List of Todos

AUGMENTATION RANK OF SATELLITES WITH BRAID PATTERN

DAVID R. HEMMINGER AND CHRISTOPHER R. CORNWELL

Abstract. Given a knot K in S^3, a question raised by Cappell and Shaneson asks if the meridional rank of K equals the bridge number of K. Using augmentations in knot contact homology we consider the persistence of equality between these two invariants under satellite operations on K with a braid pattern. In particular, we answer the question in the affirmative for a large class of iterated torus knots.

1. Introduction

Let K be an oriented knot in S^3 and denote by π_K the fundamental group of its complement $\overline{S^3 \setminus n(K)}$, with some basepoint. We call an element of π_K a *meridian* if it is represented by the oriented boundary of a disc, embedded in S^3, whose interior intersects K positively once. The group π_K is generated by meridians; the *meridional rank* of K, written $\mathrm{mr}(K)$, is the minimal size of a generating set containing only meridians.

Choose a height function $h : S^3 \to \mathbb{R}$. The *bridge number* of K, denoted $b(K)$, is the minimum of the number of local maxima of $h|_{\varphi(S^1)}$ among embeddings $\varphi : S^1 \to S^3$ which realize K.

By considering Wirtinger's presentation of π_K one can show that $\mathrm{mr}(K) \leq b(K)$ for any $K \subset S^3$. Whether the bound is equality for all knots is an open question attibuted to Cappell and Shaneson [Kir95, Prob. 1.11]. Equality is known to hold for some families of knots due to work of various authors ([BZ85, Cor13b, RZ87]).

Here we study *augmentations* of K, which are maps that arise in the study of knot contact homology. To each augmentation is associated a rank and there is a maximal rank of augmentations of a given K, called the *augmentation rank* $\mathrm{ar}(K)$. For any K the inequality $\mathrm{ar}(K) \leq \mathrm{mr}(K)$ holds (see Section 3.3). We discuss the behavior of $\mathrm{ar}(K)$ under satellite operations with a braid pattern.

To be precise, denote the group of braids on n strands by B_n and write $\hat{\beta}$ for the *braid closure* of a braid β (see Section 3, Figure 3). We write $\imath_n$ for the identity in B_n.

Throughout the paper we let $\alpha \in B_k$ and $\gamma \in B_p$ and set $K = \hat{\alpha}$. Note that $\mathrm{ar}(K) \leq k$.

Definition 1.1. Let $\imath_p(\alpha)$ be the braid in B_{kp} obtained by replacing each strand of α by p parallel copies (in the blackboard framing). Let $\bar{\gamma}$ be the inclusion of γ into B_{kp} by the map $\sigma_i \mapsto \sigma_i, 1 \leq i \leq p-1$. Set $\gamma(\alpha) = \imath_p(\alpha)\bar{\gamma}$. The *braid satellite* of K associated to α, γ is defined as $K(\alpha, \gamma) = \widehat{\gamma(\alpha)}$.

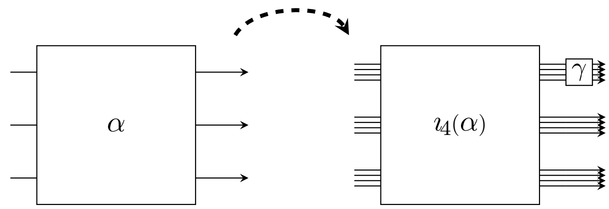


FIGURE 1. Constructing $\gamma(\alpha)$ from α; case $p = 4$.

As defined $K(\alpha, \gamma)$ depends on the choice of α – in general $\widehat{\gamma(\alpha)} \neq \hat{\gamma}(\hat{\alpha})$. However, the construction is more intrinsic if we require the index k of α to be minimal among braid representatives of K (see Section 2).

Note that if $\hat{\alpha}$ and $\hat{\gamma}$ are each a knot, $K(\alpha, \gamma)$ is also. Our principal result is the following.

Theorem 1.2. *If $\alpha \in B_k$ and $\gamma \in B_p$ are such that $\mathrm{ar}(\hat{\alpha}) = k$ and $\mathrm{ar}(\hat{\gamma}) = p$, then $\mathrm{ar}(K(\alpha, \gamma)) = kp$.*

A corollary of Theorem 1.2 involves Cappell and Shaneson's question for iterated torus knots. Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ be integral vectors. We write $T(\mathbf{p}, \mathbf{q})$ for the $(\mathbf{p}, \mathbf{q})$ *iterated torus knot*, defined as follows.

By convention take $T(\emptyset, \emptyset)$ as the unknot, then define $T(\mathbf{p}, \mathbf{q})$ inductively. Let $\hat{\mathbf{p}}, \hat{\mathbf{q}}$ be the truncated lists obtained from $\mathbf{p}, \mathbf{q}$ by removing the last integer in each. If α is a braid of minimal index such that $T(\hat{\mathbf{p}}, \hat{\mathbf{q}}) = \hat{\alpha}$ then define $T(\mathbf{p}, \mathbf{q}) = K(\alpha, (\sigma_1 \ldots \sigma_{p_n - 1})^{q_n})$.

Note that $T(\mathbf{p}, \mathbf{q})$ is a cable of $T(\hat{\mathbf{p}}, \hat{\mathbf{q}})$, but not the (p_n, q_n)-cable in the traditional Seifert framing.

Corollary 1.3. *Given integral vectors $\mathbf{p}$ and $\mathbf{q}$, suppose that $|p_i| < |q_i|$ and $gcd(p_i, q_i) = 1$ for each $1 \leq i \leq n$. Then*

$$\mathrm{ar}(T(\mathbf{p}, \mathbf{q})) = \mathrm{mr}(T(\mathbf{p}, \mathbf{q})) = b(T(\mathbf{p}, \mathbf{q})) = p_1 p_2 \ldots p_n.$$

The assumption $|p_i| < |q_i|$ is needed for the hypothesis of Theorem 1.2, that the associated braids have closures with augmentation rank equal to the braid index. This requirement is not a deficiency of our techniques; there are cables of $(n, n+1)$ torus knots which do not attain the large augmentation rank in Corollary 1.3.

Theorem 1.4. *Given $p > 1$ and $n > 1$, $\mathrm{ar}(T((n, p), (n+1, 1))) < np$.*

It is natural to wonder if the augmentation rank is multiplicative under weaker assumptions on α, γ than those in Theorem 1.2. The following is a possible generalization.

Conjecture 1.5. *Suppose $K = \hat{\alpha}$ for $\alpha \in B_k$, and that α has minimal index among braids with the same closure. Let $\gamma \in B_p$. Then* $\mathrm{ar}(K(\alpha,\gamma)) \geq \mathrm{ar}(\hat{\alpha})\,\mathrm{ar}(\hat{\gamma})$.

Remark 1.6. There are examples when the inequality of Conjecture 1.5 is strict (see Section 5).

The paper is organized as follows. Section 2 relates braid satellites to existing conventions on satellite operators. In Section 3 we give the needed background in knot contact homology, specifically Ng's cord algebra, and discuss augmentation rank and the relationship to meridional rank. Section 3.4 reviews techniques used in the proof of Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.2, its requisite supporting lemmas, and Corollary 1.3. Finally, Section 5 considers the sharpness of our results. We prove Theorem 1.4 and briefly discuss the more general case, Conjecture 1.5.

2. Satellite operators and the braid satellite

Definition 1.1 of the braid satellite $K(\alpha,\gamma)$ produces a satellite of $\hat{\alpha}$. As defined, the resulting satellite depends on the braid representative of $\hat{\alpha}$. We remark here how to avoid this ambiguity.

A neighborhood of an oriented null-homologous knot K in a 3-manifold has a standard identification with $S^1 \times D^2$ determined by an oriented Seifert surface that K bounds. Given a knot $P \subset S^1 \times D^2$, as per the usual convention, let $P(K)$ be the satellite of K with pattern P obtained with this framing.

Proposition 2.1. *Given a knot K and a braid $\gamma \in B_p$, let ω be the writhe of some minimal index closed braid representing K. Let $P \subset S^1 \times D^2$ be the braid closure of $\Delta^{2\omega}\gamma$, where Δ^2 is the full twist in B_p. Then $K(\alpha,\gamma) = P(K)$ for any minimal index braid α with $K = \hat{\alpha}$.*

Proof. The principal observation is that, since the Jones conjecture holds [DP13, LM13], the writhe of α must be ω. Thus the blackboard framing of the closure of $\imath_p(\alpha)\Delta^{-2\omega}$ agrees with the $(p,0)$-cable of K (with Seifert framing). □

We note, the satellite $T(\mathbf{p},\mathbf{q})$ corresponds to the $(p_n, p_n\omega_n + q_n)$-cable of $T(\hat{\mathbf{p}},\hat{\mathbf{q}})$, where ω_n is defined inductively by $\omega_n = p_{n-1}\omega_{n-1} + (p_{n-1}-1)q_{n-1}$ and $\omega_1 = 0$.

Concerning the bridge number of $K(\alpha,\gamma)$, a result of Schubert [] states that if K is not the unknot and $P(K)$ is a satellite such that P has winding number p, then $b(P(K)) \geq p\ b(K)$. Since $K(\alpha,\gamma) = \widehat{\gamma(\alpha)}$, it has bridge number at most kp and thus $b(K(\alpha,\gamma)) = kp$ whenever $b(\hat{\alpha}) = k$. From this we see $b(T(\mathbf{p},\mathbf{q})) = p_1p_2\cdots p_n$, provided $p_1 < q_1$.

3. Background

We review in Section 3.1 the construction of $HC_0(K)$ from the viewpoint of the combinatorial knot DGA, which was first defined in [Ng08]; our conventions are those given in [Ng12]. In Section 3.3 we discuss augmentations in knot contact homology and their rank, which gives a lower bound on the meridional rank of the knot group. Section 3.4 contains a discussion of techniques from [Cor13a] that we use to calculate the augmentation rank.

Throughout the paper we orient n-braids in B_n from left to right, labeling the strands $1, \dots, n$, with 1 the topmost and n the bottommost strand. We work with Artin's generators $\{\sigma_i^{\pm}, i = 1, \dots, n-1\}$ of B_n, where in σ_i only the i and $i+1$ strands interact, and they cross once in the manner depicted in Figure 2. Given a braid $\beta \in B_n$, the braid closure $\hat{\beta}$ of β is the link obtained

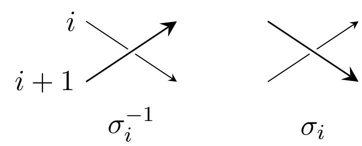


Figure 2. Generators of B_n

as shown in Figure 3. The *writhe* (or algebraic length) of β, denoted $\omega(\beta)$, is the sum of exponents of the Artin generators in a word representing β.

