List of Todos

1. Introduction

Let K be a knot in S^3. The augmentation rank $\mathrm{ar}(K)$ of K is the maximal degree of a KCH representation K. This invariant has proven to be powerful for it's difficulty to compute, allowing for a lot of progress on ^{TODO}that problem thingy (cite Cornwell)

This paper studies the augmentation rank of cables of knots.

2. Background

We begin in Section 2.1 by reviewing the construction of $HC_0(K)$ from the viewpoint of the combinatorial knot DGA, which was first defined in [Ng08]; our conventions are those given in [Ng12]. In Section 2.2 we discuss augmentations in knot contact homology and their rank, which gives a bound on the meridional rank of the knot group useful for studying the relation between meridional rank and bridge number. Finally, in Section 2.3 is a discussion of techniques from [Cor13a] that we use to calculate the augmentation rank.

2.1. **Knot contact homology.** We begin with the necessary preliminaries for defining the combinatorial knot DGA of Ng. This DGA was defined to be a calculation of knot contact homology and was shown to be so in [EENS13] (see [Ng12] for more details). Let $\mathcal{A}_n$ be the noncommutative unital algebra over $\mathbb{Z}$ generated by a_{ij}, $1 \leq i \neq j \leq n$. Let B_n be the braid group on n strands, and define $\phi : B_n \to \operatorname{Aut} \mathcal{A}_n$ by defining it on the generators of $\mathcal{A}_n$ and extending by linearity

$$\phi_{\sigma_k} : \begin{cases} a_{ij} \mapsto a_{ij} & i, j \neq k, k+1 \\ a_{k+1,i} \mapsto a_{ki} & i \neq k, k+1 \\ a_{i,k+1} \mapsto a_{ik} & i \neq k, k+1 \\ a_{k,k+1} \mapsto -a_{k+1,k} & \\ a_{k+1,k} \mapsto -a_{k,k+1} & \\ a_{ki} \mapsto a_{k+1,i} - a_{k+1,k}a_{ki} & i \neq k, k+1 \\ a_{ik} \mapsto a_{i,k+1} - a_{ik}a_{k,k+1} & i \neq k, k+1 \end{cases}$$

Let $\iota\colon B_n \to B_{n+1}$ be the inclusion that adds in an $(n+1)$th strand that doesn't interact with the others, and define $\phi^*_B \in \operatorname{Aut} \mathcal{A}_{n+1}$ by $\phi^*_B = \phi_B \circ \iota$. We then define the $n \times n$ matrices Φ^L_B and Φ^R_B with entries in $\mathcal{A}_n$ by

$$\phi^*_B(a_{i,n+1}) = \sum_{j=1}^{n} (\Phi^L_B)_{ij} a_{j,n+1}$$

$$\phi^*_B(a_{n+1,i}) = \sum_{j=1}^{n} a_{n+1,j} (\Phi^R_B)_{ji}$$

We will need a relationship that exists between Φ^L_B and Φ^R_B in order to show that an augmentation is well-defined. To this end, define an operation

$x \mapsto \overline{x}$ on $\mathcal{A}_n$ as follows: first $\overline{a_{ij}} = a_{ji}$; then, for any $x, y \in \mathcal{A}_n$, $\overline{xy} = \overline{y}\,\overline{x}$ and extend the operation linearly to $\mathcal{A}_n$.

Proposition 2.1 ([Ng05], Prop. 6.2)**.** *For a matrix of elements in $\mathcal{A}_n$, let $\overline{M}$ be the matrix such that $\left(\overline{M}\right)_{ij} = \overline{M_{ij}}$. Then for $B \in B_n$, Φ^R_B is the transpose of $\overline{\Phi^L_B}$.*

