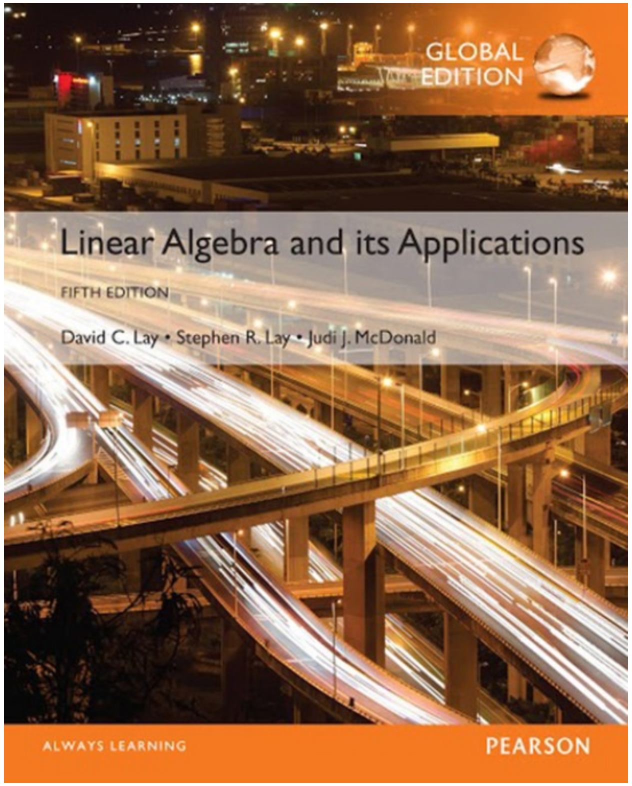
2 Matrix Algebra

2.8

SUBSPACES OF $\mathbb{R}^n$



PEARSON

SUBSPACES OF $\mathbb{R}^n$

- **Definition**: A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:
  a) The zero vector is in H.
  b) For each $\boldsymbol{u}$ and $\boldsymbol{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
  c) For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

SUBSPACES OF $\mathbb{R}^n$

- A plane through the origin is the standard way to visualize the subspace in Example 1 on the next slide. See Fig. 1 below:

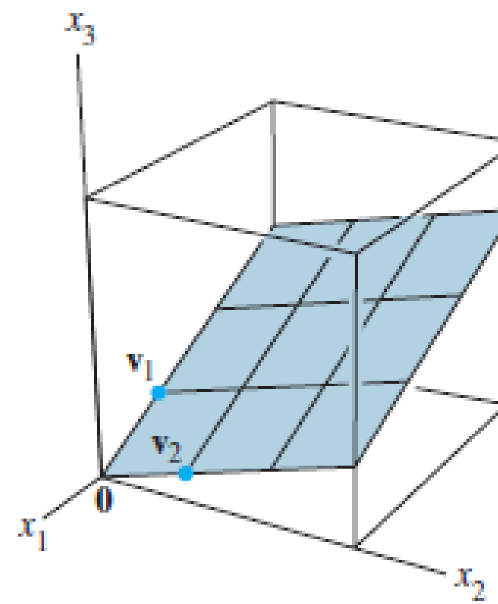


FIGURE 1

Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ as a plane through the origin.

SUBSPACES OF $\mathbb{R}^n$

- **Example 1** If v_1 and v_2 are in $\mathbb{R}^n$ and $H = \mathrm{Span}\{\mathrm{v}_1, \mathrm{v}_2\}$, then H is a subspace of $\mathbb{R}^n$. To verify this statement, note that the zero vector is in H (because $0\mathrm{v}_1 + 0\mathrm{v}_2$ is a linear combination of v_1 and v_2).
- Now take two arbitrary vectors in H, say,

$$u = s_1 v_1 + s_2 v_2 \quad \text{and} \quad v = t_1 v_1 + t_2 v_2$$

- Then

$$u + v = (s_1 + t_1)v_1 + (s_2 + t_2)v_2$$

- which shows that $\mathrm{u} + \mathrm{v}$ is a linear combination of v_1 and v_2 and hence is in H. Also, for any scalar c, the vector cu is in H, because $cu = c(s_1 v_1 + s_2 v_2) = cs_1(v_1) + cs_2(v_2)$.

