List of Todos

AUGMENTATION RANK OF SATELLITES WITH BRAID PATTERN

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too many K's

Abstract. A knot K in S^3 has a knot group that is generated by meridians of K, and the meridional rank of K is the minimal number of meridians needed to generate the group. It is an open question of Cappell and Shaneson whether the meridional rank equals the bridge number of K. We use augmentations in knot contact homology to study the persistence of this equality under satellite operations on K with braid pattern. In particular, we answer the question in the affirmative for a large class of iterated torus knots.

discussion of B unneeded here

1. Introduction

not sure how to say this

Let K be a knot in S^3, and let $B \in B_n$ be a braid closing to K. Throughout this paper we will use the framing normal to B. We denote by π_K the fundamental group of knot complement $\overline{S^3 \setminus n(K)}$. An element of π_K is a *meridian of* K if it ~~can be~~ is represented by a disc D embedded in $\mathbb{R}^3$ such that D intersects K exactly once on the interior of D. The *meridional rank* of K, written $\mathrm{mr}(K)$, is the minimal size of a meridional generating set of π_K. The *bridge number* of K, denoted $b(K)$, is the minimum number of local maxima of K taken over all embeddings of K into $\mathbb{R}^4$ S^3 with a height function.

is there a better way to say this?

It is well known that for a fixed knot K, $\mathrm{mr}(K)$ bounded above by $b(K)$, and Problem 1.11 of [**?**] asks whether $\mathrm{mr}(K) = b(K)$ for all knots K. Our main theorem answers this question for a large class of iterated torus knots as a corollary.

State Theorem first

explain

Corollary 1.1. *Let T be an iterated torus knot, and suppose it arises from taking (p_i, q_i)-cables such that $p_i < q_i$ for all i. Then $\mathrm{mr}(T) = b(T)$.*

We reach this result via the augmentation rank, a ~~powerful~~ invariant arising from knot contact homology. Let $R_0 = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$, and let $\mathcal{A}_n$ be the noncommutative unital algebra over $\mathbb{Z}$ freely generated by the $n(n-1)$ elements a_{ij}, $1 \leq i, j \leq n$. From B we define a certain ideal $\mathcal{I} \subset \mathcal{A}_n \otimes R_0$, and the *degree zero homology of the combinatorial knot DGA* is $HC_0(K) = \mathcal{A}_n \otimes R_0 / \mathcal{I}$. Since the description of $\mathcal{I}$ is fairly involved, we delay its definition until Section 2.

(Choppy Needed?)

It was shown in [**?**] that the isomorphism class of $HC_0(K)$ does not depend on the choice of B, and is thus an invariant of K. An *augmentation* of K is

(graded)

a homomorphism $\epsilon\colon \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ that descends to $HC_0(K)$, and the *rank* of ϵ is given by the rank of $\epsilon(\mathbf{A})$, where

$$\mathbf{A_{ij}} = \begin{cases} a_{ij} & i < j \\ -\mu a_{ij} & i > j \\ 1-\mu & i = j \end{cases}$$

"Knots"

The *augmentation rank* of K, written $\operatorname{ar}(K)$, is the maximum rank among augmentations of K. It is shown in [**?**] that $\operatorname{ar}(K) < \operatorname{mr}(K)$, giving the following result.

Corollary 2.4 ([**?**])**.** *Given a knot $K \subset S^3$,*

$$ar(K) \leq mr(K) \leq b(K)$$

Let $\tau_{m,l} \in B_{pk}$ be defined by $\tau_{m,l} = \sigma_m \sigma_{m+1} \cdots \sigma_{m+l-1}$, and let $\Sigma_n^{(p)} \in B_{pk}$ be defined by $\Sigma_n^{(p)} = \tau_{np,p}\tau_{np-1,p} \cdots \tau_{np-p+1,p}$ (see Figure 1). Then if $B \in B_k$ is given by the braid word $\sigma_{n_1}\sigma_{n_2} \cdots \sigma_{n_m}$, we define the *p-copy* $B^{(p)}$ of B to be $B^{(p)} = \Sigma_{n_1}^{(p)}\Sigma_{n_2}^{(p)} \cdots \Sigma_{n_m}^{(p)}$. Our main result shows that certain satellites with a braid pattern of knots with augmentation rank equal to braid index also have augmentation rank equal to bridge index.

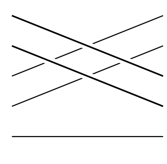

FIGURE 1. $\Sigma_1^{(2)}$

Theorem 1.2. *Let $B \in B_k$ have augmentation rank k, and let $B'' \in B_p$ have augmentation rank p. If B' is the braid B'' included into B_{pk}, then $B^{(p)}B'$ has augmentation rank pk.*

Note that if B' closes to a (p,q) torus knot, then $B^{(p)}B'$ is the (p,q)-cable of B. As a knot's bridge number is bounded above by its braid index, Corollary 2.4 implies that if a knot K has augmentation rank equal to braid index, then $\operatorname{mr}(K) = b(K)$. Thus Theorem 1.2 in conjunction with Theorem 1.3 from [**?**] gives Corollary 1.1.

In Section 2 of this paper, we give the background in knot contact homology and augmentations necessary for understand the proof the the main result. In Section 3, we define the new notation introduced and proof Theorem 1.2.

2. BACKGROUND

We review in Section 2.1 the construction of $HC_0(K)$ from the viewpoint of the combinatorial knot DGA, which was first defined in [**?**]; our conventions are those given in [**?**]. In Section 2.2 we discuss augmentations in knot lower

contact homology and their rank, which gives a bound on the meridional rank of the knot group useful for studying the relation between meridional rank and bridge number. Finally, in Section 2.3 is a discussion of techniques from [?] that we use to calculate the augmentation rank.

and

Throughout the paper we denote by B_n the n-strand braid group. We orient braids from left to right and label the strands $1, \ldots, n$, with 1 the topmost ~~to~~ n the bottommost strand. We work with the generating set $\{\sigma_i^{\pm}, i = 1, \ldots, n\}$ of B_n, where σ_i has strands i and $i+1$ that cross once in the manner depicted in Figure 2. ~~As usual~~, a braid may be closed to a link

give Artin's presentation

n-1

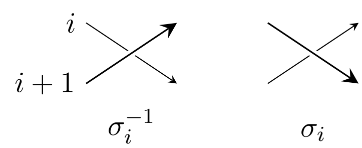

FIGURE 2. Generators of B_n

as depicted in Figure 3. The *writhe* (or algebraic sum) of ~~a braid~~ $B \in B_n$, denoted $\omega(B)$, is the sum of the exponents in a factorization of B in terms of the ~~generators~~.

