Statistics 1 Unit 1 Team 8

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Exercise 1

a)

```
> hilb <- function(n){
+   i <- 1:n
+   outer(i, i, function(i,j) 1/(i+j-1))
+ }
> hilb(6)
```

```
          [,1]      [,2]      [,3]      [,4]      [,5]       [,6]
[1,] 1.0000000 0.5000000 0.3333333 0.2500000 0.2000000 0.16666667
[2,] 0.5000000 0.3333333 0.2500000 0.2000000 0.1666667 0.14285714
[3,] 0.3333333 0.2500000 0.2000000 0.1666667 0.1428571 0.12500000
[4,] 0.2500000 0.2000000 0.1666667 0.1428571 0.1250000 0.11111111
[5,] 0.2000000 0.1666667 0.1428571 0.1250000 0.1111111 0.10000000
[6,] 0.1666667 0.1428571 0.1250000 0.1111111 0.1000000 0.09090909
```

b)

Yes, all Hilbert matrices are invertible given that they are positive definite. According to `https://www.cambridge.org/core/journals/mathematical-gazette/article/abs/on-the-inverse-of-the-hilbert-matrix/C6D50D20CBBF617C14937F22685AE8D3`, the inverse of the hilbert matrix satisfies:

$$(H_n)_{ij} = (-1)^{i+j} \binom{i+n-1}{i} \binom{j+n-1}{n-1} \binom{i+j-2}{n-i} \binom{n}{j}$$

