What we have (when $w(B)$ is odd) is a solution that gives us a matrix

$$\begin{pmatrix} -\mathrm{Id}_p & 0 \\ 0 & \mathrm{Id}_{p(k-1)} \end{pmatrix}$$

(the "1" on diagonals here being $1 \otimes 1 \in \mathcal{A}_k \otimes \mathcal{A}_p$), which comes about from the matrix $\psi(\Phi^L_{B^{(p)}B'})$ being the product of: a $k \times k$ matrix of $p \times p$ matrices with each entry being $(\Phi^L_B)_{ij}\Phi^L_{B'}$ times the $pk \times pk$ matrix with entries $1 \otimes \Phi^L_{B'}$, where we consider $B' \in B_{pk}$ (sitting in the subgroup generated by $\sigma_1, \ldots, \sigma_p$) rather than in B_p.

We run with the fact that we can get the homomorphism sending $\Phi^L_{B^{(p)}}$ to the matrix described above.

Claim: It would suffice to show that if $p < q$ are coprime and $B' = \tau_p^q$, where $\tau_p = \sigma_1 \ldots \sigma_{p-1}$, then there is a homomorphism $f : \mathcal{A}_p^{ab} \to \mathbb{C}$ so that (including B' into B_{pk}),

$$f\left(\Phi^L_{B'}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mathrm{Id}_{p-1} & 0 \\ 0 & 0 & \mathrm{Id}_{p(k-1)} \end{pmatrix}.$$

If there is such an f then note that

$$\begin{pmatrix} -\mathrm{Id}_p & 0 \\ 0 & \mathrm{Id}_{p(k-1)} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mathrm{Id}_{p-1} & 0 \\ 0 & 0 & \mathrm{Id}_{p(k-1)} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \mathrm{Id}_{pk-1} \end{pmatrix},$$

and this matrix is $\Delta(B^{(p)}B')$ since $B' = \tau_p^q$ (with p, q coprime) means that $w(B')$ and p have opposite parity, so in the case that $w(B)$ is odd, $w(B^{(p)}) + w(B')$ is odd. So it remains to prove that there is such an f.

Let p be even and let $\epsilon : \mathcal{A}_p^{ab} \to \mathbb{C}$ be such that $\epsilon(\Phi^L_{B'}) = \Delta(B')$. Define $f(a_{ij}) = (-1)^{j-i}\epsilon(a_{ij})$ and let $i' = \mathrm{perm}(B')(i)$ (this is the puncture that the monomial must start on). Fix i, j and consider a monomial $M = c_{ij}a_{i',j_1}a_{j_1,j_2}\ldots a_{j_m,j}$ in $\left(\Phi^L_{B'}\right)_{ij}$. We have used that such a monomial must arise from a product in the algebra of paths in D that begins at $i' = \mathrm{perm}(B')(i)$ and ends at j.

Now we see that $f(M) = (-1)^{\sum_{n=0}^m (j_{n+1}-j_n)}\epsilon(M) = (-1)^{j-i'}\epsilon(M)$ where $j_0 = i'$ and $j_{m+1} = j$. The power of -1 here is independent of the particular monomial chosen in $\left(\Phi^L_{B'}\right)_{ij}$ and so $f((\Phi^L_{B'})_{ij}) = \pm\epsilon((\Phi^L_{B'})_{ij})$. When $i = j$, the sign is, in fact, negative since the difference $\mathrm{perm}(B')(i) - i \bmod p$ must be invertible in $\mathbb{Z}/p$ since B' closes to a knot (here we used the particular cyclic form of $\mathrm{perm}(\tau_p^q)$). When p is even this means that the difference $i' - j$ is odd.