BERD Academy

A Connected World

Data Analysis for Real World Network Data

Latent Variable Models

08.12.2022

A different angle to cope with dependencies

- So far, ERGM allowed us to explicitly account for (and measure) network dependencies
- Another way to capture network dependencies is by making use of latent variable models
- Models within this class assume that latent (unobserved) variables Z_i are associated with each node i, and that all dependencies between edges is due to these latent variables

Definition: Latent Variable Network Models

A latent variable network model is a statistical model that relates the set of observed edges $Y = (Y_{ij})$ to a set of latent variables $Z = (Z_i)$. The actor-specific latent variables Z_i can, in general, be of any dimension and be in the discrete or continuous domain. All dependence between edges Y_{ij} and Y_{kh} is assumed to be captured by the latent variables z_i, z_j, z_k, and z_h.

$$Y_{ij} | z_i, z_j \sim F(z_i, z_j)$$

Intuition

- Nodes possess some latent attributes (e.g. unobserved group membership, positioning in a social space) which influences tie behavior
- The idea is to estimate this latent structure, to gain an understanding of it and/or control for it while doing inference on covariates
- Let us start with the simplest (and most popular) application of latent variable models… community detection

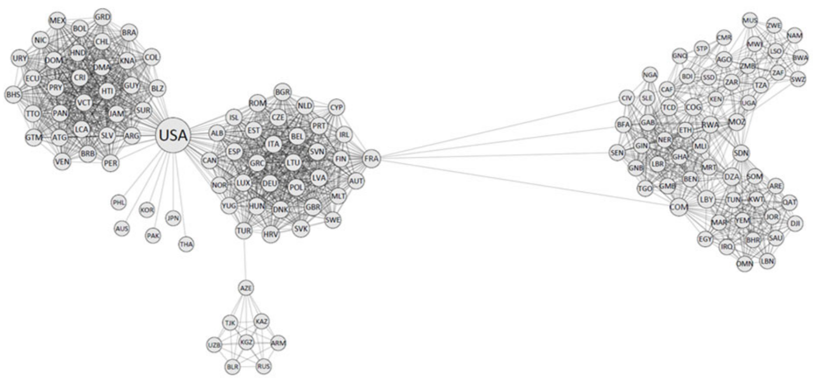
Community Detection in Networks

- Networks are often organized in smaller sub-groups
- Sometimes those subgroups are known and well defined (ex. political parties in a parliamentary network, classes in a school)
- More often that is not the case (ex. friendship circles on a facebook network, different cells in a network of terrorists)

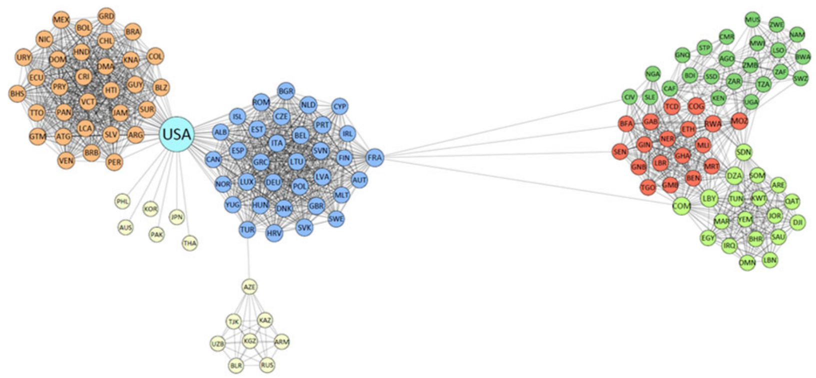
Community Detection in Networks

- We can treat the community membership as a latent (unobserved) variable and try to estimate it
- In models with built-in community structure, the probability of forming a tie within a group is typically higher than forming one between groups
- Other types of structures, such as core-periphery, are possible

We want to go from here…



…to here!



How to perform community detection?