$\sigma_i^{\pm}$, $i=1,\ldots,n-1$

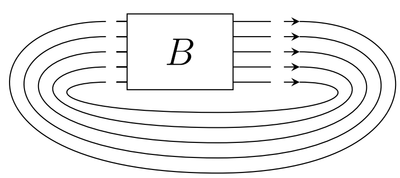

FIGURE 3. The closure of the braid B

in fact, we only define the piece in grading zero

2.1. **Knot contact homology.** Here we cover the necessary preliminaries for defining the combinatorial knot DGA of Ng. This DGA was defined in order to be a calculation of knot contact homology and was shown to be so in [?] (see [?] for more details). Let $\mathcal{A}_n$ be the noncommutative unital algebra over $\mathbb{Z}$ freely generated by a_{ij}, $1 \le i \ne j \le n$. We define a homomorphism $\phi : B_n \to \operatorname{Aut} \mathcal{A}_n$ by defining it on the generators of B_n:

$$\phi_{\sigma_k} : \begin{cases} a_{ij} \mapsto a_{ij} & i, j \ne k, k+1 \\ a_{k+1,i} \mapsto a_{ki} & i \ne k, k+1 \\ a_{i,k+1} \mapsto a_{ik} & i \ne k, k+1 \\ a_{k,k+1} \mapsto -a_{k+1,k} & \\ a_{k+1,k} \mapsto -a_{k,k+1} & \\ a_{ki} \mapsto a_{k+1,i} - a_{k+1,k} a_{ki} & i \ne k, k+1 \\ a_{ik} \mapsto a_{i,k+1} - a_{ik} a_{k,k+1} & i \ne k, k+1 \end{cases} \tag{1}$$

Let $\iota\colon B_n \to B_{n+1}$ be the inclusion $\sigma_i \mapsto \sigma_i$ so that strand $(n+1)$ does not interact with those from $B \in B_n$, and define $\phi_B^* \in \operatorname{Aut}\mathcal{A}_{n+1}$ by $\phi_B^* = \phi_B \circ \iota$. We then define the $n \times n$ matrices Φ_B^L and Φ_B^R with entries in $\mathcal{A}_n$ by

$\phi^* : B_n \to Aut(A_{n+1})$

$$\phi_B^*(a_{i,n+1}) = \sum_{j=1}^{n} (\Phi_B^L)_{ij} a_{j,n+1}$$

$$\phi_B^*(a_{n+1,i}) = \sum_{j=1}^{n} a_{n+1,j} (\Phi_B^R)_{ji}$$

Finally, ~~Letting $\omega(B)$ be the writhe of B~~, define matrices $\mathbf{A}$ and $\mathbf{\Lambda}$ by

$$\mathbf{A_{ij}} = \begin{cases} a_{ij} & i < j \\ -\mu a_{ij} & i > j \\ 1 - \mu & i = j \end{cases} \tag{2}$$

$$\mathbf{\Lambda} = \operatorname{diag}[\lambda \mu^{\omega(\mathbf{B})}, \mathbf{1}, \ldots, \mathbf{1}]. \tag{3}$$

Definition Suppose that K is the closure of $B \in B_n$ and let R_0 be the Laurent polynomial ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$. Define $\mathcal{I} \subset \mathcal{A}_n \otimes R_0$ to be the ideal generated by the entries of $\mathbf{A} - \mathbf{\Lambda} \cdot \mathbf{\Phi_B^L} \cdot \mathbf{A}$ and $\mathbf{A} - \mathbf{A} \cdot \mathbf{\Phi_B^R} \cdot \mathbf{\Lambda^{-1}}$. The *degree zero homology of the combinatorial knot DGA* is $\mathrm{HC}_0(K) = (\mathcal{A}_n \otimes R_0)/\mathcal{I}$.

Knot contact homology?

It was shown in [?] that the isomorphism class of $HC_0(K)$ is unchanged under conjugation and by positive and negative stabilization of B, hence $HC_0(K)$ is an invariant of the knot K by Markov's theorem. We only consider $HC_0(K)$ here, but there is a larger invariant, the differential graded algebra discussed in [?], ~~where the image of the differential may be generated by the same elements as $\mathcal{I}$.~~

the Markov moves

The proofs in Section 3 require a number of computations of $\phi_B(a_{ij})$ for particular braids B ~~$\in B_n$~~. Such computations are greatly benefited by an alternate description of the map ϕ_B, which follows, that we will use liberally.

Let D be a flat disk, to the right of B, with n points (punctures) where it intersects $K = \widehat{B}$ (see Figure 4). We assume the n punctures of D to be collinear, on a line that separates D into upper and lower half-disks. Denote by c_{ij} the isotopy class (fixing endpoints) of a path that is contained in the upper half-disk of D, with initial endpoint on the i^{th} strand and terminal endpoint on the j^{th} strand.

generators $[a_{ij}]$ may be identified w/ such "cords" c_{ij}

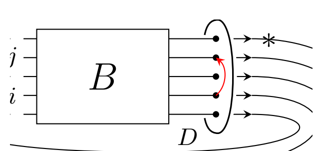

FIGURE 4. Cord c_{ij} of $K = \widehat{B}$

Considering B as a mapping class ~~element of the punctured disk~~, let $B \cdot c_{ij}$ denote the isotopy class of the path to which c_{ij} is sent. Viewing D from the left (as pictured), σ_k acts by rotating the k- and $(k+1)$-punctures an angle of π about their midpoint in counter-clockwise fashion. Consider the *algebra of paths* over $\mathbb{Z}$ generated by isotopy classes of paths in D with endpoints on punctures, modulo the relation in Figure 5 (paths depicted there are understood to agree outside the neighborhood of the puncture shown). Let $F(c_{ij}) = a_{ij}$ if $i < j$, and $F(c_{ij}) = -a_{ij}$ if $i > j$. This was shown in [?] to define an algebra map to $\mathcal{A}_n$ satisfying $F(B \cdot c_{ij}) = \phi_B(F(c_{ij}))$.

example Computation?

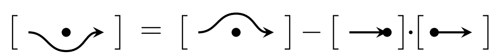

FIGURE 5. Relation in the algebra of paths

Let perm : $B_n \to S_n$ denote the homomorphism from B_n to the symmetric group sending σ_k to the simple transposition interchanging $k, k+1$. We will make use of the following ~~property of~~ ϕ_B.

Lemma 2.1. *For some $B \in B_n$ and $1 \le i \ne j \le n$, consider the element $\phi_B(a_{ij}) \in \mathcal{A}_n$ as a polynomial expression in the (non-commuting) variables $\{a_{ij}, 1 \le i \ne j \le n\}$. Writing $i' = perm(B)(i)$ and $j' = perm(B)(j)$, every non-constant monomial in $\phi_B(a_{ij})$ is a constant times $\prod_{k=0}^{l-1} a_{i_k, i_{k+1}}$, ~~where~~ for some $l \ge 1$ and $i_0 = i'$, $i_l = j'$, and $i_k \ne i_{k+1}$ for each $0 \le k \le l-1$.*

Talk about monomial in an entry of Φ_B^L. $a_{i_0 i_1} a_{i_1 i_2} \dots$ to emphasize order?

