# Today in Cryptography (5830)

Public-key encryption
The RSA permutation
PKCS#1 RSA encryption

# TLS handshake for RSA transport

Client

Server

Pick random Nc

ClientHello, MaxVer, Nc, Ciphers/CompMethods →

Pick random Ns

← ServerHello, Ver, Ns, SessionID, Cipher/CompMethod

Check CERT using CA public verification key

← CERT = (pk of bank, signature over it)

Pick random PMS
C <- E(pk,PMS)

C →

PMS <- D(sk,C)

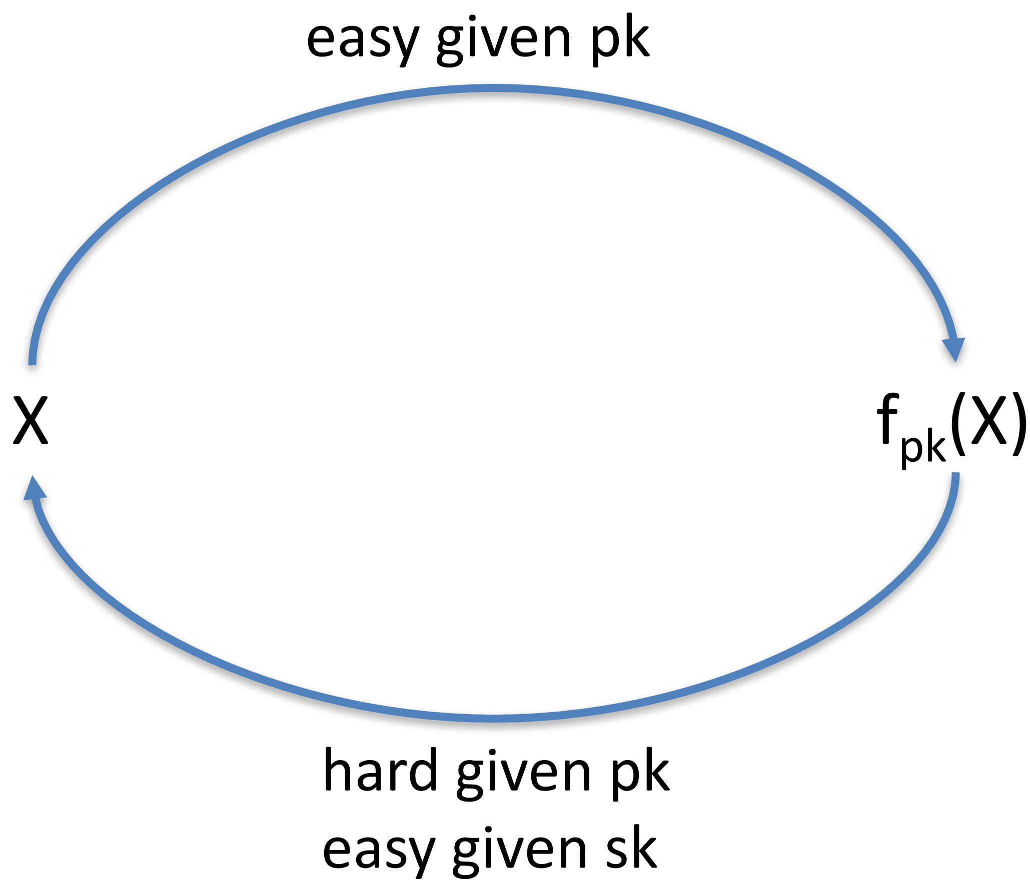
ChangeCipherSpec,
{ Finished, PRF(MS, “Client finished” || H(transcript)) } →

