Statistics 1 Unit 6 Team 8

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The provided R code and its explanation are focused on exploring the relationship between Expected Shortfall (ES) and Value at Risk (VaR) for different distributions, particularly the normal distribution and the Student's t-distribution. Let's break down the key components and findings:

Exercise 93

We start trying to determine the limit numerically:

```r
ES_over_VaR <- function(n, alpha, rfun, params, tparams = list(mu = 0, sigma = 1)){

  sim <- do.call(rfun, c(n = n, params))
  #VaR <- quantile(sim, alpha)

  if("df" %in% names(params)){
    # t-student case
    VaR <- tparams$mu + tparams$sigma * qt(alpha, df = params$df)
    q <- qt(p = alpha, df = params$df)
    ES <- tparams$mu +
      tparams$sigma * (dt(q, df = params$df)/(1-alpha)) *
      (params$df + q^2)/(params$df-1)
    5
  }else{
    # normal case
    VaR <- params$mean + params$sd * qnorm(alpha)
    q <- qnorm(p = alpha)
    ES <- params$mean + params$sd * dnorm(q)/(1 - alpha)
  }

  return(ES/VaR)
}
```

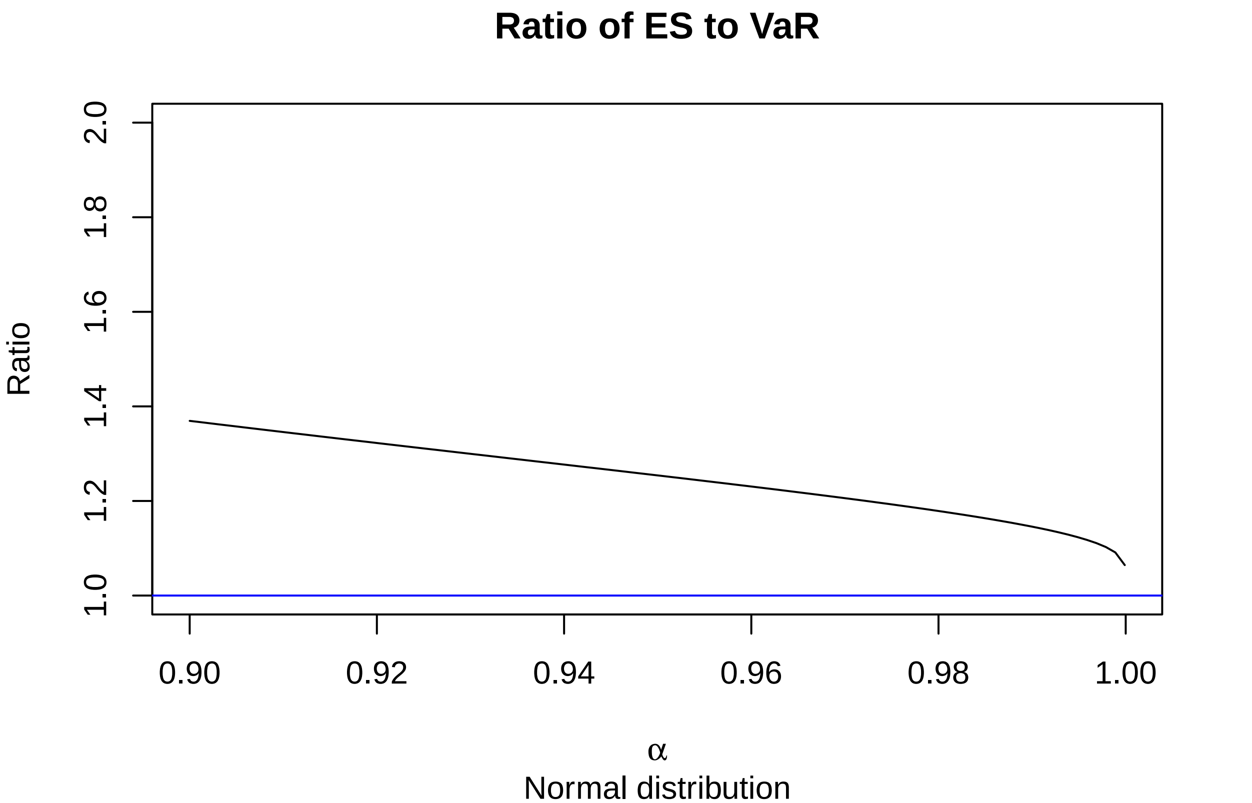
Function Definition (ES_over_VaR): The function ES_over_VaR is defined to calculate the ratio of ES to VaR for a given distribution. It takes parameters like the number of simulations (n), the confidence level (alpha), the random generation function (rfun), and parameters for the distribution (params and tparams).

```r
# Normal
alphas <- seq(from = 0.9, to = 0.9999, length.out = 100)
normal_test <- vapply(alphas, function(alpha){
  ES_over_VaR(n = 1e6,
              alpha = alpha,
              rfun = rnorm,
              params = list(mean = 0, sd = 1))
}, 0)

plot(alphas, normal_test, type = "l", ylim = c(1,2),
     main = "Ratio of ES to VaR",
     xlab = expression(alpha),
     ylab = "Ratio",
     sub = "Normal distribution")
```

Normal Distribution Test: The function is first applied to the normal distribution. A range of alpha values close to 1 (from 0.9 to 0.9999) is used to simulate the ES/VaR ratio. The plot shows that as alpha approaches 1, the ratio converges to 1. This is consistent with the theoretical expectation that for a normal distribution, the limit of ES/VaR as 1 is 1.

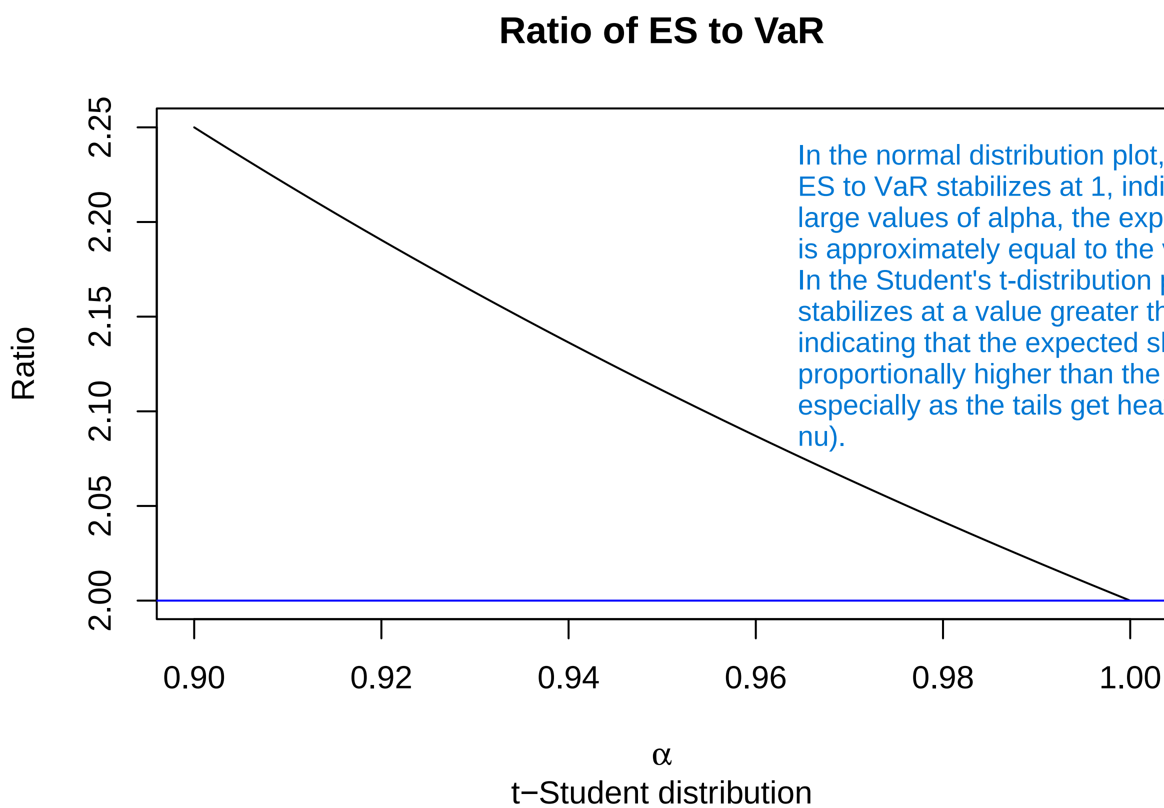
```r
abline(h = 1, col = "blue")
```




```r
# t-Student, nu = 2
nu <- 2
t_test <- vapply(alphas, function(alpha){
  ES_over_VaR(n = 1e6,
              alpha = alpha,
              rfun = rt,
              params = list(df = nu))
}, 0)

plot(alphas, t_test, type = "l",
     main = "Ratio of ES to VaR",
     xlab = expression(alpha),
     ylab = "Ratio",
     sub = "t-Student distribution")
abline(h = nu/(nu-1), col = "blue")
```