Let ω be the writhe of B, and define matrices $\mathbf{A}$ and Λ by

$$\mathbf{A_{ij}} = \begin{cases} a_{ij} & i < j \\ -\mu a_{ij} & i > j \\ 1 - \mu & i = j \end{cases} \tag{1}$$

$$\Lambda = \mathrm{diag}[\lambda\mu^{\omega}, 1, \ldots, 1]. \tag{2}$$

Definition Suppose that K is the closure of $B \in B_n$ and let R_0 be the Laurent polynomial ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$. Define $\mathcal{I} \subset \mathcal{A}_n \otimes R_0$ to be the ideal generated by the entries of $\mathbf{A} - \mathbf{\Lambda} \cdot \mathbf{\Phi^L_B} \cdot \mathbf{A}$ and $\mathbf{A} - \mathbf{A} \cdot \mathbf{\Phi^R_B} \cdot \mathbf{\Lambda^{-1}}$. The <u>degree zero homology of the combinatorial knot DGA</u> is $\mathrm{HC}_0(K) = (\mathcal{A}_n \otimes R_0)/\mathcal{I}$.

It was shown in [Ng08] that the isomorphism class of $HC_0(K)$ is unchanged under the Markov moves, and hence provides an invariant of the knot K. While we only consider $HC_0(K)$ here, it is part of the larger invariant, the combinatorial knot DGA of K, studied in [Ng08] which is a computation of the Legendrian contact homology of a Legendrian lift of K to the cosphere bundle over $\mathbb{R}^3$ ([EENS13]).

The following result, originally proved in [Ng05], on the behavior of the matrices Φ^L_B and Φ^R_B under the product in B_n will be essential to our arguments. Following language of that paper, we refer the result as the Chain Rule.

Theorem 2.2. *Let B, B' be braids in B_n. Then $\Phi^L_{BB'} = \phi_B(\Phi^L_{B'}) \cdot \Phi^L_B$ and $\Phi^R_{BB'} = \Phi^R_B \cdot \phi_B(\Phi^R_{B'})$.*

2.2. **Augmentations and augmentation rank.** Let S be a ring with 1, and consider it a differential graded algebra with grading 0 and trivial differential. An augmentation of a DGA $(\mathcal{A}, \partial)$ to $(S, 0)$ is a graded homomorphism $\epsilon : \mathcal{A} \to S$ that intertwines the differential. In the case of knot contact homology, the combinatorial knot DGA is supported in non-negative grading, implying that augmentations correspond to ring homomorphisms $HC_0(K) \to S$. We will consider only when $S = \mathbb{C}$.

Definition An <u>augmentation</u> of a cord algebra $\mathcal{C}_K$ is a homomorphism $\epsilon\colon \mathcal{C}_K \to \mathbb{C}$

A correspondence between augmentations and particular representations of the knot group were studied in [Cor13a]. Let π_K be the fundamental group of the complement of a knot $K \subset S^3$. Recall that, if we call any $g \in \pi_K$ a <u>meridian</u> if it may be represented by the boundary of an embedded disk in

S^3 that intersects K in exactly one point, then π_K is generated by meridians. We may pick any one meridian m and generate π_K by conjugates of m.

Definition For any integer $r \geq 1$ we call a homomorphism $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ a KCH representation if a meridian m of K such that $\rho(m)$ is diagonalizable and has eigenvalue 1 with multiplicity $r-1$. We call ρ a KCH irrep if it is irreducible.

In [Ng08], Ng describes an isomorphism between $HC_0(K)$ and an algebra constructed from elements of π_K. As discussed in [Ng12], by utilizing this isomorphism a KCH representation $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ induces an augmentation $\epsilon_\rho : HC_0(K) \to \mathbb{C}$. It was shown in [Cor13a] that (essentially) all augmentations arise in this fashion, and that the dimension of an inducing KCH irrep is invariant of the augmentation that can be described from the matrix $\mathbf{A}$. Specifically, if we write $\epsilon(\mathbf{A})$ for the matrix of values $(\epsilon(\mathbf{A}_{ij}))$, then we have the following theorem.

Theorem 2.3 ([Cor13a])**.** *For every augmentation $\epsilon : HC_0(K) \to \mathbb{C}$ such that $\epsilon(\mu) \neq 1$, there is a KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$, and r is the rank of $\epsilon(\mathbf{A})$.*

Considering Theorem 2.3 we make the following definition.