ChangeCipherSpec,
← { Finished, PRF(MS, “Server finished” || H(transcript’)) }

Bracket notation means contents encrypted

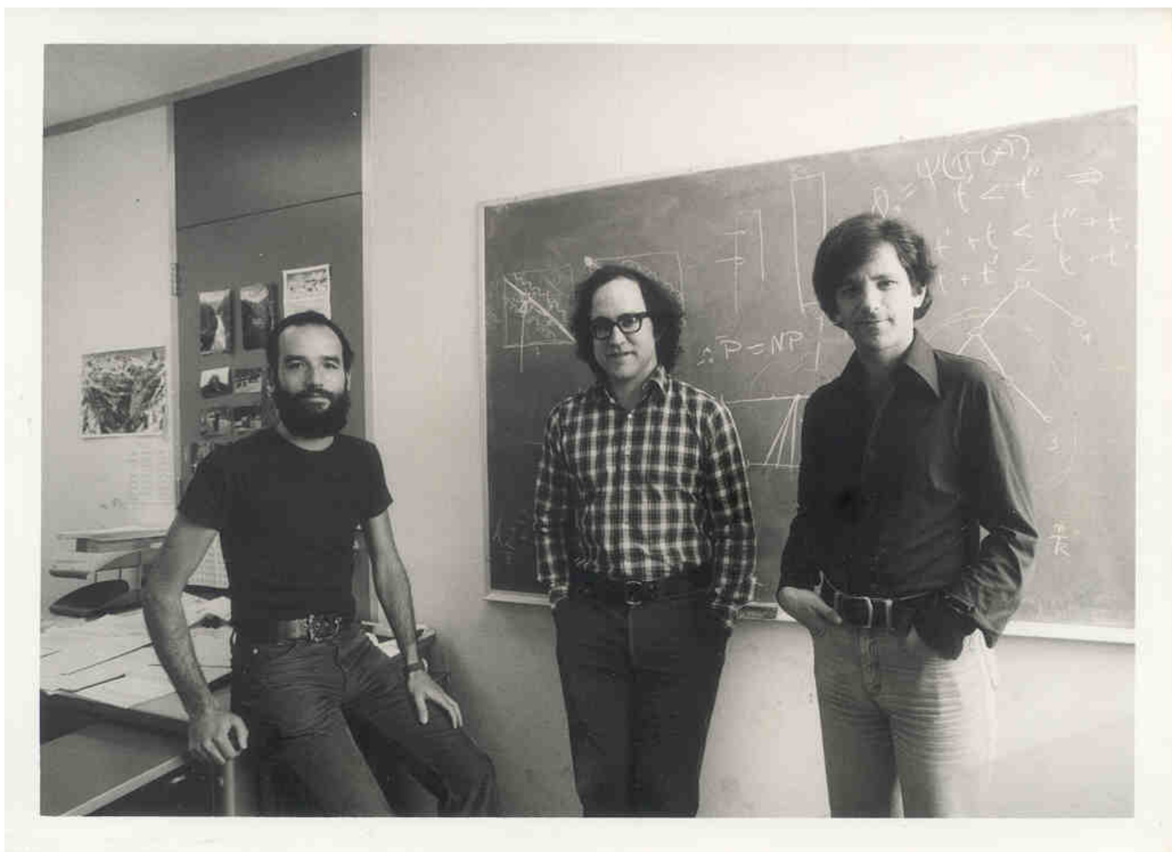
MS <- PRF(PMS, “master secret” || Nc || Ns )

# Trapdoor functions



# The RSA trapdoor function

- Rivest, Shamir, Adleman 1978
- Garnered them a Turing award



# RSA math

Let N be a positive number
Looking ahead: N = pq for large primes p,q
N will be called the modulus

p = 7, q = 13, gives N = 91

p = 17, q = 53, gives N = 901

# RSA math

Let N be a positive number
Looking ahead: N = pq for large primes p,q
N will be called the modulus

$\mathbf{Z}_N = \{0,1,2,3,\ldots, N\text{-}1\}$

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \text{ and } i < N\}$

gcd(X,Y) = 1 if greatest common divisor of X,Y is 1

# RSA math

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \}$

N = 13 $\mathbf{Z}_{13}^* = \{ 1,2,3,4,5,6,7,8,9,10,11,12 \}$

N = 15 $\mathbf{Z}_{15}^* = \{ 1,2,4,7,8,11,13,14 \}$

The size of a set S is denoted by $|S|$

Def. $\phi(N) = |\mathbf{Z}_N^*|$ (This is Euler's totient function)