c)

```
> lapply(1:10, function(i) solve(hilb(i)))
```

```
[[1]]
     [,1]
[1,]    1

[[2]]
     [,1] [,2]
[1,]    4   -6
[2,]   -6   12

[[3]]
     [,1] [,2] [,3]
[1,]    9  -36   30
[2,]  -36  192 -180
[3,]   30 -180  180

[[4]]
     [,1]  [,2]  [,3]  [,4]
[1,]   16  -120   240  -140
[2,] -120  1200 -2700  1680
[3,]  240 -2700  6480 -4200
[4,] -140  1680 -4200  2800

[[5]]
      [,1]   [,2]    [,3]    [,4]   [,5]
[1,]    25   -300    1050   -1400    630
[2,]  -300   4800  -18900   26880 -12600
[3,]  1050 -18900   79380 -117600  56700
[4,] -1400  26880 -117600  179200 -88200
[5,]   630 -12600   56700  -88200  44100

[[6]]
      [,1]    [,2]     [,3]     [,4]     [,5]     [,6]
[1,]    36    -630     3360    -7560     7560    -2772
[2,]  -630   14700   -88200   211680  -220500    83160
[3,]  3360  -88200   564480 -1411200  1512000  -582120
[4,] -7560  211680 -1411200  3628800 -3969000  1552320
[5,]  7560 -220500  1512000 -3969000  4410000 -1746360
[6,] -2772   83160  -582120  1552320 -1746360   698544

[[7]]
       [,1]     [,2]      [,3]      [,4]       [,5]       [,6]      [,7]
[1,]     49    -1176      8820    -29400      48510     -38808     12012
[2,]  -1176    37632   -317520   1128960   -1940400    1596672   -504504
[3,]   8820  -317520   2857680 -10584000   18711000  -15717240   5045040
[4,] -29400  1128960 -10584000  40320000  -72765000   62092800 -20180160
[5,]  48510 -1940400  18711000 -72765000  133402500 -115259760  37837800
```

```
[6,] -38808  1596672 -15717240  62092800 -115259760  100590336 -33297264
[7,]  12012  -504504   5045040 -20180160   37837800  -33297264  11099088

[[8]]
        [,1]      [,2]       [,3]       [,4]        [,5]        [,6]
[1,]      64     -2016      20160     -92400      221760     -288288
[2,]   -2016     84672    -952560    4656960   -11642400    15567552
[3,]   20160   -952560   11430720  -58212000   149688000  -204324119
[4,]  -92400   4656960  -58212000  304919999  -800414996  1109908794
[5,]  221760 -11642400  149688000 -800414996  2134439987 -2996753738
[6,] -288288  15567552 -204324119 1109908793 -2996753738  4249941661
[7,]  192192 -10594584  141261119 -776936154  2118916782 -3030050996
[8,]  -51480   2882880  -38918880  216215998  -594593995   856215351
            [,7]       [,8]
[1,]      192192     -51480
[2,]   -10594584    2882880
[3,]   141261119  -38918880
[4,]  -776936155  216215998
[5,]  2118916783 -594593995
[6,] -3030050996  856215352
[7,]  2175421226 -618377753
[8,]  -618377753  176679358

[[9]]
              [,1]          [,2]          [,3]          [,4]          [,5]
 [1,]  8.099993e+01 -3.239995e+03  4.157992e+04 -2.494794e+05  8.108078e+05
 [2,] -3.239995e+03  1.727997e+05 -2.494794e+06  1.596668e+07 -5.405385e+07
 [3,]  4.157992e+04 -2.494794e+06  3.841982e+07 -2.561321e+08  8.918884e+08
 [4,] -2.494794e+05  1.596668e+07 -2.561321e+08  1.756334e+09 -6.243218e+09
 [5,]  8.108078e+05 -5.405385e+07  8.918884e+08 -6.243218e+09  2.254495e+10
 [6,] -1.513508e+06  1.037834e+08 -1.748101e+09  1.243094e+10 -4.545062e+10
 [7,]  1.621615e+06 -1.135130e+08  1.942334e+09 -1.398481e+10  5.164843e+10
 [8,] -9.266368e+05  6.589418e+07 -1.141617e+09  8.302667e+09 -3.091879e+10
 [9,]  2.187892e+05 -1.575283e+07  2.756745e+08 -2.021613e+09  7.581048e+09
              [,6]          [,7]          [,8]          [,9]
 [1,]      -1513508       1621615 -9.266369e+05  2.187892e+05
 [2,]     103783367    -113513038  6.589418e+07 -1.575283e+07
 [3,]   -1748100981    1942334186 -1.141617e+09  2.756745e+08
 [4,]   12430939701  -13984805997  8.302667e+09 -2.021613e+09
 [5,]  -45450621475   51648430832 -3.091879e+10  7.581048e+09
 [6,]   92553989760 -106051443959  6.393054e+10 -1.576858e+10
 [7,] -106051443630  122367050066 -7.420509e+10  1.839668e+10
 [8,]   63930539052  -74205091248  4.522977e+10 -1.126327e+10
 [9,]  -15768581291   18396678800 -1.126327e+10  2.815818e+09

[[10]]
```

```
              [,1]          [,2]          [,3]          [,4]          [,5]
 [1,]  9.999719e+01 -4.949757e+03  7.919482e+04 -6.005529e+05       2522295
 [2,] -4.949756e+03  3.266790e+05 -5.880152e+06  4.756344e+07    -208088462
 [3,]  7.919480e+04 -5.880151e+06  1.128980e+08 -9.512635e+08    4280662450
 [4,] -6.005527e+05  4.756343e+07 -9.512634e+08  8.244246e+09  -37871868827
 [5,]  2.522294e+06 -2.080884e+08  4.280662e+09 -3.787187e+10  176734991839
 [6,] -6.305679e+06  5.350810e+08 -1.123668e+10  1.009913e+11 -477183582308
 [7,]  9.608580e+06 -8.323436e+08  1.775667e+10 -1.615857e+11  771204559101
 [8,] -8.750614e+06  7.700557e+08 -1.663323e+10  1.528915e+11 -735791094422
 [9,]  4.375282e+06 -3.898391e+08  8.505604e+09 -7.883452e+10  382044919568
[10,] -9.236661e+05  8.313037e+07 -1.828875e+09  1.706955e+10  -83214282335
              [,6]          [,7]          [,8]          [,9]         [,10]
 [1,] -6.305682e+06  9.608586e+06 -8.750620e+06  4.375286e+06 -9.236669e+05
 [2,]  5.350812e+08 -8.323439e+08  7.700561e+08 -3.898393e+08  8.313042e+07
 [3,] -1.123669e+10  1.775667e+10 -1.663324e+10  8.505608e+09 -1.828876e+09
 [4,]  1.009913e+11 -1.615857e+11  1.528915e+11 -7.883455e+10  1.706956e+10
 [5,] -4.771836e+11  7.712047e+11 -7.357912e+11  3.820450e+11 -8.321430e+10
 [6,]  1.301409e+12 -2.120813e+12  2.037577e+12 -1.064270e+12  2.330005e+11
 [7,] -2.120813e+12  3.480308e+12 -3.363622e+12  1.765901e+12 -3.883348e+11
 [8,]  2.037577e+12 -3.363622e+12  3.267520e+12 -1.723107e+12  3.804102e+11
 [9,] -1.064270e+12  1.765901e+12 -1.723107e+12  9.122340e+11 -2.020931e+11
[10,]  2.330005e+11 -3.883348e+11  3.804101e+11 -2.020931e+11  4.490964e+10
```

```
> lapply(1:10, function(n){
+   tryCatch(expr = qr.solve(hilb(n)),
+            error = function(e) paste0("For n=",
+                                       n,
+                                       " the Hilbert matrix is numerically singular."))
+ })
```

```
[[1]]
     [,1]
[1,]    1

[[2]]
     [,1] [,2]
[1,]    4   -6
[2,]   -6   12

[[3]]
     [,1] [,2] [,3]
[1,]    9  -36   30
[2,]  -36  192 -180
[3,]   30 -180  180

[[4]]
```

```
     [,1]  [,2]  [,3]  [,4]
[1,]   16  -120   240  -140
[2,] -120  1200 -2700  1680
[3,]  240 -2700  6480 -4200
[4,] -140  1680 -4200  2800

[[5]]
      [,1]   [,2]    [,3]    [,4]   [,5]
[1,]    25   -300    1050   -1400    630
[2,]  -300   4800  -18900   26880 -12600
[3,]  1050 -18900   79380 -117600  56700
[4,] -1400  26880 -117600  179200 -88200
[5,]   630 -12600   56700  -88200  44100

[[6]]
      [,1]    [,2]     [,3]     [,4]     [,5]     [,6]
[1,]    36    -630     3360    -7560     7560    -2772
[2,]  -630   14700   -88200   211680  -220500    83160
[3,]  3360  -88200   564480 -1411200  1512000  -582120
[4,] -7560  211680 -1411200  3628800 -3969000  1552320
[5,]  7560 -220500  1512000 -3969000  4410000 -1746360
[6,] -2772   83160  -582120  1552320 -1746360   698544

[[7]]
[1] "For n=7 the Hilbert matrix is numerically singular."

[[8]]
[1] "For n=8 the Hilbert matrix is numerically singular."

[[9]]
[1] "For n=9 the Hilbert matrix is numerically singular."

[[10]]
[1] "For n=10 the Hilbert matrix is numerically singular."
```

It is possible to see that, even though all Hilbert matrices are invertible, they become numerically singular quite quickly, making the inverse calculation either inaccurate or straight up not possible computationally.