Definition The rank of an augmentation $\epsilon : HC_0(K) \to \mathbb{C}$ with $\epsilon(\mu) \neq 1$ equals the rank of $\epsilon(\mathbf{A})$. Given a knot K, the augmentation rank of K, denoted $\mathrm{ar}(K)$, is the maximum of all ranks of augmentations $\epsilon : HC_0(K) \to \mathbb{C}$.

Remark The augmentation rank of a knot could be defined for augmentations into other rings, but we deal in this paper with augmentations to $\mathbb{C}$.

It is the case that $\mathrm{ar}(K)$ is well-defined. That is, given a knot K there is a bound on the maximal rank of an augmentation $\epsilon : HC_0(K) \to \mathbb{C}$ that is provided by through the correspondence $\rho \leftrightarrow \epsilon_\rho$ and fact that π_K is generated by meridians.

Theorem 2.4 ([Cor13b])**.** *Given a knot $K \subset S^3$, if $g_1, \ldots, g_d$ are meridians that generate π_K and $\rho : \pi_K \to GL_r\mathbb{C}$ is a KCH irrep then $r \leq d$.*

As in the introduction, if we denote the meridional rank of π_K by $\mathrm{mr}(K)$, then Theorem 2.4 implies that $\mathrm{ar}(K) \leq \mathrm{mr}(K)$. In addition, the geometric quantity $b(K)$ called the bridge index of K is never less than $\mathrm{mr}(K)$. Thus we have the inequality

make it a corollary to refer to later

$$\mathrm{ar}(K) \leq \mathrm{mr}(K) \leq b(K).$$

As a result, to verify for K that $\mathrm{mr}(K) = b(K)$ it suffices to find an augmentation of K with rank equal to $b(K)$. As we discuss in the next section, we will concern ourselves in this paper with a setting where $\mathrm{ar}(K) = n$ and there is a braid $B \in B_n$ which closes to K. This is a special situation, since $b(K)$ is strictly less than the braid index for many knots.

2.3. **Finding augmentations.** Throughout the paper we denote by B_n the n-strand braid group, where our braids are oriented from left to right. We will often label the strands of a braid $1, \ldots, n$, with 1 the topmost to n the bottommost strand. The group B_n has standard generators $\{\sigma_i^{\pm}, i = 1, \ldots, n\}$ which have only the i and $i+1$ strands crossing once, and in the manner depicted in the projections of Figure 1. As usual, a braid may

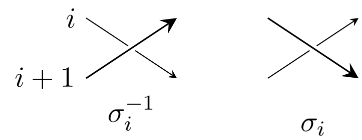

FIGURE 1. Generators of B_n

be closed to a link as depicted in Figure 2. The writhe (or algebraic sum) of a braid B, denoted $w(B)$, is the sum of the exponents in a factorization of B in terms of the standard generators.

In this paper we find augmentations that have rank equal to the braid index of the knot K. Suppose that K is the closure of $B \in B_n$ and define the diagonal matrix $\Delta(B) = \mathrm{diag}[(-1)^{w(B)}, 1, \ldots, 1]$. By considering the generators of the ideal $\mathcal{I}$ from Definition 2.1 the following statement follows from results in [Cor13a, Section 5].

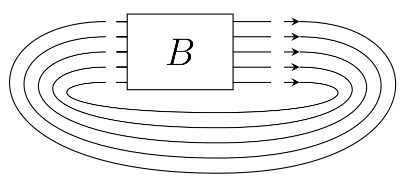

FIGURE 2. The closure of the braid B

Theorem 2.5 ([Cor13a])**.** *If K is the closure of $B \in B_n$ and has a rank n augmentation $\epsilon : HC_0(K) \to \mathbb{C}$, then*

$$\epsilon(\Phi_B^L) = \Delta(B) = \epsilon(\Phi_B^R). \tag{3}$$

Furthermore, any homomorphism $\epsilon : \mathcal{A}_n \to \mathbb{C}$ which satisfies (3) determines a rank n augmentation of K.

