Math 402/502 **Modern Geometry** **Spring 2009**

We start with a set of objects. The set will be called *the universe* and denoted $\mathcal{U}$. The elements of $\mathcal{U}$ will be called *points* and will often be denoted by the letters P, Q, and R (though the set of points may be infinite). Specific subsets of $\mathcal{U}$ will be called *lines* and we will denote lines by ℓ, ℓ_1, and ℓ_2 (though again there might be an infinite number of lines). It is usually our axioms that tell us which subsets can be lines. The only properties about the universe and its points and line are those in the axioms given to us as well as those results which we are able to prove. If ℓ is a line and P is a point of ℓ, i.e. $P \in \ell$, we instead will say that P *is on* ℓ. We also use the word incidence to indicate that a point lies on a line.

We begin with our axioms.

Axiom A *There exists at least one point and one line.*

Axiom B *Every line contains at least two points.*

Axiom C *If P and Q are two different points, there is a line which contains both P and Q.*

Axiom D *If L and M are two different lines, there is no more than one point contained in the intersection $L \cap M$.*

Definition 1 If a set $\mathcal{T}$ of points is contained in some line, then $\mathcal{T}$, as well as the points in $\mathcal{T}$, are said to be *collinear*. Otherwise, $\mathcal{T}$ (and its points) are *noncollinear*. Two lines are said to be *parallel* if they are disjoint (i.e. have empty intersection). (Notice that the definition of parallel means that a line is not parallel to itself.)

Axiom E *There exists a noncollinear set of three points.*

Theorem 1 *If P and Q are two different points, there do not exist two different lines containing both P and Q.*

Theorem 2 *Every point is contained in at least two lines.*

Definition 2 If P and Q are different points, the unique line containing points P and Q will be denoted by $\overleftrightarrow{PQ}$.

Theorem 3 *If A and B are two different points of $\overleftrightarrow{PQ}$ then $\overleftrightarrow{AB}=\overleftrightarrow{PQ}$.*

Definition 3 A geometry is said to be *elliptical* if no two lines are parallel.

A geometry is said to be *Euclidean* if given a point P not on a line ℓ there is a unique line through P that is parallel to ℓ.

A geometry is said to be *hyperbolic* if given a point P not on a line ℓ there are at least two lines through P that are parallel to ℓ.

In each of the following interpretations of the undefined terms, which of the axioms are satisfied and which are not? Tell whether each interpretation has the elliptical, Euclidean, or hyperbolic parallel property.

Example 1 "Points" are lines in 3-space, "lines" are planes in 3-space, incidence means a line lying in a plane.

Example 2 "Points" are 1-dimensional subspaces of 3-space, "lines" are 2-dimensional subspaces of 3-space, incidence means a 1-dimensional subspace of a 2-dimensional space.

Example 3 Fix the unit circle. "Points" are the ordered pairs (x, y) satisfying $x^2 + y^2 < 1$. "Lines" are chords of the circle. Incidence is the usual sense.

Example 4 †3† *Fix the unit sphere. A "point" is a set consisting of two points from the sphere that lie in a diameter of the sphere. "lines" are great circles of the sphere, that is the diameter of the circle is 2. Incidence means both points lie on the circle.*

We now assume there is a ternary relation on certain triples of points, where if A, B, C are three *different* points, we write the relation as A–B–C, and say B *is between* A *and* C. Observe that A–A–B is false.

Axiom F *If A–B–C, then B–C–A is false.*

Axiom G *If A–B–C, then C–B–A. (This says betweeness is symmetric.)*

Axiom H *If A–B–C, then $\{A, B, C\}$ is a collinear set.*

Axiom I *If $\{A, B, C\}$ is a collinear set of three different points, then one of the following is true: A–B–C, or B–C–A, or C–A–B.*

Theorem 4 †3†

(a) If A–B–C, then B–A–C is false.
(b) If A–B–C, then A–C–B is false.
(c) If A–B–C, then C–A–B is false.