Proof. Suppose a path c in D starts at puncture p and ends at puncture q. The relation in Figure 5 equates c with a sum (or difference) of another path with the same endpoints and a product of two paths, one beginning at p and the other ending at q. A finite number of applications of this relation allows one to express c as a polynomial in the $c_{pq}, 1 \le p \ne q \le n$. The result follows since the class $B \cdot c_{ij}$ is represented by a path with endpoints the i' and j' punctures.

Alternatively, the statement follows from noting that (1) defining ϕ_{σ_k} has the desired property and that $\phi : B_n \to \mathrm{Aut}(\mathcal{A}_n)$ is a homomorphism. □

2.2. **Augmentations and augmentation rank.** Let S be a ring with 1, and consider it a differential graded algebra supported in grading 0, with trivial differential. Augmentations of $(\mathcal{A}, \partial)$ are DGA maps $(\mathcal{A}, \partial) \to (S, 0)$. For our setting, if $B \in B_n$ is a braid representative of K, such a map corresponds precisely to a homomorphism $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ such that ϵ sends each generator (mentioned in 2.1) See Definition... of $\mathcal{I}$ to zero.

Number these

Definition Suppose that K is the closure of $B \in B_n$. An *augmentation* of K is a homomorphism $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ such that each element of $\mathcal{I}$ is sent by ϵ to zero.

A correspondence between augmentations and particular representations of ~~the knot group of K~~ π_K were studied in [?]. Let π_K be the fundamental group of the complement of $K \subset S^3$. An element $g \in \pi_K$ is called a *meridian* if it may be represented by the boundary of an embedded disk in S^3 that intersects K in exactly one point. Recall that π_K is generated by meridians. We may fix a meridian m and generate π_K by conjugates of m.

Definition For any integer $r \geq 1$, a homomorphism $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ is a *KCH representation* if there is a meridian m of K such that $\rho(m)$ is diagonalizable and has eigenvalue 1 with multiplicity $r-1$. We call ρ a *KCH irrep* if it is irreducible.

In [**?**], Ng describes an isomorphism between $HC_0(K)$ and an algebra constructed from elements of π_K. As discussed in [**?**]a KCH representation $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ induces an augmentation ϵ_ρ of K. Given an augementation, the first author showed how to construct a KCH representation that induces it. In fact, we have the following rephrasing of results from [**?**].

Theorem 2.2 ([**?**]). *Let $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ be an augmentation with $\epsilon(\mu) \neq 1$. There is a KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$. Furthermore, for any KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$, r equals the rank of $\epsilon(\mathbf{A})$.*

Considering Theorem 2.2 we make the following definition.

Definition The *rank* of an augmentation $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ with $\epsilon(\mu) \neq 1$ is the rank of $\epsilon(\mathbf{A})$. Given a knot K, the *augmentation rank of* K, denoted $\mathrm{ar}(K)$, is the maximum rank among augmentations of K.

Remark The augmentation rank can be defined for target rings other than $\mathbb{C}$, but this paper only considers augmentations as in ~~2.2.~~ Definition...

It is the case that $\mathrm{ar}(K)$ is well-defined. That is, given K there is a bound on the maximal rank of an augmentation of K. (regardless of B w/ $\hat{B} = K$)

Theorem 2.3 ([**?**]). *Given a knot $K \subset S^3$, if $g_1, \ldots, g_d$ are meridians that generate π_K and $\rho : \pi_K \to GL_r\mathbb{C}$ is a KCH irrep then $r \leq d$.*

As in the introduction, if we denote the meridional rank of π_K by $\mathrm{mr}(K)$, then Theorem 2.3 implies that $\mathrm{ar}(K) \leq \mathrm{mr}(K)$. In addition, the geometric quantity $b(K)$ called the bridge index of K is never less than $\mathrm{mr}(K)$. Thus we have the following corollary:

I made the inequality a corollary here

Corollary 2.4 ([**?**]). *Given a knot $K \subset S^3$,*

$$ar(K) \leq mr(K) \leq b(K)$$

As a result, to verify for K that $\mathrm{mr}(K) = b(K)$ it suffices to find an augmentation of K with rank equal to $b(K)$. As we discuss in the next section, we will concern ourselves in this paper with a setting where $\mathrm{ar}(K) = n$ and there is a braid $B \in B_n$ which closes to K. This is a special situation, since $b(K)$ is strictly less than the braid index for many knots.

2.3. **Finding augmentations.** The following theorem concerns the behavior of the matrices Φ^L_B and Φ^R_B under the product in B_n. It is an essential tool for studying $HC_0(K)$ and will be central to our arguments.

Theorem 2.5 ([**?**], Chain Rule)**.** *Let B, B' be braids in B_n. Then $\Phi^L_{BB'} = \phi_B(\Phi^L_{B'}) \cdot \Phi^L_B$ and $\Phi^R_{BB'} = \Phi^R_B \cdot \phi_B(\Phi^R_{B'})$.*

The main result of this paper concerns augmentations with rank equal to the braid index of the knot K. ~~Suppose that K is the closure of $B \in B_n$ and~~ define the diagonal matrix $\Delta(B) = \mathrm{diag}[(-1)^{w(B)}, 1, \ldots, 1]$. The following statement follows from results in [**?**, Section 5].

but the theorem is marked Cor13a?

Theorem 2.6 ([**?**])**.** *If K is the closure of $B \in B_n$ and has a rank n augmentation $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$, then*

$$\epsilon(\Phi^L_B) = \Delta(B) = \epsilon(\Phi^R_B). \tag{4}$$

Furthermore, any homomorphism $\epsilon : \mathcal{A}_n \to \mathbb{C}$ which satisfies (4) can be extended to $\mathcal{A}_n \otimes R_0$ to produce a rank n augmentation of K.

Our proof of Theorem 1.2 relies on this characterization of rank n augmentations. Suppose the knot K is the closure of $B \in B_k$ and has a rank k augmentation ϵ_k. In Section 3 we consider $B' \in B_p$ which has closure admitting a rank p augmentation ϵ_p. Applying the braid satellite construction to B, B' we obtain a satellite of K. We prove the theorem in Section 3 by describing a map from ϵ_k and ϵ_p that satisfies (4) for the braid satellite. By Theorem 4 this determines the desired rank pk augmentation.

described in the introduction...

needs to be rephrased

There is a symmetry on the matrices Φ^L_B and Φ^R_B that is relevant to the study of augmentations in this setting. Define an involution $x \mapsto \overline{x}$ on $\mathcal{A}_n$ (termed *conjugation*) as follows: first set $\overline{a_{ij}} = a_{ji}$; then, for any $x, y \in \mathcal{A}_n$, define $\overline{xy} = \overline{y}\,\overline{x}$ and extend the operation linearly to $\mathcal{A}_n$. We have the following symmetry.