Recall that $\mathcal{A}_n^{ab}$ is the quotient of $\mathcal{A}_n$ by the ideal generated by $\{xy - yx | x, y \in \mathcal{A}_n\} \cup \{a_{ij} - a_{ji} | 1 \le i \neq j \le n\}$. By Proposition 2.1 and Theorem 2.5, the existence of a homomorphism $\epsilon : \mathcal{A}_n^{ab} \to \mathbb{C}$ satisfying $\epsilon(\Phi_B^L) = \Delta(B)$ suffices to determine a rank n augmentation. In Section 3 we demonstrate that such a homomorphism exists for satellites with a braid pattern, provided one exists on both the companion and pattern braid.

It may be appropriate here to indicate that $\mathrm{ar}(K) < \mathrm{mr}(K)$ sometimes (maybe in previous subsection), and talk about the 2-cable of the trefoil that does not have $\mathrm{ar}(K, \mathbb{C}) = 4$

3. Main Result

introduce what will be done in the section: first section on notation, main result will follow from Lemma ____ and the Chain Rule

Let K be a knot and let B be a braid with closure K. Let $\tau_{m,l} \in B_{pk}$ be defined by $\tau_{m,l} = \sigma_m \sigma_{m+1} \cdots \sigma_{m+l-1}$, and let $\Sigma_n^{(p)} \in B_{pk}$ be defined by $\Sigma_n^{(p)} = \tau_{np,p}\tau_{np-1,p} \cdots \tau_{np-p+1,p}$ (see Figure 3).

TODO make clear what's on top of what?

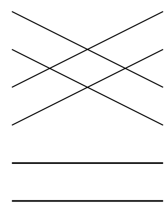

Figure 3. $\Sigma_1^{(2)}$

Then if $B \in B_k$ is given by the braid word $\sigma_{n_1}\sigma_{n_2} \cdots \sigma_{n_m}$, we define the p**-copy** $B^{(p)}$ of B to be

$$B^{(p)} = \Sigma_{n_1}^{(p)} \Sigma_{n_2}^{(p)} \cdots \Sigma_{n_m}^{(p)}$$

which depends on B

and we define $K^{(p)}$ to be the closure of that braid. We then have the following result.

Theorem 3.1. *Suppose that there exists an augmentation* $\epsilon_k\colon \mathcal{A}_k \to \mathbb{C}$ *such that* $\epsilon_k\left(\Phi_B^L\right) = \operatorname{diag}\left((\pm 1)^{\mathrm{w}(K)}, 1, \dots, 1\right)$ *and an augmentation* $\epsilon_p\colon \mathcal{A}_p \to \mathbb{C}$ *such that* $\epsilon_p\left(\Phi_{B'}^L\right) = \operatorname{diag}((\pm 1)^{w(K')}, 1, \dots, 1)$, *and furthermore that* $w(K)$ *is even. Then there exists an augmentation* $\epsilon\colon \mathcal{A}_{pk} \to \mathbb{C}$ *such that*

$B \in B_k$ and $B' \in B_p$; defined on $\mathcal{A}_k^{ab}$?

$$\epsilon\left(\Phi_{B^{(p)}B'}^L\right) = \operatorname{diag}((\pm 1)^{\mathrm{w}(K^{(p)}K')}, 1, \dots, 1)$$

Among other things, this theorem implies that many knot cables of torus knots have meridional rank equal to their bridge number. All torus knots have bridge number equal to their augmentation rank (*TODO* cite cornwell). Consider the cableof a torus knot T given by the braid sum of $T^{(p)}$ with a braid who's first p strands form a torus knot with bridge number (and therefore augmentation rank) equal to p (i.e. a (p,q) torus knot, where $\gcd(p,q) = 1$ and $p < q$). Theorem 3.1 then says that this cable has augmentation rank equal to its braid index, implying that its meridional rank is equal to its bridge number. Furthermore, we can iterate this process, taking cables of the resulting knots with augmentation rank, bridge number, and braid index all equal.