COLUMN SPACE AND NULL SPACE OF A MATRIX

- **Definition:** The **column space** of a matrix A is the set Col A of all linear combinations of the columns of A.
- If $A = [\mathrm{a}_1 \ . \ . \ . \ \mathrm{a}_n]$ with the columns of $\mathbb{R}^m$, then Col A is the same as Span$\{\mathrm{a}_1 \ . \ . \ . \ \mathrm{a}_n\}$. Example 4 shows that the column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.
- **Example 4** Let $A=\begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\mathrm{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$.

  Determine whether b is in the column space of A.

COLUMN SPACE AND NULL SPACE OF A MATRIX

- **Solution:** The vector **b** is a linear combination of the columns of A if and only if **b** can be written as A**x** for some x, that is, if and only if the equation $A\mathrm{x} = \mathrm{b}$ has a solution.
- Row reducing the augmented matrix $[A\mathbf{b}]$,

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- We conclude that $A\mathrm{x} = \mathrm{b}$ is consistent and **b** is in Col A.

COLUMN SPACE AND NULL SPACE OF A MATRIX

- **Definition:** The **null space** of a matrix A is the set Nul$A$ of all solutions of the homogenous equation $A\mathbf{x} = 0$.

- **Theorem 12**: The null space of an m x n matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system Ax = 0 of m homogenous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

- **Proof:** The zero vector is in Nul$A$ (because $A0 = 0$). To show that Nul$A$ satisfies that other two properties required for a subspace, take any **u** and **v** in Nul$A$.

COLUMN SPACE AND NULL SPACE OF A MATRIX

- That is, suppose $A\mathbf{u} = 0$ and $A\mathbf{v} = 0$. Then, by a property of matrix multiplication,

$$A(u + v) = Au + Av = 0 + 0 = 0$$

- Thus $\mathbf{u} + \mathbf{v}$ satisfies $A = 0$, and so $\mathbf{u} + \mathbf{v}$ is in Nul$A$. Also, for any scalar c, $\mathrm{A}(c\mathbf{u}) = c(\mathrm{A}\mathbf{u}) = \mathbf{x}c(0) = 0$, which shows that $c\mathbf{u}$ is in Nul$A$.

BASIS FOR A SUBSPACE

- **Definition**: A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

- **Example 5** The columns of an invertible $n \times n$ matrix form a basis for all of because they are linearly independent and span $\mathbb{R}^n$, by the Invertible Matrix Theorem.

BASIS FOR A SUBSPACE

- One such matrix is the $n \times n$ identity matrix. Its columns are denoted by $e_1, \ldots, e_n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{bmatrix}, \quad \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix},$$

- The set $\{e_1, \ldots, e_n\}$ is called the **standard basis** for $\mathbb{R}^n$. See Fig. 3 on the next slide.

BASIS FOR A SUBSPACE

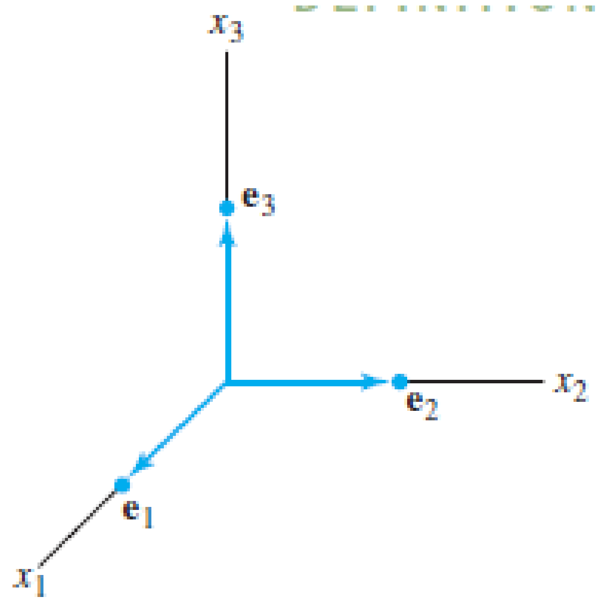


FIGURE 3
The standard basis for $\mathbb{R}^3$.

- **Theorem 13**: The pivot columns of a matrix A form a basis for the column space of A.