Theorem 2.7 ([**?**], Prop. 6.2)**.** *For a matrix of elements in $\mathcal{A}_n$, let $\overline{M}$ be the matrix such that $\left(\overline{M}\right)_{ij} = \overline{M_{ij}}$. Then for $B \in B_n$, Φ^R_B is the transpose of $\overline{\Phi^L_B}$.*

cut? maybe not if we refer to it in def of ϕ

It may be appropriate here to indicate that $\mathrm{ar}(K) < \mathrm{mr}(K)$ sometimes (maybe in previous subsection), and talk about the 2-cable of the trefoil that does not have $\mathrm{ar}(K, \mathbb{C}) = 4$

3. Main Result

figure out this two tensor products nonsense

how do I bring in equations to fit margins?

In this section, we prove our main result:

throughout this section, $B \in B_k$, $B' \in B_p$

Theorem 1.2. *Let $B \in B_k$ have augmentation rank k, and let $B'' \in B_p$ have augmentation rank p. If B' is the braid B'' included into B_{pk}, then $B^{(p)}B'$ has augmentation rank pk.*

As we saw in the introduction, Theorem 1.2 has an immediate corollary, which follows from Corollary 2.4 and Theorem 1.3 from [?]:

Corollary 1.1. *Let T be an iterated torus knot, and suppose it arises from taking (p_i, q_i)-cables such that $p_i < q_i$ for all i. Then* $\mathrm{mr}(T) = b(T)$.

To prove Theorem 1.2, we use a map $\psi\colon \mathcal{A}_{pk} \otimes \cancel{R_0} \to (\mathcal{A}_k \otimes \cancel{R_0}) \otimes (\mathcal{A}_p \otimes \cancel{R_0})$ with some useful properties ~~and Proposition 3.1.~~ Proposition 3.1 will follow from Lemmas 3.2 and 3.3, and Lemma 3.3 depends on Lemma 3.2 while Lemma 3.2 depends on Lemma 3.4. We begin with the definition of ψ and statement of Proposition 3.1, followed by the proof of Proposition 3.1 and Lemmas 3.2,3.3, and 3.4.

An important intermediate step is Prop 3.1, which we pair w/ Ch. Rule to define ϵ from ϵ_k, ϵ_p

Fix $p > 0$ ~~and let B be a braid on k~~ strands. For each $1 \le i \le pk$ define integers q_i, r_i such that $i = q_i p + r_i$, where $0 < r_i \le p$. Instrumental to the proof of Theorem 1.2 will be the map $\psi\colon \mathcal{A}_{pk} \otimes R_0 \to (\mathcal{A}_k \otimes R_0) \otimes (\mathcal{A}_p \otimes R_0)$, defined on generators as follows

$$\psi(a_{ij}) = \begin{cases} 1 \otimes a_{r_i r_j} & : q_i = q_j \\ a_{q_i+1, q_j+1} \otimes 1 & : r_i = r_j \\ 0 & : (q_i - q_j)(r_i - r_j) < 0 \\ a_{q_i+1, q_j+1} \otimes a_{r_i r_j} & : (q_i - q_j)(r_i - r_j) > 0 \end{cases}$$

We ~~also define $\psi(\mu) = \mu \otimes 1$ and $\psi(\lambda) = \lambda \otimes 1$~~. Note that $\psi(a_{ij}) \in 1 \otimes \mathcal{A}_p$ ~~or~~ and is $\neq 0$ $\psi(a_{ij}) \neq 0$ if and only if $q_i = q_j$, and that $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$. This homomorphism gives us a way of relating $\Phi^L_{B^{(p)}}$ to Φ^L_B via the following proposition.

Proposition 3.1. *For any braid B,* $\psi\left(\Phi^L_{B^{(p)}}\right) = \left(\left(\Phi^L_B\right)_{ij} \otimes 1\right) \otimes I_p$ *and* $\psi\left(\Phi^R_{B^{(p)}}\right) = \left(\left(\Phi^R_B\right)_{ij} \otimes 1\right) \otimes I_p$

fix notation

make consistent throughout paper

point out we will get $\Sigma_k(\Phi^L_B) \otimes \Sigma_p(\Phi^L_B)$

Note that here we mean the tensor product of Φ^L_B and I_p as matrices, not as linear maps, while the tensor product of $\left(\Phi^L_B \otimes I_p\right)_{ij}$ and 1 is a tensor product of algebra elements, so that if we divide the matrix $\psi(\Phi^L_{B^{(p)}})$ into k^2 $p \times p$ blocks, the ijth block is $\left(\Phi^L_B\right)_{ij} I_p$.

It turns out that instead of ψ we could have defined a simpler homomorphism $\rho\colon \mathcal{A}_{pk} \to \mathcal{A}_k$ that would take a_{ij} to $a_{q_{i+1}, q_{j+1}}$ if $r_i = r_j$ and 0 otherwise, and Proposition 3.1 would still be true (this follows from the same ideas used in the proof of Proposition 3.1). The advantage of ψ is that it doesn't send a_{ij} to 0 if $q_i = q_j$, a fact which will be important in the proof of Theorem 1.2.

necessary?

Proof of Theorem 1.2. By Theorem 2.6, if $B \in B_k$ and B' ? $\in B_p$ have augmentation ranks k and p, respectively, there exist augmentations $\epsilon_k\colon \mathcal{A}_k \otimes R_0 \to \mathbb{C}$ and $\epsilon_p\colon \mathcal{A}_p \otimes R_0 \to \mathbb{C}$ such that $\epsilon_k\left(\Phi^L_B\right) = \epsilon_k\left(\Phi^R_B\right) = \Delta(B)$ and $\epsilon_p\left(\Phi^L_{B'}\right) = \epsilon_p\left(\Phi^R_{B'}\right) = \Delta(B')$. Theorem 2.6 also implies that it suffices to prove that there exists an augmentation $\epsilon\colon \mathcal{A}_{pk} \otimes R_0 \to \mathbb{C}$ such that $\epsilon\left(\Phi^L_{B^{(p)}B'}\right) = \epsilon\left(\Phi^R_{B^{(p)}B'}\right) = \Delta(B^{(p)}B')$.