braid rep've info needed to make well-defined

prob from Kirby list should be mentioned here

Fix $p > 0$ and let B be a braid on k strands. For each $1 \le i \le pk$ define integers q_i, r_i such that $i = q_i p + r_i$, where $0 < r_i \le p$. Instrumental to the proof of Theorem 3.1 will be the map $\psi\colon \mathcal{A}_{pk}^{ab} \to \mathcal{A}_k^{ab} \otimes \mathcal{A}_p^{ab}$, defined as in $\mathcal{A}^{ab}$ we are always making $i<j$? let's state so in background

follows (note that since $a_{ij} \in \mathcal{A}^{ab}_{pk}$, $i<j$, so we must have $q_i \leq q_j$, and in particular $q_i < q_j$ in the second and third cases):

$$\psi(a_{ij}) = \begin{cases} 1 \otimes a_{r_i r_j} & : q_i = q_j \\ a_{q_i+1,q_j+1} \otimes 1 & : r_i = r_j \\ 0 & : r_i > r_j \\ a_{q_i+1,q_j+1} \otimes a_{r_i r_j} & : q_i < q_j, r_i < r_j \end{cases}$$

Note that $\psi(a_{ij}) \in 1 \otimes \mathcal{A}^{ab}_p$ if and only if $q_i = q_j$. This homomorphism gives us a way of relating $\Phi^L_{B^{(p)}}$ to Φ^L_B via the following proposition:

or $a_{ij} \in \ker \psi$

well-defined since...

Proposition 3.2. $\psi\left(\Phi^L_{B^{(p)}}\right) = \Phi^L_B \otimes I_p$

explicit about meaning here

Note that we could have instead defined a simpler homormorphism ρ that would take a_{ij} to $a_{q_{i+1},q_{j+1}}$ if $r_i = r_j$ and 0 otherwise, and Proposition 3.2 would still be true. The advantage of ψ is that it doesn't send a_{ij} to 0 if $q_i = q_j$, a fact which will be important in the proof of Theorem 3.1.

is this obvious?

Proof of Theorem 3.1. Set $\epsilon = (\epsilon_k \otimes \epsilon_p) \circ \psi$. We have that

by Chain rule thm

$$(\epsilon_k \otimes \epsilon_p) \circ \psi\left(\Phi^L_{B^{(p)}B'}\right) = (\epsilon_k \otimes \epsilon_p)\psi\left(\phi_{B^{(p)}}\left(\Phi^L_{B'}\right)\right)\psi\left(\Phi^L_{B^{(p)}}\right) \tag{4}$$

Note that since the non zero or one entries of $\Phi^L_{K'}$ are products of a_{ij} where $i < j \leq p$, $\phi_{B^{(p)}}$ takes each of the a_{ij}'s in these products to $a_{i+mp,j+mp}$ for some $0 \leq k < n$. We have that ψ takes $a_{i+mp,j+mp}$ to $1 \otimes a_{ij}$, however, so

$\Phi_{B'}$

$0 \leq m < k$ I think?; independent of i,j

$$\left(\psi\left(\phi_{B^{(p)}}\left(\Phi^L_{B'}\right)\right)\right)_{ij} = 1 \otimes (\Phi^L_{B'})_{ij}.$$

By Proposition 3.2, we have that

$$\psi\left(\Phi^L_{B^{(p)}}\right) = \Phi^L_B \otimes I_p$$

So ~~the~~ right hand side of (4) ~~is equal to~~

returning to ...

we get

$$\begin{aligned}(\epsilon_k \otimes \epsilon_p)\psi\left(\phi_{B^{(p)}}\left(\Phi^L_{B'}\right)\right)\psi\left(\Phi^L_{B^{(p)}}\right) &= (\epsilon_k \otimes \epsilon_p)\left(\Phi^L_B \otimes \Phi^L_{B'}\right) \\ &= \operatorname{diag}((-1)^{w(K)}, 1, \ldots, 1) \otimes \operatorname{diag}((-1)^{w(K')}, 1, \ldots, 1).\end{aligned}$$

But $w(K)$ is even, which also implies that $w(K^{(p)})$ is even, so the right hand side is equal to $\operatorname{diag}((-1)^{w(K^{(p)}K')}, 1, \ldots, 1)$, as desired.

□

We will use the following two lemmas in our proof of Proposition 3.2.

Lemma 3.3. $\psi(\phi_{\Sigma^{(p)}_n}(a_{ij})) = (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))$ *for all* $1 \leq n < k$, $1 \leq i, j \leq pk$.