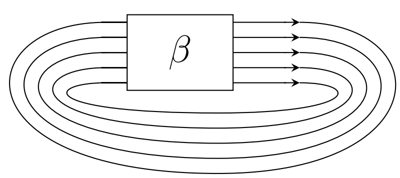


Figure 3. The braid closure of β

3.1. **Knot contact homology.** We review the construction of the combinatorial knot DGA of Ng (in fact, we discuss only the degree zero part as this will suffice for our purposes). This DGA was defined in order to be a calculation of knot contact homology and was shown to be so in [EENS13] (see [Ng12] for more details). Let $\mathcal{A}_n$ be the noncommutative unital algebra over $\mathbb{Z}$ freely generated by a_{ij}, $1 \le i \ne j \le n$. We define a homomorphism $\phi : B_n \to \operatorname{Aut} \mathcal{A}_n$ by defining it on the generators of B_n:

$$\phi_{\sigma_k}\colon \begin{cases} a_{ij} \mapsto a_{ij} & i,j \neq k,k+1 \\ a_{k+1,i} \mapsto a_{ki} & i \neq k,k+1 \\ a_{i,k+1} \mapsto a_{ik} & i \neq k,k+1 \\ a_{k,k+1} \mapsto -a_{k+1,k} & \\ a_{k+1,k} \mapsto -a_{k,k+1} & \\ a_{ki} \mapsto a_{k+1,i} - a_{k+1,k}a_{ki} & i \neq k,k+1 \\ a_{ik} \mapsto a_{i,k+1} - a_{ik}a_{k,k+1} & i \neq k,k+1 \end{cases} \tag{1}$$

Let $\iota\colon B_n \to B_{n+1}$ be the inclusion $\sigma_i \mapsto \sigma_i$ so that the $(n+1)$ strand does not interact with those from $\beta \in B_n$, and define $\phi^*_\beta \in \operatorname{Aut}\mathcal{A}_{n+1}$ by $\phi^*_\beta = \phi_{\iota(\beta)}$. We then define the $n \times n$ matrices Φ^L_β and Φ^R_β with entries in $\mathcal{A}_n$ by

$$\phi^*_\beta(a_{i,n+1}) = \sum_{j=1}^{n} (\Phi^L_\beta)_{ij} a_{j,n+1}$$

$$\phi^*_\beta(a_{n+1,i}) = \sum_{j=1}^{n} a_{n+1,j} (\Phi^R_\beta)_{ji}$$

Finally, let R_0 be the Laurent polynomial ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ and define matrices $\mathbf{A}$ and $\mathbf{\Lambda}$ over R_0 by

$$\mathbf{A}_{ij} = \begin{cases} a_{ij} & i < j \\ -\mu a_{ij} & i > j \\ 1-\mu & i = j \end{cases} \tag{2}$$

$$\mathbf{\Lambda} = \operatorname{diag}[\lambda\mu^{\omega(\beta)}, 1, \ldots, 1]. \tag{3}$$

Definition 3.1. Suppose that K is the closure of $\beta \in B_n$. Define $\mathcal{I} \subset \mathcal{A}_n \otimes R_0$ to be the ideal generated by the entries of $\mathbf{A} - \mathbf{\Lambda} \cdot \mathbf{\Phi}^{\mathbf{L}}_\beta \cdot \mathbf{A}$ and $\mathbf{A} - \mathbf{A} \cdot \mathbf{\Phi}^{\mathbf{R}}_\beta \cdot \mathbf{\Lambda}^{\mathbf{-1}}$. The *degree zero homology of the combinatorial knot DGA* is $\mathrm{HC}_0(K) = (\mathcal{A}_n \otimes R_0)/\mathcal{I}$.

3.2. **Spanning arcs.** The proofs in Sections 4 and 5 require a number of computations of ϕ_β (and of ϕ^*_β, for computing Φ^L_β) for particular braids. Such computations are benefited by an alternate description of the automorphism, which we now explain.

Definition 3.2. Given $n > 0$, let D_n be the unit disk in $\mathbb{C}$ with fixed set $P = \{P_1, \ldots, P_n\}$ of points on the real line in D. Take $P_i < P_j$ if $i < j$. A *spanning arc* of D_n is the isotopy class relative to P of an oriented path in D which begins and ends in P. We define $\mathscr{S}_n$ as the associative ring freely generated by spanning arcs of D_n modulo the ideal generated by the relation in Figure 4. Denote by $c_{ij} \in \mathscr{S}_n$ the element represented by a spanning arc contained in the upper half-disk of D beginning at P_i and ending at P_j.

which we will take to be distinct?

We understand the spanning arcs in Figure 4 to agree outside of a neighborhood of the depicted point in P.

FIGURE 4. Relation in $\mathscr{S}_n$

We consider β as a mapping class of (D, P) and denote by $\beta \cdot c$ the image of the spanning arc c. By convention σ_k acts by rotating P_k and P_{k+1} about their midpoint in counter-clockwise fashion. It was shown in [**?**, Section 2] that there is a unique, well-defined map χ which sends each spanning arc of D_n to an element of $\mathcal{A}_n$ such that

(i) $\chi(\beta \cdot c) = \phi_\beta(\chi(c))$ for any spanning arc c and $\beta \in B_n$;
(ii) $\chi(c_{ij}) = a_{ij}$ if $i < j$, $\chi(c_{ij}) = -a_{ij}$ if $i > j$.

Furthermore, χ factors through $\mathscr{S}_n$, is injective, and by the defining relation of $\mathscr{S}_n$ the value of $\phi_\beta(a_{ij})$ can be determined from (i) and (ii). This constitutes an essential technique for our calculations of ϕ_β.

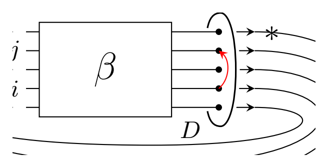


FIGURE 5. Cord c_{ij} of $K = \hat{\beta}$

Computations of Φ^L_β are carried out in likewise manner by including β into B_{n+1} and considering spanning arcs $c_{j,n+1}$, $1 \le j \le n$ of D_{n+1}. We will distinguish this situation by relabeling P_{n+1} (and corresponding indices) with the symbol $*$. In figures, we will put the point $*$ on the boundary of D.

It will be convenient for us in Section 4 to consider the free left $\mathcal{A}_n$-module $\mathcal{A}^L_n = \mathcal{A}_n\langle a_{1*}, \ldots, a_{n*}\rangle$ and right $\mathcal{A}_n$-module $\mathcal{A}^R_n = \langle a_{*1}, \ldots, a_{*n}\rangle\mathcal{A}_n$, which are each contained in $\mathcal{A}_{n+1}$. By definition, Φ^L_β (respectively Φ^R_β) is the matrix in the above basis for the $\mathcal{A}_n$-automorphism of $\mathcal{A}^L_n$ (respectively $\mathcal{A}^R_n$) determined by the image of the basis under ϕ^*_β (which differs from the non-linear map given by restricting ϕ^*_β to these submodules).

Finally, as we are considering braid satellites $K(\alpha, \gamma)$ with $\gamma \in B_p$ our perspective often considers the points in D_{kp} as k groups of p points each. We find it convenient in figures of spanning arcs in $\mathscr{S}_{kp}$ to reflect this point of view. To do so, for each $i = 0, \ldots, k-1$, we depict the points $\{P_{ip+1}, \ldots, P_{(i+1)p}\}$ by a horizontal segment, and if a spanning arc ends at P_{ip+s} for $1 \le s \le p$, it is depicted ending on the $(i+1)^{st}$ segment with a label s (see example in Figure 6).

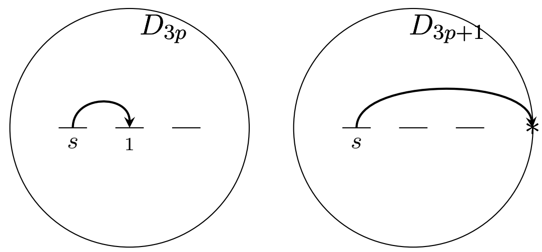


FIGURE 6. Spanning arcs $c_{s,p+1}$ and c_{s*}, $1 \leq s \leq p$.

Let perm : $B_n \to S_n$ denote the homomorphism from B_n to the symmetric group sending σ_k to the simple transposition interchanging $k, k+1$.

Lemma 3.3. *For some $\beta \in B_n$ and $1 \leq i \neq j \leq n$, consider $(\Phi^L_\beta)_{ij} \in \mathcal{A}_n$ as a polynomial expression in the (non-commuting) variables $\{a_{kl}, 1 \leq k \neq l \leq n\}$. Writing $i_0 = perm(\beta)(i)$, every monomial in $(\Phi^L_\beta)_{ij}$ is a constant times $a_{i_0 i_1} a_{i_1 i_2} \dots a_{i_{l-1},j}$ for some $l \geq 0$, the monomial being a constant if $l = 0$ and only if $i_0 = j$.*

Proof. We consider the spanning arc $\beta \cdot c_{i,*}$ which begins at i_0 and ends at $*$. Applying the relation in Figure 4 to the path equates it with a sum (or difference) of another path with the same endpoints and a product of two paths, the first beginning at i_0 and the other ending at $*$. A finite number of applications of this relation allows one to express the path as a polynomial in the $c_{kl}, 1 \leq k \neq l \leq n$ where each monomial has the form $c_{i_0 i_1} \dots c_{i_{l-1},j} c_{j,*}$ for some j. The result then follows from $\phi_\beta(a_{i,n+1}) = \phi_\beta(F(c_{i,n+1})) = F(\beta \cdot c_{i,n+1})$. $\square$

3.3. **Augmentations and augmentation rank.** Augmentations of a differential graded algebra $(\mathcal{A}, \partial)$ are graded maps $(\mathcal{A}, \partial) \to (\mathbb{C}, 0)$ that intertwine the differential (here $\mathbb{C}$ has grading zero). For our setting, if $\beta \in B_n$ is a braid representative of K, such a map corresponds precisely to a homomorphism $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ such that ϵ sends elements of $\mathcal{I}$ to zero (see Definition 3.1).

Definition 3.4. Suppose that K is the closure of $\beta \in B_n$. An *augmentation* of K is a homomorphism $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ such that each element of $\mathcal{I}$ is sent by ϵ to zero.

A correspondence between augmentations and certain representations of the knot group π_K were studied in [Cor13a]. Recall that π_K is generated by meridians, which for a knot are all conjugate. Fix some meridian m.

Definition 3.5. For any integer $r \geq 1$, a homomorphism $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ is a *KCH representation* if $\rho(m)$ is diagonalizable and has an eigenvalue of 1 with multiplicity $r - 1$. We call ρ a *KCH irrep* if it is irreducible.

In [Ng08], Ng describes an isomorphism between $HC_0(K)$ and an algebra constructed from elements of π_K. As discussed in [Ng12] a KCH representation $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ induces an augmentation ϵ_ρ of K. Given an augementation, the first author showed how to construct a KCH representation that induces it. In fact, we have the following rephrasing of results from [Cor13a].

Theorem 3.6 ([Cor13a])**.** *Let $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ be an augmentation with $\epsilon(\mu) \neq 1$. There is a KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$. Furthermore, for any KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$, the rank of $\epsilon(\mathbf{A})$ equals r.*

Considering Theorem 3.6 we make the following definition.

Definition 3.7. The *rank* of an augmentation $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ with $\epsilon(\mu) \neq 1$ is the rank of $\epsilon(\mathbf{A})$. Given a knot K, the *augmentation rank of* K, denoted $\mathrm{ar}(K)$, is the maximum rank among augmentations of K.

Remark 3.8. By Theorem 3.6 the set of ranks of augmentations of a given K does not depend on choice of braid representative.