Exercise 2

Notice that this is equivalent to solving the following system $Ax = b$:

$$\begin{pmatrix} 10^0 & 10^1 & 10^2 & 10^3 & 10^4 & 10^5 \\ 11^0 & 11^1 & 11^2 & 11^3 & 11^4 & 11^5 \\ 12^0 & 12^1 & 12^2 & 12^3 & 12^4 & 12^5 \\ 13^0 & 13^1 & 13^2 & 13^3 & 13^4 & 13^5 \\ 14^0 & 14^1 & 14^2 & 14^3 & 14^4 & 14^5 \\ 15^0 & 15^1 & 15^2 & 15^3 & 15^4 & 15^5 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} 25 \\ 16 \\ 26 \\ 19 \\ 21 \\ 20 \end{pmatrix}$$

Note A here is the Vandermonde matrix with the given values and until the power of 5.

```
> # We want to solve the system Ax = b
> b <- c(25, 16, 26, 19, 21, 20)
> A <- outer(10:15, seq_along(10:15) - 1, `^`)
> coeffs <- solve(A, b)
> coeffs
```

```
[1]  2.536100e+05 -1.025510e+05  1.650092e+04 -1.320667e+03  5.258333e+01
[6] -8.333333e-01
```

We test the coefficients to see if they do in fact work:

```
> directpoly <- function(x, coeffs){
+   vapply(x, function(x0) sum(x0^(0:(length(coeffs)-1))*coeffs), 0 )
+ }
> directpoly(10:15, coeffs)
```

```
[1] 25 16 26 19 21 20
```

Exercise 3

We start by creating the matrix with random elements:

```
> X <- matrix(runif(n = 15, min = 0, max = 1),
+             nrow = 5,
+             ncol = 3)
```

a)

```
> H <- X%*% solve( t(X)%*%X ) %*% t(X)
> H
```

```
           [,1]       [,2]      [,3]       [,4]       [,5]
[1,]  0.8355635  0.1385838 0.1747098 -0.2615680  0.1387456
[2,]  0.1385838  0.6315847 0.1555698  0.1314225 -0.4147362
[3,]  0.1747098  0.1555698 0.4499560  0.3850429  0.2109788
[4,] -0.2615680  0.1314225 0.3850429  0.5524303  0.1153400
[5,]  0.1387456 -0.4147362 0.2109788  0.1153400  0.5304656
```

```
> eig <- eigen(H)
> eig

eigen() decomposition
$values
[1] 1.000000e+00 1.000000e+00 1.000000e+00 2.259317e-15 2.075107e-15

$vectors
            [,1]        [,2]       [,3]       [,4]       [,5]
[1,]  0.84229808 -0.17010568 -0.3117074  0.3873150 -0.1200982
[2,] -0.08733724 -0.73979463 -0.2768768 -0.4749841 -0.3778961
[3,] -0.03120740  0.04364218 -0.6686385 -0.2327250  0.7041897
[4,] -0.49572081  0.08564691 -0.5471342  0.5635385 -0.3605469
[5,]  0.19024491  0.64383844 -0.2823907 -0.5026337 -0.4657186
```

b)

```
> trace <- sum(diag(H))
> eig_sum <- sum(eig$values)
> c(Trace = trace, Eigenvalue_sum = eig_sum)

         Trace Eigenvalue_sum
             3              3
```

c)

```
> determinant <- det(H)
> eig_prod <- prod(eig$values)
> c(Determinant = determinant, Eigenvalue_prod = eig_prod)

    Determinant Eigenvalue_prod
  -2.528363e-33    4.688324e-30
```

d)

We verify that each column of X is an eigenvector of H and at the same time calculate which is the corresponding eigenvalue by calculating $H \cdot x_i$ where x_i is the $i-$th column of X, and dividing (element-wise) with each element of x_i.

Note: In the case we get a 0 in some column, this might result in a NaN, but this is not a problem to show what we want to show as $0 = \lambda \cdot 0 \forall \lambda \in \mathbb{R}$.

```
> apply(X, 2, function(col){
+   Hx <- (H %*% col)
+ })

           [,1]       [,2]       [,3]
[1,] 0.05745912 0.50803248 0.64174117
```

```
[2,] 0.55428149 0.03011838 0.34679412
[3,] 0.93201058 0.64998415 0.41423154
[4,] 0.97909177 0.40377660 0.10787183
[5,] 0.18668379 0.51476637 0.09593902
```

```
> X
```

```
           [,1]       [,2]       [,3]
[1,] 0.05745912 0.50803248 0.64174117
[2,] 0.55428149 0.03011838 0.34679412
[3,] 0.93201058 0.64998415 0.41423154
[4,] 0.97909177 0.40377660 0.10787183
[5,] 0.18668379 0.51476637 0.09593902
```

```
> apply(X, 2, function(col){
+   Hx <- (H %*% col)/col
+ })
```

```
     [,1] [,2] [,3]
[1,]    1    1    1
[2,]    1    1    1
[3,]    1    1    1
[4,]    1    1    1
[5,]    1    1    1
```

We see therefore that the columns of X are eigenvectors associated to the eigenvalue 1.

Exercise 4

We use the function defined in exercise 1:

```
> H <- hilb(6)
> H
```

```
          [,1]      [,2]      [,3]      [,4]      [,5]       [,6]
[1,] 1.0000000 0.5000000 0.3333333 0.2500000 0.2000000 0.16666667
[2,] 0.5000000 0.3333333 0.2500000 0.2000000 0.1666667 0.14285714
[3,] 0.3333333 0.2500000 0.2000000 0.1666667 0.1428571 0.12500000
[4,] 0.2500000 0.2000000 0.1666667 0.1428571 0.1250000 0.11111111
[5,] 0.2000000 0.1666667 0.1428571 0.1250000 0.1111111 0.10000000
[6,] 0.1666667 0.1428571 0.1250000 0.1111111 0.1000000 0.09090909
```

We compute its eigenvalues and eigenvectors:

```
> eigen(H)
```

```
eigen() decomposition
$values
[1] 1.618900e+00 2.423609e-01 1.632152e-02 6.157484e-04 1.257076e-05
[6] 1.082799e-07

$vectors
           [,1]       [,2]       [,3]        [,4]        [,5]         [,6]
[1,] -0.7487192  0.6145448 -0.2403254 -0.06222659  0.01114432 -0.001248194
[2,] -0.4407175 -0.2110825  0.6976514  0.49083921 -0.17973276  0.035606643
[3,] -0.3206969 -0.3658936  0.2313894 -0.53547692  0.60421221 -0.240679080
[4,] -0.2543114 -0.3947068 -0.1328632 -0.41703769 -0.44357472  0.625460387
[5,] -0.2115308 -0.3881904 -0.3627149  0.04703402 -0.44153664 -0.689807199
[6,] -0.1814430 -0.3706959 -0.5027629  0.54068156  0.45911482  0.271605453
```