Lemma 3.4. $\psi\left(\Phi^L_{\Sigma^{(p)}_n}\right) = \Phi^L_{\sigma_n} \otimes I_p$

Proof of Proposition 3.2. Let $B = \sigma_{n_1} \cdots \sigma_{n_l}$, $1 \leq n < k$. We will prove the proposition by inducting on l. The base case is already taken care of by Lemma 3.4. Suppose that the proposition is true for braids of length $l-1$. Let $B' = \sigma_{n_1} \cdots \sigma_{n_{l-1}}$ Then by the Chain Rule and Lemmas 3.3 and 3.4, we have that

$$\begin{aligned}
\psi\left(\Phi^L_{B^{(p)}}\right) &= \psi\left(\phi_{B'^{(p)}}\left(\Phi^L_{\Sigma^{(p)}_{n_l}}\right) \cdot \Phi^L_{B'^{(p)}}\right) \\
&= (\phi_{B'} \otimes \mathrm{id})\left(\psi\left(\Phi^L_{\Sigma^{(p)}_{n_l}}\right)\right) \cdot \left(\Phi^L_{B'} \otimes I_p\right) \\
&= (\phi_{B'} \otimes \mathrm{id})\left(\Phi^L_{\sigma_{n_l}} \otimes I_p\right) \cdot \left(\Phi^L_{B'} \otimes I_p\right) \\
&= \Phi^L_B \otimes I_p
\end{aligned}$$

$\square$

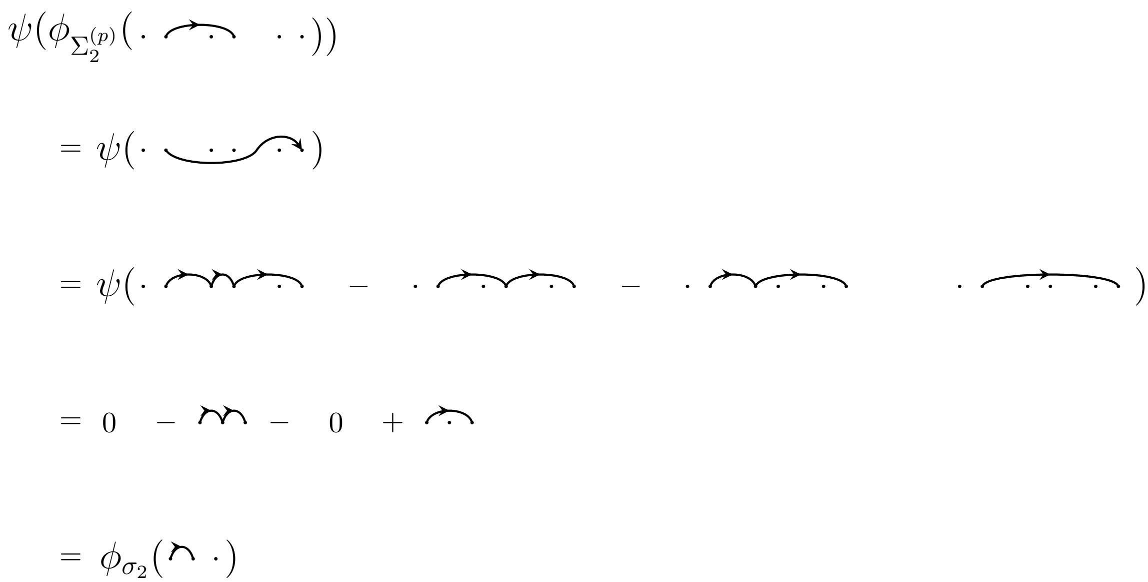

FIGURE 4. Computing $\psi(\phi_{\Sigma_2^{(p)}}(a_{24}))$

In the proof of Lemmas 3.3 and 3.4, we will make use of some calculations of $\phi_B(a_{ij})$ for simple braids B. It can easily be checked that for all $1 \leq m <$

n, $1 \leq l \leq n-m$, $i<j$:

$$\phi_{\tau_{m,l}}(a_{ij}) = \begin{cases} a_{i+1,j+1} & : m \leq i < j < m+l \\ a_{i-l,j} & : m < m+l = i < j \\ a_{i,j-l} & : i < m < m+l = j \\ a_{i+1,j-l} & : m \leq i < j = m+l \\ a_{i,j+1} - a_{i,m}a_{m,j+1} & : i < m \leq j < m+l \\ a_{i+1,j} - a_{i+1,m}a_{m,j} & : m \leq i < m+l < j \\ a_{ij} & : \text{otherwise} \end{cases} \tag{5}$$