Let $\delta\colon \mathcal{A}_p \to \mathbb{C}$ be a homomorphism, let $\pi\colon \mathbb{C}\otimes\mathbb{C} \to \mathbb{C}$ be a homomorphism defined by $\pi(a\otimes b) = ab$, and set $\epsilon = \pi\circ(\epsilon_k\otimes\delta)\circ\psi$. We will later break the theorem up into three cases depending on the parity of $w(B)$ and p and in each case define ~~δ such that $\delta(a_{ij})$ is one of $\pm\epsilon_p(a_{ij})$ in such a way~~ that ϵ is an augmentation of $B^{(p)}B'$. The Chain Rule theorem gives that

we will define a hom. δ so that $\delta(a_{ij}) = \pm\epsilon_p(a_{ij})$

the multiplication

say earlier

$$\pi\circ(\epsilon_k\otimes\delta)\circ\psi\left(\Phi^L_{B^{(p)}B'}\right) = \pi\circ(\epsilon_k\otimes\delta)\psi\left(\phi_{B^{(p)}}\left(\Phi^L_{B'}\right)\right)\psi\left(\Phi^L_{B^{(p)}}\right) \tag{5}$$

Note that since the non zero or one entries of $\Phi^L_{B'}$ are products of a_{ij} where $i < j \le p$, $\phi_{B^{(p)}}$ takes each of the a_{ij}'s in these products to $a_{i+mp,j+mp}$ for some $0 \le m < k$. We have that ψ takes $a_{i+mp,j+mp}$ to $1\otimes a_{ij}$, however, so

briefly, why (only apparent to you, me, and Lenny)

$$\psi\left(\phi_{B^{(p)}}\left(\Phi^L_{B'}\right)\right) = \left(1\otimes\left(\Phi^L_{B'}\right)_{ij}\right)$$

By Proposition 3.1, we have that

just put one

$$\psi\left(\Phi^L_{B^{(p)}}\right) = \left(\left(\Phi^L_B\right)_{ij}\otimes 1\right)\otimes I_p = \left(\left(\Phi^L_B\otimes I_p\right)_{ij}\otimes 1\right)$$

So returning to the right hand side of (5) we get

$$\begin{aligned}\pi\circ(\epsilon_k\otimes\delta)\left(\psi\left(\phi_{B^{(p)}}\left(\Phi^L_{B'}\right)\right)\psi\left(\Phi^L_{B^{(p)}}\right)\right) &= \pi\circ(\epsilon_k\otimes\delta)\left(\left(1\otimes\left(\Phi^L_{B'}\right)_{ij}\right)\left(\left(\Phi^L_B\otimes I_p\right)_{ij}\otimes 1\right)\right)\\ &= \delta\left(\Phi^L_{B'}\right)\left(\Delta(B)\otimes I_p\right)\end{aligned}$$

So it suffices to find an augmentation δ such that the right hand side is equal to $\Delta(B^{(p)}B')$. If $w(B)$ is even, then we simply let $\delta = \epsilon_p$. Since $w(B)$ is even we know that $w(B^{(p)})$ is also even and that $\Delta(B) = I_k$. Since $\epsilon_p\left(\Phi^L_{B'}\right) = \Delta(B')$, it follows that the right hand side is equal to $\Delta(B^{(p)}B')$.

ϵ

Now suppose that $w(B)$ is odd. In a moment we will define $g\colon \{1,\dots,p\} \to \{\pm1\}$ for each of the cases for when p is even or odd, but for now let $\delta(a_{ij}) = g(i)g(j)\epsilon_k(a_{ij})$. Fix i, j and consider a monomial M in $\left(\Phi^L_{B'}\right)_{ij}$. Since B' is a braid on p strands included into B_{pk}, if $i > p$ or $j > p$ then M is 0 or 1 and $\delta(M) = M$. If $i, j \le p$, such a monomial must arise from a product in the algebra of paths in D that begins at $i' = \mathrm{perm}(B')(i)$ and ends at j, so $M = c_{ij}a_{i',j_1}a_{j_1,j_2}\dots a_{j_m,j}$ for some $j_1,\dots j_m \in \{1,\dots,p\}$, unless $i' = j$, in which case it is possible that $M = c_{ij}$. We then see that

refer to proposition

$$\delta(M) = g(i')g(j)\left(\prod_{k=1}^{m} g(j_k)^2\right)\epsilon_p(M) = g(i')g(j)\epsilon_p(M)$$

Or $\delta(M) = M = g(i')g(j)\epsilon_p(M)$ in the case that $i' = j$ and $M = c_{ij}$. Since this is true for each monomial M chosen in $\left(\Phi^L_{B'}\right)_{ij}$, we have that

$$\delta\left(\left(\Phi^L_{B'}\right)_{ij}\right) = g(i')g(j)\epsilon_p\left(\left(\Phi^L_{B'}\right)_{ij}\right)$$

Now let $x_1 = 1$, and $x_l = \mathrm{perm}(B')(x_{l-1})$ for $1 < l \leq p$. Since the first p strands of B' close to a knot, $\mathrm{perm}(B')$ is given by the p-cycle $(x_1 x_2 \ldots x_p)$.

Suppose p is even. Then we let $g(x_1) = 1$, and $g(x_l) = -g(x_{l-1})$ for $1 < l \leq p$. Since p is even, $w(B^{(p)})$ is even and therefore the opposite parity of $w(B)$. Our definition of g gives that $\delta\left(\left(\Phi^L_{B'}\right)_{ii}\right) = -\epsilon\left(\left(\Phi^L_{B'}\right)_{ii}\right)$ for $i \leq p$, so

$$\delta\left(\Phi^L_{B'}\right) = \begin{pmatrix} (-1)^{w(B')+1} & 0 & 0 \\ 0 & -I_{p-1} & 0 \\ 0 & 0 & I_{(k-1)p} \end{pmatrix}$$

and therefore

$$\delta\left(\Phi^L_{B'}\right)(\Delta(B) \otimes I_p) = \mathrm{diag}[(-1)^{w(B)+w(B')+1}, 1 \ldots 1] = \Delta(B^{(p)}B')$$

as desired.

