1 Problem Definitions and Statement

Definition 1. A *point cloud* A over $\mathbb{R}^n$ is a sequence of $|A| = k \in \mathbb{N}^+$ vectors in R^n, that is,

$$A = (\mathbf{a_1}, \ldots, \mathbf{a_k}) \quad \forall i \in [k],\, \mathbf{a_i} \in \mathbb{R}^n$$

Definition 2. The *centroid* of a point cloud A is defined to be

$$\bar{\mathbf{a}} = \frac{1}{|A|} \sum_{\mathbf{a} \in A} \mathbf{a}$$

Definition 3. A *rigid-body transformation* on a point cloud A is an affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ determined by an rotation matrix $R \in \mathbb{R}^{n \times n}$ and translation $t \in \mathbb{R}^n$ and given by

$$T : \mathbf{a} \mapsto R(\mathbf{a} - \bar{\mathbf{a}}) + \bar{\mathbf{a}} + \mathbf{t}$$

that is, a rotation around the centroid of followed by a translation.

The Point-Cloud Registration Problem is, given two point clouds A, B and a pairing function $\varphi(T, \cdot)$ based on a rigid-body transformation T such that $\forall \mathbf{a} \in A,\, \varphi(T, \mathbf{a}) \in B$, to determine the rigid-body transformation T that optimizes

$$\min_T \sum_{\mathbf{a} \in A} \|T(\mathbf{a}) - \varphi(T, \mathbf{a})\|^2 \tag{1}$$

Definition 4. The *point-to-point metric* for two point clouds A, B is given by

$$\varphi(T, \mathbf{a}) = \operatorname*{argmin}_{b \in B} \|T(\mathbf{a}) - \mathbf{b}\|^2 \tag{2}$$

From here on, we will use the point-to-point metric unless stated otherwise.

2 Solution

In general, there is no closed-form solution that minimizes (1). When there is no translation, Horn (1987) derives a closed-form solution; therefore, if an initial odometry estimate is precise enough, this translation can be applied to all the points in the point cloud, producing a problem instance with zero translation.

2.1 Approximation for $\mathbb{R}^2$

Consider the situation where the point clouds A and B (with n and m points respectively) are over $\mathbb{R}^2$. First, let us simplify the cost function $\mathcal{L}$.

$$
\begin{aligned}
\mathcal{L} &= \min_T \sum_{\mathbf{a}\in A} \|T(\mathbf{a}) - \varphi(T, \mathbf{a})\|^2 \\
\implies & \min_T \sum_{\mathbf{a}\in A} \|R\mathbf{a}' + \bar{\mathbf{a}} + \mathbf{t} - \varphi(T, \mathbf{a})\|^2 && \text{let } \mathbf{a}' = \mathbf{a} - \bar{\mathbf{a}} \\
\implies & \min_T \sum_{\mathbf{a}\in A} \|R\mathbf{a}' + \mathbf{t} - \varphi(T, \mathbf{a})\|^2 && \text{since } \bar{\mathbf{a}} \text{ is constant}
\end{aligned}
$$

In $\mathbb{R}^2$, the rotation matrix R takes the form

$$
R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}
$$

where it is described by a single parameter θ. Therefore, the task is to optimize $\mathcal{L}$ over the parameter space (θ, t_x, t_y). We take $\mathbf{b}$ to be $\varphi(T, \mathbf{a})$ where context permits.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial t_x} &= \sum_{\mathbf{a}\in A} 2t_x + 2a'_x\cos(\theta) - 2a'_y\sin(\theta) - 2b_x \\
\frac{\partial \mathcal{L}}{\partial t_y} &= \sum_{\mathbf{a}\in A} 2t_y + 2a'_x\sin(\theta) + 2a'_y\cos(\theta) - 2b_y \\
\frac{\partial \mathcal{L}}{\partial \theta} &= \sum_{\mathbf{a}\in A} 2(a'_x\sin(\theta) + a'_y\cos(\theta))(b_x - t_x - a'_x\cos(\theta) + a'_y\sin(\theta)) \\
&\qquad + 2(a'_x\cos(\theta) - a'_y\sin(\theta))(t_y + a'_x\sin(\theta) + a'_y\cos(\theta) - b_y)
\end{aligned}
$$

The roots of $\partial\mathcal{L}/\partial t_x$ and $\partial\mathcal{L}/\partial t_y$ can easily be solved in closed form;

$$
\begin{aligned}
t^*_x &= \frac{1}{n}\sum_{\mathbf{a}\in A} -a'_x\cos(\theta) + a'_y\sin(\theta) + b_x \\
t^*_y &= \frac{1}{n}\sum_{\mathbf{a}\in A} -a'_x\sin(\theta) - a'_y\cos(\theta) + b_y
\end{aligned}
$$

or

$$
\mathbf{t}^* = \frac{1}{n}\sum_{\mathbf{a}\in A} \mathbf{b} - R\mathbf{a}'
$$