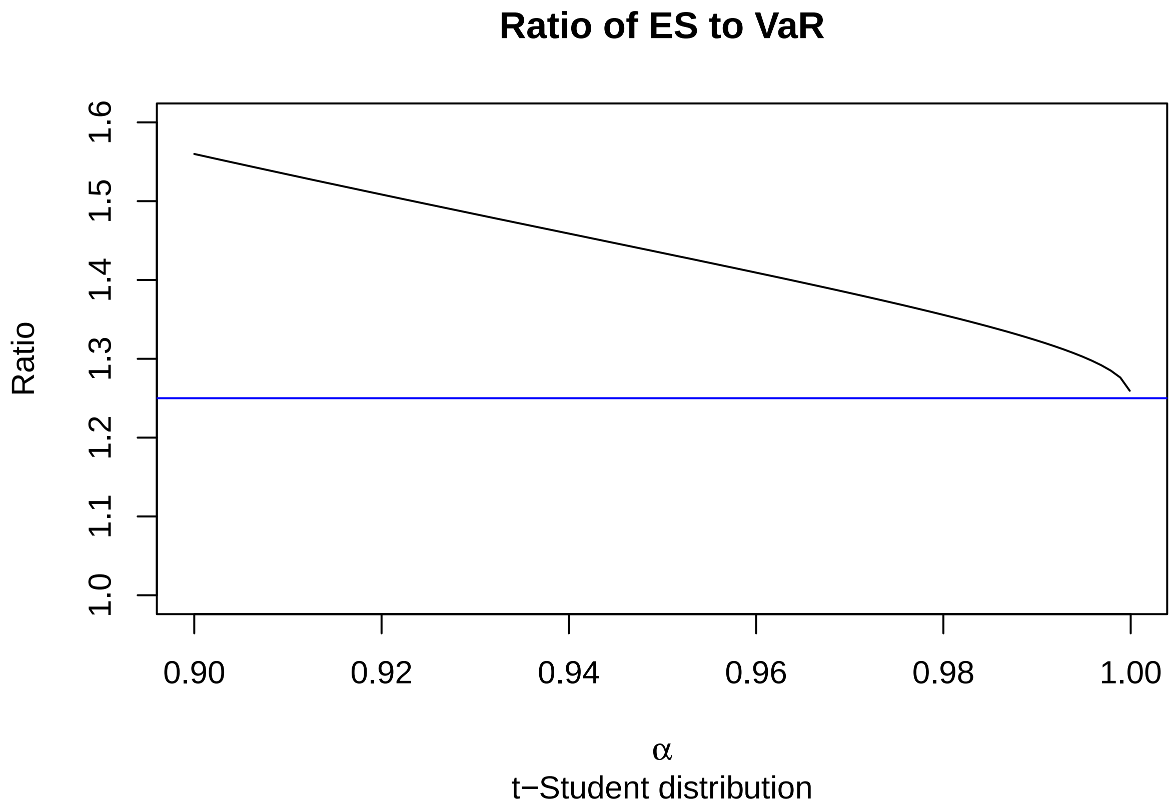
Student's t-Distribution Test: The function is then applied to the Student's t-distribution with different degrees of freedom (nu). For nu = 2 and nu = 5, the plots show that as alpha approaches 1, the ES/VaR ratio converges to nu/(nu - 1). This is in line with the theoretical expectation for the Student's t-distribution.



In the normal distribution plot, the ratio of ES to VaR stabilizes at 1, indicating that for large values of alpha, the expected shortfall is approximately equal to the value at risk. In the Student's t-distribution plots, the ratio stabilizes at a value greater than 1, indicating that the expected shortfall is proportionally higher than the value at risk, especially as the tails get heavier (lower nu).

```r
# t-Student, nu = 5
nu <- 5
t_test <- vapply(alphas, function(alpha){
  ES_over_VaR(n = 1e6,
              alpha = alpha,
              rfun = rt,
              params = list(df = nu))
}, 0)

plot(alphas, t_test, type = "l",
     ylim = c(1, 1.6),
     main = "Ratio of ES to VaR",
     xlab = expression(alpha),
     ylab = "Ratio",
     sub = "t-Student distribution")
abline(h = nu/(nu-1), col = "blue")
```



We proceed to prove the given limits. For that, let's first notice the following: For a continuous random variable X from the location-scale family that has an invertible cdf with location parameter μ and scale parameter σ^2, we have that $VaR_\alpha = \mu + \sigma F_{\hat{X}}^{-1}(\alpha)$, where $\hat{X}$ is its "standardized'' version of X. This follows from the definition, since:

$$\begin{aligned} VaR_\alpha &= inf\{x \in \mathbb{R} | P(X > x) \leq 1 - \alpha\} \\ &= inf\{x \in \mathbb{R} | P(X \leq x) \geq \alpha\} \\ &= inf\{x \in \mathbb{R} | F_X(x) \geq \alpha\} \\ &= inf\{x \in \mathbb{R} | x \geq F_X^{-1}(\alpha)\} \\ &= F_X^{-1}(\alpha) \end{aligned}$$

Now, because X is of the location-scale family with the parameters given above, $F_X^{-1}(\alpha) = x \iff \alpha = F_X(x) = F_{\hat{X}}\left(\frac{x-\mu}{\sigma}\right)$. Then, $F_{\hat{X}}^{-1}(\alpha) = \frac{x-\mu}{\sigma} \implies x = \mu + \sigma F_{\hat{X}}^{-1}(\alpha)$. This is useful because both the normal and the t-Student distribution fulfill the requirements for this to hold.

Normal case

Using what we proved above, notice that for $X \sim N(\mu, \sigma^2)$ we have $VaR_\alpha = \mu + \sigma\Phi^{-1}(\alpha)$. Now let $\Phi(x) = \alpha \iff x = \Phi^{-1}(\alpha)$, and notice $\alpha \to 1- \iff x \to \infty$. In that case, we can assume x is large enough that we can use the provided approximation $\Phi(x) \approx 1 - \phi(x)/x$ and value for ES_α to get that:

$$
\begin{aligned}
\lim_{\alpha \to 1-} \frac{ES_\alpha}{VaR_\alpha} &= \lim_{\alpha \to 1-} \frac{\mu + \sigma \left(\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha} \right)}{\mu + \sigma \Phi^{-1}(\alpha)} \\
&= \lim_{x \to \infty} \frac{\mu + \frac{\sigma \phi(x)}{1-\Phi(x)}}{\mu + \sigma x} \\
&= \lim_{x \to \infty} \frac{\mu + \frac{\sigma \phi(x)}{1-(1-\phi(x)/x)}}{\mu + \sigma x} \\
&= \lim_{x \to \infty} \frac{\mu + x \frac{\sigma \phi(x)}{\phi(x)}}{\mu + \sigma x} \\
&= \lim_{x \to \infty} \frac{\mu + \sigma x}{\mu + \sigma x} \\
&= 1
\end{aligned}
$$

t-Student case

For the t-Student case we can apply the same argument for location-scale families, given that if we have $X \sim t(\nu, \mu, \sigma^2)$, then $X \sim \mu + \sigma T_\nu$, where T_ν is standard t-distributed with ν degrees of freedom. Then $VaR_\alpha = \mu + \sigma F_{t_\nu}^{-1}(\alpha)$. Now, also note that $F_{t_\nu}(x) = \int_{-\infty}^{x} f_{t_\nu}(y)dy \implies 1 - F_{t_\nu}(x) = \int_x^\infty f_{t_\nu}(y)dy$

$$
\begin{aligned}
1 - F_{t_\nu}(x) &= \int_x^\infty f_{t_\nu}(y)dy \\
&\approx \int_x^\infty \frac{C(\nu)}{y^{\nu+1}} dy \\
&= C(\nu) \int_x^\infty y^{-\nu-1} dy \\
&= C(\nu) \left. \frac{y^{-\nu}}{\nu} \right|_x^\infty \\
&= C(\nu) \frac{x^{-\nu}}{\nu}
\end{aligned}
$$

Again, as in the normal case, let x such that $x = F_{t_\nu}^{-1}(\alpha) = x \iff \alpha = F_{t_\nu}(x)$. Then:

The exercise also includes a theoretical proof for these limits. It uses the properties of the location-scale family of distributions and the behavior of their cumulative distribution functions (CDFs) and quantile functions as alpha approaches 1.