$\phi(13) = 12$

$\phi(15) = 8$

$\mathbf{Z}_{\phi(15)}^* = \mathbf{Z}_8^* = \{ 1,3,5,7 \}$

# RSA math

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \}$

Fact. For any a,N with N > 0, there exists unique q,r such that

$$a = Nq + r \qquad \text{and} \qquad 0 \le r < N$$

$$17 \bmod 15 = 2 \qquad\qquad 105 \bmod 15 = 0$$

Def. $a \bmod N = r \in \mathbf{Z}_N$

Def. $a \equiv b \pmod N$ iff $(a \bmod N) = (b \bmod N)$

Operations work in natural way:

$$a \bullet b \bmod N \qquad\qquad a+b \bmod N$$

# RSA math

$\mathbf{Z}_N^*$ = { i | gcd(i,N) = 1 }

($\mathbf{Z}_N^*$ , •) is a **group** where • denotes multiplication mod N

Group is a set and operator (G,•) that satisfy:

1. *Closure*: for all a,b ∈ G it holds that a•b ∈ G
2. *Associativity*: for all a,b,c ∈ G it holds that a•(b•c) = (a•b)•c
3. *Identity*: Exists $\mathrm{I}$ ∈ G s.t. for all a ∈ G a•$\mathrm{I}$ = a
4. *Inverses*: for a ∈ G there exists $a^{-1}$ ∈ G s.t. $a \bullet a^{-1} = \mathrm{I}$

Abelian group is additionally commutative:

for all a,b ∈ G it holds that a•b = b•a

# RSA math

$\mathbf{Z}_N^*$ = { i | gcd(i,N) = 1 }

($\mathbf{Z}_N^*$ , •) is a **group**

Group is a set and operator (G,•)that satisfy:

1. Closure: for all a,b ∈ G it holds that a•b ∈ G
2. Associativity: for all a,b,c ∈ $\mathbf{Z}_N^*$ it holds that a•(b•c) = (a•b)•c
3. Identity: Exists $\mathrm{I}$ ∈ $\mathbf{Z}_N^*$ s.t. for all a ∈ $\mathbf{Z}_N^*$ a•$\mathrm{I}$ = a
4. Inverses: for a ∈ $\mathbf{Z}_N^*$ there exists $a^{-1}$ ∈ $\mathbf{Z}_N^*$ s.t. a•$a^{-1}$ = $\mathrm{I}$

# RSA math

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \}$

$(\mathbf{Z}_N^*, \bullet)$ is a **group**

$$\mathbf{Z}_{15}^* = \{ 1,2,4,7,8,11,13,14 \}$$

$2 \bullet 7 \equiv 14 \pmod{15}$

$4 \bullet 8 \equiv 2 \pmod{15}$

Closure: for any $a,b \in \mathbf{Z}_N^*$ $\quad a \bullet b \bmod N \in \mathbf{Z}_N^*$

Def. $\quad a^i \bmod N = \underbrace{a \bullet a \bullet a \bullet \ldots \bullet a}_{i \text{ times}} \bmod N$

# Some needed algorithms

| **Algorithm** | **Running time (n = log N)** |
|---|---|
| Modular multiplication<br>ab mod N | $\mathcal{O}(n^2)$ |
| Modular exponentation<br>$a^i$ mod N | $\mathcal{O}(n^3)$ |
| Modular inverse<br>$a^{-1}$ mod N | $\mathcal{O}(n^2)$ |

# Textbook exponentiation

Let G be a group.
How do we compute $h^x$ for any $h \in G$?

```
Exp(h,x)
X' = h
For i = 2 to x  do
      X' = X'•h
Return X'
```

Requires time $O(|G|)$ in worst case.

```
SqrAndMulExp(h,x)
b_k,…,b_0 = x
f = 1
For i = k down to 0 do
      f = f^2
      If b_i = 1 then
            f = f•h
Return f
```

Requires time $O(k)$ multiplies and squares in worst case.

```
SqrAndMulExp(h,x)
b_k,…,b_0 = x
f = 1
For i = k down to 0 do
    f = f^2
    If b_i = 1 then
        f = f*h
Return f
```

$$x = \sum_{b_i \neq 0} 2^i$$

$$h^x = h^{\sum_{b_i \neq 0} 2^i} = \prod_{b_i \neq 0} h^{2^i}$$

$$h^{11} = h^{8+2+1} = h^8 \bullet h^2 \bullet h$$

$b_3 = 1$ $\quad f_3 = 1 \bullet h$

$b_2 = 0$ $\quad f_2 = h^2$

$b_1 = 1$ $\quad f_1 = (h^2)^2 \bullet h$

$b_0 = 1$ $\quad f_0 = (h^4 \bullet h)^2 \bullet h = h^8 \bullet h^2 \bullet h$

Don't implement this algorithm:
side-channel attacks

# RSA math

$\mathbf{Z}_N^* = \{\, i \mid \gcd(i,N) = 1 \,\}$

Claim: Suppose $e,d \in \mathbf{Z}_{\phi(N)}^*$ satisfying $ed \bmod \phi(N) = 1$ then for any $x \in \mathbf{Z}_N^*$ we have that