We now compute the inverse:

```
> H_inv <- solve(H)
> H_inv

      [,1]    [,2]     [,3]     [,4]     [,5]     [,6]
[1,]    36    -630     3360    -7560     7560    -2772
[2,]  -630   14700   -88200   211680  -220500    83160
[3,]  3360  -88200   564480 -1411200  1512000  -582120
[4,] -7560  211680 -1411200  3628800 -3969000  1552320
[5,]  7560 -220500  1512000 -3969000  4410000 -1746360
[6,] -2772   83160  -582120  1552320 -1746360   698544
```

And its eigenvalues and eigenvectors:

```
> eigen(H_inv)

eigen() decomposition
$values
[1] 9.235320e+06 7.954970e+04 1.624040e+03 6.126880e+01 4.126079e+00
[6] 6.177034e-01

$vectors
             [,1]        [,2]        [,3]       [,4]       [,5]       [,6]
[1,]  0.001248194 -0.01114432 -0.06222659 -0.2403254 -0.6145448 -0.7487192
[2,] -0.035606643  0.17973276  0.49083921  0.6976514  0.2110825 -0.4407175
[3,]  0.240679080 -0.60421221 -0.53547692  0.2313894  0.3658936 -0.3206969
[4,] -0.625460387  0.44357472 -0.41703769 -0.1328632  0.3947068 -0.2543114
[5,]  0.689807199  0.44153664  0.04703402 -0.3627149  0.3881904 -0.2115308
[6,] -0.271605453 -0.45911482  0.54068156 -0.5027629  0.3706959 -0.1814430
```

Now, let's examine the eigenvalues of H and H^{-1}:

```
> eigen(H)$values
```

```
[1] 1.618900e+00 2.423609e-01 1.632152e-02 6.157484e-04 1.257076e-05
[6] 1.082799e-07
```

```
> eigen(H_inv)$values
```

```
[1] 9.235320e+06 7.954970e+04 1.624040e+03 6.126880e+01 4.126079e+00
[6] 6.177034e-01
```

```
> eigen(H)$values*eigen(H_inv)$values
```

```
[1] 1.495106e+07 1.927974e+04 2.650680e+01 3.772616e-02 5.186793e-05
[6] 6.688490e-08
```

In theory, eigenvalues of the inverse matrix should equal the multiplicative inverse of the eigenvalues of the eigenvalues of the original matrix. However, it is possible to see this is not the case in this example. This is due to the ill-conditioning of the problem.

Exercise 5

a)

We define P:

```
> circulant <- function(probs){
+   n <- length(probs)
+
+   indx <- c(rep(1:n %% (n+1), n-1), 1)
+
+   indx <- indx[rep(1:n, n) + rep( seq(from = 0,
+                                      to = (n-1)^2,
+                                      by = n-1),
+                                  each = n )]
+
+   matrix(probs[indx], nrow = n, ncol = n, byrow = T)
+ }
> P <- circulant(1:4/10)
> P
```

```
     [,1] [,2] [,3] [,4]
[1,]  0.1  0.2  0.3  0.4
[2,]  0.4  0.1  0.2  0.3
[3,]  0.3  0.4  0.1  0.2
[4,]  0.2  0.3  0.4  0.1
```

We verify the rows add up to 1:

```
> apply(P, 1, sum)
```

```
[1] 1 1 1 1
```

b)

```
> matPow <- function(A, n){
+
+   eig <- eigen(A)
+
+   Pmat <- eig$vectors
+
+   res <- lapply(n, function(i) Re(Pmat %*% diag(eig$values^i) %*% solve(Pmat)) )
+
+   names(res) <- paste0("A^", n)
+
+   res
+ }
> matPow(P, c(1,2,3,5,10,20))
```

```
$`A^1`
     [,1] [,2] [,3] [,4]
[1,]  0.1  0.2  0.3  0.4
[2,]  0.4  0.1  0.2  0.3
[3,]  0.3  0.4  0.1  0.2
[4,]  0.2  0.3  0.4  0.1

$`A^2`
     [,1] [,2] [,3] [,4]
[1,] 0.26 0.28 0.26 0.20
[2,] 0.20 0.26 0.28 0.26
[3,] 0.26 0.20 0.26 0.28
[4,] 0.28 0.26 0.20 0.26

$`A^3`
      [,1]  [,2]  [,3]  [,4]
[1,] 0.256 0.244 0.240 0.260
[2,] 0.260 0.256 0.244 0.240
[3,] 0.240 0.260 0.256 0.244
[4,] 0.244 0.240 0.260 0.256

$`A^5`
        [,1]    [,2]    [,3]    [,4]
[1,] 0.25056 0.25072 0.24928 0.24944
[2,] 0.24944 0.25056 0.25072 0.24928
[3,] 0.24928 0.24944 0.25056 0.25072
[4,] 0.25072 0.24928 0.24944 0.25056

$`A^10`
          [,1]      [,2]      [,3]      [,4]
[1,] 0.2500000 0.2500016 0.2500000 0.2499983
```

```
[2,] 0.2499983 0.2500000 0.2500016 0.2500000
[3,] 0.2500000 0.2499983 0.2500000 0.2500016
[4,] 0.2500016 0.2500000 0.2499983 0.2500000

$`A^20`
     [,1] [,2] [,3] [,4]
[1,] 0.25 0.25 0.25 0.25
[2,] 0.25 0.25 0.25 0.25
[3,] 0.25 0.25 0.25 0.25
[4,] 0.25 0.25 0.25 0.25
```

c)

Notice we want to find x that solves the system $(P' - I)x = 0$, and such that $\sum x_i = 1$.

