1 Problem Definitions and Statement

Definition 1. A point cloud A over \mathbb{R}^n is a sequence of $|A| = k \in \mathbb{N}^+$ vectors in \mathbb{R}^n , that is,

$$A = (a_1, \dots, a_k) \quad \forall i \in [k], a_i \in \mathbb{R}^n$$

Definition 2. The *centroid* of a point cloud A is defined to be

$$\bar{a} = \frac{1}{|A|} \sum_{a \in A} a$$

Definition 3. A rigid-body transformation on a point cloud A is an affine map $T: \mathbb{R}^n \to \mathbb{R}^n$ determined by an rotation matrix $R \in \mathbb{R}^{n \times n}$ and translation $t \in \mathbb{R}^n$ and given by

$$T: a \mapsto R(a - \bar{a}) + \bar{a} + t$$

that is, a rotation around the centroid of followed by a translation.

The Point-Cloud Registration Problem is, given two point clouds A,B and a pairing function $\phi(T,\cdot)$ based on a rigid-body transformation T such that $\forall a\in A,\, \phi(T,a)\in B,$ to determine the rigid-body transformation T that optimizes

$$\min_{T} \sum_{a \in A} ||T(a) - \phi(T, a)||^2 \tag{1}$$

Definition 4. The *point-to-point metric* for two point clouds A, B is given by

$$\phi(T, a) = \underset{b \in B}{\operatorname{argmin}} \|T(a) - b\|^2 \tag{2}$$

2 Solution

In general, there is no closed-form solution that minimizes (1). When there is no translation, Horn (1987) derives a closed-form solution; therefore, if an initial odometry estimate is precise enough, this translation can be applied to all the points in the point cloud, producing a problem instance with zero translation.

Therefore, we must resort to heuristics

$$\begin{split} & \min_{T} \sum_{a \in A} \|T(a) - \phi(T, a)\|^2 \\ &= \min_{T} \sum_{a \in A} \|Ra' + \bar{a} + t - \phi(T, a)\|^2 \\ &= \min_{T} \sum_{a \in A} \|Ra' + t - \phi(T, a)\|^2 \\ &= \min_{T} \sum_{a \in A} \|Ra' + t - \phi(T, a)\|^2 \end{aligned} \qquad (\bar{a} \text{ is constant})$$

$$-2b_x + 2t_x + 2(a'_x)\cos(\theta) - 2(a'_y)\sin(\theta) -2b_y + 2t_y + 2(a'_x)\sin(\theta) + 2(a'_y)\cos(\theta) (-2(a'_x)\sin(\theta) - 2(a'_y)\cos(\theta))(-b_x + t_x + (a'_x)\cos(\theta) - (a'_y)\sin(\theta)) + (2(a'_x)\cos(\theta) - 2(a'_y)\sin(\theta))(-b_y + (a'_x)\cos(\theta) - (a'_y)\sin(\theta)) + (a'_x)\cos(\theta) - (a'_y)\sin(\theta) + (a'_x)\cos(\theta) + (a'_x)\cos$$