$$(x^e)^d \bmod N = x$$

$$\begin{aligned}(x^e)^d \bmod N &= x^{(ed \bmod \phi(N))} \bmod N \\ &= x^1 \bmod N \\ &= x \bmod N\end{aligned}$$

First equality is by Euler's Theorem

# RSA math

$\mathbf{Z}_N^*$ = { i | gcd(i,N) = 1 }

Claim: Suppose e,d $\in$ $\mathbf{Z}_{\phi(N)}^*$ satisfying ed mod $\phi$(N) = 1
then for any x $\in$ $\mathbf{Z}_N^*$ we have that

$$(x^e)^d \bmod N = x$$

$\mathbf{Z}_{15}^*$ = { 1,2,4,7,8,11,13,14 }     $\mathbf{Z}_{\phi(15)}^*$ = { 1,3,5,7 }

e = 3 , d = 3 gives ed mod 8 = 1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| x | 1 | 2 | 4 | 7 | 8 | 11 | 13 | 14 |
| $x^3$ mod 15 | 1 | 8 | 4 | 13 | 2 | 11 | 7 | 14 |
| $y^3$ mod 15 | 1 | 2 | 4 | 7 | 8 | 11 | 13 | 14 |

# The RSA trapdoor permutation

$pk = (N,e)$ $\quad$ $sk = (N,d)$ $\quad$ with $ed \bmod \phi(N) = 1$

$f_{N,e}(x) = x^e \bmod N$ $\quad$ $g_{N,d}(y) = y^d \bmod N$

$$\begin{tikzcd} X \arrow[rr, bend left, "\text{easy given N,e}"] & & f_{pk}(X) \arrow[ll, bend left, "\substack{\text{hard given N,e} \\ \text{easy given N,d}}"] \end{tikzcd}$$

# The RSA trapdoor permutation

$pk = (N,e)$ $sk = (N,d)$ with $ed \bmod \phi(N) = 1$

$f_{N,e}(x) = x^e \bmod N$ $g_{N,d}(y) = y^d \bmod N$

But how do we find suitable $N,e,d$ ?

If $p,q$ distinct primes and $N = pq$ then $\phi(N) = (p-1)(q-1)$

Why?

$$\begin{aligned}
\phi(N) &= |\{1,\ldots,N-1\}| - |\{ip : 1 \le i \le q-1\}| - |\{iq : 1 \le i \le p-1\}| \\
&= N-1 - (q-1) - (p-1) \\
&= pq - p - q + 1 \\
&= (p-1)(q-1)
\end{aligned}$$

# The RSA trapdoor permutation

$pk = (N,e)$ $sk = (N,d)$ with $ed \bmod \phi(N) = 1$

$f_{N,e}(x) = x^e \bmod N$ $g_{N,d}(y) = y^d \bmod N$

But how do we find suitable $N,e,d$ ?