When writing this system out we get the following:

$$\begin{pmatrix} -0.9 & 0.4 & 0.3 & 0.2 \\ 0.2 & -0.9 & 0.4 & 0.3 \\ 0.3 & 0.2 & -0.9 & 0.4 \\ 0.4 & 0.3 & 0.2 & -0.9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We want a nontrivial solution with non-negative elements that add to 1. Note $0.9 = 0.4 + 0.3 + 0.2$. Therefore all equations are satisfied if $x_1 = x_2 = x_3 = x_4$. Also, $\sum x_i = 1 \Leftrightarrow \sum x_1 = 1 \Leftrightarrow 4 \cdot x_1 = 1 \Leftrightarrow x_1 = \frac{1}{4}$.

Therefore $x_1 = x_2 = x_3 = x_4 = \frac{1}{4}$. We verify this:

```
> t(P) %*% rep(0.25, 4)

     [,1]
[1,] 0.25
[2,] 0.25
[3,] 0.25
[4,] 0.25
```

It is possible to see that the x found is to what the powers of P seem to converge both column and row-wise. Particularly the rows P^10 seem to be very close to x. And it is possible to see that in the case of P^20 they are equal.

Exercise 8

Since L_1 and L_2 are non-singular and lower triangular, every diagonal element must be non-zero. This means that the whole matrix is itself lower triangular (note that this is a slight abuse of notation given the partitioned linear system might not be square). Therefore, it is possible to solve the system in the following way:

1. Solve $L_1 \cdot x = b$ through forward solve.

2. Once x is known, then $B \cdot x$ is a known vector.

3. Therefore it's enough to solve $L_2 \cdot y = c - B \cdot x$, also with a forward solve.

Exercise 9

a)

Note that by construction of M_k, the first non-zero entry of the k-th column that is not 1 will be $-\mu_{k+1}$, and will be located in the $k+1$-th element of this column. All other non-zero elements (i.e all other μ_i's) will be below this. Therefore, M_k is a lower triangular matrix with all diagonal elements equal to 1.

Also, since M_k is lower triangular $det(M_k) = \prod_{i=1}^{n} m_{ii} = \prod_{i=1}^{n} 1 = 1$. Therefore, M_k is non-singular.

b)

Let $m_k = (0, 0, ..., \mu_{k+1}, ..., \mu_n)^t$ and $e_k = (0, 0, ..., 0, 1, 0, ..., 0)^t$, where the 1 is in the k-th position. Then $A := m_k e_k^t$ is an $n \times n$ matrix for which the i, j-th element is given by: $a_{ij} = (m_k)_i \cdot (e_k)_j$. Note that $(e_k)_j = 0$ if $j \neq k$ and $(e_k)_k = 1$. Also, $(m_k)_i = 0$ for all $i \leq k$, and $(m_k)_i = \mu_i$ for $i \geq k+1$.

Therefore $a_{ij} = 0$ for all $j \neq k$, $a_{ik} = 0$ for all $i \leq k$, and finally $a_{ik} = \mu_i$ for all $i \geq k+1$. In other words:

$$A = m_k e_k^t = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n & \cdots & 0 \end{pmatrix}$$

And it is clear to see that:

$$I - m_k e_k^t = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & 0 & \cdots & -\mu_{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\mu_n & \cdots & 1 \end{pmatrix} = M_k$$

c)

Note that:

$$\begin{aligned}(I - m_k e_k^t) \cdot (I + m_k e_k^t) &= I - m_k e_k^t + (I - m_k e_k^t) m_k e_k^t \\ &= I - m_k e_k^t + m_k e_k^t - m_k e_k^t m_k e_k^t \\ &= I - m_k e_k^t m_k e_k^t \\ &= I - (m_k e_k^t)^2\end{aligned}$$

Note that (using the last part's notation) $(m_k e_k^t)^2 = A^2$. Now:

$$\begin{aligned}(A^2)_{ij} &=< a_i, a^j > \\ &= \sum_{m=1}^{n} a_{im} \cdot a_{mj}\end{aligned}$$

Recall that $a_{ij} = 0$ for all $j \neq k$, $a_{ik} = 0$ for all $i \leq k$, and finally $a_{ik} = \mu_i$ for all $i \geq k+1$. Notice that in the sum, not all of them can hold at the same time, since we would need $j = m = k$ for the column to have non-zero entries, but this would mean the row of the second term would also be k, but $a_{kk} = 0$. Thus, $A^2 = 0$.

We have shown then that

$$(I - m_k e_k^t) \cdot (I + m_k e_k^t) = I$$

.

Therefore

$$M_k^{-1} = I + m_k e_k^t$$

d)

Using what we have found in the previous parts, note that:

$$\begin{aligned}M_k M_l &= (I - m_k e_k')(I - m_l e_l') \\ &= (I - m_k e_k') - (I - m_k e_k') m_l e_l' \\ &= I - m_k e_k' - m_l e_l' + m_k e_k' m_l e_l'\end{aligned}$$

Now, notice that

$$(m_k e_k' m_l e_l')_{ij} = \sum_{r=1}^{n} (m_k e_k')_{ir} (m_l e_l')_{rj}$$

However, notice that for $(m_k e_k')_{ir}$ to be different from 0 we need $r = k$. If this happens, then $(m_l e_l')_{rj} = 0$ since $k < l$. If it doesn't then clearly $(m_k e_k')_{ir} = 0$.

Then:

$$(m_k e_k' m_l e_l')_{ij} = 0 \qquad \forall i, j \in \{1, ..., n\}$$

Therefore:

$$M_k M_l = I - m_k e_k' - m_l e_l'$$

Exercise 10

Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We try to find a lower triangular and a upper triangular matrix L and U, such that $A = LU$. Let

$$L = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \qquad U = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$$

By matrix multiplication, LU can be expressed as:

$$LU = \begin{bmatrix} ad & ae \\ bd & be + cf \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then, we know that $ad = 0$ only if $a = 0$ or $d = 0$. However, if either of those is 0, then $ae = 0$ or $bd = 0$, which shows that matrix A can not be expressed by LU decomposition.

