Conversion Between Itô and Stratonovich Stochastic Integrals

1. Definitions of Stochastic Integrals

Consider a stochastic process X_t driven by a Wiener process W_t and a function $f(X_t)$. The stochastic integral can be formulated in two different ways:

Itô Integral (Left-Point Rule)

$$\int_0^T f(X_t)\, dW_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(X_{t_i})\Delta W_i$$

where t_i is the left endpoint of each partition interval.

Stratonovich Integral (Midpoint Rule)

$$\int_0^T f(X_t) \circ dW_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} f\left(\frac{X_{t_i} + X_{t_{i+1}}}{2}\right) \Delta W_i.$$

2. Expansion Using Taylor's Theorem

Using a first-order Taylor expansion around X_{t_i}:

$$f\left(\frac{X_{t_i} + X_{t_{i+1}}}{2}\right) = f(X_{t_i}) + \frac{1}{2} f'(X_{t_i})(X_{t_{i+1}} - X_{t_i}) + O(\Delta t).$$

From the stochastic differential equation:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

we approximate the discrete change:

$$X_{t_{i+1}} - X_{t_i} = \mu(X_{t_i})\Delta t + \sigma(X_{t_i})\Delta W_i.$$

3. Substituting into the Stratonovich Integral

Substituting this into the Stratonovich sum,

$$\sum_{i=0}^{n-1} f(X_{t_i})\Delta W_i + \sum_{i=0}^{n-1} \frac{1}{2} f'(X_{t_i})\sigma(X_{t_i})(\Delta W_i^2).$$

Using the property of Brownian motion $\mathbb{E}[\Delta W_i^2] = \Delta t$, we obtain:

$$\sum_{i=0}^{n-1} f(X_{t_i})\Delta W_i + \sum_{i=0}^{n-1} \frac{1}{2} f'(X_{t_i})\sigma(X_{t_i})\Delta t.$$

4. Final Conversion Formula

Taking the limit $n \to \infty$, we obtain the conversion formula:

$$\int_0^T f(X_t) \circ dW_t = \int_0^T f(X_t)\, dW_t + \frac{1}{2} \int_0^T f'(X_t)\sigma(X_t)\, dt.$$

Thus, the **Stratonovich integral** differs from the **Itô integral** by the correction term:

$$\frac{1}{2} \int_0^T f'(X_t)\sigma(X_t) dt.$$