Heavier tails in a distribution indicate a greater likelihood of extreme values or outliers. In financial terms, this means a higher risk of extreme losses (or gains, depending on the context).In summary, this exercise numerically and theoretically explores how the ratio of expected shortfall to value at risk behaves as the confidence level approaches 1 for different distributions. It demonstrates that for the normal distribution, this ratio approaches 1, while for the Student's t-distribution, it approaches nu/(nu - 1), reflecting the heavier tails of the t-distribution.

$$
\begin{aligned}
\lim_{\alpha\to 1-}\frac{ES_\alpha}{VaR_\alpha} &= \lim_{\alpha\to 1-}\frac{\mu+\sigma\left(\frac{f_{t_\nu}(F_{t_\nu}^{-1}(\alpha))}{1-\alpha}\right)\left(\frac{\nu+(F_{t_\nu}^{-1}(\alpha))^2}{\nu-1}\right)}{\mu+\sigma\Phi^{-1}(\alpha)} \\
&= \lim_{x\to\infty}\frac{\mu+\sigma\left(\frac{f_{t_\nu(x)}}{C(\nu)\frac{x^{-\nu}}{\nu}}\right)\left(\frac{\nu+x^2}{\nu-1}\right)}{\mu+\sigma x} \\
&= \lim_{x\to\infty}\frac{\mu+\sigma\frac{\nu}{x}\frac{\nu+x^2}{\nu-1}}{\mu+\sigma x} \\
&= \lim_{x\to\infty}\frac{\mu}{\mu+\sigma x}+\lim_{x\to\infty}\sigma\frac{\nu}{\nu-1}\frac{\left(\frac{\nu+x^2}{x}\right)}{\mu+\sigma x} \\
&= \sigma\frac{\nu}{\nu-1}\lim_{x\to\infty}\frac{\nu+x^2}{\mu x+\sigma x^2} \\
&= \sigma\frac{\nu}{\nu-1}\lim_{x\to\infty}\frac{\frac{\nu}{x^2}+1}{\frac{\mu}{x}+\sigma} \\
&= \sigma\frac{\nu}{\nu-1}\frac{1}{\sigma} \\
&= \frac{\nu}{\nu-1}
\end{aligned}
$$

Heavier tails in a distribution indicate a greater likelihood of extreme values or outliers. In financial terms, this means a higher risk of extreme losses (or gains, depending on the context).

In summary, heavier tails in a distribution signify a greater risk of extreme outcomes. In financial risk management, this translates to a need for more cautious and comprehensive approaches to assess and mitigate risk, especially the risk of extreme losses.

Exercise 94

Recall that for a generalized Pareto distribution, the cdf is given by:

$$
F(x)=\begin{cases}1-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-1/\xi} & \xi\neq 0\\ 1-\exp\left(-\frac{x-\mu}{\sigma}\right) & \xi=0\end{cases}
$$

This means that we have two cases:

Case 1: $\xi \neq 0$

The value of provides insight into the likelihood and severity of extreme events.A positive indicates a long-tailed distribution where extreme events are more probable.A negative suggests a distribution with a natural upper limit, beyond which values are not possible.

If $\xi\neq 0$ then $F(x)=\alpha \iff 1-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-1/\xi}=\alpha \iff x=\mu+\sigma\frac{(1-\alpha)^{-\xi}-1}{\xi}$. Then:

$$
VaR_\alpha=\mu+\sigma\frac{(1-\alpha)^{-\xi}-1}{\xi}
$$

Then, we calculate the expected shortfall as follows:

Value at Risk (VaR):
Definition: VaR is a statistical measure that quantifies the level of financial risk within a firm, portfolio, or position over a specific time frame. It represents the maximum loss that is expected to be exceeded with a given probability, known as the confidence level, over a specified period under normal market conditions.
Example: If a portfolio has a 1-day 5% VaR of $1 million, it means that there is a 5% chance that the portfolio will lose more than $1 million in a single day.
Usage: VaR is widely used by banks, securities firms, and corporate risk managers to gauge the amount of assets needed to cover potential losses. It helps in risk assessment and regulatory compliance.
Expected Shortfall (ES):Definition: Also known as Conditional Value at Risk (CVaR), ES is a risk measure that quantifies the expected loss in the worst-case scenario (beyond the VaR threshold). It is the average loss during the worst %% of cases.
Example: If a portfolio has a 1-day 5% ES of $2 million, it means that, on average, the portfolio expects to lose $2 million or more on the worst 5% of days.
Advantages over VaR: ES provides information about the size of extreme losses, which VaR does not capture. It is also a coherent risk measure, addressing some of the criticisms of VaR, such as not being sub-additive.
Comparison and Relationship:VaR gives a threshold that losses are not expected to exceed, but it doesn’t tell anything about the magnitude of losses beyond this threshold.
ES, on the other hand, provides an average of the losses that occur beyond the VaR threshold, offering a clearer picture of the tail risk.
In practice, both measures are often used together for a more comprehensive risk assessment.

In summary, this exercise involves calculating key risk metrics for the Generalized Pareto Distribution and understanding their behavior as the probability of extreme losses increases. The analysis of the shortfall-to-quantile ratio is particularly important for understanding the severity of potential losses in the tail of the distribution.

$$
\begin{aligned}
ES_\alpha &= \frac{1}{1-\alpha}\int_\alpha^1 VaR_\omega d\omega \\
&= \frac{1}{1-\alpha}\int_\alpha^1 \mu + \sigma\frac{(1-\omega)^{-\xi}-1}{\xi} d\omega \\
&= \mu - \frac{\sigma}{\xi} + \frac{\sigma}{\xi(1-\alpha)}\int_\alpha^1 (1-\omega)^{-\xi} d\omega \\
&\text{Taking } u = 1-\omega \implies du = -d\omega \\
&= \mu - \frac{\sigma}{\xi} + \frac{\sigma}{\xi(1-\alpha)}\int_0^{1-\alpha} u^{-\xi} du \\
&= \mu - \frac{\sigma}{\xi} + \frac{\sigma}{\xi(1-\alpha)}\left[\frac{u^{-\xi+1}}{-\xi+1}\right]_0^{1-\alpha} \\
&= \mu - \frac{\sigma}{\xi} + \frac{\sigma}{\xi(1-\alpha)}\frac{(1-\alpha)^{1-\xi}}{1-\xi} \\
&= \mu + \frac{\sigma}{\xi}\left(\frac{(1-\alpha)^{1-\xi}}{(1-\alpha)(1-\xi)} - 1\right) \\
&= \mu + \frac{\sigma}{\xi}\left(\frac{(1-\alpha)^{-\xi}}{(1-\xi)} - 1\right) \\
&= \frac{1}{1-\xi}\left[\mu(1-\xi) + \frac{\sigma}{\xi}\left((1-\alpha)^{-\xi} - (1-\xi)\right)\right] \\
&= \frac{1}{1-\xi}\left[\mu - \mu\xi + \frac{\sigma}{\xi}((1-\alpha)^{-\xi}-1) + \sigma\right] \\
&= \frac{1}{1-\xi}\left[VaR_\alpha + \sigma - \mu\xi\right]
\end{aligned}
$$

Before calculating the limit, first note that:

$$
\lim_{a\to 1-} VaR_\alpha = \lim_{a\to 1-} \mu + \frac{\sigma}{\xi(1-\alpha)^\xi} - \frac{\sigma}{\xi} = \begin{cases} \infty & \xi \in (0,1) \\ \mu - \frac{\sigma}{\xi} & \xi < 0 \end{cases}
$$

This means that:

$$
\begin{aligned}
\lim_{a\to 1-} \frac{ES_\alpha}{VaR_\alpha} &= \lim_{a\to 1-} \frac{1}{1-\xi}\left[1 + \frac{\sigma - \xi\mu}{VaR_\alpha}\right] \\
&= \begin{cases} \frac{1}{1-\xi} & \xi \in (0,1) \\ \frac{1}{1-\xi}\left[1 + \frac{\sigma-\xi\mu}{\mu - \frac{\alpha}{\xi}}\right] = 1 & \xi < 0 \end{cases}
\end{aligned}
$$

Case 2: $\xi = 0$

If $\xi = 0$ then $F(x) = \alpha \iff 1 - \exp(-\frac{x-\mu}{\sigma}) = \alpha \iff x = \mu - \sigma\ln(1-\alpha)$. Then:

$$
VaR_\alpha = \mu - \sigma\ln(1-\alpha)
$$

exp dis

Then, we calculate the expected shortfall as follows:

$$
\begin{aligned}
ES_\alpha &= \frac{1}{1-\alpha}\int_\alpha^1 VaR_\omega d\omega \\
&= \frac{1}{1-\alpha}\int_\alpha^1 \mu - \sigma\ln(1-\omega)d\omega \\
&= \mu - \frac{\sigma}{1-\alpha}\int_\alpha^1 \ln(1-\omega)d\omega \\
&\text{Taking } u = 1-\omega \implies du = -d\omega \\
&= \mu - \frac{\sigma}{1-\alpha}\int_0^{1-\alpha} \ln(u)du \\
&= \mu - \frac{\sigma}{1-\alpha}\left[u(\ln(u)-1)\right|_0^{1-\alpha} \\
&= \mu - \frac{\sigma}{1-\alpha}((1-\alpha)(\ln(1-\alpha)-1)) \\
&= \mu - \sigma(\ln(1-\alpha)-1) \\
&= VaR_\alpha + \sigma
\end{aligned}
$$