Exercise 11

($\Rightarrow$)

Suppose $rank(A) = 1$. Then all rows of A are linearly dependent. Let $i \in \{1, ..., n\}$ such that the i-th row of A A_i is a non-zero vector. This row has to exist because otherwise A would be the 0 matrix, which has rank 0. Then there exist $\alpha_1, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_n$ such that:

$$A = \begin{pmatrix} \alpha_1 \cdot A_i \\ \vdots \\ \alpha_{i-1} \cdot A_i \\ A_i \\ \alpha_{i+1} \cdot A_i \\ \vdots \\ \alpha_n \cdot A_i \end{pmatrix}$$

Take $u = (\alpha_1, ..., \alpha_{i-1}, 1, \alpha_{i+1}, ..., \alpha_n)^t$ and $v = A_i$. Then $uv' = A$, and clearly u, v are non-zero vectors.

($\Leftarrow$)

Let $u, v \in \mathbb{R}^n$ be non-zero vectors. Then:

$$
\begin{aligned}
A &= uv' \\
&= \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} (v_1, ..., v_n) \\
&= \begin{pmatrix} u_1 v_1 & \cdots & u_1 v_n \\ \vdots & \ddots & \vdots \\ u_n v_1 & \cdots & u_n v_n \end{pmatrix}
\end{aligned}
$$

Let $i, k \in \{1, ..., n\}$ such that $u_i \neq 0$, and $i \neq k$. Then take $\alpha = \frac{u_k}{u_i}$, and notice that $A_k = \alpha \cdot A_i$. Therefore all rows of A are linearly dependent.

Therefore $rank(A) = 1$.

Exercise 12

a)

For this proof we will follow the same logic as in question 24. For that, note that:

$$det(uv' - \lambda I) = det(-(\lambda I - uv')) = (-1)^n det(\lambda I - uv')$$

Therefore, using the result in exercise 24, $(u'v - \lambda)(\lambda)^{n-1} = (-1)^n det(\lambda I - uv')$.

Evaluating in $\lambda = 1$, we have that $u'v - 1 = (-1)^n det(I - uv')$. We can see the determinant is 0 (and therefore $I - uv'$ singular) if and only if $u'v = 1$.

b)

We want to show $\exists \sigma$ such that $(I - uv')(I - \sigma uv') = I$. First, notice that if $uv' \neq 0$ then $\sigma \neq 0$. If $uv' = 0$ then we just get the identity matrix, for which the inverse is itself, so take $\sigma = 0$.

Now, assume $uv' \neq 0$. Then:

$$
\begin{aligned}
(I - uv')(I - \sigma uv') &= I - uv' - \sigma uv'(I - uv') \\
&= I - uv' - \sigma uv' + \sigma uv'uv' \\
&= I - uv' - \sigma uv' + \sigma(v'u)uv' \\
&= I - (I + \sigma I - \sigma(v'u)I)uv'
\end{aligned}
$$

Notice we want $I + \sigma I - \sigma(v'u)I = 0 \Leftrightarrow 1 + \sigma - \sigma v'u = 0$.

$$
\begin{aligned}
1+\sigma-\sigma v'u=0 &\Leftrightarrow 1+\sigma=\sigma(v'u)\\
&\Leftrightarrow \frac{1}{\sigma}+1=(v'u)\\
&\Leftrightarrow \sigma=\frac{1}{v'u-1}
\end{aligned}
$$

Note that this coincides with what we found in part a), since if $v'u=1$, then σ does not exist and therefore A would not have an inverse.

c)

Yes, M_k are elementary matrices. In this case, $u=m_k$, $v=e_k$, and $\sigma=\frac{1}{m_k'e_k-1}$ as long as $m_k'e_k\neq 1$.

Exercise 13

Note that, multiplying on the right by $A-uv'$ we get:

$$
\begin{aligned}
&[A^{-1}+A^{-1}u(1-v'A^{-1}u)^{-1}v'A^{-1}](A-uv')\\
&=A^{-1}(A-uv')+A^{-1}u(1-v'A^{-1}u)^{-1}v'A^{-1}(A-uv')\\
&=I-A^{-1}uv'+A^{-1}u(1-v'A^{-1}u)^{-1}v'-A^{-1}u(1-v'A^{-1}u)^{-1}v'A^{-1}uv'\\
&=I-A^{-1}u[I+(1-v'A^{-1}u)^{-1}v'A^{-1}uI-(1-v'A^{-1}u)^{-1}I]v'\\
&=I-A^{-1}u[I+(1-v'A^{-1}u)^{-1}(v'A^{-1}u-1)I]v'\\
&=I-A^{-1}u[I-(1-v'A^{-1}u)^{-1}(1-v'A^{-1}u)I]v'\\
&=I-A^{-1}u[I-I]v'\\
&=I-0\\
&=I
\end{aligned}
$$

Therefore:

$$(A-uv')^{-1}=A^{-1}+A^{-1}u(1-v'A^{-1}u)^{-1}v'A^{-1}$$

Exercise 14

Similarly to the previous exercise, note that multiplying on the right by $A-UV'$ we get:

$$
\begin{aligned}
&[A^{-1} + A^{-1}U(I - V'A^{-1}U)^{-1}V'A^{-1}](A - UV') \\
&= A^{-1}(A - UV') + A^{-1}U(I - V'A^{-1}U)^{-1}V'A^{-1}(A - UV') \\
&= I - A^{-1}UV' + A^{-1}U(I - V'A^{-1}U)^{-1}V' - A^{-1}U(I - V'A^{-1}U)^{-1}V'A^{-1}UV' \\
&= I - A^{-1}U[I + (I - V'A^{-1}U)^{-1}V'A^{-1}U - (I - V'A^{-1}U)^{-1}]V' \\
&= I - A^{-1}U[I + (I - V'A^{-1}U)^{-1}(V'A^{-1}U - I)]V' \\
&= I - A^{-1}U[I - (I - V'A^{-1}U)^{-1}(I - V'A^{-1}U)]V' \\
&= I - A^{-1}U[I - I]V' \\
&= I - 0 \\
&= I
\end{aligned}
$$

Therefore:

$$(A - UV')^{-1} = A^{-1} + A^{-1}U(I - V'A^{-1}U)^{-1}V'A^{-1}$$

Exercise 22

a)

We have to prove that if λ is an eigenvalue of A_{11}, then it is also an eigenvalue of A. (Hint: let u be the corresponding eigenvector of A_{11}, and determine an $(n-k)$-vector v such that $[u', v']'$ is an eigenvector of A with eigenvalue λ.)