Theorem 5 *If $\{A, B, C\}$ is a collinear set of three different points, then exactly one of the following is true: A–B–C, or B–C–A, or C–A–B.*

Definition 4 Suppose that A and B are two points.
(a) $\overline{AB}$=
(b) $\overset{\circ\!-\!\circ}{AB}$=

Definition 5 Suppose A and B are two different points.
(a) $\overrightarrow{AB}$=
(b) $\overset{\circ\!\longrightarrow}{AB}$=

Theorem 6 *Let A and B be two different points. $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$.*

Theorem 7 *Let A and B be two different points. $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.*

Question 1 *Suppose A–C–B and $D \in \overleftrightarrow{AC}$ ($D \neq A, B, C$). Is $D \in (\overrightarrow{CA} \cup \overrightarrow{CB})$?*

Definition 6 Suppose L is a line and A and B are points not on L. We say A and B are *on the same side of L* if either $A = B$ or if $\overline{AB} \cap L = \emptyset$. If A and B are different points and $\overline{AB} \cap L \neq \emptyset$ then we say A and B are on opposite sides. (Observe that if A and B are not on L then they are either on the same side or on opposite sides.)

Axiom J *For any line L and points A, B, C:*
(i) If A and B are on the same side of L, and B and C are on the same side of L, then A and C are on the same side of L.
(ii) If A and B are on opposite sides of L, and B and C are on opposite sides, then A and C are on the same side of L.

Axiom K *If A and B are two different points, then there is a point C such that A–B–C.*

Theorem 8 *Suppose A, B, C are non-collinear and ℓ is a line intersecting $\overline{AB}$ at a point between A and B. Then ℓ intersects either $\overline{AC}$ or $\overline{BC}$. Furthermore if $C \notin \ell$ then ℓ does not intersect both segments.*

Definition 7 Suppose $\{A, B, C\}$ is a non-collinear set of points. Define $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$, that is, the set of points lying in $\overrightarrow{BA}$ or in $\overrightarrow{BC}$ (or in both). ($\angle ABC$ is called the *angle ABC*.) Observe by commutativity of union $\angle ABC = \angle CBA$.

We define the *interior of the angle $\angle ABC$* to be the set of points D for which: a) D and C are on the same side of $\overleftrightarrow{AB}$, and b) D and A are on the same side of $\overleftrightarrow{BC}$.

Theorem 9 *Suppose D is in the interior of $\angle ABC$. Then every point in $\overset{\circ\!\longrightarrow}{BD}$ is also in the interior of $\angle ABC$. Moreover, if D–B–E then E is not in the interior of $\angle ABC$.*

Theorem 10 *Suppose A, B, C are non-collinear and $D \in \overleftrightarrow{AC}$. D is in the interior of $\angle ABC$ if and only if A–D–C.*

Theorem 11 *If D is in the interior of $\angle ABC$ and D is also in the interior of $\angle BCA$, then D is in the interior of $\angle CAB$.*

Theorem 12 **[The Crossbar Theorem]** *Suppose D is in the interior of $\angle ABC$. Then $\overline{AC} \cap \overrightarrow{BD} \neq \emptyset$.*

Theorem 13 *Every angle has nonempty interior.*

Theorem 14 *Suppose A and B are two different points. Then there is a point C such that A–C–B. Moreover, there are infinite number of points between them.*

Question 2 *Can a line be contained in the interior of an angle?*

We now assume we have an undefined relation $\cong$ on the set of all closed segments, written $\overline{AB} \cong \overline{QR}$, and said "$\overline{AB}$ is *congruent* to $\overline{QR}$".