This means that:

$$
\begin{aligned}
\lim_{a\to 1-}\frac{ES_\alpha}{VaR_\alpha} &= \lim_{a\to 1-}\frac{VaR_\alpha+\sigma}{VaR_\alpha} \\
&= 1 + \lim_{a\to 1-}\frac{\sigma}{\mu-\sigma\ln(1-\alpha)} \\
&= 1
\end{aligned}
$$

Exercise 96

a) Independence copula

Let $U_1, ..., U_d \overset{iid}{\sim} \mathcal{U}(0,1)$. Then note that by independence:

simply the product of the individual probabilities, which for uniform distributions is the product of the ui values.

$$P(U_1 \le u_1, ..., U_d \le u_d) = \prod_{i=1}^d P(U_i \le u_i) = \prod_{i=1}^d (u_i \mathbb{I}_{u_i\in[0,1]} + \mathbb{I}_{u_i>1})$$

joint probability

Then for $(u_1, ..., u_d) \in [0,1]^d$ we have that $C(u_1, ..., u_d) = \prod_{i=1}^d u_i$

b) Comonotonicity copula

Note that

joint probability

$$P(U_1 \le u_1, ..., U_1 \le u_d) = P(U_1 \le \min\{u_1, ..., u_d\}) = \min\{u_1, ..., u_d\}\mathbb{I}_{\min\{u_1,...,u_d\}\in[0,1]} + \mathbb{I}_{\min\{u_1,...,u_d\}>1}$$

Then for $(u_1, ..., u_d) \in [0,1]^d$ we have that $C(u_1, ..., u_d) = \min\{u_1, ..., u_d\}$

is the probability that 1U 1 is less than or equal to the minimum of all u i

.

This probability is non-zero only when 1u 1 is greater than or equal to 121u 2 , and it is equal to 1(12)u 1(1u 2) in that case.

c) Countermonotonicity copula

Let $U_2 = 1 - U_1$. Note that, given $u_1, u_2 \in [0,1]$

joint probability

$$\begin{aligned}
P(U_1 \leq u_1, U_2 \leq u_2) &= P(U_1 \leq u_1, 1 - U_1 \leq u_2) \\
&= P(U_1 \leq u_1, 1 - u_2 \leq U_1) \\
&= P(1 - u_2 \leq U_1 \leq u_1) \\
&= [u_1 - (1 - u_2)]\mathbb{I}_{\{u_1 \geq 1 - u_2\}}
\end{aligned}$$

In summary, these copulas represent different types of dependencies between random variables: independence, perfect positive dependence, and perfect negative dependence. The independence copula is a product of the individual probabilities, the comonotonicity copula is the minimum of the probabilities, and the countermonotonicity copula is a specific function reflecting the perfect negative correlation.

Then for $(u_1, u_2) \in [0,1]^2$ we have that $C(u_1, u_2) = [u_1 - (1 - u_2)]\mathbb{I}_{\{u_1 \geq 1 - u_2\}}$

Exercise 98

Recall that from the law of total probability we have that:

$$P(X_1 = 1) = P(X_1 = 1, X_2 = 1) + P(X_1 = 1, X_2 = 0) = \frac{3}{8} + \frac{2}{8} = \frac{5}{8}$$

$$P(X_2 = 1) = P(X_1 = 1, X_2 = 1) + P(X_1 = 0, X_2 = 1) = \frac{3}{8} + \frac{2}{8} = \frac{5}{8}$$

Then, $P(X_1 = 0) = 1 - P(X_1 = 1) = 1 - P(X_2 = 1) = P(X_2 = 0) = \frac{3}{8}$. This means that:

In summary, while we can calculate the marginal distributions and some specific values of the copula, finding all possible copulas for this discrete distribution is complex and may not yield a unique or simple solution.

$$F_{X_1}(x) = F_{X_2}(x) = \begin{cases} 0 & x < 0 \\ \frac{3}{8} & x \in [0,1) \\ 1 & x \geq 1 \end{cases}$$

However, determining all possible copulas for this distribution is not straightforward due to the discrete nature of the distribution and the fact that copulas typically describe dependencies in continuous distributions. In practice, for discrete distributions like this Bernoulli distribution, the copula is often approximated or simplified.

Now, we want to find all copulas C such that $P(X_1 \leq x_1, X_2 \leq x_2) = C(F_{X_1}(x_1), F_{X_2}(x_2))$. Note that in particular, we need for $x_1, x_2 \in [0,1)$ that:

$$C(F_{X_1}(x_1), F_{X_2}(x_2)) = C\left(\frac{3}{8}, \frac{3}{8}\right) = P(X_1 \leq x_1, X_2 \leq x_2) = P(X_1 = 0, X_2 = 0) = \frac{1}{8}$$

Exercise 99

Let X and Y be continuous random variables with cdfs F_X and F_Y , respectively, and copula C. Let $U \sim \mathcal{U}(0,1)$.

a)

both less than or equal to t

$$\begin{aligned}
P(\max\{X, Y\} \leq t) &= P(X \leq t, Y \leq t) \\
&= P(F_X^{-1}(U) \leq t, F_Y^{-1}(U) \leq t) \\
&= P(U \leq F_X(t), U \leq F_Y(t)) \\
&= C(F_X(t), F_Y(t))
\end{aligned}$$

same distri

copula links marg dis to their joint dist

b)

$$\begin{aligned} P(\min\{X,Y\} \leq t) &= P(\{X \leq t\} \cup \{Y \leq t\}) \\ &= P(X \leq t) + P(Y \leq t) - P(X \leq t, Y \leq t) \\ &\text{Using the definition of cdf and part a)} \\ &= F_X(t) + F_Y(t) - C(F_X(t), F_Y(t)) \end{aligned}$$

intersection prob from part a

Exercise 102

a)

Before proving what is requested, first notice that:

norm distr syymetric around 0

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(-x)^2}{2}\right) = \phi(-x)$$

Now, also notice that:

$$\begin{aligned} \Phi(-x) &= \int_{-\infty}^{-x} \phi(y)dy \\ &= 1 - \int_{-x}^{\infty} \phi(y)dy \\ &\text{Taking } -u = y \implies dy = -du \\ &= 1 - \int_{-\infty}^{x} \phi(-u)du \\ &= 1 - \Phi(x) \end{aligned}$$

cdf of the standard normal dis

Using the above, fix $i \in \{1,2\}$, then:

$$\begin{aligned} F_{X_i}(x) &= P(X_i \leq x) \\ &= P(\varepsilon_i X \leq x) \\ &= P(\varepsilon_i X \leq x | \varepsilon_i = 1)P(\varepsilon_i = 1) + P(\varepsilon_i X \leq x | \varepsilon_i = -1)P(\varepsilon_i = -1) \\ &= \frac{1}{2}P(X \leq x) + \frac{1}{2}P(-X \leq x) \\ &= \frac{1}{2}\Phi(x) + \frac{1}{2}P(X > -x) \\ &= \frac{1}{2}\Phi(x) + \frac{1}{2}(1 - \Phi(-x)) \\ &= \frac{1}{2}\Phi(x) + \frac{1}{2}(1 - (1 - \Phi(x))) \\ &= \Phi(x) \end{aligned}$$

Since the above argument works for $i = 1, 2$, we have proven $X_1 \sim N(0,1)$ and $X_2 \sim N(0,1)$. Now, let's prove they are not correlated:

$$
\begin{aligned}
\rho(X_1, X_2) &= \frac{Cov(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}} \\
&= Cov(X_1, X_2) \\
&= E[X_1 X_2] - E[X_1]E[X_2] \\
&= E[\varepsilon_1 \varepsilon_2 X^2] - E[\varepsilon_1 X]E[\varepsilon_2 X] \\
&\text{Due to independence} \\
&= E[\varepsilon_1]E[\varepsilon_2]E[X^2] - E[\varepsilon_1]E[X]E[\varepsilon_2]E[X] \\
&= 0
\end{aligned}
$$

b)

Notice both X_1 and X_2 depend on X. If they were independent, then we would have that $P(X_1 \in A | X_2 \in B) = P(X_1 \in A)$. Take A $(0, \infty)$ and $B = \{0\}$, and notice that, since $\varepsilon_2 X = 0 \iff X = 0$ and $\varepsilon_1 X \neq 0 \iff X \neq 0$, then $\{\varepsilon_1 X > 0\} \cap \{\varepsilon_2 X = 0\} = \{X > 0\} \cap \{X = 0\} = \emptyset$. Then:

$$
P(X_1 > 0 | X_2 = 0) = 0 \neq P(X_1 > 0) = 1 - P(X_1 \leq 0) = 1 - \Phi(0) = \frac{1}{2}
$$