Let $w = \begin{bmatrix} u' \\ v' \end{bmatrix}$, then we have to prove that $Aw = \lambda w$. We also have from the task that $A_{11}u = \lambda u$. Then,

$$A[u, v]' = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A_{11}u + A_{12}v \\ A_{22}v \end{bmatrix} = \begin{bmatrix} \lambda u + A_{12}v \\ A_{22}v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}.$$

Then, we get the following equations: $\lambda u + A_{12}v = \lambda u$, and $A_{22}v = \lambda v$. From the first equation, we get that $v = 0$. Then, using this equality, we get

$$A \begin{bmatrix} u \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} u \\ 0 \end{bmatrix}$$

So, λ is an eigenvalue of A.

b)

The problem asks us to show that if λ is an eigenvalue of A_{22} (but not of A_{11}), then it is also an eigenvalue of A. To do this, we need to find a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. We can use the hint provided in the problem statement to find this vector.

Since A_{22} is an (n-k) $\times$ (n-k) matrix, it has n - k eigenvalues. Let λ_2 be one of these eigenvalues. Since λ_2 is not an eigenvalue of A_{11}, $\det(A_{11} - \lambda_2 I) \neq 0$, meaning $(A_{11} - \lambda_2 I)$ is invertible. We can solve the equation $A_{11}\mathbf{u} + A_{12}\mathbf{v} = \lambda_2\mathbf{u}$ for $\mathbf{u}$ to get:

$$\mathbf{u} = -(A_{11} - \lambda_2 I)^{-1} A_{12}\mathbf{v}$$

where I is the identity matrix of size $k \times k$. We can choose $\mathbf{v}$ to be any nonzero vector in $\mathbb{R}^{n-k}$ since the choice of $\mathbf{v}$ does not affect the eigenvalue of $[\mathbf{u}, \mathbf{v}]$.

Now, let's compute $A\mathbf{x}$ for the vector $[\mathbf{u}, \mathbf{v}]$:

$$A[\mathbf{u}, \mathbf{v}]^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} A_{11}\mathbf{u} + A_{12}\mathbf{v} \\ A_{22}\mathbf{v} \end{bmatrix}$$

Since $\mathbf{u}$ is given by $\mathbf{u} = -(A_{11} - \lambda_2 I)^{-1} A_{12}\mathbf{v}$, we have:

$$A_{11}\mathbf{u} + A_{12}\mathbf{v} = A_{11}(-(A_{11} - \lambda_2 I)^{-1} A_{12}\mathbf{v}) + A_{12}\mathbf{v} =$$

$$-\lambda_2(A_{11} - \lambda_2 I)^{-1} A_{12}\mathbf{v} + A_{12}\mathbf{v} = (A_{12} - \lambda_2 I)(A_{11} - \lambda_2 I)^{-1} A_{12}\mathbf{v}$$

Substituting this into the expression for $A[\mathbf{u}, \mathbf{v}]^T$, we get:

$$A[\mathbf{u}, \mathbf{v}]^T = \begin{bmatrix} A_{11}\mathbf{u} + A_{12}\mathbf{v} \\ A_{22}\mathbf{v} \end{bmatrix} = \begin{bmatrix} (A_{12} - \lambda_2 I)(A_{11} - \lambda_2 I)^{-1} A_{12}\mathbf{v} \\ \lambda_2\mathbf{v} \end{bmatrix}$$

We can see that $[\mathbf{u}, \mathbf{v}]$ is an eigenvector of A with eigenvalue λ_2. Therefore, if λ_2 is an eigenvalue of A_{22} (but not of A_{11}), then it is also an eigenvalue of A.

c)

If λ is an eigenvalue of A with corresponding eigenvector [u', v']' where u is a k-vector,show that λ is an eigenvalue ofA_{11} with corresponding eigenvector u or an eigenvalue of A_{22} with corresponding eigenvector v.

$$A \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A_{11}u + A_{12}v \\ A_{22}v \end{bmatrix} = \begin{bmatrix} \lambda u \\ \lambda v \end{bmatrix}.$$

From this we get the following equations: $A_{11}u + A_{12}v = \lambda u$, and $A_{22}v = \lambda v$. For u to be an eigenvalue, v must be 0. Then, u is the eigenvector corresponding to A_{11}. Otherwise, if v is non zero, then the first equation implies that u could not be an eigenvalue of A_{11} if v is not in the kernel of A_{12} since $A_{11}u + A_{12}v = \lambda u$

$$A \begin{bmatrix} u \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} u \\ 0 \end{bmatrix}$$

So, u is an eigenvector of A_{11} or v is an eigenvector of A_{22}

d)

($\Rightarrow$) Is evident from c).
($\Leftarrow$) Is evident from a) and b).

Exercise 24

Let's first deal with the edge case where $u = 0$ or $v = 0$. In this case, $uv' = 0$ (in the matrix sense), and also $u'v = 0$. Therefore, $det(I + uv') = det(I) = 1 = 1 + u'v$.

Now, assume $u \neq 0$ and $v \neq 0$. Recall from exercise 11 that uv' has rank 1. This means that uv' has at most one non-zero eigenvalue.

Note that $uv'u = u(v'u) = (v'u)u$. Therefore u is an eigenvector of uv' associated to the eigenvalue $\lambda = v'u$. Let's now see that if a non-zero eigenvalue exists, then it has to be $v'u$:

Case 1: If $v'u \neq 0$ then this is evident.