Axiom L *The relation "$\cong$" is an equivalence relation. That is: If A and B are two different points and C and D are two different points and E and F are two different points, then all three of the following are true:*
- *(a) $\overline{AB} \cong \overline{AB}$* *(reflexive)*
- *(b) If $\overline{AB} \cong \overline{CD}$ then $\overline{CD} \cong \overline{AB}$* *(symmetric)*
- *(c) If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.* *(transitive)*

Axiom M *If A and B are two different points and C and D are two different points, then there is a point E on the open ray $\overset{\circ}{\overrightarrow{CD}}$ such that $\overline{AB} \cong \overline{CE}$.*

Axiom N *If A–B–C then it is not true that $\overline{AB} \cong \overline{AC}$.*

Axiom O *If A–B–C, A'–B'–C', $\overline{AB} \cong \overline{A'B'}$, and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.*

Theorem 15 *(Segment Subtraction) If A–B–C, D–E–F, $\overline{AB} \cong \overline{DE}$, and $\overline{AC} \cong \overline{DF}$, then $\overline{BC} \cong \overline{EF}$.*

Theorem 16 *If A–B–C and $\overline{AC} \cong \overline{DF}$, then there is a point E such that D–E–F and $\overline{AB} \cong \overline{DE}$.*

Definition 8 $\overline{AB} < \overline{CD}$ means there is a point X such that $\overline{AB} \cong \overline{CX}$ and C–X–D.

Theorem 17 *It is false that $\overline{AB} < \overline{AB}$.*

Theorem 18 *If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} < \overline{EF}$ then $\overline{AB} < \overline{EF}$.*

Theorem 19 *If $\overline{AB} < \overline{CD}$ and $\overline{CD} \cong \overline{EF}$ then $\overline{AB} < \overline{EF}$.*

Theorem 20 *If $\overline{AB} < \overline{CD}$ and $\overline{CD} < \overline{EF}$ then $\overline{AB} < \overline{EF}$.*

Theorem 21 *If A and B are two different points and C and D are two different points, then one and only one of the following is true:*
- *(a) $\overline{AB} < \overline{CD}$,*
- *(b) $\overline{AB} \cong \overline{CD}$, or*
- *(c) $\overline{CD} < \overline{AB}$.*

We are now given another undefined relation on pairs of angles, also called "is congruent to", and written $\angle ABC \cong \angle DEF$.

Axiom L'. The relation $\cong$ for angles is symmetric, reflexive, and transitive.

Axiom M'. If A, B, C are non-collinear, and D, E, F are non-collinear, then there is a point G on the same side of $\overleftrightarrow{DE}$ as F such that $\angle ABC \cong \angle DEG$.

Axiom N'. If D is in the interior of $\angle ABC$, then it is *not* true that $\angle ABC \cong \angle ABD$.

Axiom O'. If D is in the interior of $\angle ABC$ and D' is in the interior of $\angle A'B'C'$, and $\angle ABD \cong \angle A'B'D'$ and $\angle DBC \cong \angle D'B'C'$, then $\angle ABC \cong \angle A'B'C'$.

Axiom P'. If D is in the interior of $\angle ABC$, and $\angle ABC \cong \angle A'B'C'$, then there is a point D' in the interior of $\angle A'B'C'$ such that $\angle ABD \cong \angle A'B'D'$.

Definition 9 $\angle ABC < \angle DEF$ means that there is a point X in the interior of $\angle DEF$ such that $\angle ABC \cong \angle DEX$.

Theorem 22 *It is false that* $\angle ABC < \angle ABC$.

Theorem 23 *If* $\angle ABC \cong \angle DEF$ *and* $\angle DEF < \angle GHI$, *then* $\angle ABC < \angle GHI$.

Theorem 24 *If* $\angle ABC < \angle DEF$ *and* $\angle DEF \cong \angle GHI$, *then* $\angle ABC < \angle GHI$.

Theorem 25 *If* $\angle ABC < \angle DEF$ *and* $\angle DEF < \angle GHI$, *then* $\angle ABC < \angle GHI$.