Therefore X_1 and X_2 are not independent.

c)

Recall for $u, v \in [0,1]$, we want C such that $C(u,v) = P(\Phi(X_1) \leq u, \Phi(X_2) \leq v)$. By law of total probabilities we have:

$$
P(\Phi(X_1) \leq u, \Phi(X_2) \leq v) = \overbrace{\sum_{i=\{-1,1\}} \sum_{j=\{-1,1\}}}^{\text{total probability}} P(\Phi(X_1) \leq u, \Phi(X_2) \leq v | \varepsilon_1 = i, \varepsilon_2 = j) P(\varepsilon_1 = i, \varepsilon_2 = j)
$$

By independence and the fact that for $i = 1, 2$ we know $P(\varepsilon_i = \pm 1) = \frac{1}{2}$, we get that:

$$
\begin{aligned}
P(\Phi(X_1) \leq u, \Phi(X_2) \leq v) =& \frac{1}{4} P(\Phi(X) \leq u, \Phi(X) \leq v) + \frac{1}{4} P(\Phi(-X) \leq u, \Phi(-X) \leq v) \\
&+ \frac{1}{4} P(\Phi(-X) \leq u, \Phi(X) \leq v) + \frac{1}{4} P(\Phi(X) \leq u, \Phi(-X) \leq v)
\end{aligned}
$$

We will now find an expression for each term:

$$
\begin{aligned}
P(\Phi(X) \leq u, \Phi(X) \leq v) &= P(\Phi(X) \leq \min\{u, v\}) \\
&= P(X \leq \Phi^{-1}(\min\{u, v\})) \\
&= \Phi(\Phi^{-1}(\min\{u, v\})) \\
&= \min\{u, v\}
\end{aligned}
$$

$$
\begin{aligned}
P(\Phi(-X) \leq u, \Phi(-X) \leq v) &= P(\Phi(-X) \leq \min\{u, v\}) \\
&= P(1 - \Phi(X) \leq \min\{u, v\}) \\
&= P(1 - \min\{u, v\} \leq \Phi(X)) \\
&= 1 - P(\Phi(X) < 1 - \min\{u, v\}) \\
&= 1 - P(X < \Phi^{-1}(1 - \min\{u, v\})) \\
&= 1 - \Phi(\Phi^{-1}(1 - \min\{u, v\})) \\
&= \min\{u, v\}
\end{aligned}
$$

For the following two terms, recall that since P is a probability measure then P is non-negative. For that reason we must take into account cases when the inverses of the cdf might cause trouble.

$$
\begin{aligned}
P(\Phi(-X) \leq u, \Phi(X) \leq v) &= P(1 - \Phi(X) \leq (u), \Phi(X) \leq v) \\
&= P(1 - u \leq \Phi(X), \Phi(X) \leq v) \\
&= P(1 - u \leq \Phi(X) \leq v) \\
&= P(\Phi^{-1}(1 - u) \leq X \leq \Phi^{-1}(v)) \\
&= \Phi(\Phi^{-1}(v)) - \Phi(\Phi^{-1}(1 - u)) \\
&= \begin{cases} v - 1 + u & \text{If } v - 1 + u \geq 0 \\ 0 & \text{Otherwise} \end{cases} \\
&= \max\{v + u - 1, 0\}
\end{aligned}
$$

In summary, the solution demonstrates that 1X 1and2X 2 are normally distributed and uncorrelated but not independent. The copula is derived by considering the dependence structure imposed by X on1X1and 2X 2
and using the properties of the normal distribution.

By symmetry we have that $P(\Phi(X) \leq u, \Phi(-X) \leq v) = max\{v + u - 1, 0\}$. Therefore:

$$
C(u, v) = \frac{1}{2}\min\{u, v\} + \frac{1}{2}\max\{v + u - 1, 0\}
$$

Exercise 103

For model B, $X_1 \sim N(0, 1)$ is evident since $X_1 = Z$. Now, following the same logic as in exercise 102), we see $X_2 \sim N(0, 1)$:

$$
\begin{aligned}
F_{X_2}(x) &= P(X_2 \leq x) \\
&= P(\varepsilon Z \leq x) \\
&= P(\varepsilon Z \leq x | \varepsilon = 1)P(\varepsilon = 1) + P(\varepsilon Z \leq x | \varepsilon = -1)P(\varepsilon = -1) \\
&= \frac{1}{2}P(Z \leq x) + \frac{1}{2}P(-Z \leq x) \\
&= \frac{1}{2}\Phi(x) + \frac{1}{2}P(Z > -x) \\
&= \frac{1}{2}\Phi(x) + \frac{1}{2}(1 - \Phi(-x)) \\
&= \frac{1}{2}\Phi(x) + \frac{1}{2}(1 - (1 - \Phi(x))) \\
&= \Phi(x)
\end{aligned}
$$

The fact that they are not correlated follows again the logic from exercise 102):

$$
\begin{aligned}
\rho(X_1, X_2) &= \frac{Cov(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}} \\
&= Cov(X_1, X_2) \\
&= E[X_1X_2] - E[X_1]E[X_2] \\
&= E[\varepsilon Z^2] - E[Z]E[\varepsilon Z] \\
&\text{Due to independence} \\
&= E[\varepsilon]E[Z^2] - E[\varepsilon]E[Z]^2 \\
&= 0
\end{aligned}
$$

And finally, since both are standard normal distributions, from exercise 102) we also get that:

$$
C(u, v) = \frac{1}{2}\min\{u, v\} + \frac{1}{2}\max\{v + u - 1, 0\}
$$

dependence among x1 and x2 in B

Now, we compute the VaR for both models. For model A note that due to independence, $X_1 + X_2 \sim N(0, 2)$, and as we saw in exercise 93) $VaR_\alpha = \sqrt{2}\Phi^{-1}(1 - \alpha)$. We verify this numerically:

```
# Model A
alphas <- c(0.9, 0.95, 0.99, 0.999)
pnl_A <- rnorm(n = 10000)+rnorm(n = 10000)
VaR_A <- vapply(alphas,
                function(alpha) quantile(pnl_A, 1-alpha),
                0)
VaR_A
```

```
## [1] -1.815827 -2.310473 -3.255411 -4.342060
```

```
# We compare against the closed form provided above
sqrt(2)*qnorm(1-alphas)
```

```
## [1] -1.812388 -2.326174 -3.289953 -4.370248
```

For model B, since there is no independence we calculate the VaR numerically, as we cannot provide a closed form with the information provided for this homework:

```
# Model B
Z <- rnorm(n = 10000)
X1 <- Z
X2 <- -1 + 2*rbinom(n = 10000, size = 1, prob = 1/2)

pnl_B <- X1 + X2
VaR_B <- vapply(alphas,
                function(alpha) quantile(pnl_B, 1-alpha),
                0)
VaR_B
```

```
## [1] -1.834347 -2.296292 -3.078521 -3.808006
```

```
data.frame(alpha = alphas,
           VaR_Model_A = VaR_A,
           VaR_Model_B = VaR_B)
```

The computed VaRs for both models are compared for various
 levels.
The comparison shows how the dependence structure in Model B affects the aggregate risk (VaR) of the portfolios compared to the independent case in Model A.

```
##   alpha VaR_Model_A VaR_Model_B
## 1 0.900   -1.815827   -1.834347
## 2 0.950   -2.310473   -2.296292
## 3 0.990   -3.255411   -3.078521
## 4 0.999   -4.342060   -3.808006
```

Model B's VaR is generally lower than Model A's VaR for higher levels.
This suggests that the comonotonic and countermonotonic relationships in Model B (perfectly correlated or perfectly inversely correlated) lead to a different risk profile compared to the independent case in Model A.
The dependence structure in Model B can lead to scenarios where losses in one portfolio are offset by gains in the other, reducing the overall risk (VaR) under certain conditions

Exercise 107

We write an algorithm to simulate from the copula, first generating multivariate normal distribution:

```
rmvnorm <- function(n, m, Sigma){
  X <- matrix(rnorm(n*nrow(Sigma)), nrow = nrow(Sigma), ncol = n)

  A <- t(chol(Sigma))

  t(A %*% X + m)
}
```

And then using Φ:

```
gaussian_copula <- function(n, mean, P){
  Z <- rmvnorm(n, mean, P)

  t(apply(Z, 1, pnorm))
}
```