Next suppose that p is odd. Then we let $g(x_1) = g(x_2) = 1$ and $g(x_l) = -g(x_{l-1})$ for $2 < l \leq p$. Since p is odd, $w(B^{(p)})$ is odd and therefore the same parity of $w(B)$. Our definition of g gives that $\delta\left(\left(\Phi^L_{B'}\right)_{11}\right) = \epsilon\left(\left(\Phi^L_{B'}\right)_{11}\right)$ and $\delta\left(\left(\Phi^L_{B'}\right)_{ii}\right) = -\epsilon\left(\left(\Phi^L_{B'}\right)_{ii}\right)$ for $1 < i \leq p$, so

$$\delta\left(\Phi^L_{B'}\right) = \begin{pmatrix} (-1)^{w(B')} & 0 & 0 \\ 0 & -I_{p-1} & 0 \\ 0 & 0 & I_{(k-1)p} \end{pmatrix}$$

and therefore

$$\delta\left(\Phi^L_{B'}\right)(\Delta(B) \otimes I_p) = \mathrm{diag}[(-1)^{w(B)+w(B')}, 1 \ldots 1] = \Delta(B^{(p)}B')$$

as desired. Similarly, we have that

$$\pi \circ (\epsilon_k \otimes \delta) \circ \psi\left(\Phi^R_{B^{(p)}B'}\right) = \pi \circ (\epsilon_k \otimes \delta)\left(\left(\left(\Phi^R_B \otimes I_p\right)_{ij} \otimes 1\right)\left(1 \otimes \left(\Phi^R_{B'}\right)_{ij}\right)\right)$$

but since $\epsilon_k\left(\Phi^L_B\right) = \epsilon_k\left(\Phi^R_B\right)$ and $\epsilon_p\left(\Phi^L_{B'}\right) = \epsilon_p\left(\Phi^R_{B'}\right)$, in each case above we have

$$\pi \circ (\epsilon_k \otimes \delta)\left(\left(\left(\Phi^R_B \otimes I_p\right)_{ij} \otimes 1\right)\left(1 \otimes \left(\Phi^R_{B'}\right)_{ij}\right)\right) = \pi \circ (\epsilon_k \otimes \delta)\left(\left(\left(\Phi^L_B \otimes I_p\right)_{ij} \otimes 1\right)\left(1 \otimes \left(\Phi^L_{B'}\right)_{ij}\right)\right)$$
$$= \Delta\left(B^{(p)}B'\right)$$

no need to write this out, say it

Which completes the proof. $\square$

New Subsection

$\phi_{\sigma_2} \otimes \mathrm{id}$

We will use the following two lemmas in our proof of Proposition 3.1. Figure 6 demonstrates an example for Lemma 3.2, showing that $\psi(\phi_{\Sigma_2^{(2)}}(a_{24})) = \phi_{\sigma_2}(\psi(a_{24}))$. Note that in the figure we condense elements such as $a_{13} \otimes 1$ to a_{13} in order to make the notation cleaner. you also write products of algebra elements in one figure

Lemma 3.2. *$\psi(\phi_{\Sigma_n^{\pm(p)}}(a_{ij})) = (\phi_{\sigma_n^{\pm 1}} \otimes \mathrm{id})(\psi(a_{ij}))$ for all $1 \leq n < k$, $1 \leq i, j \leq pk$.*

draw commutative diagram?

Lemma 3.3. $\psi\left(\Phi^L_{\Sigma_n^{\pm(p)}}\right) = \left(\left(\Phi^L_{\sigma_n^{\pm 1}}\right)_{ij} \otimes 1\right) \otimes I_p$

Call it "Base Case"

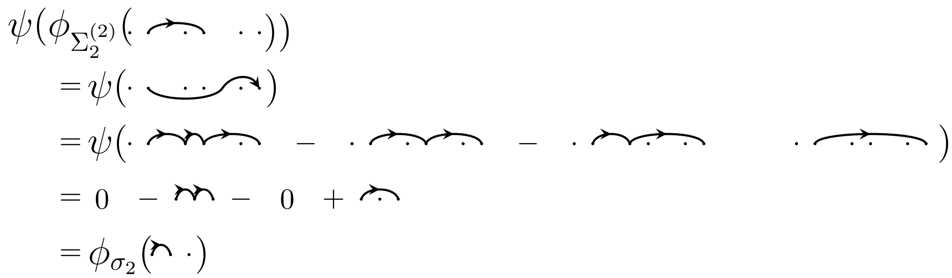

FIGURE 6. Computing $\psi(\phi_{\Sigma_2^{(p)}}(a_{24}))$

induction

Proof of Proposition 3.1. Let $B = \sigma_{n_1}^{q_1} \cdots \sigma_{n_r}^{q_r}$, where $1 \le n_i < k$ and $q_i = \pm 1$. We will prove the proposition by ~~inducting~~ on r. The base case is already taken care of by Lemma 3.3. Suppose that the proposition holds for braids of length $r-1$. Let $B' = \sigma_{n_1}^{q_1} \cdots \sigma_{n_{r-1}}^{q_{r-1}}$ Then by the Chain Rule and Lemmas 3.2 and 3.3, we have that

$$\begin{aligned}
\psi\left(\Phi^L_{B^{(p)}}\right) &= \psi\left(\phi_{B'^{(p)}}\left(\Phi^L_{\Sigma_{n_r}^{q_r(p)}}\right)\cdot \Phi^L_{B'^{(p)}}\right) \\
&= (\phi_{B'} \otimes \mathrm{id})\left(\psi\left(\Phi^L_{\Sigma_{n_r}^{q_r(p)}}\right)\right)\cdot\left(\left(\Phi^L_{B'}\right)_{ij}\otimes 1\right)\otimes I_p \\
&= (\phi_{B'} \otimes \mathrm{id})\left(\left(\left(\Phi^L_{\sigma_{n_r}^{q_r}}\right)_{ij}\otimes 1\right)\otimes I_p\right)\cdot\left(\left(\Phi^L_{B'}\right)_{ij}\otimes 1\right)\otimes I_p \\
&= \left(\left(\Phi^L_B\right)_{ij}\otimes 1\right)\otimes I_p
\end{aligned}$$

Which implies that $\psi\left(\Phi^R_{B^{(p)}}\right) = \left(\left(\Phi^R_B\right)_{ij}\otimes 1\right)\otimes I_p$ as well, since $\Phi^R_B = \overline{\Phi^L_B}^t$ and $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$. □

In the proof of Lemmas 3.2 and 3.3, we will make use of some calculations of $\phi_B(a_{ij})$ for simple braids B. Recall that $\tau_{m,l} = \sigma_m\sigma_{m+1}\cdots\sigma_{m+l-1}$. It can easily be checked that for all $1 \le m < n$, $1 \le l \le n-m$, $i < j$

$$\phi_{\tau_{m,l}}(a_{ij}) = \begin{cases} a_{i+1,j+1} & : m \le i < j < m+l \\ a_{i-l,j} & : m < m+l = i < j \\ a_{i,j-l} & : i < m < m+l = j \\ a_{i+1,j-l} & : m \le i < j = m+l \\ a_{i,j+1} - a_{i,m}a_{m,j+1} & : i < m \le j < m+l \\ a_{i+1,j} - a_{i+1,m}a_{m,j} & : m \le i < m+l < j \\ a_{ij} & : \text{otherwise} \end{cases} \tag{6}$$