If $p,q$ distinct primes and $N = pq$ then $\phi(N) = (p-1)(q-1)$

Given $\phi(N)$, choose $e \in \mathbf{Z}^*_{\phi(N)}$ and calculate

$$d = e^{-1} \bmod \phi(N)$$

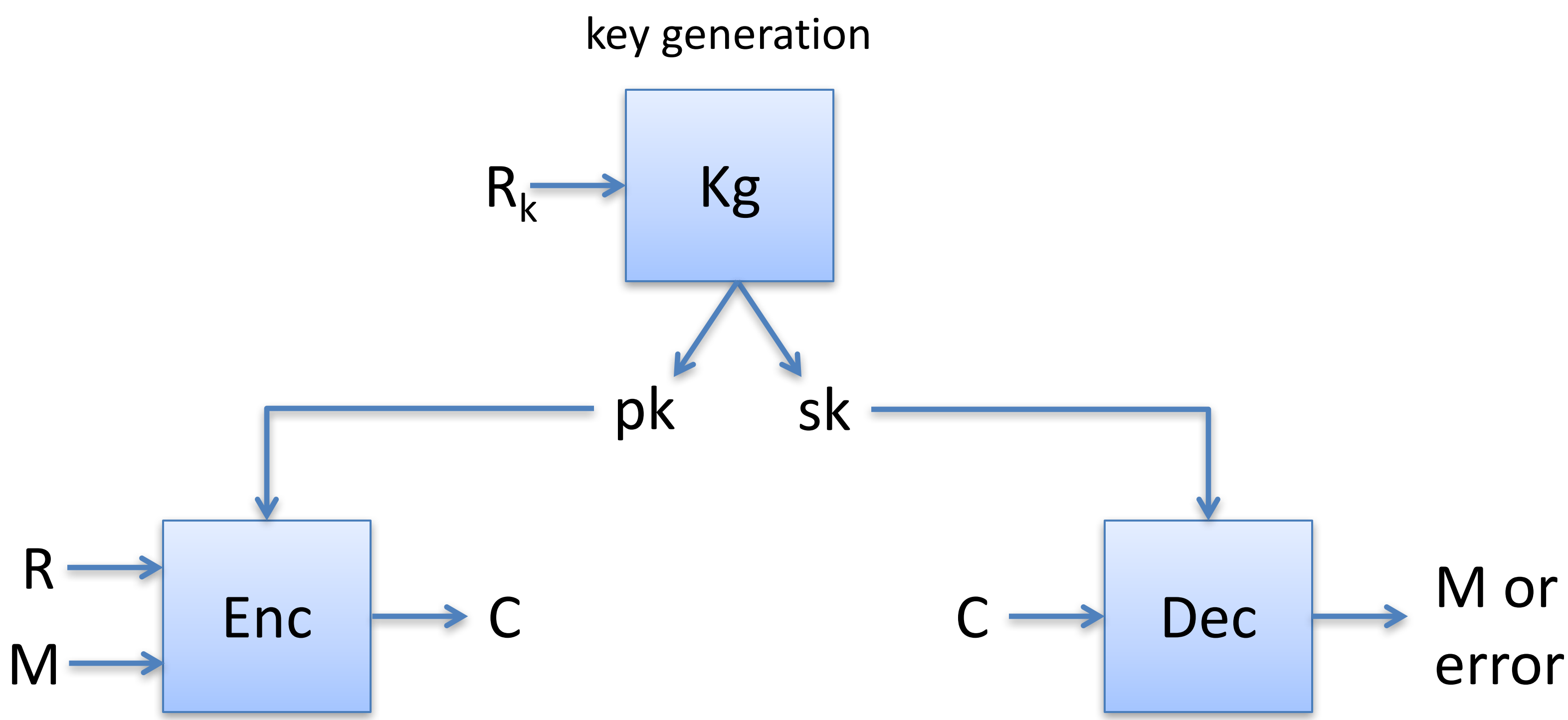
How to find suitable p,q prime?

Choose random numbers and test primality

# Summary

- Find 2 large primes p, q . Let $N = pq$
  - random integers + primality testing
- Choose e (usually 65,537)
  - Compute d using $\phi(N) = (p-1)(q-1)$
- pk = (N,e) and sk = (N,d)
  - Often store p,q with sk to use Chinese Remainder Theorem

# Public-key encryption




C is a ciphertext

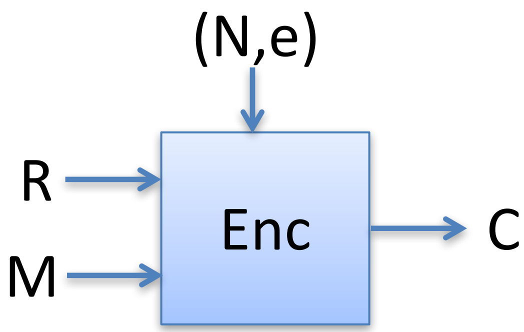
Correctness: D( sk , E(pk,M,R) ) = M with probability 1 over randomness used