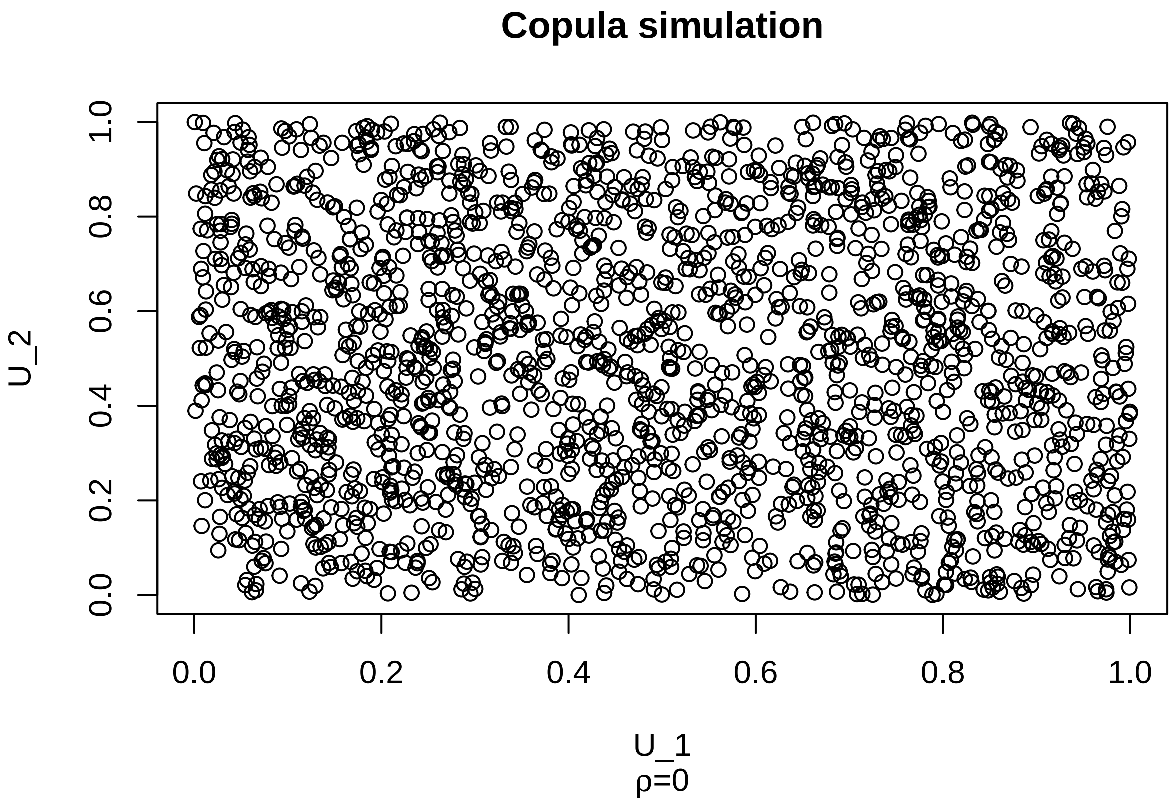
```
# Case 1: cov = 0
mean <- c(0,0)
P <- matrix(c(1,0,0,1), nrow = 2)
n <- 2000

gc_0 <- gaussian_copula(n, mean, P)
plot(gc_0,
     main = "Copula simulation",
     xlab = expression(U_1),
     ylab = expression(U_2),
     sub = expression(paste(rho,"=0")))
```

It's a type of copula derived from the multivariate normal distribution. It captures the dependence structure between variables.

Correlation Matrix (P): Determines the dependence structure in the Gaussian copula. The off-diagonal elements represent the correlation between different variables.
Simulation Process:
Generate a multivariate normal distribution with mean zero and correlation matrix P.
Transform this to a copula by applying the standard normal CDF () to each component.

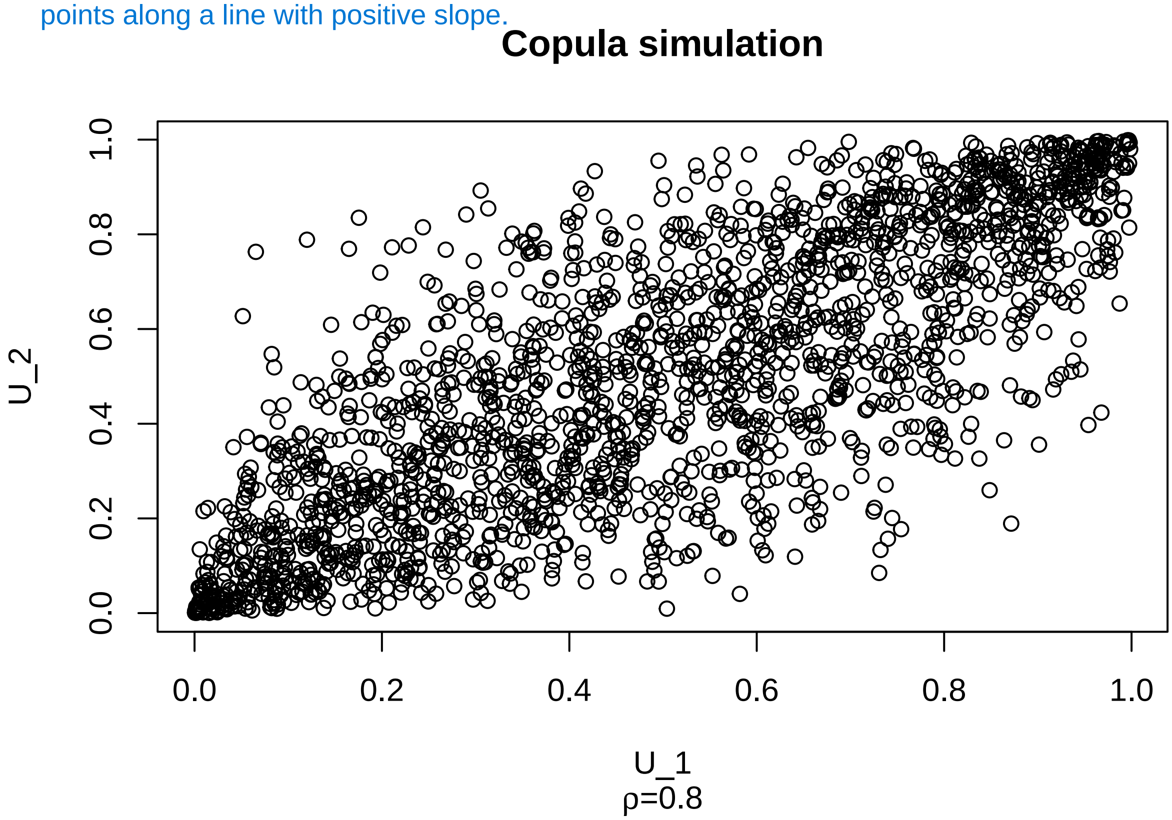
The variables are independent, shown by a lack of any clear pattern in the scatter plot




```r
# Case 2: cov = 0.8
P <- matrix(c(1,0.8,0.8,1), nrow = 2)

gc_0.8 <- gaussian_copula(n, mean, P)
plot(gc_0.8,
     main = "Copula simulation",
     xlab = expression(U_1),
     ylab = expression(U_2),
     sub = expression(paste(rho,"=0.8")))
```

