

1 Problem Definitions and Statement

Definition 1. A *point cloud* A over \mathbb{R}^n is a sequence of $|A| = k \in \mathbb{N}^+$ vectors in \mathbb{R}^n , that is,

$$A = (\mathbf{a}_1, \dots, \mathbf{a}_k) \quad \forall i \in [k], \mathbf{a}_i \in \mathbb{R}^n$$

Definition 2. The *centroid* of a point cloud A is defined to be

$$\bar{\mathbf{a}} = \frac{1}{|A|} \sum_{\mathbf{a} \in A} \mathbf{a}$$

Definition 3. A *rigid-body transformation* on a point cloud A is an affine map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ determined by an rotation matrix $R \in \mathbb{R}^{n \times n}$ and translation $\mathbf{t} \in \mathbb{R}^n$ and given by

$$T : \mathbf{a} \mapsto R(\mathbf{a} - \bar{\mathbf{a}}) + \bar{\mathbf{a}} + \mathbf{t}$$

that is, a rotation around the centroid of followed by a translation.

The POINT-CLOUD REGISTRATION PROBLEM is, given two point clouds A, B and a pairing function $\phi(T, \cdot)$ based on a rigid-body transformation T such that $\forall \mathbf{a} \in A, \phi(T, \mathbf{a}) \in B$, to determine the rigid-body transformation T that optimizes

$$\min_T \sum_{\mathbf{a} \in A} \|T(\mathbf{a}) - \phi(T, \mathbf{a})\|^2 \quad (1)$$

Definition 4. The *point-to-point metric* for two point clouds A, B is given by

$$\phi(T, \mathbf{a}) = \operatorname{argmin}_{\mathbf{b} \in B} \|T(\mathbf{a}) - \mathbf{b}\|^2 \quad (2)$$

From here on, we will use the point-to-point metric unless stated otherwise.

2 Solution

In general, there is no closed-form solution that minimizes (1). When there is no translation, Horn (1987) derives a closed-form solution; therefore, if an initial odometry estimate is precise enough, this translation can be applied to all the points in the point cloud, producing a problem instance with zero translation.

2.1 Approximation for \mathbb{R}^2

Consider the situation where the point clouds A and B (with n and m points respectively) are over \mathbb{R}^2 . First, let us simplify the cost function \mathcal{L} .

$$\begin{aligned}\mathcal{L} &= \min_T \sum_{\mathbf{a} \in A} \|T(\mathbf{a}) - \phi(T, \mathbf{a})\|^2 \\ \implies \min_T \sum_{\mathbf{a} \in A} \|R\mathbf{a}' + \bar{\mathbf{a}} + \mathbf{t} - \phi(T, \mathbf{a})\|^2 &\quad \text{let } \mathbf{a}' = \mathbf{a} - \bar{\mathbf{a}} \\ \implies \min_T \sum_{\mathbf{a} \in A} \|R\mathbf{a}' + \mathbf{t} - \phi(T, \mathbf{a})\|^2 &\quad \text{since } \bar{\mathbf{a}} \text{ is constant}\end{aligned}$$

In \mathbb{R}^2 , the rotation matrix R takes the form

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

where it is described by a single parameter θ . Therefore, the task is to optimize \mathcal{L} over the parameter space (θ, t_x, t_y) . We take \mathbf{b} to be $\phi(T, \mathbf{a})$ where context permits.

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial t_x} &= \sum_{\mathbf{a} \in A} 2t_x + 2a'_x \cos(\theta) - 2a'_y \sin(\theta) - 2b_x \\ \frac{\partial \mathcal{L}}{\partial t_y} &= \sum_{\mathbf{a} \in A} 2t_y + 2a'_x \sin(\theta) + 2a'_y \cos(\theta) - 2b_y \\ \frac{\partial \mathcal{L}}{\partial \theta} &= 2(a'_x \sin(\theta) + a'_y \cos(\theta))(b_x - t_x - a'_x \cos(\theta) + a'_y \sin(\theta)) \\ &\quad + 2(a'_x \cos(\theta) - a'_y \sin(\theta))(t_y + a'_x \sin(\theta) + a'_y \cos(\theta) - b_y)\end{aligned}$$

The roots of $\partial \mathcal{L} / \partial t_x$ and $\partial \mathcal{L} / \partial t_y$ can easily be solved in closed form;

$$\begin{aligned}t_x^* &= \frac{1}{n} \sum_{\mathbf{a} \in A} -a'_x \cos(\theta) + a'_y \sin(\theta) + b_x \\ t_y^* &= \frac{1}{n} \sum_{\mathbf{a} \in A} -a'_x \sin(\theta) - a'_y \cos(\theta) + b_y\end{aligned}$$

or

$$\mathbf{t}^* = \frac{1}{n} \sum_{\mathbf{a} \in A} \mathbf{b} - R\mathbf{a}'$$