- Many heuristic methods available (see Fortunato & Hric, 2016)
- Most popular is modularity maximization: assign node to groups in a way that maximises some target function (“modularity”)
- Other methods based on matrix factorization (i.e. spectral decomposition)

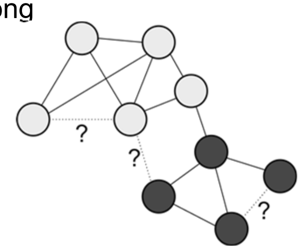
Heuristics - Pros and Cons

- **Pros:**
  - Fast
  - Good at “pure” community detection
- **Cons:**
  - Not a statistical model: no uncertainty estimations, no theoretical guarantees
  - Usually not possible/straightforward to include covariates
- Simplest statistical approach: **The Stochastic Blockmodel**

Stochastic Blockmodels

Stochastic Blockmodels - Main ideas

- A probabilistic model for networks (the edges are random)
- Each node belongs to one (unobserved) class or “block”
- The probability of any two nodes to connect depends solely on the blocks to which the two nodes belong



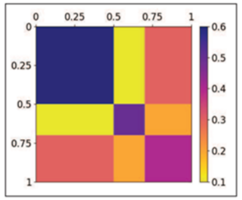
The Stochastic Blockmodel

We assume the conditional probability of a tie Y_{ij} to follow:

$$Y_{ij} | z_i, z_j \sim Bernoulli(p(z_i, z_j))$$

With $p(z_i,z_j)$ governed by a block-probability matrix $\boldsymbol{P}$.

$$\left.\begin{array}{l} \boldsymbol{\pi} = (0.5, 0.2, 0.3) \\ \boldsymbol{P} = \begin{pmatrix} 0.6 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.2 \\ 0.3 & 0.2 & 0.4 \end{pmatrix} \end{array}\right\} \iff$$



From De Nicola et al., 2022

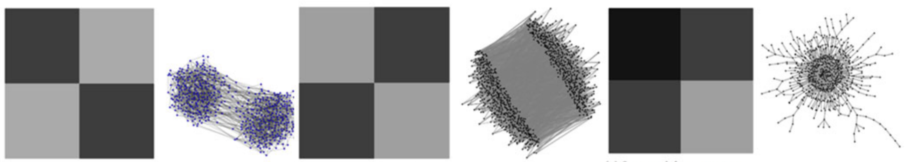
Stochastic Blockmodel - Estimation

- Everything looks very simple, but…
- Block-memberships are unknown, and need to be estimated!
- The complete data likelihood is untreatable

- Need to solve a complex estimation problem. Some routes:
  - Variational inference
  - Vertex-switching algorithms
  - MCEM algorithms
  - …

The great thing about the SBM

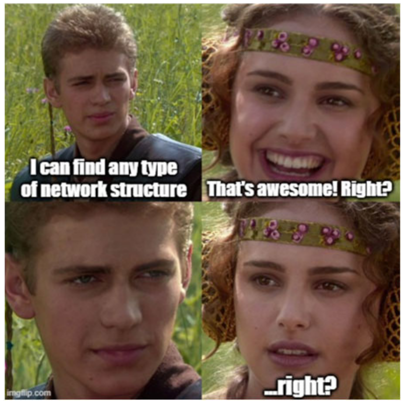
- Unlike pure “community detection” algorithms, able to find any type of structure, beyond classic communities