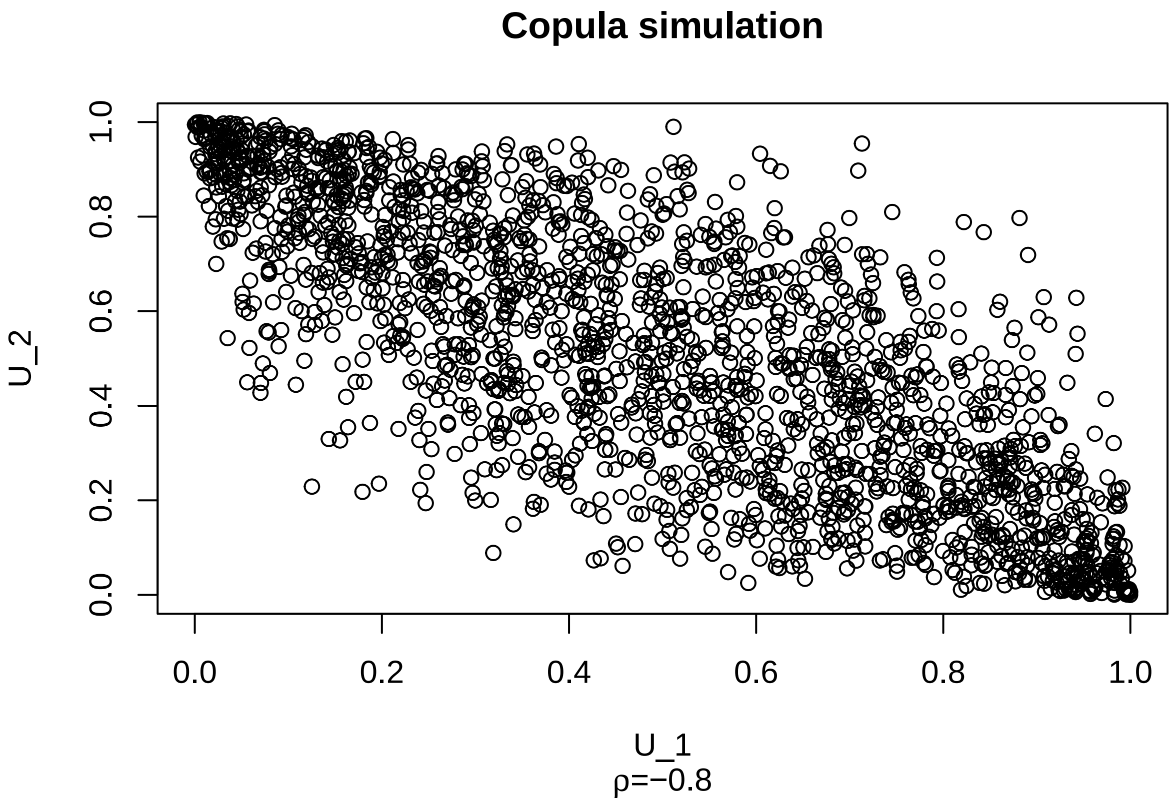
Positive correlation is evident. As one variable increases, so does the other, resulting in a concentration of points along a line with positive slope.




```r
# Case 3: cov = -0.8
P <- matrix(c(1,-0.8,-0.8,1), nrow = 2)

gc_m0.8 <- gaussian_copula(n, mean, P)
plot(gc_m0.8,
     main = "Copula simulation",
     xlab = expression(U_1),
     ylab = expression(U_2),
     sub = expression(paste(rho,"=-0.8")))
```

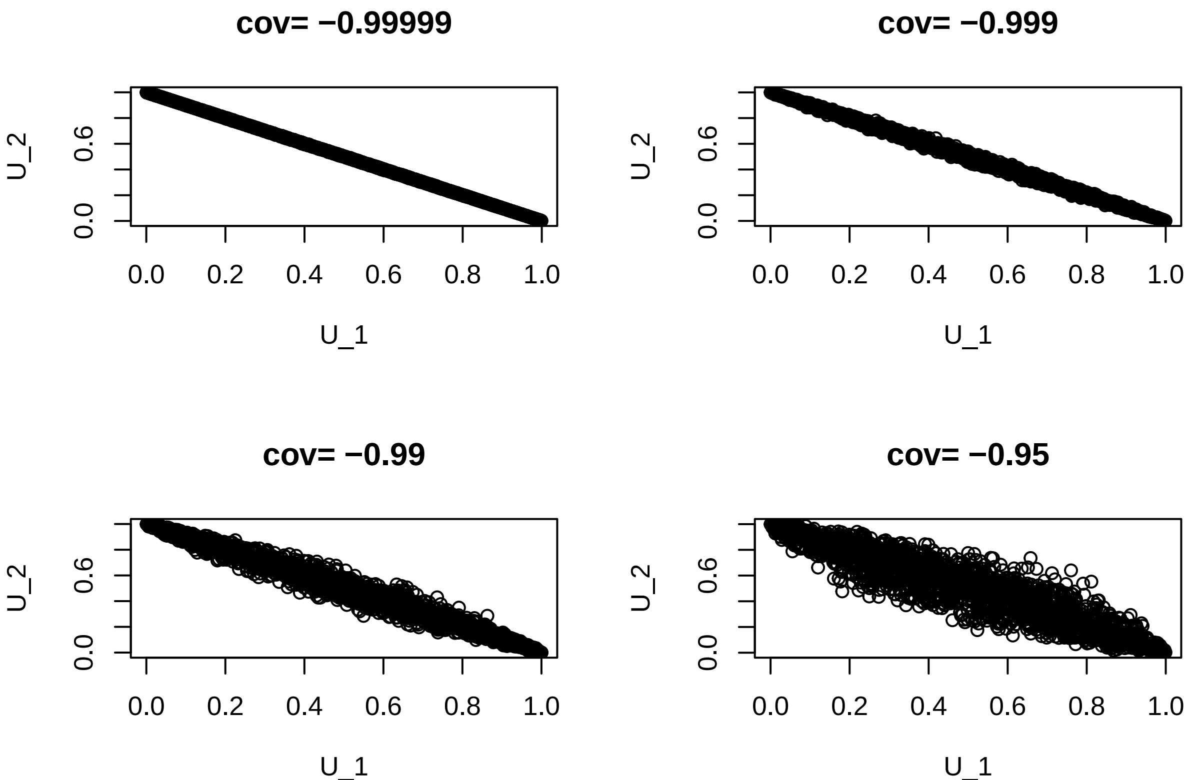
: Negative correlation is shown. As one variable increases, the other decreases, leading to a concentration of points along a line with negative slope.



We notice that the correlation, and thus the matrix P affects the behavior of the copula in the sense that it creates a correlation between the elements of the copula. This copula's correlation's sign and magnitude are directly proportional to the sign and magnitude of rho, and thus is determined by the P matrix. Now, to look at the behavior for the covariance close to ± 1, we plot multiple values of covariance:

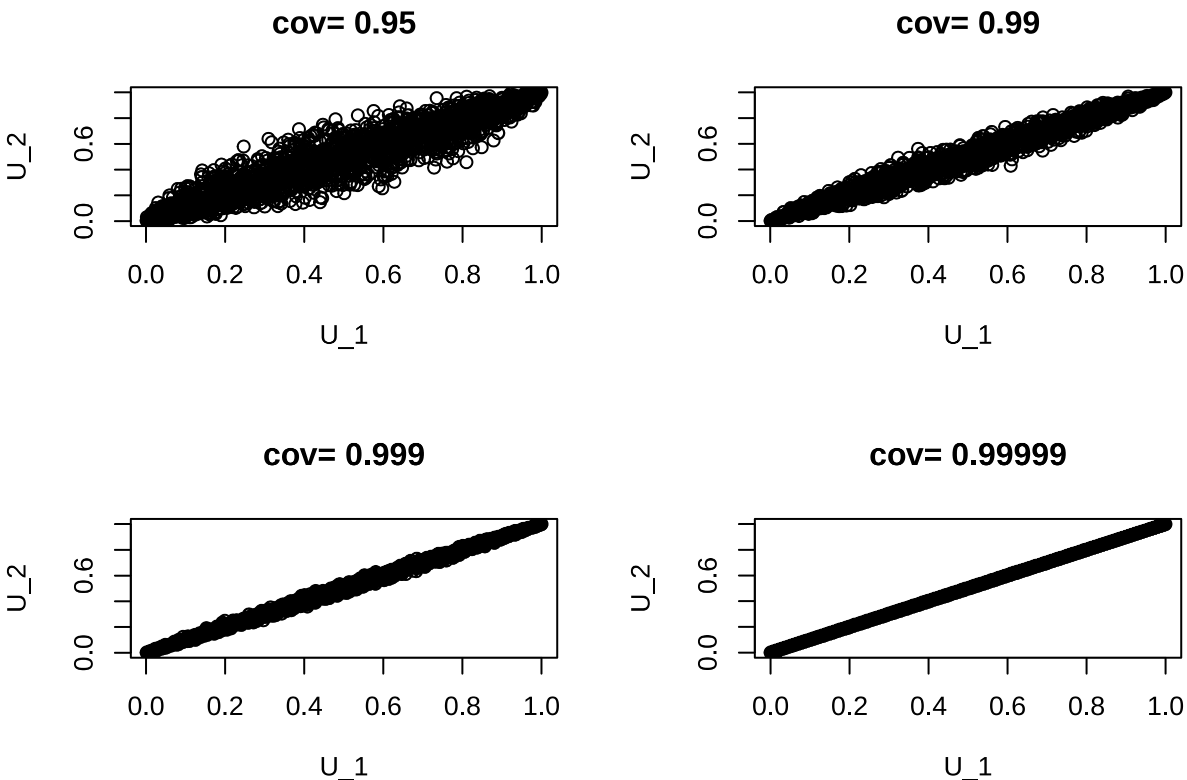
```
rhos <- c(-0.99999, -0.999, -0.99, -0.95)
par(mfrow = c(2,2))

for(rho in rhos){
  P <- matrix(c(1,rho,rho,1), nrow = 2)
  plot(gaussian_copula(n, mean, P),
       main = paste("cov=", rho),
       xlab = expression(U_1),
       ylab = expression(U_2))
}
```




```r
rhos <- c(0.95, 0.99, 0.999, 0.99999)
par(mfrow = c(2,2))

for(rho in rhos){
  P <- matrix(c(1,rho,rho,1), nrow = 2)
  plot(gaussian_copula(n, mean, P),
       main = paste("cov=", rho),
       xlab = expression(U_1),
       ylab = expression(U_2))
}
```



It is possible to see that as *rho* tends to ± 1, we get an almost perfectly linear relation between the components of the copula, where the slope of this relation changes based on the sign of the correlation. This coincides with the behavior described above.