It is the case that $\mathrm{ar}(K)$ is well-defined. That is, given K there is a bound on the maximal rank of an augmentation of K.

Theorem 3.9 ([Cor13b])**.** *Given a knot $K \subset S^3$, if $g_1, \ldots, g_d$ are meridians that generate π_K and $\rho : \pi_K \to GL_r\mathbb{C}$ is a KCH irrep then $r \leq d$.*

As in the introduction, if we denote the meridional rank of π_K by $\mathrm{mr}(K)$, then Theorem 3.9 implies that $\mathrm{ar}(K) \leq \mathrm{mr}(K)$. In addition, the geometric quantity $b(K)$ called the bridge index of K is never less than $\mathrm{mr}(K)$. Thus we have the following corollary.

Corollary 3.10 ([Cor13b])**.** *Given a knot $K \subset S^3$,*

$$ar(K) \leq mr(K) \leq b(K)$$

Hence to verify that $\mathrm{mr}(K) = b(K)$ it suffices to find a rank $b(K)$ augmentation of K. Herein we concern ourselves with a setting where $\mathrm{ar}(K) = n$ and there is a braid $\beta \in B_n$ which closes to K. This is a special situation, since $b(K)$ is strictly less than the braid index for many knots.

3.4. **Finding augmentations.** The following theorem concerns the behavior of the matrices Φ^L_β and Φ^R_β under the product in B_n. It is an essential tool for studying $HC_0(K)$ and is central to our arguments.

Theorem 3.11 ([Ng05], Chain Rule)**.** *Let β_1, β_2 be braids in B_n. Then $\Phi^L_{\beta_1\beta_2} = \phi_{\beta_1}(\Phi^L_{\beta_2}) \cdot \Phi^L_{\beta_1}$ and $\Phi^R_{\beta_1\beta_2} = \Phi^R_{\beta_1} \cdot \phi_{\beta_1}(\Phi^R_{\beta_2})$.*

Another property of Φ^L_β and Φ^R_β that is important to us is the following symmetry. Define an involution $x \mapsto \overline{x}$ on $\mathcal{A}_n$ (termed *conjugation*) as follows: first set $\overline{a_{ij}} = a_{ji}$; then, for any $x, y \in \mathcal{A}_n$, define $\overline{xy} = \overline{y}\,\overline{x}$ and extend the operation linearly to $\mathcal{A}_n$.

Theorem 3.12 ([Ng05], Prop. 6.2)**.** *For a matrix of elements in $\mathcal{A}_n$, let $\overline{M}$ be the matrix such that $\left(\overline{M}\right)_{ij} = \overline{M_{ij}}$. Then for $\beta \in B_n$, Φ^R_β is the transpose of $\overline{\Phi^L_\beta}$.*

The main result of the paper concerns augmentations with rank equal to the braid index of K. Define the diagonal matrix $\Delta(\beta) = \mathrm{diag}[(-1)^{w(\beta)}, 1, \dots, 1]$. From Section 5 of [Cor13a] we have the following.

Theorem 3.13 ([Cor13a])**.** *If K is the closure of $\beta \in B_n$ and has a rank n augmentation $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$, then*

$$\epsilon(\Phi^L_\beta) = \Delta(\beta) = \epsilon(\Phi^R_\beta). \tag{4}$$

Furthermore, any homomorphism $\epsilon : \mathcal{A}_n \to \mathbb{C}$ which satisfies (4) can be extended to $\mathcal{A}_n \otimes R_0$ to produce a rank n augmentation of K.

4. Main Result

how do I bring in equations to fit margins?

The proof of Theorem 1.2 relies heavily on the characterization presented in Theorem 3.13. We define a homomorphism $\psi : \mathcal{A}_{kp} \to \mathcal{A}_k \otimes \mathcal{A}_p$ which, given $\alpha \in B_k$, suitably simplifies $\Phi^L_{\imath_p(\alpha)}$ and $\Phi^R_{\imath_p(\alpha)}$ when applied to the entries; given also our $\gamma \in B_p$, this and Theorem 3.11 allow us to construct a map that is "close to" the tensor of an augmentation of $\hat{\alpha}$ and an augmentation of $\hat{\gamma}$, composed with ψ, and such that the composition satisfies (4) for $\beta = \gamma(\alpha)$.

Section 4.1 begins with the intermediate result, Proposition 4.1, followed by the proofs of Theorem 1.2 and Corollary 1.3. In Section 4.2 we prove Lemmas 4.2 and 4.3, which are needed to prove Proposition 4.1.

4.1. Proof of main result.

We recall the statement of Theorem 1.2.

Theorem 1.2. If $\alpha \in B_k$ and $\gamma \in B_p$ are such that $\mathrm{ar}(\hat{\alpha}) = k$ and $\mathrm{ar}(\hat{\gamma}) = p$, then $\mathrm{ar}(K(\alpha, \gamma)) = kp$.

For $1 \le i \le kp$, write $i = q_i p + r_i$, where $1 \le r_i \le p$ and $0 \le q_i \le k - 1$. For each generator $a_{ij}, 1 \le i \ne j \le kp$, define

$$\psi(a_{ij}) = \begin{cases} 1 \otimes a_{r_i r_j} & : q_i = q_j \\ a_{q_i+1, q_j+1} \otimes 1 & : r_i = r_j \\ 0 & : (q_i - q_j)(r_i - r_j) < 0 \\ a_{q_i+1, q_j+1} \otimes a_{r_i r_j} & : (q_i - q_j)(r_i - r_j) > 0 \end{cases}, \tag{5}$$

which determines an algebra map $\psi \colon \mathcal{A}_{pk} \to \mathcal{A}_k \otimes \mathcal{A}_p$. Note, if we extend conjugation to $\mathcal{A}_k \otimes \mathcal{A}_p$ by applying it to each factor, then $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$. We have the following proposition.

Proposition 4.1. *For any braid α, $\psi\left(\Phi^L_{\imath_p(\alpha)}\right) = \Phi^L_\alpha \otimes I_p$ and $\psi\left(\Phi^R_{\imath_p(\alpha)}\right) = \Phi^R_\alpha \otimes I_p$*

We briefly comment on the statement, using notation introduced in Section 3.1. The tensor product over $\mathbb{Z}$ of $\mathcal{A}^L_k$ and $\mathcal{A}^L_p$ is a left $(\mathcal{A}_k \otimes \mathcal{A}_p)$-module with canonical basis $\{a_{i*} \otimes a_{j*}\}$. By $\Phi^L_\alpha \otimes I_p$ we mean the matrix in this basis for the $(\mathcal{A}_k \otimes \mathcal{A}_p)$-linear map equal to the tensor product of the map corresponding to Φ^L_α with the identity on $\mathcal{A}^L_p$. Similarly for $\mathcal{A}^R_k$ and $\mathcal{A}^R_p$.

The proof of Proposition 4.1 relies heavily on the following technical lemma, whose proof we delay until Section 4.2.

Lemma 4.2. $\psi(\phi_{\imath_p(\alpha)}(a_{ij})) = (\phi_\alpha \otimes \mathrm{id})(\psi(a_{ij}))$ *for any* $\alpha \in B_k$

Proof of Proposition 4.1. Extend ψ to a map $\tilde{\psi} : \mathcal{A}^L_{pk} \to \mathcal{A}^L_k \otimes \mathcal{A}^L_p$ that takes one canonical basis to another: $\tilde{\psi}(a_{i*}) = a_{q_i+1,*} \otimes a_{r_i,*}$ for any $1 \leq i \leq pk$. The proposition readily follows from Lemma 4.2. Fixing $\alpha \in B_k$ and $1 \leq i \leq kp$, we have

$$\begin{aligned}
\left(\sum_{l=1}^{k} \left(\Phi^L_\alpha\right)_{q_i+1,l} a_{l*}\right) \otimes a_{r_i*} &= (\phi_\alpha \otimes \mathrm{id})\, \tilde{\psi}\,(a_{i*}) \\
&= \tilde{\psi}\left(\phi_{\imath_p(\alpha)}\,(a_{i*})\right) \\
&= \sum_{j=1}^{kp} \psi\left(\left(\Phi^L_{\imath_p(\alpha)}\right)_{ij}\right)\left(a_{q_j+1,*} \otimes a_{r_j*}\right).
\end{aligned}$$

Hence $\psi\left(\left(\Phi^L_{\imath_p(\alpha)}\right)_{ij}\right) = 0$ if $r_i \neq r_j$ and $\psi\left(\left(\Phi^L_{\imath_p(\alpha)}\right)_{ij}\right) = \left(\Phi^L_\alpha\right)_{q_i+1,q_j+1}$ if $r_i = r_j$, since for each $1 \leq l \leq k$ there is exactly one j satisfying $r_j = r_i$ and $q_j + 1 = l$. We conclude $\psi\left(\Phi^L_{\imath_p(\alpha)}\right) = \Phi^L_\alpha \otimes I_p$. Since $\Phi^R_\alpha = \overline{\Phi^L_\alpha}^t$ and $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$, we have that $\psi\left(\Phi^R_{\imath_p(\alpha)}\right) = \Phi^R_\alpha \otimes I_p$ as well. $\square$

we only need $\psi \circ \phi_{\imath_p(\alpha)}$ and $(\phi_\alpha \otimes \mathrm{id}) \circ \psi$ to agree on $a_{i*}, 1 \leq i \leq kp$

Proof of Theorem 1.2. By Theorem 3.13 there exist augmentations $\epsilon_k\colon \mathcal{A}_k \otimes R_0 \to \mathbb{C}$ and $\epsilon_p\colon \mathcal{A}_p \otimes R_0 \to \mathbb{C}$, for the closures of α, γ respectively, such that $\epsilon_k\left(\Phi^L_\alpha\right) = \epsilon_k\left(\Phi^R_\alpha\right) = \Delta(\alpha)$ and $\epsilon_p\left(\Phi^L_\gamma\right) = \epsilon_p\left(\Phi^R_\gamma\right) = \Delta(\gamma)$. Theorem 3.13 also implies that it suffices to prove that there exists an augmentation $\epsilon\colon \mathcal{A}_{pk} \otimes R_0 \to \mathbb{C}$ such that $\epsilon\left(\Phi^L_{\gamma(\alpha)}\right) = \epsilon\left(\Phi^R_{\gamma(\alpha)}\right) = \Delta(\gamma(\alpha))$.

Below we will define a homomorphism $\delta\colon \mathcal{A}_p \to \mathbb{C}$ such that for each generator a_{ij} we have $\delta(a_{ij}) = \pm\epsilon_p(a_{ij})$, the sign depending on the parity of $w(\alpha)$ and p. Let $\pi\colon \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ be the multiplication $a \otimes b \mapsto ab$. Our desired map is defined by $\epsilon = \pi \circ (\epsilon_k \otimes \delta) \circ \psi$.