Theorem 26 *If* A, B, C *are non-collinear and* D, E, F *are non-collinear then one and only one of the following is true:*
(a) $\angle ABC < \angle DEF$,
(b) $\angle DEF < \angle ABC$, *or*
(c) $\angle ABC \cong \angle DEF$.

Definition 10 Given three non-collinear points A, B, C the *triangle*, denoted $\triangle ABC$, is the union of the segments $\overline{AB} \cup \overline{BC} \cup \overline{AC}$. Observe that $\triangle ABC = \triangle BAC = \triangle BCA$ and so forth. We say $\triangle ABC \cong \triangle DEF$ if corresponding sides are congruent and corresponding angles are congruent.

Axiom P (SAS) *Given triangles* $\triangle ABC$ *and* $\triangle DEF$ *if* $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, *and* $\angle ABC \cong \angle DEF$, *then* $\triangle ABC \cong \triangle DEF$.

Theorem 27 *Given* $\triangle ABC$ *and* $\overline{DE} \cong \overline{AB}$ *there is a point* F *such that* D, E, F *are non-collinear and* $\triangle ABC \cong \triangle DEF$.

Definition 11 If the angles $\angle BAD$ and $\angle CAD$ satisfy that $\{B, A, C\}$ is a collinear set and $\overrightarrow{AC} \neq \overrightarrow{AB}$, then the angles are called *supplementary*. If an angle is congruent to a supplementary angle then the angle is called a *right angle*. Finally, two lines ℓ and m are said to be *perpendicular* if they intersect at a point A and there are other points $B \in \ell$, $C \in m$ such that $\angle BAC$ is a right angle.

Theorem 28 *If in* $\triangle ABC$, *we have* $\overline{AB} \cong \overline{BC}$, *then* $\angle BAC \cong \angle BCA$.

Theorem 29 *Supplements of congruent angles are congruent.*

Proposition 12 *A supplement of a right angle is a right angle.*

Theorem 30 *An angle congruent to a right angle is a right angle.*

Definition 13 Let A–X–B and D–X–F. Also suppose that $\overleftrightarrow{AB} \neq \overleftrightarrow{DF}$. The angles $\angle AXD$ and $\angle FXB$ are called *vertical angles.*

Corollary 14 *Vertical angles are congruent.*

Theorem 31 *For every line* ℓ *and a point* P *not on* ℓ *there is a line through* P *which is perpendicular to* ℓ. *Therefore, right angles exist.*

Theorem 32 *(ASA) Given triangles* $\triangle ABC$ *and* $\triangle DEF$ *suppose that* $\angle BAC \cong \angle EDF$, $\angle BCA \cong \angle EFD$, *and* $\overline{AC} \cong \overline{DF}$. *Then* $\triangle ABC \cong \triangle DEF$.

Theorem 33 *If in* $\triangle ABC$, $\angle ABC \cong \angle ACB$, *then* $\overline{AB} \cong \overline{AC}$.

Theorem 34 *(SSS)*

Theorem 35 *Suppose* A–B–C *and* A'–B'–C' *and* D *is a point not on* $\overleftrightarrow{AB}$ *(similarly for* D'*). If* $\overline{AD} \cong \overline{A'D'}$, $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, $\overline{BD} \cong \overline{B'D'}$, *then* $\overline{CD} \cong \overline{C'D'}$.

Theorem 36 *Any two right angles are congruent.*

Definition 15 Given two distinct lines ℓ_1 and ℓ_2 a third line ℓ which intersects both ℓ_1 and ℓ_2 in distinct points is called a *transversal*. We also say ℓ_1 and ℓ_2 are *cut* by the transversal ℓ.

Definition 16 Given two distinct lines ℓ_1 and ℓ_2 and a transversal ℓ let A_1 be the point where ℓ crosses ℓ_1 and A_2 be the point where ℓ crosses ℓ_2. Give the definition of alternate interior angles.