Exercise 108

We first implement the simulation of a multivariate t random variable:

```
rmvt <- function(n, nu, m, Sigma){

  AZ <- rmvnorm(n, 0, Sigma)

  W <- nu/rchisq(n, df = nu)

  t( apply(sqrt(W)*AZ, 1, function(Xi) Xi+m))
}
```

Using that we implement the copula simulation:

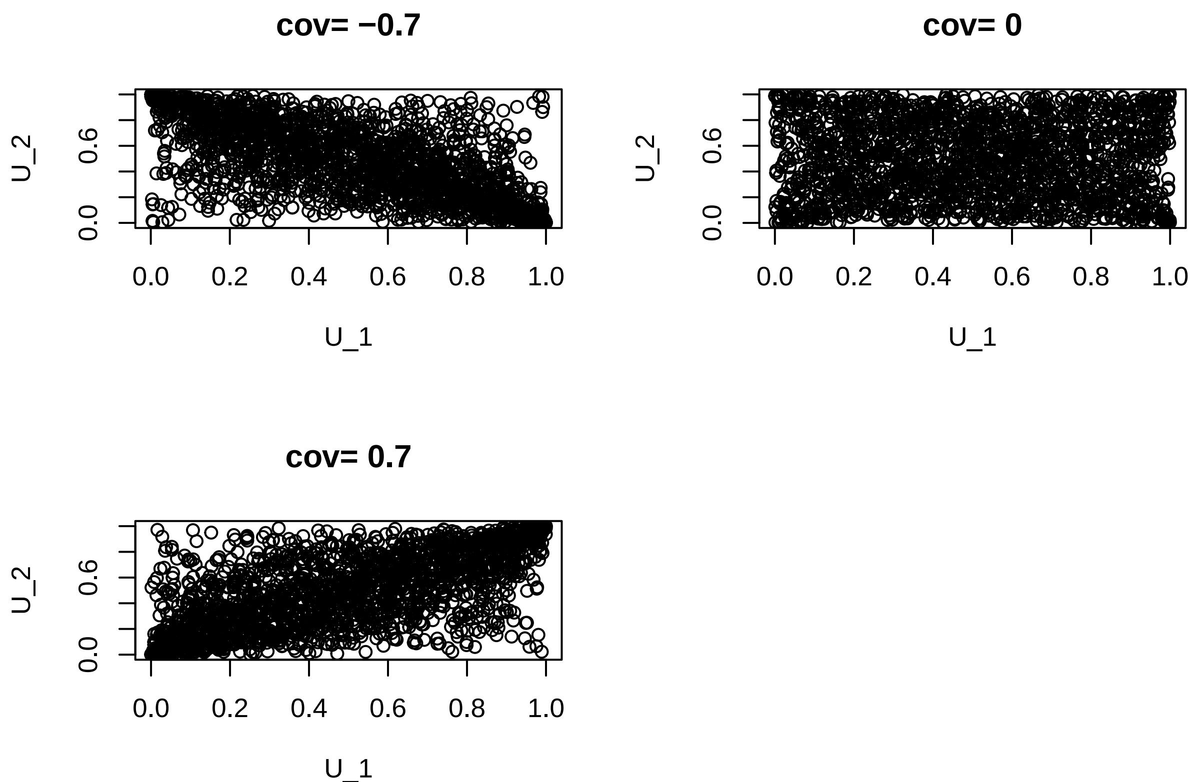
```
t_copula <- function(n, nu, mean, P){
  Z <- rmvt(n, nu, mean, P)

  t(apply(Z, 1, function(row) pt(row, df = nu)))
}
```

For $\nu = 3$

```
rhos <- c(-0.7, 0, 0.7)
par(mfrow = c(2,2))
```

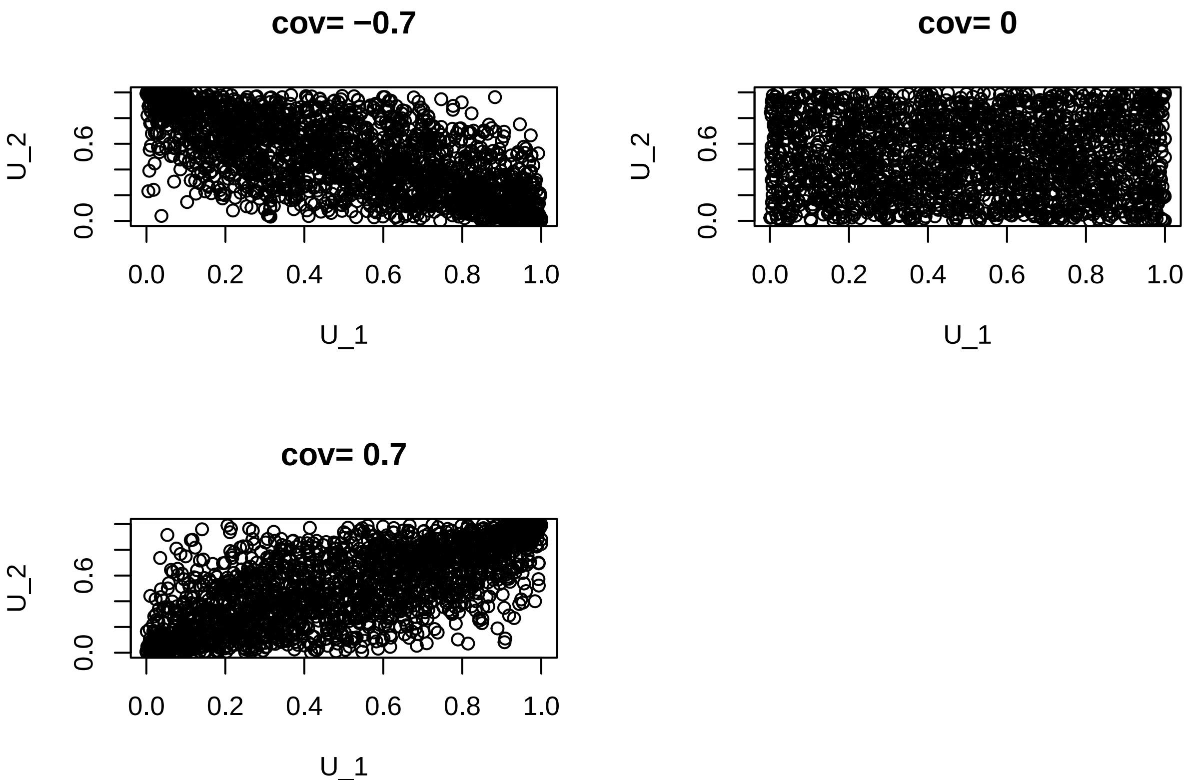
```r
for(rho in rhos){
  P <- matrix(c(1,rho,rho,1), nrow = 2)
  plot(t_copula(n, nu = 3, mean = c(0,0), P),
       main = paste("cov=", rho),
       xlab = expression(U_1),
       ylab = expression(U_2))
}
```



For $\nu = 10$

```r
rhos <- c(-0.7, 0, 0.7)
par(mfrow = c(2,2))

for(rho in rhos){
  P <- matrix(c(1,rho,rho,1), nrow = 2)
  plot(t_copula(n, nu = 10, mean = c(0,0), P),
       main = paste("cov=", rho),
       xlab = expression(U_1),
       ylab = expression(U_2))
}
```



Similar to the case of the Gaussian copula, we see that the sign and magnitude of the correlation directly generates a correlation in the case of the copulas. This is seen by the concentration and the "tilting" of the scatterplot, indicating a slightly more linear relationship between the variables (note this is a slight abuse of notation).

General Observations:
Influence of Correlation: Both Gaussian and t copulas show how the correlation matrix P directly influences the dependence structure between variables. Positive correlations lead to a positive linear relationship, negative correlations to a negative linear relationship, and zero correlation results in no discernible pattern.
Tail Behavior: The t copula, especially with lower degrees of freedom, shows heavier tails compared to the Gaussian copula. This is important in risk modeling where extreme values are of interest.
Theory and Concepts:
Copulas in Risk Modeling: These exercises demonstrate how copulas can model different types of dependencies between variables, which is crucial in financial risk assessment and portfolio management.
Sklar's Theorem: This theorem underpins these exercises, stating that any multivariate distribution can be expressed in terms of its marginals and a copula that captures the dependence structure.
In summary, these exercises illustrate the practical application of copulas in modeling complex dependencies and the impact of parameters like correlation and degrees of freedom on the dependence structure. The scatter plots visually represent these dependencies, showing how closely variables are related and the nature of their relationship.