# PKCS #1 RSA encryption

Kg outputs $(N,e),(N,d)$ where $|N|_8 = n$

Let $B = \{0,1\}^8 / \{00\}$ be set of all bytes except 00

Want to encrypt messages of length $|M|_8 = m$

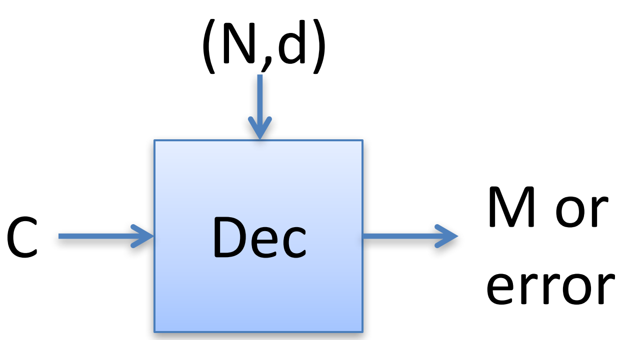



<u>Enc((N,e), M, R)</u>

pad = first $n - m - 3$ bytes from R that are in B

$X = 00 \,||\, 02 \,||\, pad \,||\, 00 \,||\, M$

Return $X^e \bmod N$




<u>Dec((N,d), C )</u>

$X = C^d \bmod N$ ; $aa||bb||w = X$

If $(aa \neq 00)$ or $(bb \neq 02)$ or $(00 \notin w)$

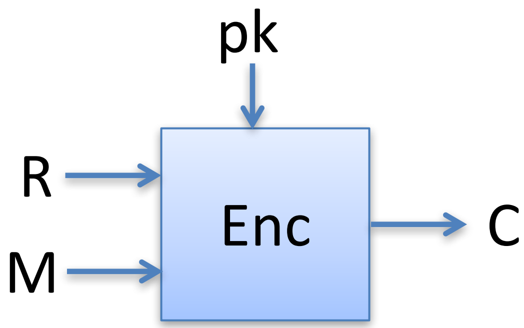
Return error

$pad \,||\, 00 \,||\, M = w$

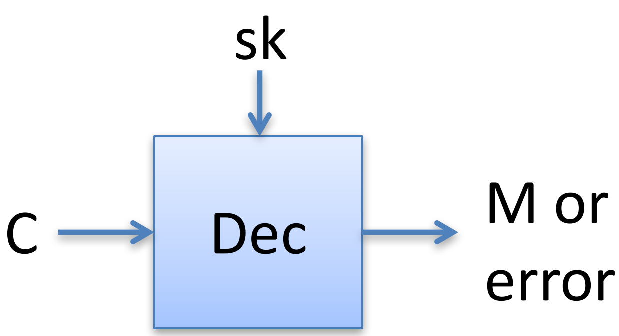
Return M

# Hybrid encryption

Kg outputs (pk,sk)




<u>Enc(pk, M, R)</u>
K||R1||R2 = R
C1 = Enc(pk,K,R1)
C2 = Enc(K,M,R2)
Return (C1,C2)




<u>Dec(sk, (C1,C2) )</u>
K = Dec(sk,C1)
M = Dec(K,C2)
Return M

# TLS handshake for RSA transport

Bank customer

Bank

Pick random Nc — ClientHello, MaxVer, Nc, Ciphers/CompMethods →

← ServerHello, Ver, Ns, SessionID, Cipher/CompMethod — Pick random Ns

Check CERT using CA public verification key ← CERT = (pk of bank, signature over it)

Pick random PMS
C <- E(pk,PMS)

C →

PMS <- D(sk,C)

ChangeCipherSpec,
{ Finished, PRF(MS, “Client finished” || H(transcript)) } →

ChangeCipherSpec,
{ Finished, PRF(MS, “Server finished” || H(transcript’)) } ←

Bracket notation means contents encrypted

MS <- PRF(PS, “master secret” || Nc || Ns )

# Security of RSA PKCS#1

- Passive adversary sees (N,e),C
- Attacker would like to invert C
- Possible attacks?