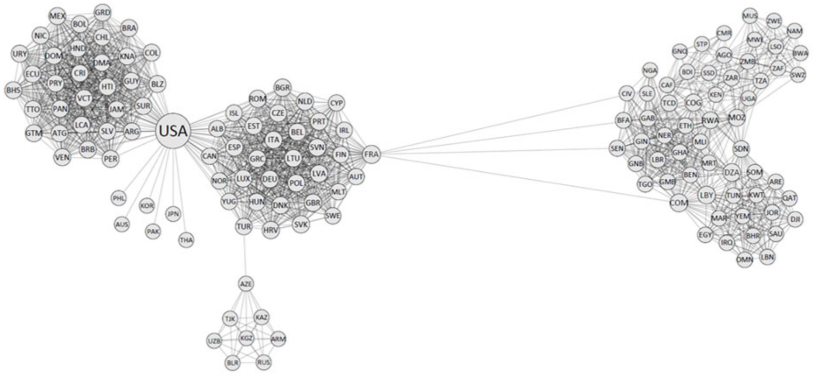
(a) Community structure. (b) Disassortative structure. (c) Core–periphery structure.

From Fortunato & Hric, 2016

…but is it always a good thing?

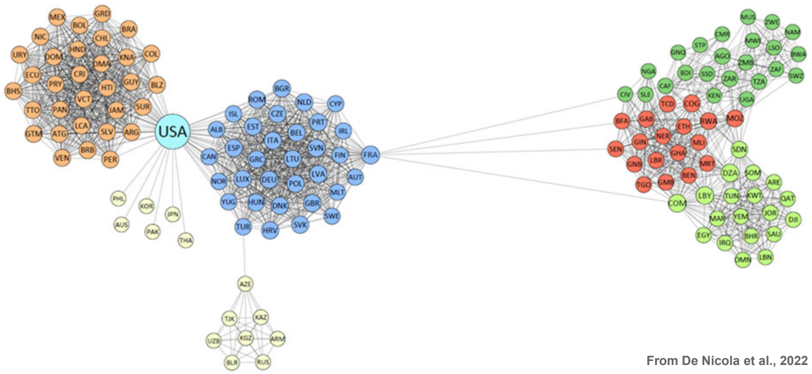


Example 1: Alliances Network



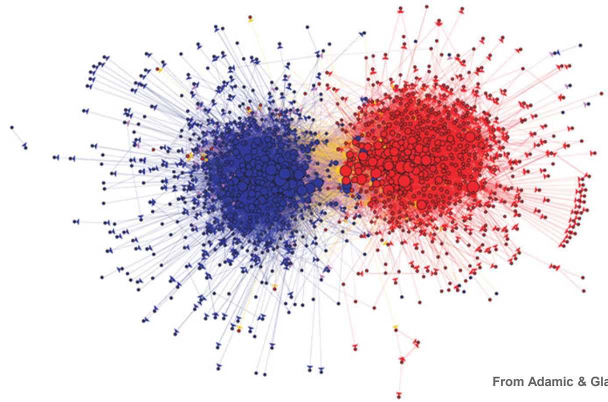
What will the SBM find? (K=7)

…(almost) classical community structure!



From De Nicola et al., 2022

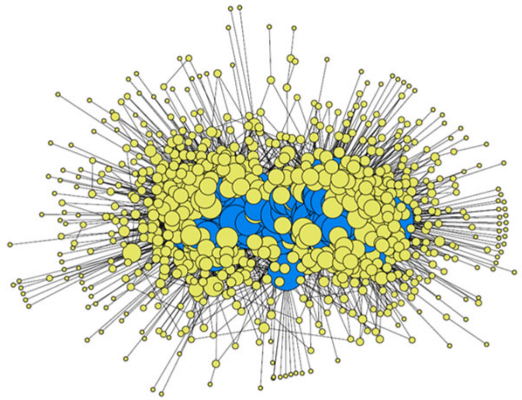
Example: Political blogs network



From Adamic & Glance, 2005

What will the SBM find? (K=2)

….a core-periphery structure?!?