Definition 17 Given the triangle $\triangle ABC$ the angles $\angle ABC$, $\angle BCA$, and $\angle CAB$ are called the *interior angles of* $\triangle ABC$. A triangle is called a *right triangle* if one of its interior angles is a right angle. Suppose $\triangle ABC$ is a right triangle and $\angle ABC$ is a right angle. The segment $\overline{AC}$ is called a *hypotenuse* of the right triangle.

Theorem 37 *If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are parallel.*

Theorem 38 *Two different lines perpendicular to the same line are parallel. Hence, in Theorem 31, the line through P and perpendicular to ℓ is unique.*

Theorem 39 *Given a line ℓ and a point P not on ℓ there exists at least one line ℓ' through P which is parallel to ℓ. (Note: We knew this already but here it is very clearly: (SAS) implies our geometry is either Euclidean or Hyperbolic.)*

Theorem 40 *Let $\triangle ABC$ and D be a point such that B–C–D. Then $\angle ACD > \angle ABC$. Also, $\angle ACD > \angle BAC$. Is it true that $\angle ACD > \angle ACB$?*

Theorem 41 *(SAA)*

Theorem 42 *Every segment has a unique midpoint.*

Theorem 43 *Every angle has a unique bisector. (A bisector of the angle $\angle ABC$ is a ray $\overrightarrow{BD}$ where D is in the interior of the angle and $\angle ABD \cong \angle DBC$.)*

Theorem 44 *In a triangle $\triangle ABC$, $\overline{AB} \leq \overline{BC}$ if and only if $\angle C \leq \angle A$.*

Theorem 45 *Given triangles $\triangle ABC$ and $\triangle DEF$, if $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, then $\angle B < \angle E$ if and only if $\overline{AC} < \overline{DF}$.*

Theorem 46 *Suppose A–B–C and D is not on $\overleftrightarrow{AC}$. If $\overleftrightarrow{DC}$ is perpendicular to $\overleftrightarrow{AC}$, then $\overline{CD} < \overline{BD} < \overline{AD}$.*

Theorem 47 *Given triangle $\triangle PQR$ and any point S between Q and R show that either $\overline{PS} < \overline{PQ}$ or $\overline{PS} < \overline{PR}$.*

Theorem 48 *Construct a triangle which is not isosceles.*

Definition 18 Let O be a point. Given a segment $\overline{AB}$ we define the *circle centered at O and of radius $\overline{AB}$* to be the following set:

$$\gamma = \{D : \overline{OD} \cong \overline{AB}\}.$$

The *interior of* γ is defined as

$$\text{int}\gamma = \{D : \overline{OD} < \overline{AB}\}.$$

The *exterior* of γ is

$$ext\gamma = \{D : \overline{AB} < \overline{OD}\}.$$

Two points $A, B \in \gamma$ are said to form a *diameter* if A–O–B.

Theorem 49 *Suppose γ is a circle centered at O of radius $\overline{AB}$. If $X, Y \in \gamma$ form a diameter then O is the midpoint of $\overline{XY}$. Furthermore every circle has a diameter.*

Theorem 50 *Suppose γ is a circle centered at O of radius $\overline{AB}$. Let $X, Y \in \gamma$ be distinct points so that X, Y do not form a diameter. Let M be the midpoint of $\overline{XY}$. Show that $\overleftrightarrow{OM}$ is perpendicular to $\overleftrightarrow{XY}$.*

Theorem 51 *Suppose γ is a circle centered at O of radius $\overline{AB}$. Let $X, Y \in \gamma$ be distinct points and let M be the midpoint of $\overline{XY}$. Let ℓ be any line through M which is perpendicular to $\overline{XY}$. Prove that $O \in \ell$.*

Theorem 52 *Prove in Euclidean geometry that a triangle inscribed in a semi-circle is a right triangle.*

Definition 19 The four points A, B, C, D form a *quadrilateral* if A, B, C are non-collinear, A, C, D are non-collinear, and D is in the interior of $\angle ABC$. The quadrilateral $\square ABCD$ is the union of the segments $\overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$. The *interior* of the quadrilateral $\square ABCD$ is the union of the interior of triangle ABD, the interior of triangle BDC, and $\overset{\circ\!-\!\circ}{BD}$. We call the quadrilateral a *convex quadrilateral* if C is in the interior of $\angle BAD$.