The Chain Rule theorem gives that

$$\pi \circ (\epsilon_k \otimes \delta) \circ \psi\left(\Phi^L_{\gamma(\alpha)}\right) = \pi \circ (\epsilon_k \otimes \delta)\psi\left(\phi_{\imath_p(\alpha)}\left(\Phi^L_{\tilde{\gamma}}\right)\right)\psi\left(\Phi^L_{\imath_p(\alpha)}\right) \tag{6}$$

Consider how the homomorphism $\phi_{\imath_p(\alpha)}$ acts on spanning arcs. For c_{ij} with $1 \leq i \neq j \leq p$, since the punctures $P_1, \ldots, P_p$ are moved as one block by the action of $\imath_p(\alpha)$, there is an $0 \leq m < k$ so that $\phi_{\imath_p(\alpha)}(a_{ij}) = a_{i+mp,j+mp}$ for each i, j in this range. As $\psi(a_{i+mp,j+mp}) = 1 \otimes a_{ij}$,

$$\psi\left(\phi_{\imath_p(\alpha)}\left(\Phi^L_{\bar{\gamma}}\right)\right) = \left(1 \otimes \left(\Phi^L_{\bar{\gamma}}\right)_{ij}\right)$$

Note that while the entries of $\Phi^L_{\bar{\gamma}}$ are elements of $\mathcal{A}_{kp}$, all of them lie in the image of the natural inclusion of $\mathcal{A}_p$ into $\mathcal{A}_{kp}$, so we regard the entries of the matrix on the right hand side as elements of $\mathcal{A}_k \otimes \mathcal{A}_p$. Returning to the right hand side of (6), by Proposition 4.1 we have

$$\begin{aligned}\pi \circ (\epsilon_k \otimes \delta)\left(\psi\left(\phi_{\imath_p(\alpha)}\left(\Phi^L_{\bar{\gamma}}\right)\right)\psi\left(\Phi^L_{\imath_p(\alpha)}\right)\right) &= \pi \circ (\epsilon_k \otimes \delta)\left(\left(1 \otimes \left(\Phi^L_{\bar{\gamma}}\right)_{ij}\right)\left(\Phi^L_\alpha \otimes I_p\right)\right) \\ &= \delta\left(\Phi^L_{\bar{\gamma}}\right)\pi\left(\Delta(\alpha) \otimes I_p\right).\end{aligned}$$

We are done if we define δ so that $\delta\left(\Phi^L_{\bar{\gamma}}\right)\pi\left(\Delta(\alpha) \otimes I_p\right) = \Delta(\gamma(\alpha))$. When $w(\alpha)$ is even $w(\imath_p(\alpha))$ is also, and further $\Delta(\alpha) = I_k$. Letting $\delta = \epsilon_p$ makes

$$\delta\left(\Phi^L_{\bar{\gamma}}\right)\pi\left(\Delta(\alpha) \otimes I_p\right) = \epsilon_p\left(\Phi^L_{\bar{\gamma}}\right) = \Delta(\bar{\gamma}) = \Delta(\gamma(\alpha)).$$

Suppose $w(\alpha)$ is odd. We define $g \colon \{1, \ldots, p\} \to \{\pm 1\}$ as follows. Let $x_1 = 1$, and $x_l = \mathrm{perm}(\bar{\gamma})(x_{l-1})$ for $1 < l \leq p$. Since the first p strands of $\bar{\gamma}$ close to a knot, $\mathrm{perm}(\bar{\gamma})$ is given by the p-cycle $(x_1 x_2 \ldots x_p)$. If p is even, we let $g(x_1) = 1$, and $g(x_l) = -g(x_{l-1})$ for $1 < l \leq p$. If p is odd, let $g(x_1) = g(x_2) = 1$ and $g(x_l) = -g(x_{l-1})$ for $2 < l \leq p$.

Define $\delta : \mathcal{A}_p \to \mathbb{C}$ by setting $\delta(a_{ij}) = g(i)g(j)\epsilon_p(a_{ij})$ for $1 \leq i \neq j \leq p$. Fix i, j and consider a monomial M of $\left(\Phi^L_{\bar{\gamma}}\right)_{ij}$, which is constant if $i > p$ or $j > p$. For $i, j \leq p$, writing $i_0 = \mathrm{perm}(\bar{\gamma})(i)$, Proposition 3.3 implies $M = c_{ij} a_{i_0,j_1} a_{j_1,j_2} \ldots a_{j_m,j}$ for some $j_1, \ldots j_m \in \{1, \ldots, p\}$, possibly being constant if $i_0 = j$, implying that

$$\delta(M) = g(i_0)g(j)\left(\prod_{k=1}^{m} g(j_k)^2\right)\epsilon_p(M) = g(i_0)g(j)\epsilon_p(M).$$

For M a constant, $\delta(M) = M = g(i_0)g(j)\epsilon_p(M)$ since $i_0 = j$. This holds for each monomial, thus

$$\delta\left(\left(\Phi^L_{\bar{\gamma}}\right)_{ij}\right) = g(i_0)g(j)\epsilon_p\left(\left(\Phi^L_{\bar{\gamma}}\right)_{ij}\right).$$

When p is even, $w(\imath_p(\alpha))$ is also even and so the opposite parity of $w(\alpha)$. Our definition of g gives $\delta\left(\left(\Phi^L_{\bar{\gamma}}\right)_{ii}\right) = -\epsilon\left(\left(\Phi^L_{\bar{\gamma}}\right)_{ii}\right)$ for $i \leq p$. Thus

$$\delta\left(\Phi^L_{\bar{\gamma}}\right) = \begin{pmatrix} (-1)^{w(\bar{\gamma})+1} & 0 & 0 \\ 0 & -I_{p-1} & 0 \\ 0 & 0 & I_{(k-1)p} \end{pmatrix}$$

and therefore

$$\delta\left(\Phi^L_{\bar\gamma}\right)(\Delta(\alpha)\otimes I_p)=\mathrm{diag}[(-1)^{w(\alpha)+w(\bar\gamma)+1},1\dots 1]=\Delta(\gamma(\alpha))$$

as desired.

When p is odd, $w(\imath_p(\alpha))$ is odd and therefore the same parity of $w(\alpha)$. Our definition of g gives that $\delta\left(\left(\Phi^L_{\bar\gamma}\right)_{11}\right)=\epsilon\left(\left(\Phi^L_{\bar\gamma}\right)_{11}\right)$ and $\delta\left(\left(\Phi^L_{\bar\gamma}\right)_{ii}\right)=-\epsilon\left(\left(\Phi^L_{\bar\gamma}\right)_{ii}\right)$ for $1<i\le p$, so

$$\delta\left(\Phi^L_{\bar\gamma}\right)=\begin{pmatrix}(-1)^{w(\bar\gamma)} & 0 & 0\\ 0 & -I_{p-1} & 0\\ 0 & 0 & I_{(k-1)p}\end{pmatrix}$$

and therefore

$$\delta\left(\Phi^L_{\bar\gamma}\right)(\Delta(\alpha)\otimes I_p)=\mathrm{diag}[(-1)^{w(\alpha)+w(\bar\gamma)},1\dots 1]=\Delta(\gamma(\alpha))$$

as desired.

There is little difference in the proof that $\epsilon(\Phi^R_{\gamma(\alpha)})=\Delta(\gamma(\alpha))$, except that monomials in $(\Phi^R_{\bar\gamma})_{ij}$ are of the form $c_{ij}a_{i,j_1}a_{j_1,j_2}\cdots a_{j_k,j'}$ where $j'=\mathrm{perm}(\bar\gamma)(j)$. Applying Theorem 3.13 now completes the proof. □

Proof of Corollary 1.3. We prove the corollary by induction on the dimensions of the vectors $\mathbf{p}$ and $\mathbf{q}$. If $\mathbf{p}$ and $\mathbf{q}$ have one entry, then $T(\mathbf{p},\mathbf{q})$ is simply the (p_1,q_1)-torus knot, and by Theorem 1.3 from [Cor13b] we have $\mathrm{ar}(T(\mathbf{p},\mathbf{q}))=p_1$.

Suppose that $\mathbf{p}$ and $\mathbf{q}$ have n entries and $\mathrm{ar}(T(\hat{\mathbf{p}},\hat{\mathbf{q}}))=p_1p_2\cdots p_{n-1}$. Choose a braid $\alpha\in B_{p_1p_2\cdots p_{n-1}}$ such that $\hat\alpha=T(\hat{\mathbf{p}},\hat{\mathbf{q}})$, and let $\gamma=(\sigma_1\dots\sigma_{p_n-1})^{q_n}$. Theorem 1.3 from [Cor13b] implies that $\mathrm{ar}(\gamma)=p_n$, and since $T(\hat{\mathbf{p}},\hat{\mathbf{q}})=K(\alpha,\gamma)$, Theorem 1.2 gives the desired result. □

4.2. **Supporting Lemmas.** We now prove Lemma 4.2. Figure 7 demonstrates an example for Lemma 4.2, showing that $\psi(\phi_{\imath_2(\sigma_2)}(a_{24}))=\phi_{\sigma_2}\otimes\mathrm{id}(\psi(a_{24}))$. Note that in the figure we condense elements such as $a_{13}\otimes 1$ to a_{13} and include products of algebra elements on a single set of points in order to make the notation cleaner.

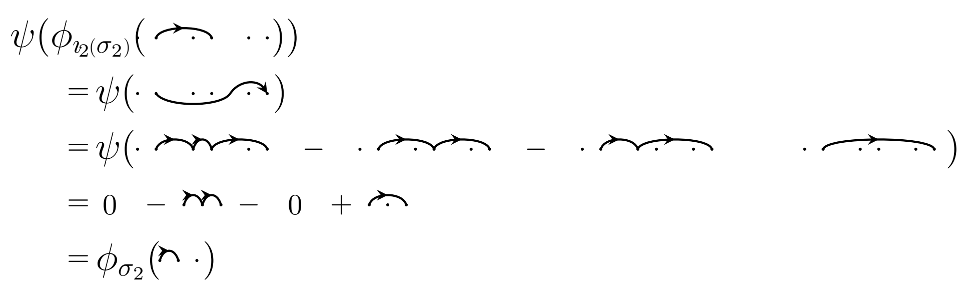


FIGURE 7. Computing $\psi(\phi_{\imath_p(\sigma_2)}(a_{24}))$

In the proof of Lemma 4.2, we will make use of some calculations of $\phi_\alpha(a_{ij})$ for simple braids α. Define $\tau_{m,l} = \sigma_m\sigma_{m+1}\cdots\sigma_{m+l-1}$. We leave it to the reader to check that for all $1 \leq m < n$, $1 \leq l \leq n-m$, $i < j$

$$\phi_{\tau_{m,l}}(a_{ij}) = \begin{cases} a_{i+1,j+1} & : m \leq i < j < m+l \\ a_{i-l,j} & : m < m+l = i < j \\ a_{i,j-l} & : i < m < m+l = j \\ -a_{i+1,j-l} & : m \leq i < j = m+l \\ a_{i,j+1} - a_{i,m}a_{m,j+1} & : i < m \leq j < m+l \\ a_{i+1,j} - a_{i+1,m}a_{m,j} & : m \leq i < m+l < j \\ a_{ij} & : \text{otherwise} \end{cases} \tag{7}$$