We also make the following definition
Let $X \subseteq \{1,\ldots,n\}$, and write the elements of a subset $Y \subseteq X$ as $y_1 < \ldots < y_k$. Define

$$A(i,j,X) = \sum_{Y\subseteq X} (-1)^{|Y|} a_{iy_1}a_{y_1y_2}\cdots a_{y_kj}$$

and

$$A'(i,j,X) = \sum_{Y \subseteq X} (-1)^{|Y|} a_{iy_k} a_{y_k y_{k-1}} \cdots a_{y_1 j}$$

and have the following lemma

Lemma 3.4. *Suppose $i < j$. Let $\kappa_{m,l} = \tau_{m+l-1,p}\tau_{m+l-2,p}\cdots\tau_{m,p}$, and let $X_{m,l} = \{m, \ldots, m+l-1\}$. Then*

check

$$\phi_{\kappa_{m,l}}(a_{ij}) = \begin{cases} a_{i-p,j-p} & : m+p \le i < j < m+l+p \\ a_{i-p,j} & : m+p \le i < m+l+p \le j \\ a_{i,j-p} & : i < m < m+p \le j < m+l+p \\ a_{i+l,j+l} & : m \le i < j < m+p \\ A'(i+l, j-p, X_{m,l} \setminus (j-p)) & : m \le i < m+p \le j < m+l+p \\ A(i, j+l, X_{m,l}) & : i < m \le j < m+p < m+l+p \\ A'(i+l, j, X_{m,l}) & : m \le i < m+p < m+l+p \le j \\ a_{ij} & : \text{otherwise} \end{cases}$$

Note that letting $l = p$ and $m = (n-1)p+1$ gives us $\phi_{\Sigma_n^{(p)}}(a_{ij})$ when $i < j$ as a special case. Letting $X_n^{(p)} = \{(n-1)p+1, \ldots, np\}$, we have

Do you ever use the calculation for anything but this?

$$\phi_{\Sigma_n^{(p)}}(a_{ij}) = \begin{cases} a_{i-p,j-p} & : np < i < j \le (n+1)p \\ a_{i-p,j} & : np < i \le (n+1)p < j \\ a_{i,j-p} & : i \le (n-1)p < np < j \le (n+1)p \\ a_{i+p,j+p} & : (n-1)p < i < j \le np \\ A'(i+p, j-p, X_n^{(p)} \setminus (j-p)) & : (n-1)p < i \le np < j \le (n+1)p \\ A(i, j+p, X_n^{(p)}) & : i \le (n-1)p < j \le np < (n+1)p \\ A'(i+p, j, X_n^{(p)}) & : (n-1)p < i \le np < (n+1)p < j \\ a_{ij} & : \text{otherwise} \end{cases}$$

is this right?

Proof of Lemma 3.2. Note that if $\psi(\phi_{\Sigma_n^{(p)}}(a_{ij})) = (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))$, then

$$(\phi_{\sigma_n} \otimes \mathrm{id})\left(\psi\left(\phi_{\Sigma_n^{-(p)}}(a_{ij})\right)\right) = \psi(a_{ij})$$

So $\psi \circ \phi_{\Sigma_n^{-(p)}}$ is the ~~inverse~~ function of $(\phi_{\sigma_n} \otimes \mathrm{id})$, and therefore

ψ is not injective...

$$\psi\left(\phi_{\Sigma_n^{-(p)}}(a_{ij})\right) = \left(\phi_{\sigma_n^{-1}} \otimes \mathrm{id}\right)(a_{ij})$$

Furthermore, $\phi_B(\overline{a_{ij}}) = \overline{\phi_B(a_{ij})}$ and $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$, so it suffices to prove the lemma for $\Sigma_n^{(p)}$ in the case where $i < j$.

does this need justification?

With these restrictions, we then break the statement up into the cases from Lemma 3.4, from which the first four cases as well as the last case can be checked easily. Consider the sixth case. Lemma 3.4 gives that

$$\psi\left(\phi_{\Sigma_n^{(p)}}(a_{ij})\right) = \sum_{Y \subseteq \{np-p+1, \ldots, np\}} (-1)^{|Y|} \psi\left(a_{iy_1} a_{y_1 y_2} \cdots a_{y_k, j+p}\right)$$

do I need to explain what I'm doing here?

Let $\alpha_i = np - p + r_i$. Note that if $y_1 < \alpha_i$ then $\psi(a_{iy_1}) = 0$, and if $y_k > \alpha_j$ then $\psi(a_{y_kj}) = 0$, so the sum on the right hand side can be taken over $Y \subseteq \{\alpha_i, \alpha_i + 1, \ldots, \alpha_j\}$. Then we manipulate the sum to get

$$\begin{aligned}
&\sum_{Y\subseteq\{\alpha_i,\ldots,\alpha_j\}} (-1)^{|Y|}\psi\left(a_{iy_1}a_{y_1y_2}\cdots a_{y_k,j+p}\right)\\
&= \psi\left(a_{i,j+p} - a_{i,\alpha_i}a_{\alpha_i,j+p}\right)\\
&\quad + \sum_{y=\alpha_i+1}^{\alpha_j}\ \sum_{Y\subseteq\{y+1,\ldots,\alpha_j\}} (-1)^{|Y|+1}\psi\left(a_{iy}a_{yy_1}\cdots a_{y_k,j+p}\right) + (-1)^{|Y|}\psi\left(a_{i,\alpha_i}a_{\alpha_i,y}a_{yy_1}\cdots a_{y_k,j+p}\right)\\
&= \psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right)\\
&\quad + \sum_{y=\alpha_i+1}^{\alpha_j}\ \sum_{Y\subseteq\{y+1,\ldots,\alpha_j\}} (-1)^{|Y|}\psi\left(a_{i,\alpha_i}a_{\alpha_i,y} - a_{iy}\right)\psi\left(a_{yy_1}\cdots a_{y_k,j+p}\right)
\end{aligned}$$

should make accessible, quickly see which case... of which statement

Note that $r_i = r_{\alpha_i}$ and since we're in the sixth case we have $(n-1)p < j \le np$, so $q_{\alpha_i} = q_y$. Thus $\psi(a_{i,\alpha_i}) = a_{q_i+1,q_{\alpha_i}+1} \otimes 1 = a_{q_i+1,q_y+1} \otimes 1$ and $\psi(a_{\alpha_i,y}) = 1 \otimes a_{r_{\alpha_i},r_y} = 1 \otimes a_{r_i,r_y}$, so we have

$$\psi(a_{i,\alpha_i}a_{\alpha_i,y} - a_{iy}) = \left(a_{q_i+1,q_y+1} \otimes 1\right)\left(1 \otimes a_{r_i,r_y}\right) - a_{q_i+1,q_y+1} \otimes a_{r_i,r_y} = 0$$