From Karrer & Newman, 2012

The feature/bug of SBM for social networks

- The classical SBM implicitly assumes the degree structure *within blocks* to be relatively homogeneous
- But many real world social networks exhibit extremely skewed degree distributions
- This leads the SBM to very often find core-periphery structures, as opposed to classical assortative communities

From Karrer & Newman, 2012

Degree-corrected SBM

- Karrer & Newman (2012) introduced the idea of degree correction
- The probability of an edge depends not only on block-membership, but also explicitly on node-specific heterogeneity parameters (i.e. node degree):

From Karrer & Newman, 2012

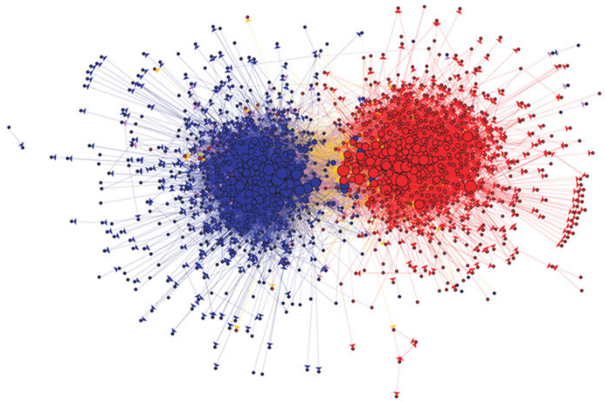
Degree-corrected SBM

- Karrer & Newman (2012) introduced the idea of degree correction
- The probability of an edge depends not only on block-membership, but also explicitly on node-specific heterogeneity parameters (i.e. node degree):

$$\lambda_{ij} = \exp\{\gamma_i + \gamma_j + \omega_{z_i z_j}\}$$

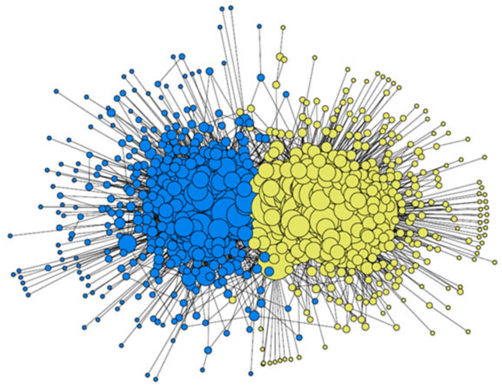
From Karrer & Newman, 2012

Back to political blogs



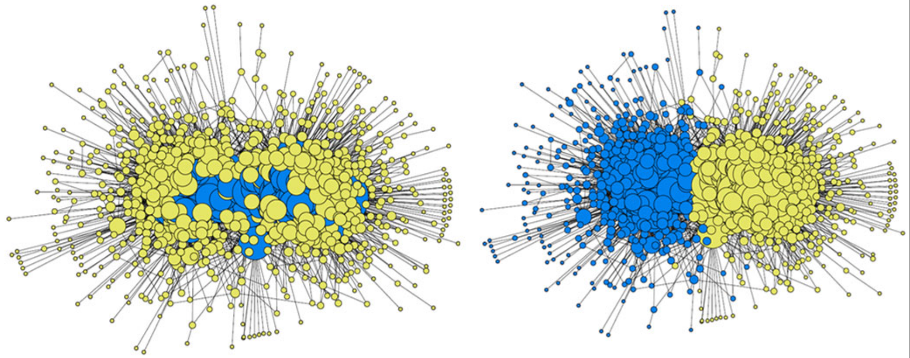
What will the degree-corrected SBM find?

…assortative communities!



From Karrer & Newman, 2012

SBM vs DCSBM



From Karrer & Newman, 2012

SBMs: Variants and Extensions

- Classical SBM is a very simple model, many other variants and extensions exist
- Variants aimed at finding specific types of network structures
- Some of the most prominent ones:
  - Mixed membership SBM (Airoldi et al., 2008)
  - Hierarchical SBM (Peixoto, 2012)
  - Mixture of experts SBM (Gormley & Murphy, 2010)

SBM as a model class: Features

- Good for finding different types of community structure
- Principle, likelihood based methods, with all the perks that come with it
- Relatively fast estimation routines exist
- A lot of software available openly available

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- Good for finding different types of community structure
- Principle, likelihood based methods, with all the perks that come with it
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All in all, a solid tool for finding discrete structures in different types of networks

SBM as a model class: Limitations

- Discrete → too simplistic
- Not straightforward to include covariates
- Number of communities K needs to be inputted
  - Several ways to estimate it data-driven
  - Still requires some prior assumptions (far from being solved)

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Can we address these?