We also make the following definitions. Let $W, X \subseteq \{1,\ldots,n\}$, and write the elements of a subset $Y \subseteq X$ as $y_1 < \ldots < y_k$. Define

$$A(i,j,X) = \sum_{Y\subseteq X} (-1)^{|Y|} a_{iy_1}a_{y_1y_2}\cdots a_{y_kj}$$

and

$$A'(i,j,X) = \sum_{Y\subseteq X} (-1)^{|Y|} a_{iy_k}a_{y_ky_{k-1}}\cdots a_{y_1j}$$

and lastly let $r = \min X$ and define

$$B'(i,j,X,X_{r,s}) = \sum_{Y\subseteq X, y_1\neq r+s} c_Y a_{iy_k}a_{y_ky_{k-1}}\cdots a_{y_1j}$$

Where we set $c_Y = (-1)^{|Y|+1}$ if $Y \cap X_{r,s+1} = \emptyset$, and $c_Y = (-1)^{|Y|}$ if $Y \cap X_{r,s} \neq \emptyset$. We have the following lemma

Lemma 4.3. *Let $X_n^{(p)} = \{(n-1)p+1,\ldots,np\}$, let $X_{m,l} = \{m,\ldots,m+l-1\}$, and let $Y = X_{(n-1)p+1,j-np-1}$ We have*

$$\phi_{\iota_p(\sigma_n)}(a_{ij}) = \begin{cases} a_{i-p,j-p} & : i,j \in X_{n+1}^{(p)} \\ a_{i-p,j} & : j > (n+1)p, i \in X_{n+1}^{(p)} \\ a_{i,j-p} & : i \leq (n-1)p, j \in X_{n+1}^{(p)} \\ a_{i+p,j+p} & : i,j \in X_n^{(p)} \\ B'(i+p,j-p,X_n^{(p)},Y) & : i \in X_n^{(p)}, j \in X_{n+1}^{(p)} \\ A(i,j+p,X_n^{(p)}) & : i \leq (n-1)p, j \in X_n^{(p)} \\ A'(i+p,j,X_n^{(p)}) & : j > (n+1)p, i \in X_n^{(p)} \\ a_{ij} & : \text{otherwise} \end{cases}$$

Proof of Lemma 4.2. We will prove that for all $1 \leq n < k$

$$\psi \circ \phi_{\iota_p(\sigma_n)} = \left(\phi_{\sigma_n}^{\pm 1} \otimes \mathrm{id}\right) \circ \psi$$

As the map $B_k \to \mathrm{Aut}(\mathcal{A}_k \otimes \mathcal{A}_p)$ given by $\alpha \mapsto \phi_\alpha \otimes \mathrm{id}$ is a homomorphism, this suffices. Note that if $\psi(\phi_{\iota_p(\sigma_n)}(a_{ij})) = (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))$, then

$$\psi(a_{ij}) = \psi\left(\phi_{\iota_p(\sigma_n)}\left(\phi_{\iota_p(\sigma_n)^{-1}}\left(\alpha_{ij}\right)\right)\right) = (\phi_{\sigma_n} \otimes \mathrm{id})\left(\psi\left(\phi_{\iota_p(\sigma_n)^{-1}}\left(a_{ij}\right)\right)\right)$$

And applying $\left(\phi_{\sigma_n^{-1}} \otimes \mathrm{id}\right)$ to both sides gives

$$\psi\left(\phi_{\imath_p(\sigma_n)^{-1}}\left(a_{ij}\right)\right) = \left(\phi_{\sigma_n^{-1}} \otimes \mathrm{id}\right)\psi(a_{ij})$$

Furthermore, $\phi_\beta(\overline{a_{ij}}) = \overline{\phi_\beta(a_{ij})}$ for any braid β and $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$, so it suffices to prove the lemma for $\imath_p(\sigma_n)$ in the case where $i < j$. With these restrictions, we then break the statement up into the cases from Lemma 4.3. The first four of these as well as the last case can be checked easily. Consider the sixth case. Lemma 4.3 gives that

$$\psi\left(\phi_{\imath_p(\sigma_n)}(a_{ij})\right) = \sum_{Y \subseteq \{np-p+1,\ldots,np\}} (-1)^{|Y|}\psi\left(a_{iy_1}a_{y_1y_2}\cdots a_{y_k,j+p}\right)$$

Let $\alpha_i = np - p + r_i$. Note that if $np - p + 1 \leq y_1 < \alpha_i$ then $\psi(a_{iy_1}) = 0$, and if $np \geq y_k > \alpha_j$ then $\psi(a_{y_kj}) = 0$, so the sum on the right hand side can be taken over $Y \subseteq \{\alpha_i, \alpha_i + 1, \ldots, \alpha_j\}$. Then we manipulate the sum to get

$$\begin{aligned}
&\sum_{Y \subseteq \{\alpha_i,\ldots,\alpha_j\}} (-1)^{|Y|}\psi\left(a_{iy_1}a_{y_1y_2}\cdots a_{y_k,j+p}\right)\\
&= \psi\left(a_{i,j+p} - a_{i,\alpha_i}a_{\alpha_i,j+p}\right)\\
&\quad + \sum_{y=\alpha_i+1}^{\alpha_j}\sum_{Y \subseteq \{y+1,\ldots,\alpha_j\}} (-1)^{|Y|+1}\psi\left(a_{iy}a_{yy_1}\cdots a_{y_k,j+p}\right) + (-1)^{|Y|}\psi\left(a_{i,\alpha_i}a_{\alpha_i,y}a_{yy_1}\cdots a_{y_k,j+p}\right)\\
&= \psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right)\\
&\quad + \sum_{y=\alpha_i+1}^{\alpha_j}\sum_{Y \subseteq \{y+1,\ldots,\alpha_j\}} (-1)^{|Y|}\psi\left(a_{i,\alpha_i}a_{\alpha_i,y} - a_{iy}\right)\psi\left(a_{yy_1}\cdots a_{y_k,j+p}\right)
\end{aligned}$$

Note that $r_i = r_{\alpha_i}$ and since we're in the sixth case we have $(n-1)p < j \leq np$, so $q_{\alpha_i} = q_y$. Thus $\psi(a_{i,\alpha_i}) = a_{q_i+1,q_{\alpha_i}+1} \otimes 1 = a_{q_i+1,q_y+1} \otimes 1$ and $\psi(a_{\alpha_i,y}) = 1 \otimes a_{r_{\alpha_i},r_y} = 1 \otimes a_{r_i,r_y}$, so we have

$$\psi(a_{i,\alpha_i}a_{\alpha_i,y} - a_{iy}) = \left(a_{q_i+1,q_y+1} \otimes 1\right)\left(1 \otimes a_{r_i,r_y}\right) - a_{q_i+1,q_y+1} \otimes a_{r_i,r_y} = 0$$

Thus the right hand side reduces to

$$\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right)$$

Remark 4.4. The fact that $\psi(a_{i,\alpha_i}a_{\alpha_i,y} - a_{iy}) = 0$ and ψ behaves similarly for the analogous terms in cases 5 and 7 is the key to this proof working, and ψ is defined the way it is mainly so that this will be true. As we hinted at earlier, the homomorphism $\rho\colon \mathcal{A}_{pk} \to \mathcal{A}_k$ defined to send a_{ij} to a_{q_i+1,q_j+1} if $r_i = r_j$ and to 0 otherwise would also send these terms to 0, so Proposition 4.1 would still be true with ρ used in the place of ψ. We will need ψ for the proof of the main result, however.

Note that, since we're in the sixth case, $q_j + 1 = n$. If $r_i = r_j$, then

$$\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right) = \left(a_{q_i+1,n+1} - a_{q_i+1,n}a_{n,n+1}\right)\otimes 1 = (\phi_{\sigma_n}\otimes \mathrm{id})(\psi(a_{ij}))$$

If $r_i < r_j$, then

$$\begin{aligned}\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right) &= \left(a_{q_i+1,n+1}\otimes a_{r_ir_j} - a_{q_i+1,n}a_{n,n+1}\otimes a_{r_ir_j}\right)\\ &= \left(a_{q_i+1,n+1} - a_{q_i+1,n}a_{n,n+1}\right)\otimes a_{r_ir_j}\\ &= (\phi_{\sigma_n}\otimes \mathrm{id})(\psi(a_{ij}))\end{aligned}$$

Finally, if $r_i > r_j$, then

$$\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right) = 0 = (\phi_{\sigma_n}\otimes \mathrm{id})(\psi(a_{ij}))$$

Consider the seventh case. Lemma 4.3 gives that

$$\psi\left(\phi_{\iota_p(\sigma_n)}(a_{ij})\right) = \sum_{Y\subseteq\{np-p+1,\dots,np\}} (-1)^{|Y|}\psi\left(a_{i+p,y_k}a_{y_{k-1}y_{k-2}}\cdots a_{y_1,j}\right)$$

Note that if $\alpha_i < y_k \leq np$ then $\psi(a_{i+p,y_k}) = 0$, and if $np-p+1 \leq y_1 < \alpha_j$ then $\psi(a_{y_1j}) = 0$, so the sum on the right hand side can be taken over $Y \subseteq \{\alpha_j, \alpha_j+1, \dots, \alpha_i\}$. Then we manipulate the sum to get

$$\begin{aligned}&\sum_{Y\subseteq\{\alpha_j,\dots,\alpha_i\}} (-1)^{|Y|}\psi\left(a_{i+p,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j}\right)\\ &= \psi\left(a_{i+p,j} - a_{i+p,\alpha_i}a_{\alpha_i,j}\right)\\ &\quad+\sum_{y=\alpha_j}^{\alpha_i-1}\sum_{Y\subseteq\{\alpha_j,\dots,y-1\}} (-1)^{|Y|+1}\psi\left(a_{i+p,y}a_{y,y_k}\cdots a_{y_1,j}\right) + (-1)^{|Y|}\psi\left(a_{i+p,\alpha_i}a_{\alpha_i,y}a_{yy_k}\cdots a_{y_1,j}\right)\\ &= \psi\left(a_{i+p,j} - a_{i+p,\alpha_i}a_{\alpha_ij}\right)\\ &\quad+\sum_{y=\alpha_j}^{\alpha_i-1}\sum_{Y\subseteq\{\alpha_j,\dots,y-1\}} (-1)^{|Y|}\psi\left(a_{i+p,\alpha_i}a_{\alpha_i,y} - a_{i+p,y}\right)\psi\left(a_{yy_k}\cdots a_{y_1,j}\right)\end{aligned}$$

Note that $r_i = r_{\alpha_i}$ and since we're in the seventh case, $q_{\alpha_i} = q_i$. Thus $\psi(a_{i+p,\alpha_i}) = a_{q_i+2,q_i+1}\otimes 1$ and $\psi(a_{\alpha_i,y}) = 1\otimes a_{r_i,r_y}$, so we have

$$\psi(a_{i+p,\alpha_i}a_{\alpha_i,y} - a_{i+p,y}) = \left(a_{q_i+2,q_i+1}\otimes 1\right)\left(1\otimes a_{r_i,r_y}\right) - a_{q_i+2,q_i+1}\otimes a_{r_i,r_y} = 0$$

Thus the right hand side reduces to

$$\psi\left(a_{i+p,j} - a_{i+p,\alpha_i}a_{\alpha_i,j}\right)$$

Note that, since we're in the seventh case, $q_i + 1 = n$ and $q_j > q_i + 2$. If $r_i = r_j$, then

$$\psi\left(a_{i+p,j} - a_{i+p,\alpha_i}a_{\alpha_i,j}\right) = \left(a_{n+1,q_j+1} - a_{n+1,n}a_{n,q_j+1}\right)\otimes 1 = (\phi_{\sigma_n}\otimes \mathrm{id})(\psi(a_{ij}))$$