Thus the right hand side reduces to

$$\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right)$$

cases 5, 7

remove

Remark The fact that $\psi(a_{i,\alpha_i}a_{\alpha_i,y} - a_{iy}) = 0$ and ψ behaves similarly for the analogous terms in the other cases is the key to this proof working, and ψ is defined the way it is ~~mainly~~ so that this will be true. ~~As we hinted at earlier, the homomorphism $\rho\colon \mathcal{A}_{pk} \to \mathcal{A}_k$ defined to send a_{ij} to a_{q_i+1,q_j+1} if $r_i = r_j$ and to 0 otherwise would also send these terms to 0, so Proposition 3.1 would still be true with ρ used in the place of ψ. We will need ψ for the proof of the main result, however.~~

Note that, since we're in the sixth case, $q_j + 1 = n$. If $r_i = r_j$, then

$$\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right) = \left(a_{q_i+1,n+1} - a_{q_i+1,n}a_{n,n+1}\right)\otimes 1 = (\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))$$

If $r_i < r_j$, then

$$\begin{aligned}
\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right) &= \left(a_{q_i+1,n+1}\otimes a_{r_ir_j} - a_{q_i+1,n}a_{n,n+1}\otimes a_{r_ir_j}\right)\\
&= \left(a_{q_i+1,n+1} - a_{q_i+1,n}a_{n,n+1}\right)\otimes a_{r_ir_j}\\
&= (\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))
\end{aligned}$$

Finally, if $r_i > r_j$, then

$$\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right) = 0 = (\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))$$

The proof for the seventh case goes exactly as the proof for the sixth case except with all i's replaced with $i + p$, all $(j + p)$'s replaced with j, all y_i's replaced with y_{k+1-i}, and with α_i and α_j swapped. The proof for the fifth case goes exactly as the proof for the seventh, except that $j-p$ is removed from the set that Y is a subset of in all the sums. $\square$

check this

Proof of Lemma 3.3. First we will prove the lemma for $\Sigma_n^{(p)}$. ~~We can~~ extend the definition of ψ to be from the free module over $\mathcal{A}_{pk}$ generated by $\{a_{i*} | 1 \leq i \leq pk\}$ to the free module over $\mathcal{A}_k \otimes \mathcal{A}_p$ generated by $\{a_{i*} | 1 \leq i \leq k\}$ by defining $\psi(a_{i*}) = a_{i*}$ and extending ~~by linearity~~. Then the statement of the lemma is equivalent to saying that for all $1 \leq i \leq pk$, the coefficient of a_{j*} in $\psi\left(\phi_{\Sigma_n^{(p)}}(a_{i*})\right)$ is equal to 0 unless $r_j = r_i$, in which case it is equal to the coefficient of a_{q_j*} in $\phi_{\sigma_n}(a_{q_i*})$. If $q_i + 1 \neq n$, this fact can be easily checked. In the case that $q_i + 1 = n$, we have that

check

algebra morphism or module morphism?

symbols

to an algeb. morphism

say

$i = (n-1)p + r_i = \alpha_i$

$$\psi\left(\phi_{\Sigma_n^{(p)}}(a_{i*})\right) = \psi\left(A(i+p, *, \{np-p+1, \dots, np\}\right)$$

which is equal to

$$\psi(a_{i+p,*} - a_{i+p,\alpha_i} a_{\alpha_i,*}) = a_{i+p,*} - (a_{q_i,q_i+1} \otimes 1) a_{\alpha_i,*}$$

n+1?

by the same argument that was used in Lemma 3.2. The coefficients of the a_{j*} are equal to the coefficients of the a_{q_j*} in $\phi_{\sigma_n}(a_{q_i*})$, so we have $\psi\left(\Phi^L_{\Sigma_n^{(p)}}\right) = \left(\left(\Phi^L_{\sigma_n}\right)_{ij} \otimes 1\right) \otimes I_p$.

Using this fact, the Chain Rule, and Lemma 3.2, we have

$$\begin{aligned}
\left((I_{pk})_{ij} \otimes 1\right) &= \psi\left(\Phi^L_{\Sigma_n^{(p)} \Sigma_n^{-(p)}}\right) \\
&= \psi\left(\phi_{\Sigma_n^{-(p)}}\left(\Phi^L_{\Sigma_n^{(p)}}\right)\right) \psi\left(\Phi^L_{\Sigma_n^{-(p)}}\right) \\
&= \left(\phi_{\sigma_n^{-1}} \otimes \mathrm{id}\right) \left(\left((\Phi_{\sigma_n})_{ij} \otimes 1\right) \otimes I_p\right) \psi\left(\Phi^L_{\Sigma_n^{-(p)}}\right)
\end{aligned}$$

But note that the Chain Rule also gives that $\left(\left(\Phi^L_{\sigma_n^{-1}}\right)_{ij} \otimes 1\right) \otimes I_p$ is the inverse of $\left(\phi_{\sigma_n^{-1}} \otimes \mathrm{id}\right)\left(\left((\Phi_{\sigma_n})_{ij} \otimes 1\right) \otimes I_p\right)$, so

$$\psi\left(\Phi^L_{\Sigma_n^{-(p)}}\right) = \left(\left(\Phi^L_{\sigma_n^{-1}}\right)_{ij} \otimes 1\right) \otimes I_p$$

which completes the proof. $\square$

Proof of Lemma 3.4. check

The first four cases as well as the eighth can be easily checked. We will prove the remaining cases by induction on l. Consider the sixth case. The base case is covered by (6). For the inductive step, we have that

$$\begin{aligned}
\phi_{\kappa_{m,l}}(a_{ij}) &= \phi_{\tau_{m+l-1,p}}\left(\phi_{\kappa_{m,l-1}}(a_{ij})\right) \\
&= \sum_{Y\subseteq\{m,\dots,m+l-2\}} (-1)^{|Y|}\phi_{\tau_{m+l-1,p}}\left(a_{i,y_1}a_{y_1y_2}\cdots a_{y_k,j+l-1}\right) \\
&= \sum_{Y\subseteq\{m,\dots,m+l-2\}} (-1)^{|Y|}a_{iy_1}a_{y_1y_2}\cdots a_{y_{k-1}y_k}\left(a_{y_k,j+l} - a_{y_k,m+l-1}a_{m+l-1,j+l}\right) \\
&= \sum_{Y\subseteq\{m,\dots,m+l-1\}} (-1)^{|Y|}a_{i,y_1}a_{y_1y_2}\cdots a_{y_k,j+l} \\
&= A(i, j+l, X_{m,l})
\end{aligned}$$

is this clear/can it be shortened?

The proof of the seventh case goes exactly as the proof of the sixth, with all i's replaced with $i+l$, j's replaced with $j-l$, and y_i's replaced with y_{k-i+1}. The proof of the fifth case goes exactly as the proof of the seventh, except with the element $j-p$ removed from the set Y is a subset of in all of the sums. $\square$