Proposition 20 *Suppose the quadrilateral $\square ABCD$ is convex. Prove that the interior of quadrilateral $\square ABCD$ is the same as the interior of the quadrilateral $\square BADC$.*

Example 5 *Given an example of a quadrilateral $\square ABCD$ that is not convex.*

Definition 21 Let S be a set. We say S is a *convex set* if whenever $A, B \in S$, then $\overline{AB} \subseteq S$.

Theorem 53 *Prove that the intersection of two convex sets is again a convex set.*

Theorem 54 *Consider the quadrilateral $\square ABCD$. It is a convex quadrilateral if and only if its interior is a convex set.*

Theorem 55 *The diagonals of a convex quadrilateral intersect.*

Definition 22 The quadrilateral $\square ABCD$ is called a *Saccheri quadrilateral* if $\angle B$ and $\angle C$ are right angles while $\overline{AB} \cong \overline{CD}$.

Theorem 56 *Saccheri quadrilaterals exist.*

Theorem 57 *Suppose $\square ABCD$ is a Saccheri quadrilateral. Then $\angle A \cong \angle D$ and $\overline{AC} \cong \overline{BD}$.*

Theorem 58 *Suppose $\square ABCD$ and $\square A'B'C'D'$ are Saccheri quadrilateral and that $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$. Then $\angle A \cong \angle A'$, $\angle D \cong \angle D'$ and $\overline{AD} \cong \overline{A'D'}$.*

Definition 23 The quadrilateral $\square ABCD$ is called a *Lambert quadrilateral* if three of its angles are right angles. A *rectangle* is a quadrilateral with four right angles.

Theorem 59 *Lambert quadrilaterals exist.*

Theorem 60 *Suppose $\square ABCD$ is a Lambert quadrilateral. Prove that the fourth angle is never obtuse.*

Theorem 61 *If $\square ABCD$ is a rectangle, then the pairs of opposite sides are congruent. Conclude that a rectangle is a Saccheri quadrilateral.*

Theorem 62 *Suppose $\square ABCD$ is a Saccheri quadrilateral and a Lambert quadrilateral (not clear whether such a thing exists) if and only if it is a rectangle.*

Theorem 63 *Suppose $\square ABCD$ is a rectangle. Prove that $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$.*

Theorem 64 *Suppose $\square ABCD$ is a quadrilateral and $\angle B$ and $\angle C$ are right angles. $\overline{AB} < \overline{CD}$ if and only if $\angle ADC < \angle BAD$. Moreover, $\overline{AB} \cong \overline{CD}$ if and only if $\angle ADC \cong \angle BAD$.*

Definition 24 We now assume that there is a way of assigning to each pair of distinct points A, B a unique positive real number which we shall denote by $\ell(AB)$ (and call this the *length of the segment* $\overline{AB}$) such that the following properties hold:

1. $\ell(AB) = \ell(CD)$ if and only if $\overline{AB} \cong \overline{CD}$.
2. A–B–C if and only if $\ell(AC) = \ell(AB) + \ell(BC)$.
3. For each positive real number r there is a segment $\overline{AB}$ such that $\ell(AB) = r$.

Definition 25 We now assume that there is a way of assigning to each angle $\angle ABC$ a unique positive real number which we shall denote by $\angle ABC^\circ$ (and call this the *angle measure* of $\angle ABC$ or the *degree of* $\angle ABC$) such that the following properties hold:

1. for any angle $\angle ABC$, $0 < \angle ABC^\circ < 180$.
2. $\angle ABC^\circ = \angle DEF^\circ$ if and only if $\angle ABC \cong \angle DEF$.