Case 2: If $v'u = 0$, then u, v are orthogonal. Assume that there exist a non-zero $x \in \mathbb{R}^n$ and $\lambda \neq 0$ such that $uv'x = \lambda x$. Then, multiplying by v' on the left, we have that $v'uv'x = v'\lambda x$, but because of orthogonality this implies that $0 = \lambda v'x$. But since $\lambda \neq 0$, then $v'x = 0$, so $uv'x = 0$ (Contradiction!).

Therefore, $\lambda = v'u$ is the only potential non-zero eigenvalue. This means that we can write the characteristic polynomial of uv' as:

$$det(uv' - \lambda I) = (v'u - \lambda) \cdot \lambda^{n-1}$$

This means that the eigenvalues of uv' are $u'v$ with multiplicity 1 and 0 with multiplicity $n - 1$. Now, notice that is x is an eigenvector of uv' associated to λ then $(I + uv')x = Ix + uv'x = x + \lambda x = (1 + \lambda)x$. Therefore the eigenvalues of $I + uv'$ are 1 plus the eigenvalues of uv'.

Therefore: $det(I + uv') = (1 + u'v) \cdot \prod_{i=1}^{n-1} 1 = 1 + u'v$.

Exercise 25

Let

$$H = I - 2\frac{vv'}{v'v}$$

Notice that $(vv')_{ij} = v_i \cdot v_j = (vv')_{ji}$, so vv' is symmetric.

Then, since it corresponds to only a scalar multiplication, then $2\frac{vv'}{v'v}$ is also a symmetric matrix. Finaly, as $I - 2\frac{vv'}{v'v}$ only changes elements of the diagonal, and the sign of all off diagonal elements, then H is symmetric.

Now, notice that:

$$
\begin{aligned}
HH' &= \left(I - 2\frac{vv'}{v'v}\right)\left(I - 2\frac{vv'}{v'v}\right) \\
&= I - 2\frac{vv'}{v'v} - 2\frac{vv'}{v'v} + 4\frac{vv'vv'}{v'vv'v} \\
&= I - 4\frac{vv'}{v'v} + 4\frac{vv'}{v'v} \\
&= I
\end{aligned}
$$

Thus, H is an orthogonal matrix. Therefore, if λ is an eigenvalue of H, then $\lambda = \pm 1$. Notice that when applying the transformation to a vector, we get $Hx = (I - 2\frac{vv'}{v'v})x = x - 2\frac{vv'}{v'v}x = x - 2\frac{v'x}{v'v}v$, which means we are reflecting the vector x with respect to the orthogonal hyperplane generated by v. That is, the middle-point between x and the resulting vector after applying the transformation is exactly the orthogonal projection of x onto the hyperplane.

Exercise 27

Let's prove it by induction over the degree of the polynomial:

Base case: $n = 1$:

Then $C = (-\gamma_0) \Rightarrow det(C - zI) = det(-\gamma_0 - z) = -\gamma_0 - z = (-1) \cdot (\gamma_0 + z) = (-1)^1 p(z)$. So it holds.

Inductive step: Assume it holds for $n - a$, let's prove it for n:

$$
det(C - zI) = \begin{pmatrix} -z & 0 & \cdots & 0 & -\gamma_0 \\ 1 & -z & \cdots & 0 & -\gamma_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\gamma_{n-1} - z \end{pmatrix}
$$

Let's use Laplace's formula expanding over the first row. Let $A := C - zI$:

$$
\begin{aligned}
det(C - zI) &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} det(A_{-1j}) \\
&= (-1)^2(-z)det(A_{-11}) + (-1)^{n+1}(-\gamma_0) \\
&= (-z)(-1)^{n-1}(\gamma_1 + \gamma_2 z + ... + \gamma_{n-1} z^{n-2} + z^{n-1}) + (-1)^{1+n}(-\gamma_0) \\
&= (-1)^n(\gamma_1 z + \gamma_2 z^2 + ... + \gamma_{n-1} z^{n-1} + z^n) + (-1)^{n+1}(-\gamma_0) \\
&= (-1)^n(\gamma_1 z + \gamma_2 z^2 + ... + \gamma_{n-1} z^{n-1} + z^n) + (-1)^n \gamma_0 \\
&= (-1)^n(\gamma_0 + \gamma_1 z + \gamma_2 z^2 + ... + \gamma_{n-1} z^{n-1} + z^n) \\
&= (-1)^n p(z)
\end{aligned}
$$

Let's now calculate the roots of $p(z) = 24 - 40z + 35z^2 - 13z^3 + z^4$ using the companion matrix:

```
> C <- cbind(rbind(0, diag(1, nrow = 3)), c(-24,40,-35,13))
> eigen(C)$values

[1] 9.8274224+0.0000000i 1.8047699+0.0000000i 0.6839038+0.9409769i
[4] 0.6839038-0.9409769i

> p <- function(x) 24 - 40*x + 35*x^2 - 13*x^3 + x^4
> p(eigen(C)$values)

[1]  0.000000e+00+0.000000e+00i -7.460699e-14+0.000000e+00i
[3] -3.241851e-14-5.151435e-14i -3.241851e-14+5.151435e-14i
```

Exercise 29

```
> vec <- function(A){
+   c(A)
+ }
> A <- hilb(5)
> vec(A)

 [1] 1.0000000 0.5000000 0.3333333 0.2500000 0.2000000 0.5000000 0.3333333
 [8] 0.2500000 0.2000000 0.1666667 0.3333333 0.2500000 0.2000000 0.1666667
[15] 0.1428571 0.2500000 0.2000000 0.1666667 0.1428571 0.1250000 0.2000000
[22] 0.1666667 0.1428571 0.1250000 0.1111111
```

Exercise 30

```
> vech <- function(A){
+   vec(A[lower.tri(A, diag = T)])
+ }
> vech(A)

 [1] 1.0000000 0.5000000 0.3333333 0.2500000 0.2000000 0.3333333 0.2500000
 [8] 0.2000000 0.1666667 0.2000000 0.1666667 0.1428571 0.1428571 0.1250000
[15] 0.1111111
```