We also make the following definition
Let $X \subseteq \{1,\ldots,n\}$, and write the elements of a subset $Y \subseteq X$ as $y_1 < \ldots < y_k$ Define

$$A(i,j,X) = \sum_{Y \subseteq X} (-1)^{|Y|} a_{iy_1}a_{y_1y_2}\cdots a_{y_kj}$$

and

$$A'(i,j,X) = \sum_{Y \subseteq X} (-1)^{|Y|} a_{iy_k}a_{y_ky_{k-1}}\cdots a_{y_1j}$$

and have the following lemma

Lemma 3.5. *Let $\kappa_{m,l} = \tau_{m+l-1,p}\tau_{m+l-2,p}\cdots\tau_{m,p}$, and let $X_{m,l} = \{m-p,\ldots,m+l-p-1\}$. Then* TODO

$$\phi_{\kappa_{m,l}}(a_{ij}) = \begin{cases} a_{i-p,j-p} & : m+p \leq i < j < m+l+p \\ a_{i-p,j} & : m+p \leq i < m+l+p \leq j \\ a_{i,j-p} & : i < m < m+p \leq j < m+l+p \\ a_{i+l,j+l} & : m \leq i < j < m+p \\ A'(i+l,j-p,X_{m,l} \setminus (j-p)) & : m \leq i < m+p \leq j < m+l+p \\ A(i,j+l,X_{m,l}) & : i < m \leq j < m+p < m+l+p \\ A'(i+l,j,X_{m,l}) & : m \leq i < m+p < m+l+p \leq j \\ a_{ij} & : \text{otherwise} \end{cases}$$

Note that letting $l=p$ and $m=p(n-1)+1$ gives us $\phi_{\Sigma_n}(a_{ij})$ as a special case of this lemma.

Proof of Lemma 3.3. The first four cases as well as the last case from Lemma 3.5 can be checked easily. Consider the sixth case. Let $\alpha_i = np-p+r_i$, and we have that

$$
\begin{aligned}
&\psi\left(\phi_{\Sigma_n^{(p)}}(a_{ij})\right)\\
&=\psi\left(A(i,j+p,\{np-p+1,\ldots,np\})\right)\\
&=\psi\left(\sum_{Y\subseteq\{np-p+1,\ldots,np\}}(-1)^{|Y|}a_{iy_1}a_{y_1y_2}\cdots a_{y_k,j+p}\right)\\
&=\sum_{Y\subseteq\{np-p+1,\ldots,np\}}(-1)^{|Y|}\psi\left(a_{iy_1}a_{y_1y_2}\cdots a_{y_k,j+p}\right)\\
&=\sum_{Y\subseteq\{\alpha_i,\ldots,\alpha_j\}}(-1)^{|Y|}\psi\left(a_{iy_1}a_{y_1y_2}\cdots a_{y_k,j+p}\right)\\
&=\psi\left(a_{i,j+p}-a_{i,\alpha_i}a_{\alpha_i,j+p}\right)\\
&\quad+\sum_{y=\alpha_i+1}^{\alpha_j}\sum_{Y\subseteq\{y+1,\ldots,\alpha_j\}}(-1)^{|Y|+1}\psi\left(a_{iy}a_{yy_1}\cdots a_{y_k,j+p}\right)+(-1)^{|Y|}\psi\left(a_{i,\alpha_i}a_{\alpha_i,y}a_{yy_1}\cdots a_{y_k,j+p}\right)\\
&=\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right)\\
&\quad+\sum_{y=\alpha_i+1}^{\alpha_j}\sum_{Y\subseteq\{y+1,\ldots,\alpha_j\}}(-1)^{|Y|}\psi\left(a_{i,\alpha_i}a_{\alpha_i,y}-a_{iy}\right)\psi\left(a_{yy_1}\cdots a_{y_k,j+p}\right)\\
&=\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right)
\end{aligned}
$$

