\mathbf{v}_1, \mathbf{v}_2\}$.
- We conclude that H and Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ are actually the set of vectors.
- It follows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of H since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

BASIS FOR COL B

- **Example 8:** Find a basis for Col B, where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Solution:** Each nonpivot column of B is a linear combination of the pivot columns.
- In fact, $\mathbf{b}_2 = 4\mathbf{b}_1$ and $\mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$.
- By the Spanning Set Theorem, we may discard $\mathbf{b}_2$ and $\mathbf{b}_4$, and $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ will still span Col B.

BASIS FOR COL *B*

- Let

$$S = \{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- Since $\mathbf{b}_1 \neq 0$ and no vector in S is a linear combination of the vectors that precede it, S is linearly independent. (Theorem 4).

- Thus S is a basis for Col B.

BASES FOR NUL A AND COL A

- **Theorem 6:** The pivot columns of a matrix A form a basis for Col A.

- **Proof:** Let B be the reduced echelon form of A.
- The set of pivot columns of B is linearly independent, for no vector in the set is a linear combination of the vectors that precede it.
- Since A is row equivalent to B, the pivot columns of A are linearly independent as well, because any linear dependence relation among the columns of A corresponds to a linear dependence relation among the columns of B.

BASES FOR NUL A AND COL A

- For this reason, every nonpivot column of A is a linear combination of the pivot columns of A.
- Thus the nonpivot columns of a may be discarded from the spanning set for Col A, by the Spanning Set Theorem.
- This leaves the pivot columns of A as a basis for Col A.

- **Warning:** The pivot columns of a matrix A are evident when A has been reduced only to echelon form.
- But, be careful to use the pivot columns of A itself for the basis of Col A.

BASES FOR NUL A AND COL A

- Row operations can change the column space of a matrix.
- The columns of an echelon form B of A are often not in the column space of A.

- **Two Views of a Basis**
- When the Spanning Set Theorem is used, the deletion of vectors from a spanning set must stop when the set becomes linearly independent.
- If an additional vector is deleted, it will not be a linear combination of the remaining vectors, and hence the smaller set will no longer span V.

TWO VIEWS OF A BASIS

- Thus a basis is a spanning set that is as small as possible.

- A basis is also a linearly independent set that is as large as possible.

- If S is a basis for V, and if S is enlarged by one vector—say, $\mathbf{w}$—from V, then the new set cannot be linearly independent, because S spans V, and $\mathbf{w}$ is therefore a linear combination of the elements in S.