If $r_i < r_j$, then

$$\begin{aligned}\psi\left(a_{i+p,j} - a_{i+p,\alpha_i}a_{\alpha_i,j}\right) &= \left(a_{n+1,q_j+1} \otimes a_{r_ir_j} - a_{n+1,n}a_{n,q_j+1} \otimes a_{r_ir_j}\right) \\ &= (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))\end{aligned}$$

Finally, if $r_i > r_j$, then

$$\psi\left(a_{i+p,j} - a_{i+p,\alpha_i}a_{\alpha_i,j}\right) = 0 = (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))$$

Consider the fifth case. Lemma 4.3 gives that

$$\begin{aligned}\psi\left(\phi_{\bar{\bar{p}}\sigma_n}(a_{ij})\right) = &\sum_{Y\subseteq\{np-p+1,\dots,np\}, Y\cap X_{np-p+1,j-np-1}\neq\emptyset} (-1)^{|Y|}\psi(a_{i+p,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j-p}) \\ - &\sum_{Y\subseteq\{np-p+1,\dots,np\}, Y\cap X_{np-p+1,j-np}=\emptyset} (-1)^{|Y|}\psi(a_{i+p,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j-p})\end{aligned}$$

Note that if $\alpha_i < y_k \leq np$ then $\psi(a_{i+p,y_k}) = 0$, so if $r_i < r_j$, then all terms in the the second sum on the right hand side are sent to zero, and manipulating the first sum gives

$$\begin{aligned}\psi\left(\phi_{\bar{\bar{p}}\sigma_n}(a_{ij})\right) &= \sum_{Y\subseteq\{np-p+1,\dots,\alpha_i\}} (-1)^{|Y|}\psi(a_{i+p,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j-p}) \\ &= \psi\left(-a_{i+p,\alpha_i}a_{\alpha_i,j-p}\right) \\ &\quad + \sum_{y=np-p+1}^{\alpha_i-1}\sum_{Y\subseteq\{np-p+1,\dots,y-1\}} (-1)^{|Y|}\psi\left(a_{i+p,\alpha_i}a_{\alpha_i,y} - a_{i+p,y}\right)\psi\left(a_{yy_k}\cdots a_{y_1,j-p}\right) \\ &= \psi\left(-a_{i+p,\alpha_i}a_{\alpha_i,j-p}\right) \\ &= -a_{q_i+2,q_i+1} \otimes a_{r_i,r_j} \\ &= (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))\end{aligned}$$

The third equality holds because $\psi\left(a_{i+p,\alpha_i}a_{\alpha_i,y} - a_{i+p,y}\right) = 0$. If $r_i = r_j$, then all of the terms in the second sum are sent to zero except for $-a_{i+p,j-p}$, giving

$$\begin{aligned}\psi\left(\phi_{\bar{\bar{p}}\sigma_n}(a_{ij})\right) &= \sum_{Y\subseteq\{np-p+1,\dots,\alpha_i\}} (-1)^{|Y|}\psi(a_{i+p,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j-p}) \\ &= \psi\left(-a_{i+p,j-p}\right) \\ &\quad + \sum_{y=np-p+1}^{\alpha_i-1}\sum_{Y\subseteq\{np-p+1,\dots,y-1\}} (-1)^{|Y|}\psi\left(a_{i+p,\alpha_i}a_{\alpha_i,y} - a_{i+p,y}\right)\psi\left(a_{yy_k}\cdots a_{y_1,j-p}\right) \\ &= \psi\left(-a_{i+p,j-p}\right) \\ &= -a_{q_i+2,q_i+1} \otimes 1 \\ &= (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))\end{aligned}$$

Finally, if $r_i > r_j$, then using the notation above for $B'(i+p, j-p, X_n^{(p)}, X_{(n-1)p+1,j-np-1})$ we see that for a given $Y \subseteq X_n^{(p)}$ that if $\alpha_i \notin Y$, then $c_Y = -c_{Y\cup\{\alpha_i\}}$ since $\alpha_i \notin X_{(n-1)p+1,j-np-1}$. We then have that

$$\begin{aligned}
\psi\left(\phi_{\bar{\bar{p}}\sigma_n}(a_{ij})\right) &= \sum_{Y\subseteq\{np-p+1,\ldots,\alpha_i\}} c_Y \psi(a_{i+p,y_k} a_{y_k y_{k-1}} \cdots a_{y_1,j-p}) \\
&= \sum_{y=np-p+1}^{\alpha_i-1} \sum_{Y\subseteq\{np-p+1,\ldots,y-1\}} c_Y \psi\left(a_{i+p,\alpha_i} a_{\alpha_i,y} - a_{i+p,y}\right) \psi\left(a_{y y_k} \cdots a_{y_1,j-p}\right) \\
&= 0 \\
&= (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))
\end{aligned}$$

□

Proof of Lemma 4.3. We will prove a more general statement than the one presented in Lemma 4.3. Let $\kappa_{m,l} = \tau_{m+l-1,p}\tau_{m+l-2,p}\cdots\tau_{m,p}$. We will prove that if $i < j$ and $l \leq p$ then

$$\phi_{\kappa_{m,l}}(a_{ij}) = \begin{cases}
a_{i-p,j-p} & : i, j \in X_{m+p,l} \\
a_{i-p,j} & : j \geq m+l+p, i \in X_{m+p,l} \\
a_{i,j-p} & : i < m, j \in X_{m+p,l} \\
a_{i+l,j+l} & : i, j \in X_{m,l} \\
B'(i+l, j-p, X_{m,l}, X_{m,j-m-p}) & : i \in X_{m,l}, j \in X_{m+p,l} \\
A(i, j+l, X_{m,l}) & : i < m, j \in X_{m,l} \\
A'(i+l, j, X_{m,l}) & : j \geq m+l+p i \in X_{m,l} \\
a_{ij} & : \text{otherwise}
\end{cases}$$

Letting $l = p$ and $m = (n-1)p + 1$ then gives us Lemma 4.3 as a special case. The first four cases as well as the eighth can be easily checked. We will prove the remaining cases by induction on l. Consider the sixth case. The base case is covered by (7). For the inductive step, we have that

$$\begin{aligned}
\phi_{\kappa_{m,l}}(a_{ij}) &= \phi_{\tau_{m+l-1,p}}\left(\phi_{\kappa_{m,l-1}}(a_{ij})\right) \\
&= \sum_{Y\subseteq\{m,\ldots,m+l-2\}} (-1)^{|Y|} \phi_{\tau_{m+l-1,p}}\left(a_{i,y_1} a_{y_1 y_2} \cdots a_{y_k,j+l-1}\right) \\
&= \sum_{Y\subseteq\{m,\ldots,m+l-2\}} (-1)^{|Y|} a_{iy_1} a_{y_1 y_2} \cdots a_{y_{k-1} y_k}\left(a_{y_k,j+l} - a_{y_k,m+l-1} a_{m+l-1,j+l}\right) \\
&= \sum_{Y\subseteq\{m,\ldots,m+l-1\}} (-1)^{|Y|} a_{i,y_1} a_{y_1 y_2} \cdots a_{y_k,j+l} \\
&= A(i, j+l, X_{m,l})
\end{aligned}$$

Where the second equality holds because $l \leq p$.

Consider the seventh case. The base case is covered by (7). For the inductive step, we have that

$$\begin{aligned}
\phi_{\kappa_{m,l}}(a_{ij}) &= \phi_{\tau_{m+l-1,p}}\left(\phi_{\kappa_{m,l-1}}(a_{ij})\right)\\
&= \sum_{Y\subseteq\{m,\ldots,m+l-2\}} (-1)^{|Y|}\phi_{\tau_{m+l-1,p}}\left(a_{i+l-1,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j}\right)\\
&= \sum_{Y\subseteq\{m,\ldots,m+l-2\}} (-1)^{|Y|}\left(a_{i+l,y_k}-a_{i+l,m+l-1}a_{m+l-1,y_k}\right)a_{y_ky_{k-1}}\cdots a_{y_1j}\\
&= \sum_{Y\subseteq\{m,\ldots,m+l-1\}} (-1)^{|Y|}a_{i+l,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j}\\
&= A'(i+l,j,X_{m,l})
\end{aligned}$$

Where in the second equality we make use of the facts that $l\leq p$, $j\geq m+l+p$, and that $\phi_{\tau_{m+l-1,p}}(a_{i,y_k})=\overline{\phi_{\tau_{m+l-1,p}}(a_{y_k,i})}$.

Consider the fifth case. We have from the seventh case that

$$\phi_{\kappa_{m,j-m-p}}(a_{ij}) = A'(i+j-m-p,j,X_{m,j-m-p})$$

We then have that

$$\begin{aligned}
&\phi_{\tau_{m+(j-m-p),p}}(A'(i+j-m-p,j,X_{m,j-m-p}))\\
&= \sum_{Y\subseteq\{m,\ldots,j-p-1\}} (-1)^{|Y|}\phi_{\tau_{j-p,p}}\left(a_{i+j-m-p,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j}\right)\\
&= -a_{i+j-m-p+1,j-p}\\
&+ \sum_{Y\subseteq\{m,\ldots,j-p-1\},Y\neq\emptyset} (-1)^{|Y|}\left(a_{i+j-m-p+1,y_k}-a_{i+j+m-p+1,j-p}a_{j-p,y_k}\right)a_{y_ky_{k-1}}\cdots a_{y_2y_1}a_{y_1,j-p}\\
&= B'(i+j-m-p+1,j-p,X_{m,j-m-p+1},X_{m,j-m-p})
\end{aligned}$$

For $l>j-m-p+1$, $l\leq p$, we have

$$\begin{aligned}
&\tau_{m+l-1,p}(B'(i+l-1,j-p,X_{m,l-1},X_{m,j-m-p}))\\
&= \sum_{Y\subseteq\{m,\ldots,m+l-2\},Y\cap X_{m,j-m-p}\neq\emptyset} (-1)^{|Y|}\tau_{m+l-1,p}(a_{i+l-1,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j-p})\\
&- \sum_{Y\subseteq\{m,\ldots,m+l-2\},Y\cap X_{m,j-m-p+1}=\emptyset} (-1)^{|Y|}\tau_{m+l-1,p}(a_{i+l-1,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j-p})\\
&= \sum_{Y\subseteq\{m,\ldots,m+l-2\},Y\cap X_{m,j-m-p}\neq\emptyset} (-1)^{|Y|}(a_{i+l,y_k}-a_{i+l,m+l-1}a_{m+l-1,y_k})a_{y_ky_{k-1}}\cdots a_{y_1,j-p}\\
&- \sum_{Y\subseteq\{m,\ldots,m+l-2\},Y\cap X_{m,j-m-p+1}=\emptyset} (-1)^{|Y|}(a_{i+l,y_k}-a_{i+l,m+l-1}a_{m+l-1,y_k})a_{y_ky_{k-1}}\cdots a_{y_1,j-p}\\
&= B'(i+l,j-p,X_{m,l},X_{m,j-m-p})
\end{aligned}$$

giving the desired result. □

5. Comments on augmentation rank and multiplicativity

The section is in two parts. First we prove Theorem 1.4, showing some cables of torus knots have augmentation rank less than bridge number. In the second part we discuss how this result, and some computational evidence, might fit into Conjecture 1.5.