Inverting RSA : given $N, e, y$ find $x$ such that $x^e \equiv y \pmod{N}$

$\uparrow$ EASY — because $f^{-1}(y) = y^d \bmod N$

Know $d$

$\uparrow$ EASY — because $d = e^{-1} \bmod \varphi(N)$

Know $\varphi(N)$

$\uparrow$ EASY — because $\varphi(N) = (p-1)(q-1)$

Know $p, q$

$\uparrow$ ? — Learning p,q from N is the factoring problem

Know $N$

We don't know if inverse is true, whether inverting RSA implies ability to factor

# Factoring composites

- What is p,q for N = 901?

```
Factor(N):
for i = 2 , … ,  sqrt(N) do
     if N mod i = 0 then
          p = i
          q = N / p
          Return (p,q)
```

Woops… we can always factor

But not always efficiently:
Run time is sqrt(N)

$\mathcal{O}(\mathrm{sqrt}(N)) = \mathcal{O}(e^{0.5\ \ln(N)})$

# Factoring composites

| **Algorithm** | **Time to factor N** |
|---|---|
| Naïve | $\mathcal{O}(e^{0.5 \ln(N)})$ |
| Quadratic sieve (QS) | $\mathcal{O}(e^{c})$<br>$c = d (\ln N)^{1/2} (\ln \ln N)^{1/2}$ |
| Number Field Sieve (NFS) | $\mathcal{O}(e^{c})$<br>$c = 1.92 (\ln N)^{1/3} (\ln \ln N)^{2/3}$ |

# Factoring records

| **Challenge** | **Year** | **Algorithm** | **Time** |
|---|---|---|---|
| RSA-400 | 1993 | QS | 830 MIPS years |
| RSA-478 | 1994 | QS | 5000 MIPS years |
| RSA-515 | 1999 | NFS | 8000 MIPS years |
| RSA-768 | 2009 | NFS | ~2.5 years |
| RSA-512 | 2015 | NFS | $75 on EC2 / 4 hours |

RSA-x is an RSA challenge modulus of size x bits

# Security of RSA PKCS#1

- Passive adversary sees (N,e),C
- Attacker would like to invert C
- Possible attacks?
  - Pick |N| > 1024 and factoring will fail
  - Active attacks?

# Bleichanbacher attack

I've just learned some information about $C_1^d \bmod N$

$C_1$ →

← padding error?

$C_2$ →

← padding error?

…

<u>Dec((N,d), C )</u>

$X = C^d \bmod N$ ; $aa||bb||w = X$

If $(aa \neq 00)$ or $(bb \neq 02)$ or $(00 \notin w)$

Return error

$pad \,||\, 00 \,||\, M = w$

Return M

We can take a target C and decrypt it using a sequence of chosen ciphertexts $C_1, \ldots, C_q$ where $q \approx 1$ million

[Bardou et al. 2012] q = 9400 ciphertexts on average

# Response to this attack

- Ad-hoc fix: Don't leak whether padding was wrong or not
  - This is harder than it looks (timing attacks, control-flow side channel attacks, etc.)
- Better:
  - use chosen-ciphertext secure encryption
  - OAEP is common choice

# Summary

- RSA is example of trapdoor one-way function
  – Security conjectured. Relies on factoring being hard
- RSA security scales somewhat poorly with size of primes
- RSA PKCS#1 v1.5 is insecure due to padding oracle attacks. Don't use it in new systems.
  – Use OAEP instead

Client

# TLS handshake for Diffie-Hellman Key Exchange

Server

Pick random Nc

ClientHello, MaxVer, Nc, Ciphers/CompMethods →

Pick random Ns

ServerHello, Ver, Ns, SessionID, Cipher/CompMethod ←

Pick random x

$X = g^x$

CERT = ($pk_s$ , signature over it) ←

$p$ , $g$ , $X$ , $\sigma$ = Sign($sk_s$, $p$ || $g$ || $X$) ←

Check CERT using CA public verification key

Check σ

Pick random y

$Y = g^y$

Y →

PMS = $g^{xy}$

PMS = $g^{xy}$

ChangeCipherSpec,
{ Finished, PRF(MS, “Client finished” || H(transcript)) } →

ChangeCipherSpec,
{ Finished, PRF(MS, “Server finished” || H(transcript’)) } ←

Bracket notation means contents encrypted

MS <- PRF(PMS, “master secret” || Nc || Ns )

# Diffie-Hellman math

Let p be a large prime number
Fix the group G = $\mathbf{Z}_p^*$ = {1,2,3,…, p-1}

Then G is *cyclic*. This means one can give a member $g \in G$, called the generator, such that

$$G = \{ g^0, g^1, g^2, \ldots, g^{p-1} \}$$

Example: p = 7. Is 2 or 3 a generator for $\mathbf{Z}_7^*$ ?