Latent Space Models

Continuous Latent Variables

- It is quite natural to generalize the idea of discrete communities into continuous ones

- Hoff et al. (2002) propose to “map” the network into a Euclidean latent social space, where the distance between two nodes determines their probability of being connected

The Latent Distance Model

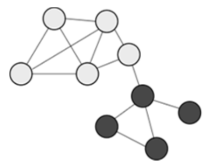
- Postulates that the actors are located in a latent social space
- The closer they are in this space, the more likely they are to connect
- Specifically, log-odds of a tie between nodes *i* and *j* given by:

$$\eta_{i,j} = \log \text{odds}(y_{i,j} = 1 | z_i, z_j, x_{i,j}, \alpha, \beta)$$
$$= \alpha + \beta' x_{i,j} - |z_i - z_j|.$$

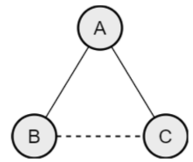
Properties

- Does a good job at representing patterns that are typical of social networks, such as:

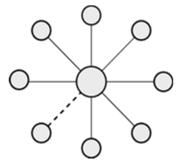
Homophily



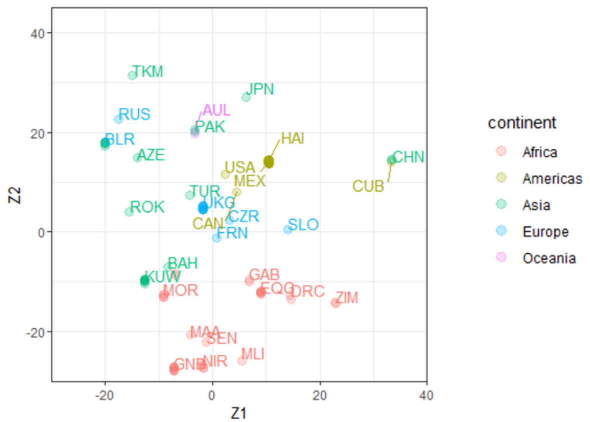
Triadic Closure



Preferential attachment



Example: Alliances network



The latent position cluster model

- We can allow for model-based clustering of the latent positions, to also get communities (see Handcock et al., 2007)
- Assume the positions to come from a mixture distribution:

$$Z_i \overset{\text{i.i.d.}}{\sim} \sum_{g=1}^{G} \lambda_g \, \text{MVN}_d \left(\mu_g, \sigma_g^2 I_d\right) \quad i = 1, \ldots, n$$

Further extension

- We can also control for the actors' different propensity to form ties (see Krivitsky et al., 2009)
- Add node-specific random effects:

$$\eta_{i,j} = \sum_{k=1}^{p} \beta_k x_{k,i,j} - \|Z_i - Z_j\| + \delta_i + \gamma_j$$

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- Add node-specific random effects:

$$\eta_{i,j} = \sum_{k=1}^{p} \beta_k x_{k,i,j} - \|Z_i - Z_j\| + \delta_i + \gamma_j, \quad \text{with}$$

$$\delta_i \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma_\delta^2) \quad i = 1, \dots, n$$

$$\gamma_i \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma_\gamma^2) \quad i = 1, \dots, n$$

Network of COVID-19 Twitter elites

- Start from database with all tweets about COVID-19
- Rank tweets by their popularity (likes + retweets + replies)
- A user is "elite" if they have a tweet on COVID-19 with popularity > 2000
- Result: 1024 tweets by a total of 363 users