Note that, since we're in the sixth case, $q_j+1=n$. If $r_i=r_j$, then

$$\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right)=\left(a_{q_i+1,n+1}-a_{q_i+1,n}a_{n,n+1}\right)\otimes 1=(\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))$$

If $r_i<r_j$, then

$$
\begin{aligned}
\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right)&=\left(a_{q_i+1,n+1}\otimes a_{r_ir_j}-a_{q_i+1,n}a_{n,n+1}\otimes a_{r_ir_j}\right)\\
&=\left(a_{q_i+1,n+1}-a_{q_i+1,n}a_{n,n+1}\right)\otimes a_{r_ir_j}\\
&=(\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))
\end{aligned}
$$

Finally, if $r_i>r_j$, then

$$\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right)=0=(\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))$$

The proof for the seventh case goes exactly as the proof for the sixth case except with all i's replaced with $i+p$, all $(j+p)$'s replaced with j, all y_i's replaced with y_{k+1-i}, and with α_i and α_j swapped. The proof for the fifth case goes exactly as the proof for the seventh, except that $j-p$ is removed from the set that Y is a subset of in all the sums.

$\square$

Proof of Lemma 3.4. We can extend the definition of ψ to be from the free module over $\mathcal{A}_{pk}$ generated by $\{a_{i*}|1\leq i\leq pk\}$ to the free module over $\mathcal{A}_k\otimes\mathcal{A}_p$ generated by $\{a_{i*}|1\leq i\leq k\}$ by defining $\psi(a_{i*})=a_{i*}$ and extending by linearity. Then the statement of the lemma is equivalent to saying that for all $1 \leq i \leq pk$, the coefficient of a_{j*} in $\psi\left(\phi_{\Sigma_n^{(p)}}(a_{i*})\right)$ is equal to 0 unless $r_j = r_i$, in which case it is equal to the coefficient of $a_{q_j *}$ in $\phi_{\sigma_n}(a_{q_i *})$. If $q_i + 1 \neq n$, this fact can be easily checked. In the case that $q_i + 1 = n$, we have that

$$\psi\left(\phi_{\Sigma_n^{(p)}}(a_{i*})\right) = \psi\left(A(i+p, *, \{np - p + 1, \dots, np\}\right)$$

which is equal to

$$\psi(a_{i+p,*} - a_{i+p,\alpha_i} a_{\alpha_i,*}) = a_{i+p,*} - a_{q_i,q_i+1} a_{\alpha_i,*}$$

by the same argument that was used in Lemma 3.3. The coefficients of the a_{j*} are equal to the coefficients of the $a_{q_j *}$ in $\phi_{\sigma_n}(a_{q_i *})$, so we're done. □

Proof of Lemma 3.5. TODO The first four cases as well as the eighth can be easily checked. We will prove the remaining cases by induction on l. Consider the sixth case. The base case is covered by (5). For the inductive step, we have that

$$\begin{aligned}
\phi_{\kappa_{m,l}}(a_{ij}) &= \phi_{\tau_{m,p}}\left(\phi_{\kappa_{m+1,l-1}}(a_{ij})\right) \\
&= \phi_{\tau_{m,p}}\left(A(i-l+1, j, X_{m+1,l-1})\right) \\
&= \sum_{Y \subseteq \{m+p+1,\dots,m+l+p-1\}} (-1)^{|Y|} \phi_{\tau_{m,p}}\left(a_{i-l+1,y_1} a_{y_1 y_2} \cdots a_{y_k,j}\right) \\
&= \sum_{Y \subseteq \{m+p+1,\dots,m+l+p-1\}} (-1)^{|Y|} \left(a_{i-l,y_1} - a_{i-l,m+p} a_{m+p,y_1}\right) a_{y_1 y_2} \cdots a_{y_k,j} \\
&= \sum_{Y \subseteq \{m+p,\dots,m+l+p-1\}} (-1)^{|Y|} a_{i-l,y_1} a_{y_1 y_2} \cdots a_{y_k,j} \\
&= A(i-l, j, X_{m,l})
\end{aligned}$$

□

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