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $2^x$ mod 7 | 1 | 2 | 4 | 1 | 2 | 4 | 1 |
| $3^x$ mod 7 | 1 | 3 | 2 | 6 | 4 | 5 | 1 |

# Textbook exponentiation

Let G be cyclic group.

How do we compute $h^x$ for any $h \in G$?

<u>ModExp(h,x)</u>

```
X' = h
For i = 2 to x  do
      X' = X'*h
Return X'
```

Requires time O(|G|) in worst case.

<u>SqrAndMulExp(h,x)</u>

$b_k,\ldots,b_0 = x$

$f = 1$

For i = k down to 0 do

  $f = f^2 \mod N$

  If $b_i = 1$ then

    $f = f*h$

Return f

Requires time O(k) multiplies and squares in worst case.

```
SqrAndMulExp(h,x)
b_k,…,b_0 = x
f = 1
For i = k down to 0 do
    f = f^2  mod N
    If b_i = 1 then
        f = f*h
Return f
```

$$x = \sum_{b_i \neq 0} 2^i$$

$$h^x = h^{\sum_{b_i \neq 0} 2^i} = \prod_{b_i \neq 0} h^{2^i}$$

$$h^{11} = h^{8+2+1} = h^8 \bullet h^2 \bullet h$$

$b_3 = 1$ $\quad f_3 = 1 \bullet h$

$b_2 = 0$ $\quad f_2 = h^2$

$b_1 = 1$ $\quad f_1 = (h^2)^2 \bullet h$

$b_1 = 1$ $\quad f_0 = (h^4 \bullet h)^2 \bullet h = h^8 \bullet h^2 \bullet h$

# The discrete log problem

Fix a cyclic group G with generator g

Pick x at random from $\mathbf{Z}_{|G|}$

Give adversary g, $X = g^x$ . Adversary's goal is to compute x

$$\begin{tikzcd} x \arrow[r, bend left, "\text{easy}"] & g^x \arrow[l, bend left, "\text{hard}"] \end{tikzcd}$$

# The discrete log problem

Fix a cyclic group G with generator g

Pick x at random from $\mathbf{Z}_{|G|}$

Give adversary g, $X = g^x$ . Adversary's goal is to compute x

<u>$\mathcal{A}$(X):</u>
for i = 2 , … , |G|-1 do
    if $X = g^i$ then
        Return i

Very slow for large groups!
$\mathcal{O}(|G|)$

Baby-step giant-step is better:
$\mathcal{O}(|G|^{0.5})$
Nothing faster is known for some groups.

# Diffie-Hellman Key Exchange

$X$ →

← $Y$

Pick random x from $\mathbf{Z}_{|G|}$
$X = g^x$

Pick random y from $\mathbf{Z}_{|G|}$
$Y = g^y$

$K = H(Y^x)$

$K = H(X^y)$

Get the same key. Why? $Y^x = g^{yx} = g^{xy} = X^y$

What type of security does this protocol provide?

# Computational Diffie-Hellman Problem

Fix a cyclic group G with generator g

Pick x,y both at random $\mathbf{Z}_{|G|}$

Give adversary g, $X = g^x$ , $Y = g^y$.
Adversary must compute $g^{xy}$

For most groups, best known algorithm finds discrete log of X or Y.
But we have no proof that this is best approach.

Client

# TLS handshake for Diffie-Hellman Key Exchange

Server

| | | |
|---|---|---|
| Pick random Nc | ClientHello, MaxVer, Nc, Ciphers/CompMethods → | |
| Check CERT using CA public verification key | ← ServerHello, Ver, Ns, SessionID, Cipher/CompMethod | Pick random Ns |
| | ← CERT = ($pk_s$ , signature over it) | Pick random x |
| Check σ | ← p , g , X , σ = Sign($sk_s$, p \|\| g \|\| X) | $X = g^x$ |
| Pick random y<br>$Y = g^y$ | Y → | PMS = $g^{xy}$ |
| PMS = $g^{xy}$ | ChangeCipherSpec,<br>{ Finished, PRF(MS, “Client finished” \|\| H(transcript)) } → | |
| | ← ChangeCipherSpec,<br>{ Finished, PRF(MS, “Server finished” \|\| H(transcript’)) } | |

Bracket notation means contents encrypted

MS <- PRF(PMS, “master secret” || Nc || Ns )