Application to COVID-19 Twitter elites

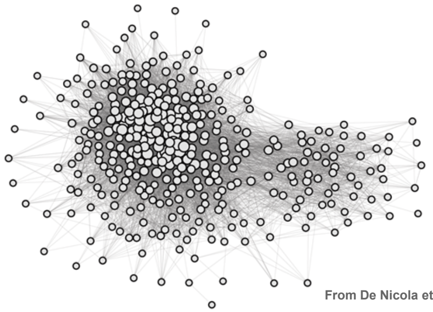
- Start from database with all tweets in German about COVID-19
- Rank tweets by their popularity (likes + retweets + replies)
- A user is "elite" if they have a tweet on COVID-19 with popularity > 2000
- Result: 1024 tweets by a total of 363 users

text	author	tweet_popularity
Wir haben keinen einzigen #COVID19 Pati ...	Ricardo Lange	29,422
Der Bundesgesundheitsminister fordert so ...	Jens Clasen	25,852
Über Freiheit und Eigenverantwortung spr ...	Dunja Hayali ❤️🇩🇪🇪🇺🧠	25,832
Kosten einer BioNTech-Impfdosis: 19,95K ...	Krankenpflegel	25,725
Das Letzte, was das Coronavirus sieht, b ...	Fabian Köster	25,205
"Der Weg hierher und hier raus ist ein h ...	Christian Drosten	21,368
Wir stecken tief in der Schuld unserer P ...	Prof. Karl Lauterbach	21,208
Echt stark, wie gut wir Covid-19 im Grif ...	Cornelius W. M. Oettle	20,367
(1) Nachdem ich mich heute bei der dpa z ...	Carsten Watzl	19,434
Um das noch einmal ganz klar zu sagen: ...	Jens Clasen	19,415

Network of COVID-19 Twitter elites

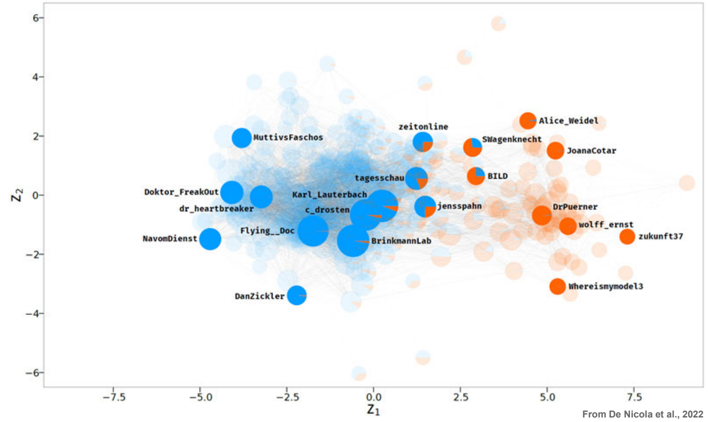
- We naturally define an edge from user A to user B if A follows B on Twitter
- Resulting network **of 363 users has** 12182 directed edges **(9.2% density)**

SENDER	RECEIVER
c_drosten	Karl_Lauterbach
Karl_Lauterbach	c_drosten
jensspahn	c_drosten
BrinkmannLab	Flying__Doc
Alice_Weidel	JoanaCotar
……	……



From De Nicola et al., 2022

The latent social space of COVID-19 elites



From De Nicola et al., 2022

LSM as a model class: Features

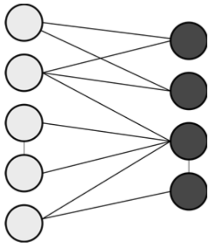
- Good for a more nuanced view than discrete SBM
- In our example: Beyond polarization on social media
- Great for graphical representation: positions have a probabilistic meaning
- Uncertainty quantification
- Possible to incorporate covariates (but interpretation changes greatly)