5.1. Cables of $(n, n+1)$ torus knots.

Theorem 1.4. Given $p > 1$ and $n > 1$, we have

$$\mathrm{ar}(T((n,p),(n+1,1))) < np.$$

Remark 5.1. The remarks at the end of Section 2 imply that the bridge number of $T((n,p),(n+1,1))$ is np.

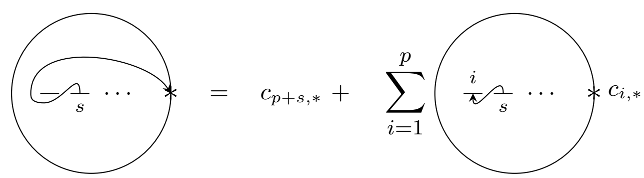


Figure 8. $\imath_p(\tau) \cdot c_{s*}$, $1 \le s \le p$ as an element of $\mathscr{S}_{pn+1}$

Proof. Let $\tau = \sigma_1 \dots \sigma_{n-1} \in B_n$ and set $\alpha = \tau^{n+1}$, which has the $(n, n+1)$ torus knot as its braid closure. We have $T((n,p),(n+1,1)) = K(\alpha,\gamma)$ for $\gamma = \sigma_1 \dots \sigma_{p-1} \in B_p$. Write $(\Phi^L_{\gamma(\alpha)})_i$ for the i^{th} row of $\Phi^L_{\gamma(\alpha)}$.

The structure of the proof is the following. By Theorem 3.13 we must prove there is no homomorphism $\epsilon : \mathcal{A}_{np} \to \mathbb{C}$ such that $\epsilon(\Phi^L_{\gamma(\alpha)}) = \Delta(\gamma(\alpha))$. Note that, since $\overline{\gamma}$ is in the image of the inclusion $B_p \hookrightarrow B_{np}$ given by $\sigma_i \mapsto \sigma_i$, Theorem 3.11 implies that $(\Phi^L_{\gamma(\alpha)})_i = (\Phi^L_{\imath_p(\alpha)})_i$ for $p < i \le np$. Hence, were such an ϵ to exist then $\epsilon((\Phi^L_{\imath_p(\alpha)})_i) = \mathbf{e}_i$ for $p < i \le np$.

We will see that $(\Phi^L_{\overline{\gamma}})_p = \mathbf{e}_1$, implying that the entry $(\Phi^L_{\imath_p(\alpha)})_{1p}$ agrees with a diagonal entry of $\Phi^L_{\gamma(\alpha)}$. We then calculate that, for any ϵ satisfying $\epsilon((\Phi^L_{\imath_p(\alpha)})_i) = \mathbf{e}_i$ for $p < i \le np$, we must have $\epsilon((\Phi^L_{\imath_p(\alpha)})_{1p}) = 0$. This contradicts $\epsilon(\Phi^L_{\gamma(\alpha)}) = \Delta(\gamma(\alpha))$ and proves the result.

In consideration of the above, for the remainder of the proof $\epsilon : \mathcal{A}_{np} \to \mathbb{C}$ denotes a homomorphism with the property $\epsilon((\Phi^L_{\imath_p(\alpha)})_i) = \mathbf{e}_i$ for $p < i \le np$.

To prove that $\epsilon((\Phi^L_{\imath_p(\alpha)})_{1p}) = 0$ we first demonstrate, in **I** below, that $\epsilon((\Phi^L_{\imath_p(\alpha)})_{1p}) = -\epsilon(a_{p+1,p})$. This is followed in **II** by a proof that $\epsilon(a_{p+1,p}) = 0$, which completes the proof of the theorem (the equality $(\Phi^L_{\overline{\gamma}})_p = \mathbf{e}_1$ is derived in the process).

I. For $k \in \mathbb{Z}$, consider matrices $\Phi^L_{\imath_p(\mathcal{T})^k}$ and partition them into an $n \times n$ array of $p \times p$ submatrices. In notation, define for $1 \le i, j \le n$ the $p \times p$ matrix Ψ^k_{ij} so that

$$\Phi^L_{\imath_p(\mathcal{T})^k} = \begin{pmatrix} \Psi^k_{11} & \cdots & \Psi^k_{1n} \\ & \ddots & \\ \Psi^k_{n1} & \cdots & \Psi^k_{nn} \end{pmatrix}.$$

We claim that

(a) the $(n-1)p \times (n-1)p$ submatrix $(\Psi^1_{ij})_{i<n, j>1}$ is the identity matrix;
(b) Ψ^1_{n1} is the $p \times p$ identity matrix;
(c) Ψ^1_{nj} is the zero matrix for $j > 1$;
(d) finally, $(\Phi^L_{\gamma})_p = (1, 0, \ldots, 0)$.

Verification of the claim is left to the reader. As an example, (a) requires identities in $\mathscr{S}_{np+1}$ (which are passed through to $\mathcal{A}_{np+1}$ by χ) similar to the identity in Figure 8, which can be used to calculate Ψ^1_{1j} for $1 \le j \le n$. Also, (d) can be deduced from (b) and (c) in the case that the Ψ^1_{ij} are of size 1×1.

By Theorem 3.11 we have $\Phi^L_{\imath_p(\mathcal{T})^{k+1}} = \phi_{\imath_p(\mathcal{T})}(\Phi^L_{\imath_p(\mathcal{T})^k})\Phi^L_{\imath_p(\mathcal{T})}$. Thus by the above claim, for $1 \le j < n$,

$$\Psi^{k+1}_{i,j+1} = \phi_{\imath_p(\mathcal{T})}(\Psi^k_{ij}). \tag{8}$$

Theorem 3.11 also shows $\Phi^L_{\imath_p(\mathcal{T})^{k+1}} = \phi_{\imath_p(\mathcal{T})^k}(\Phi^L_{\imath_p(\mathcal{T})})\Phi^L_{\imath_p(\mathcal{T})^k}$. Hence

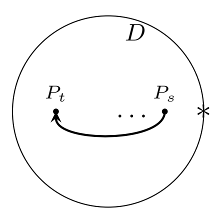
$$\Psi^{k+1}_{nj} = \Psi^k_{1j} \tag{9}$$

for all $1 \le j \le n$ and also, for $1 \le i < n$,

$$\Psi^{k+1}_{ij} = \Psi^k_{i+1,j} + \phi_{\imath_p(\mathcal{T})^k}(\Psi^1_{i1})\Psi^k_{1j}.$$

Taking $k = n$, equations (8) and (9) thus imply

$$\Psi^{n+1}_{ij} = \phi^{-1}_{\imath_p(\mathcal{T})}(\Psi^{n+1}_{i+1,j+1}) + \phi_{\imath_p(\mathcal{T})^n}(\Psi^1_{i1})\Psi^{n+1}_{nj}.$$



$\chi^{-1}(w_{st})$

FIGURE 9. The spanning arc for w_{st}

Hence $\epsilon(\Psi^{n+1}_{ij}) = \epsilon(\phi^{-1}_{\imath_p(\mathcal{T})}(\Psi^{n+1}_{i+1,j+1}))$ for $1 \le j < n$, since $\epsilon(\Psi^{n+1}_{nj}) = \mathbf{0}$ by assumption. Utilizing (8) and (9) again we find that, for $i \ge j$,

$$\epsilon(\Psi^{n+1}_{ij}) = \epsilon(\phi^{-n+i}_{\imath_p(\mathcal{T})}(\Psi^{n+1}_{n,j+(n-i)})) = \epsilon(\Psi^{i+1}_{nj}) = \epsilon(\Psi^i_{1j}).$$

Taking $s = 1$ in Figure 8, we see that the $(1, p)$-entry of Ψ^1_{11} is $\chi(c_{p+1,p}) = -a_{p+1,p}$. And so $\epsilon((\Phi^L_{\imath_p(\alpha)})_{1p}) = \epsilon((\Psi^{n+1}_{11})_{1p}) = -\epsilon(a_{p+1,p})$, which we were to show in **I**.

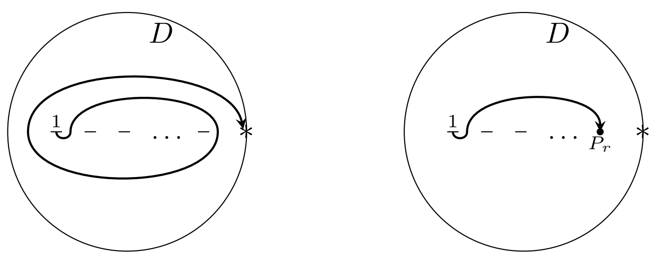
II. We must show that $\epsilon(a_{p+1,p}) = 0$. To do so we consider $\phi^*_{\imath_p(\alpha)}(a_{(n-1)p+1,*})$ in $\mathcal{A}^L_{np} \subset \mathcal{A}_{np+1}$ which, similar to above, we understand through the corresponding spanning arc (see Figure 10). Our assumption that $\epsilon((\Phi^L_{\imath_p(\alpha)})_{(n-1)p+1}) = \mathbf{e}_{(n-1)p+1}$ means that if we write $\phi^*_{\imath_p(\alpha)}(a_{(n-1)p+1,*})$ in the basis $\{a_{1,*}, \ldots, a_{np,*}\}$ of $\mathcal{A}^L_{np}$ then ϵ sends all but $(n-1)p + 1$ coefficient to zero.

For $p < r \leq np$, define $v_r \in \mathcal{A}_{np}$ such that $\chi^{-1}(v_r)$ is the spanning arc shown on the right in Figure 10, which ends at P_r. Define w_{st} so that (as shown in Figure 9) $\chi^{-1}(w_{st})$ is contained in the lower half of D, and begins at P_s and ends at P_t (with $w_{st} = 1$ if $s = t$).

In **I** we showed $\epsilon(\Phi^{n+1}_{ij}) = \epsilon(\Psi^i_{1j})$ for any $i \geq j$. This has an important consequence for elements of the form $w_{ip+1,j}$. The entries of Ψ^i_{1j} are computed from $\imath_p(\tau)^i \cdot c_{s,*}$ where $1 \leq s \leq p$ (Figure 8 shows the case $i = 1$). Take $s = 1$. Let $1 \leq i \leq n-1$ and $1 \leq j = (q-1)p + r \leq ip$ (for some $1 \leq r \leq p$). Then the $(1, r)$-entry of Ψ^i_{1q} is $w_{ip+1,j}$. For $1 < i \leq n-1$, our assumption on ϵ implies

$$\epsilon(w_{ip+1,j}) = \epsilon((\Psi^{n+1}_{iq})_{1r}) = \delta_{iq}\delta_{1r} = \delta_{(i-1)p+1,j}, \tag{10}$$

where δ is the Kronecker-delta.