3. If x is in the interior of $\angle ABC$, then $\angle ABC^\circ = \angle ABX^\circ + \angle XBC^\circ$.
4. For each positive real number r between 0 and 180, there is an angle $\angle ABC$ such that $\angle ABC^\circ = r$.
5. If $\angle ABC$ and $\angle ABD$ are supplementary, then $\angle ABC^\circ + \angle ABD^\circ = 180$.

An angle is called *acute* if its measure is strictly between 0 and 90. If the angle measure is strictly between 90 and 180 then we call the angle *obtuse*.

Theorem 65 *An angle is a right angle if and only if its angle measure is 90.*

Theorem 66 *If $\angle A$ and $\angle B$ are two angles, then $\angle A^\circ < \angle B^\circ$ if and only if $\angle A < \angle B$.*

Theorem 67 *If $\overline{AB}$ and $\overline{CD}$ are two segments, then $\ell(AB) < \ell(CD)$ if and only if $\overline{AB} < \overline{CD}$.*

Theorem 68 *(Triangle Inequality) For any three non-collinear points, A, B, C, $\ell(AC) < \ell(AB) + \ell(BC)$.*

Theorem 69 (Saccheri-Legendre) *The sum of the angle measures of three angles in any triangle is less than or equal to 180.*

Corollary 26 *The sum of of the degree measures of the angles in any convex quadrilateral is at most 360.*

Corollary 27 *Suppose $\square ABCD$ is a Lambert quadrilateral with $\angle B$, $\angle C$, and $\angle D$ all right angles. Then $\angle D$ is never obtuse.*

Corollary 28 *Suppose $\square ABCD$ is a Saccheri quadrilateral. Then $\angle A$ and $\angle D$ are either right angles or they are acute.*

Theorem 70 *Suppose $\square ABCD$ is a Saccheri quadrilateral. Let M and N be the midpoints of $\overline{BC}$ and $\overline{AD}$, respectively. Then $\overleftrightarrow{MN} \perp \overleftrightarrow{BC}$ and $\overleftrightarrow{MN} \perp \overleftrightarrow{AD}$. Furthermore, $\overline{MN} \leq \overline{AB}$ and $\overline{MN} \leq \overline{CD}$.*

Theorem 71 *Suppose $\square ABCD$ is a Saccheri quadrilateral. Then $\ell(BC) \leq \ell(AD)$. Furthermore, $\angle BAC^\circ \leq \angle ACD^\circ$.*

Definition 29 Recall the following postulates:

Euclid's 5^{th} Postulate: *If two lines are cut by a transversal so that the sum of the degree measures of the two interior angles on one side of the transversal is less than 180, then the two lines meet on that side of the transversal.*

Playfair's Parallel Postulate: *Given a point P not on a line ℓ there is a unique line through P that is parallel to ℓ.*

Hilbert's Parallel Postulate: *Given a point P not on a line ℓ there is at most one line through P that is parallel to ℓ.*

We already know that in neutral geometry there is at least one parallel through a point not on a line. Therefore, Playfair's Postulate and Hilbert's Postulate are logically equivalent in neutral geometry.

Theorem 72 *The following statements are logically equivalent.*

1. *Euclid's 5^{th} Postulate.*
2. *Playfair's Parallel Postulate.*
3. *Hilbert's Parallel Postulate.*
4. *If two lines are parallel, then the two lines cut by any transversal have a pair of congruent alternate interior angles.*
5. *Given two lines ℓ_1 and ℓ_2 cut by the transversal ℓ satisfy that $\ell_1 || \ell_2$, $\ell \perp \ell_1$, then $\ell \perp \ell_2$.*
6. *The angle sum of every triangle is 180.*
7. *There is a triangle whose angle sum is 180.*
8. *Every Saccheri quadrilateral is a rectangle.*
9. *A rectangle exists.*