LSM as a model class: Chief limitation

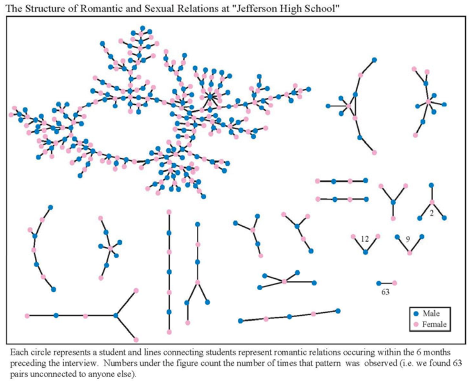
- Fails at capturing disassortative structures

LSM as a model class: Chief limitation

- Fails at capturing disassortative structures
  - **Example:**



A heterophilic networks



From Bearman et al., 2004

LSM: Terms of use

- Best to use LSMs only when reasonable to believe that the network is (mostly) homophilic and triadic
- That is not always known a priori
- Real world network can also display mixed patterns:

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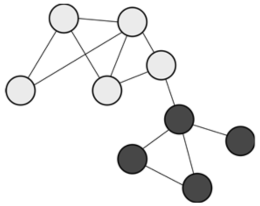

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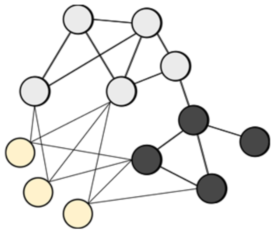

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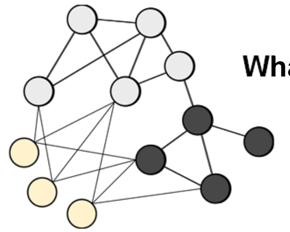

What can we do about this?

Additive and Multiplicative Effect Models

AME: Motivation and framework

- Network data often exhibit dependencies of different orders:
  - First order: Node-specific heterogeneity
  - Second order: Reciprocity
  - Third order: Triadic effects
- The AME network model (Hoff, 2021) is designed to capture all of these type of dependencies simoultaneously.

The AME Network Model

- The AME Probit model specifes the probability of a tie as:

$$\mathbb{P}(Y_{ij} = 1|W) = \boldsymbol{\Phi}(\theta^{\top} x_{ij} + e_{ij}),$$

- Where:
  - $\boldsymbol{\Phi}$ is the standard normal cumulative distribution function
  - $\theta^{\top} x_{ij}$ accommodates the inclusion of covariates
  - e_{ij} can be viewed as a structured residual

The AME Network Model

- The structured error e_{ij} is a function of the latent variables:

$$e_{ij} = a_i + b_j + u_i v_j + \varepsilon_{ij}$$

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$$e_{ij} = a_i + b_j + u_i v_j + \varepsilon_{ij}$$

- a_i and b_j are zero-mean additive effects for sender i and receiver j, which account for first order dependencies
  - More specifically:

$$(a_1, b_1), \ldots, (a_n, b_n) \overset{\text{i.i.d.}}{\sim} N_2(0, \Sigma_1), \quad \text{with} \quad \Sigma_1 = \begin{pmatrix} \sigma_a & \sigma_{ab} \\ \sigma_{ab} & \sigma_b \end{pmatrix}$$

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$$e_{ij} = a_i + b_j + u_i v_j + \varepsilon_{ij}$$

- ε_{ij} is a zero-mean residual term, accounting for second order dependency, i.e. reciprocity
  - More specifically:

$$\{(\varepsilon_{ij}, \varepsilon_{ji}) : i < j\} \overset{\text{i.i.d.}}{\sim} N_2(0, \Sigma_2), \quad \text{with} \quad \Sigma_2 = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

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$$e_{ij} = a_i + b_j + u_i v_j + \varepsilon_{ij}$$