$\chi^{-1}(v_r)$

FIGURE 10. The spanning arc $\chi^{-1}(\phi^*_{\imath_p(\alpha)}(a_{(n-1)p+1}))$; row $(n-1)p + 1$ of $\Phi^L_{\imath_p(\alpha)}$

For $p < j \leq np$, the coefficient of $a_{j,*}$ in $\phi^*_{\imath_p(\alpha)}(a_{(n-1)p+1,*})$ can be written as

$$x_j := \langle \phi^*_{\imath_p(\alpha)}(a_{(n-1)p+1,*}), a_{j,*} \rangle = \sum_{r=j}^{np} v_r w_{rj}. \tag{11}$$

Claim: For $p < j \leq np$, if $j = (n-i)p + 1$ then $\epsilon(v_j) = (-1)^{i+1}$ and $\epsilon(v_j) = 0$ otherwise.

Proof of Claim. The proof uses induction on i, proving the statement for each $(n-i)p + 1 \leq j \leq (n-i+1)p$.

For $i = 1$, by assumption $\epsilon(x_j) = 0$ for $(n-1)p + 1 < j \le np$. Note that $x_{np} = v_{np}$. Thus $\epsilon(v_{np}) = \epsilon(x_{np}) = 0$. This, applied to $j = np - 1$, then $j = np-2$, and so on, implies that $\epsilon(v_j) = \epsilon(x_j) = 0$ for $(n-1)p+1 < j \le np$. Furthermore, we get that $\epsilon(v_{(n-1)p+1}) = \epsilon(x_{(n-1)p+1}) = 1$.

Now suppose for some $1 < i \le n-1$ that $(n-i)p + 1 \le j \le (n-i+1)p$. Assuming the induction hypothesis (one case at a time, $j = (n-i+1)p$, then $j = (n-i+1)p - 1$, and so on) we have

$$0 = \epsilon(x_j) = \sum_{r=j}^{np} \epsilon(v_r)\epsilon(w_{rj}) = \epsilon(v_j) + \sum_{k=1}^{i-1} (-1)^{k+1} \epsilon(w_{(n-k)p+1,j}).$$

Recalling (10), $\epsilon(w_{(n-k)p+1,j})$ is the Kronecker-delta $\delta_{(n-k-1)p+1,j}$ (provided $n-k > 1$), and thus $\epsilon(v_j) = 0$ for each case when $j > (n-i)p+1$. When $j = (n-i)p+1$ we get that

$$0 = \epsilon(v_j) + (-1)^i \epsilon(w_{(n-i+1)p+1,(n-i)p+1}) = \epsilon(v_j) + (-1)^i$$

as claimed. □

We finish the proof by considering $\langle \phi^*_{\imath_p(\alpha)}(a_{(n-1)p+1,*}), a_{p,*}\rangle$; the spanning arc corresponding to $\phi^*_{\imath_p(\alpha)}(a_{(n-1)p+1,*})$ indicates a small difference to the previous coefficients. We have

$$x_p := \langle \phi^*_{\imath_p(\alpha)}(a_{(n-1)p+1,*}), a_{p,*}\rangle = \sum_{r=p+1}^{np} v_r w_{rp}.$$

Applying our previous claim, (10), and $w_{p+1,p} = -a_{p+1,p}$ we have

$$\begin{aligned} 0 = \epsilon(x_p) = \sum_{i=1}^{n-1} (-1)^{n-i+1} \epsilon(w_{ip+1,p}) &= (-1)^n \epsilon(w_{p+1,p}) \\ &= (-1)^{n-1} \epsilon(a_{p+1,p}). \end{aligned}$$

This implies the desired result and finishes the proof of the theorem.

□

5.2. **Augmentation rank does not multiply.** As discussed in Section 2 the braid satellite $K(\alpha, \gamma)$ depends only on γ and the closure $\hat{\alpha}$ if α has minimal braid index. Letting ω denote the writhe of such a braid, we write P for the closure $\widehat{\Delta^\omega \gamma}$, as in Section 2.

Theorem 5.2. *For any braid α with $K = \hat{\alpha}$ and $\gamma \in B_p$, there are p KCH irreps $\sigma : \pi_{K(\alpha,\gamma)} \to GL_r\mathbb{C}$ for each KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$. In particular,* $\mathrm{ar}(K(\alpha,\gamma)) \ge \mathrm{ar}(K)$.

Proof. Consider a neighborhood $n(K)$ of K that contains $K(\alpha, \gamma)$. Write $T = \partial(n(K))$. Choose the basepoint of $\pi_{K(\alpha,\gamma)}$ on T. Then inclusion makes $\pi_1(T)$ a subgroup and $\pi_{K(\alpha,\gamma)}$ is isomorphic to the product of π_K and π_P amalgamated along $\pi_1(T)$.

Let m_1 be the meridian of K determined by a based loop contained in T that is contractible in $n(K)$. Suppose that $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ is an irreducible KCH representation with $\widetilde{M} = \rho(m_1) = \mathrm{diag}[\widetilde{\mu}_0, 1, \dots, 1]$ for some $\widetilde{\mu}_0 \in \mathbb{C} \setminus \{0\}$. Choose any p^{th} root μ_0 of $\widetilde{\mu}_0$.

Consider a collection of meridians $m_1, \dots, m_r$ of K that generate π_K. For each $1 \le i \le r$ there are p meridians $m_{i1}, \dots, m_{ip}$ of $K(\alpha, \gamma)$ such that $m_{i1}m_{i2} \dots m_{ip} = m_i$. Set $\sigma(m_{1j}) = \mathrm{diag}[\mu_0, 1, \dots, 1] = M$ for $1 \le j \le p$. Then, for each $1 < i \le r$ find $w_i \in \pi_K$ so that $m_i = w_i m_1 w_i^{-1}$ and set $\sigma(m_{ij}) = \rho(w_i)M\rho(w_i)^{-1}$ for $1 \le j \le p$.

Due to the braid pattern of $K(\alpha, \gamma)$, $\pi_{K(\alpha,\gamma)}$ has a presentation so that each relation has the form $xm_{i,j}x^{-1} = w_i m_{1,k} w_i^{-1}$, where x is a word in $\{m_{i1}^{\pm}, \dots, m_{ip}^{\pm}\}$ and $1 \le j, k \le p$, $1 \le i \le r$. Thus $\sigma : \pi_{K(\alpha,\gamma)} \to \mathrm{GL}_r\mathbb{C}$ is a well-defined KCH representation. Moreover, the image of σ contains that of ρ, implying it is irreducible and that $\mathrm{ar}(K(\alpha, \gamma)) \ge \mathrm{ar}(K)$. □

We remark that $\mathrm{ar}(K(\alpha, \gamma)) \ge \mathrm{ar}(P)$ also, for $P = \widehat{\Delta^{2\omega}\gamma}$. This follows from Proposition 2.1 and the existence of a surjection $\pi_{K(\alpha,\gamma)} \to \pi_P$, preserving peripheral structures (see Proposition 3.4 in [SW06], for example).

Oddly, the product $\mathrm{ar}(K)\,\mathrm{ar}(P)$ does not relate well to $\mathrm{ar}(K(\alpha, \gamma))$: from Theorem 1.4 we find examples where $\mathrm{ar}(K(\alpha, \gamma)) < \mathrm{ar}(K)\,\mathrm{ar}(P)$ and from Theorem 1.2 there are examples with $\mathrm{ar}(K(\alpha, \gamma)) > \mathrm{ar}(K)\,\mathrm{ar}(P)$ (take $\alpha = \sigma_1^3$ and $\gamma = \sigma_1^{-5}$, for example). However, to our knowledge the statement of Conjecture 1.5 could hold.

There are cases where $\mathrm{ar}(K(\alpha, \gamma))$ is strictly larger than $\mathrm{ar}(K)\,\mathrm{ar}(\hat{\gamma})$. One example can be found from the $(2, 11)$-cable of the $(2, 5)$ torus knot. By finding a solution to (4) for $\alpha = \sigma_1^5 \in B_2$ and $\gamma = \sigma_1 \in B_2$, we can compute that $\mathrm{ar}(K(\sigma_1^5, \sigma_1)) = 4$, even though $\mathrm{ar}(\hat{\sigma_1^5}) = 2$ and $\mathrm{ar}(\hat{\sigma_1}) = 1$. Unfortunately, any more examples of cables of torus knots (not covered by Theorems 1.2 and 1.4) seem outside of our computational abilities.

We end with a remark on computational observations. By the inequalities in (3.10) if a knot has bridge number less than its minimal braid index n, it cannot have augmentation rank equal to n. Take a minimal index braid representative of such a knot, and multiply that braid by successively higher powers of $\Delta^2 \in B_n$, testing in each instance if the closure has augmentation rank equal to n. In examples, the power of Δ^2 need not be very high, compared to n, before a braid with augmentation rank n appears. Also, once such an augmentation appears, it seems to persist.

Dehornoy introduced a total, left-invariant order on B_n. It was shown in [MN04] that there is a constant m_n such that if $\alpha > \Delta^{2m_n}$ (or $\alpha^{-1} > \Delta^{2m_n}$) then α does not admit a Birman-Menasco template, and thus is a minimal index representative of $K = \hat{\alpha}$ by the MTWS [**?**].

Question. For a given braid index n, is there a number m_n so that $\mathrm{ar}(\hat{\alpha}) = n$ for any $\alpha \in B_n$ greater than Δ^{2m_n} in Dehornoy's order?

References

[BZ85] M. Boileau and H. Zieschang. Nombre de ponts et générateurs méridiens des entrelacs de Montesinos. *Commentarii Mathematici Helvetici*, 60:270–279, 1985. 10.1007/BF02567413.

[Cor13a] C. Cornwell. KCH representations, augmentations, and A-polynomials, 2013. arXiv: 1310.7526.

[Cor13b] C. Cornwell. Knot contact homology and representations of knot groups. arXiv: 1303.4943, 2013.

[DP13] I. Dynnikov and M. Prasolov. Bypasses for rectangular diagrams. Proof of Jones' conjecture and related questions. *Proc. Moscow Math. Soc.*, 74(1):115–173, 2013.

[EENS13] T. Ekholm, J. Etnyre, L. Ng, and M. Sullivan. Knot contact homology. *Geom. Topol.*, 17:975–1112, 2013.

[Kir95] R. (Ed.) Kirby. Problems in low-dimensional topology. In *Proceedings of Georgia Topology Conference, Part 2*, pages 35–473. Press, 1995.

[LM13] D.J. LaFountain and W.W. Menasco. Embedded annuli and Jones' Conjecture. arXiv: 1302.1247, 2013.

[MN04] A.V. Malyutin and N.Y. Netsvetaev. Dehornoy's ordering on the braid group and braid moves. *St. Petersburg Math. J.*, 15(3):437–448, 2004.

[Ng05] L. Ng. Knot and braid invariants from contact homology I. *Geom. Topol.*, 9:247–297, 2005.

[Ng08] L. Ng. Framed knot contact homology. *Duke Math. J.*, 141(2):365–406, 2008.

[Ng12] L. Ng. A topological introduction to knot contact homology, 2012. arXiv: 1210.4803.

[RZ87] Markus Rost and Heiner Zieschang. Meridional generators and plat presentations of torus links. *J. London Math. Soc. (2)*, 35(3):551–562, 1987.

[SW06] D.S. Silver and W. Whitten. Knot group epimorphisms. *J. Knot Theory Ramifications*, 15(2):153–166, 2006.