Sec 5a)

Task 1

a) $$g(E) = \frac{d\Omega(E)}{dE}$$

$\Omega(E)$ is phase space volume enclosed by energy constraint $H \leq E$

$H \leq E$ describes 4D hypersphere in phase space

$$H = \frac{P_x^2 + P_y^2}{2m} + \frac{1}{2}m\omega^2(x^2+y^2) \leq E$$

↪ transform to action angle coords

$$E = E_x + E_y \; ; \; I_x = \frac{E_x}{\omega} \; , \; I_y = \frac{E_y}{\omega}$$

$$\Omega(E) = \int_{H \leq E} d^2x\, d^2p \equiv \int_{H \leq E} \frac{dI_x\, dI_y}{(2\pi\hbar)^2}$$

$$\Omega(E) = \frac{(2\pi)^2}{\omega^2}\int_0^E E'\, dE'$$

$$\Omega(E) = \frac{4\pi^2}{\omega^2}\frac{E^2}{2} = \frac{2\pi^2 E^2}{\omega^2}$$

$$g(E) = \frac{d\Omega(E)}{dE} = \frac{4\pi^2 E}{\omega^2}$$

b) $$Z(\beta) = \int_0^\infty g(E)e^{-\beta E}dE \; : \; \beta = \frac{1}{k_BT}$$

$$Z(\beta) = \frac{4\pi^2}{\omega^2}\int_0^\infty E e^{-\beta E} dE$$

$\int_0^\infty x e^{-ax} = \frac{1}{a^2}$ for $a>0$

$$Z(\beta) = \frac{4\pi^2}{\omega^2\beta^2}$$

c) $H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2} m\omega^2 (x^2+y^2) + \lambda (x^2+y^2)^2$ $\quad r^2 = x^2+y^2$

$H = \frac{p_x^2+p_y^2}{2m} + \frac{1}{2} m\omega^2 r^2 + \lambda r^4$ $\quad U(r) = \frac{1}{2} m\omega^2 r^2 + \lambda r^4$

$H \leq E \equiv \frac{p_x^2+p_y^2}{2m} \leq E - U(r) \longrightarrow P_{max}(r) = \sqrt{2m(E-U(r))} \rightarrow$ momentum space area : $\pi (P_{max})^2$

$\pi (2m(E-U(r)))$

$\Omega(E) = \int_0^{2\pi}\int_0^{r_{max}} \pi [2m(E-U(r))] r\,dr\,d\theta$

$= 2m\pi \int_0^{2\pi} d\theta \int_0^{r_{max}} r(E-U(r))dr$

$2\pi \qquad r(E-(\frac{1}{2}m\omega^2 r^2 + \lambda r^4))dr$

$4\pi^2 m \int_0^{r_{max}} [E - \frac{1}{2}m\omega^2 r^2 + \lambda r^4] r\,dr$ → need to evaluate "r_{max}"

$Er - \frac{1}{2}m\omega^2 r^3 + \lambda r^5$

$[\frac{1}{2}Er^2 - \frac{1}{8}m\omega^2 r^4 + \frac{1}{6}\lambda r^6]_0^{r_{max}}$

we know r_{max} is when $E = U(r)$

$\lambda r_{max}^4 + \frac{1}{2} m\omega^2 r_{max}^2 - E = 0$

$r_{max}^2 = "x" = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-\frac{1}{2}m\omega^2 \pm \sqrt{(\frac{1}{2}m\omega^2)^2 - 4\lambda(-E)}}{2\lambda}$

$\Omega(E) = 2\pi^2 m (E r_{max}^2 - \frac{1}{4} m\omega^2 r_{max}^2 + \frac{1}{3}\lambda r_{max}^6)$

$r_{max_+}^2 = \frac{-\frac{1}{2}m\omega^2 \cancel{\pm} \sqrt{\frac{m^2\omega^4}{4} + 4\lambda E}}{2\lambda}$

$g(E) = \frac{d\Omega(E)}{dE} = 2\pi^2 m r_{max}^2$

$= \cancel{2}\pi^2 m \left[\frac{-\frac{1}{2}m\omega^2 + \sqrt{\frac{m^2\omega^4}{4} + 4\lambda E}}{\cancel{2}\lambda}\right]$

$= \frac{\pi^2 m}{\lambda}\left(-\frac{1}{2}m\omega^2 + \sqrt{\frac{m^2\omega^4}{4} + 4\lambda E}\right)$

$g(E) = \frac{\pi^2 m}{2\lambda}\left(-m\omega^2 + \sqrt{m^2\omega^4 + 16\lambda E}\right)$

Task 2

a) $L = \frac{1}{2}(m_1+m_2)L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2L_2^2\dot{\theta}_2^2 + m_2L_1L_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + (m_1+m_2)gL_1\cos\theta_1$

$+m_2gL_2\cos\theta_2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_i}\right) - \frac{\partial L}{\partial\theta_i} = 0$$

$$\begin{bmatrix}(m_1+m_2)L_1^2 & m_2L_1L_2\cos(\theta_1-\theta_2)\\ m_2L_1L_2\cos(\theta_1-\theta_2) & m_2L_2^2\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} = \begin{bmatrix}-m_2L_1L_2\sin(\theta_1-\theta_2)\dot{\theta}_2^2-(m_1+m_2)gL_1\sin\theta_1\\ m_2L_1L_2\sin(\theta_1-\theta_2)\dot{\theta}_1^2-m_2gL_2\sin\theta_2\end{bmatrix}$$

b) generalized momenta:

$$P_i = \frac{\partial L}{\partial\dot{\theta}_i}$$

$$p_1 = \frac{\partial L}{\partial\dot{\theta}_1} = (m_1+m_2)L_1^2\dot{\theta}_1 + m_2L_1L_2\cos(\theta_1-\theta_2)\dot{\theta}_2$$

$$P_2 = \frac{\partial L}{\partial\dot{\theta}_2} = m_2L_2^2\dot{\theta}_2 + m_2L_1L_2\cos(\theta_1-\theta_2)\dot{\theta}_1$$

$$\begin{bmatrix}p_1\\ p_2\end{bmatrix} = \underbrace{\begin{bmatrix}(m_1+m_2)L_1^2 & m_2L_1L_2\cos(\theta_1-\theta_2)\\ m_2L_1L_2\cos(\theta_1-\theta_2) & m_2L_2^2\end{bmatrix}}_{M}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

$$\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} = M^{-1}\begin{bmatrix}p_1\\ p_2\end{bmatrix}$$

$$H = p_1\dot{\theta}_1 + p_2\dot{\theta}_2 - L$$

$$H = \frac{1}{2}[p_1\ p_2]M^{-1}\begin{bmatrix}p_1\\ p_2\end{bmatrix} + (m_1+m_2)gL_1\cos\theta_1 + m_2gL_2\cos\theta_2$$

matrix: $H = \frac{1}{2}p^TM^{-1}p + U(\theta_1,\theta_2)$; $p = \begin{bmatrix}p_1\\ p_2\end{bmatrix}$, $U(\theta_1,\theta_2) = -(m_1+m_2)gL_1\cos\theta_1 - m_2gL_2\cos\theta_2$