- u_i and v_j are d-dimensional multiplicative “latent positions” accounting for third order dependencies, with

$$(u_1, v_1), \ldots, (u_n, v_n) \sim \mathcal{N}_{2d}(0, \Sigma_3)$$

AME: Pros and cons

Advantages:

- Incredibly flexible, able to represent many network structures
- Has been shown to generalize both SBM and LSM

Disadvantages:

- Incredibly complex, estimation slow
- Multiplicative latent space is not as interpretable nor good for representation as the LSM

AME: How to use

- Given it's complexity AME is a suboptimal choice when focus is on interpretability and visualization of the latent structure
- To the contrary, it is an ideal fit when underlying network dependencies are unkown, and the focus is on estimating covariate effects controlling for the network structure
- Many such cases, especially in the social sciences

Application: Historical forex network

- In 1900, every financial center featured a foreign exchange market were bankers bought and sold foreign currency against the domestic one.
- Foreign exchange market activity was monitored in local bulletins, which allowed Flandreau and Jobst (2005) to collect a global dataset.

Application: Historical forex network

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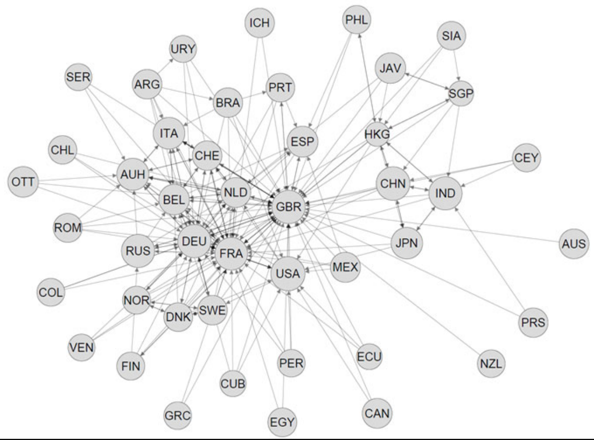
- Foreign exchange market activity was monitored in local bulletins, which allowed Flandreau and Jobst (2005) to collect a global dataset.

- This gives rise to a directed network, where:
  - Countries are nodes
  - A directed edge i → j is present if currency from country j is traded within the financial center of country i

Historical forex network



Application: Historical forex network

- Interested in understanding the effect of several factors on currency trade between countries, which is an important indicator of economic influence.
- **Nodal covariates:**
  - Gold standard, GDP per-capita, democracy index
- **Dyadic covariates:**
  - Distance, reciprocal trade volume

Historical forex network

		AME	Classical Probit
	Intercept	-4.845 (5.310)	-3.211 (1.580)*
Sender	Gold standard	-0.629 (0.397)	-0.354 (0.155)*
	log-GDP per-capita	-0.453 (0.419)	-0.259 (0.152)
	Democracy index	-0.033 (0.064)	-0.025 (0.026)
	Currency coverage	1.418 (0.405)***	0.470 (0.137)***
Receiver	Gold standard	-0.599 (0.667)	-0.468 (0.191)*
	log-GDP per-capita	0.426 (0.703)	0.240 (0.159)
	Democracy index	0.121 (0.102)	0.066 (0.019)***
	Currency coverage	2.734 (0.691)***	1.363 (0.181)***
Dyadic	Distance	-1.019 (0.151)***	-0.471 (0.064)***
	log-trade volume	0.488 (0.081)***	0.346 (0.036)***

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

Takeaways

- **Dependencies matter!**
- Latent variable models are one way to measure them and/or take them into account
- There is no single best: different networks require different approaches
- Different research questions also are answered with different models

Latent Variable Models: Overview

- **Stochastic blockmodels** are good to find group structures of different kinds, and can capture stochastic equivalence
- **Latent distance models** are great for representing and understanding networks in which nodes that behave similarly tend to connect to each other frequently
- **AME models** are optimal when the focus is on measuring the effect of exogenous covariates, while controlling for the network

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Thank you for your attention!

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Questions?

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