We want to show that a (p,q) torus knot, $p<q$ has a rank p augmentation ϵ with the property that $\epsilon(a_{ij})=\epsilon(a_{ji})$ for all $1\le i\ne j\le p$. In fact, a much stronger statement is true. Define n by $p/2=n$ if p is even and $(p-1)/2=n$ if p is odd and consider a set of variables $a_1,\dots,a_n$. For $i\ne j$ and $1\le c\le n$, if $c=|i-j| \mod p$ then define $[a_{ij}]=a_c$. On the other hand, if $-|i-j| \mod p=c$ (and $c\ne p/2$) then define $[a_{ij}]=-a_c$. For example, if $p=4$ then $[a_{12}]=[a_{21}]=[a_{23}]=[a_{32}]=[a_{34}]=[a_{43}]=x_1$, $[a_{14}]=[a_{41}]=-x_1$, and $[a_{13}]=[a_{31}]=[a_{24}]=[a_{42}]=x_2$.

Let $\sigma_1,\dots,\sigma_{p-1}$ be the standard generators of B_p and write $\tau_p=\sigma_1\dots\sigma_{p-1}$ so that the closure of τ_p^q is the (p,q) torus link. We note that for any a_{ij} we have that $[\phi_{\tau_p}(a_{ij})]=[a_{ij}]$. Indeed, if $i,j<p$ then $\phi_{\tau_p}(a_{ij})=a_{i+1,j+1}$ and $[a_{ij}]=[a_{i+1,j+1}]=[\phi_{\tau_p}(a_{ij})]$. On the other hand, $[\phi_{\tau_p}(a_{ip})]=[-a_{i+1,1}]=-[a_{i+1,1}]=[a_{ip}]$ since $p-i=-i \mod p$. Also $[\phi_{\tau_p}(a_{pi})]=[a_{pi}]$ since $[a_{ip}]=[a_{pi}]$ and $[a_{i+1,1}]=[a_{1,i+1}]$. Thus ϕ_{τ_p} determines a well-defined map on the quotient algebra (written $[\mathcal{A}_p]$) that is the image of $a_{ij}\mapsto[a_{ij}]=\pm a_{|i-j|}$. In fact, it descends to the identity map on $[\mathcal{A}_p]$. Making everything commutative, note that $[\mathcal{A}_p]$ is a quotient of $\mathcal{A}_p^{ab}$.

We work with the algebra $[\mathcal{A}_p]$, but will write, for example, Φ_B^L though we understand the entries to be the classes in $[\mathcal{A}_p]$ corresponding to entries in the typical matrix of this name over $\mathcal{A}_p$. Note that we have

$$\Phi^L_{\tau_p}=\begin{pmatrix} -a_1 & & & \\ \vdots & & \mathrm{Id}_{p-1} & \\ -a_{p-1} & & & \\ 1 & 0 & \dots & 0\end{pmatrix}.$$

Let us write $Q^k_{ij}(\mathbf{a})$ for the polynomial in the variables $\mathbf{a}=(a_1,\dots,a_{p-1})$ given by the (i,j) entry of $\Phi^L_{\tau_p^k}$ (more precisely the quotient map applied to $\Phi^L_{\tau_p^k}$). Using the factorization $\tau_p^k=\tau_p\tau_p^{k-1}$, the chain rule, that ϕ_{τ_p} descends to the identity map, and the form of $\Phi^L_{\tau_p}$ above, we see that if $j>1$ then $Q^k_{ij}(\mathbf{a})=Q^{k-1}_{i,j-1}(\mathbf{a})$ and that

$$Q^k_{i1}(\mathbf{a})=Q^{k-1}_{ip}(\mathbf{a})-\sum_{j=1}^{p-1}Q^{k-1}_{ij}(\mathbf{a})a_j. \tag{1}$$

We could also use the factorization $\tau_p^{k-1}\tau_p$ of τ_p^k and the chain rule to get that $Q^k_{pj}(\mathbf{a})=Q^{k-1}_{1j}(\mathbf{a})$ and that if $i<p$ then

$$Q^k_{ij}(\mathbf{a})=Q^{k-1}_{i+1,j}(\mathbf{a})-a_iQ^{k-1}_{1j}(\mathbf{a}). \tag{2}$$

We first note that for $j>1$ we have $Q^k_{1j}(\mathbf{a})=Q^{k-1}_{1,j-1}(\mathbf{a})=Q^k_{p,j-1}(\mathbf{a})$. Thus if we find a map that sends $Q^k_{p,j-1}(\mathbf{a})$ to 0 for $1<j\le p$ then by (2) we would have that $Q^k_{i,j-1}(\mathbf{a})$ and $Q^{k-1}_{i+1,j-1}(\mathbf{a})$ have the same image for $1<j\le p$ under this map. Thus, so do $Q^k_{i,j-1}(\mathbf{a})$ and $Q^k_{i+1,j}(\mathbf{a})$ for $1<j\le p$

(by using the equality from the $j > 1$ case before (1). Since we have sent $Q^k_{p,j-1}(\mathbf{a}) = Q^k_{1j}(\mathbf{a})$ to 0 for $1 < j \leq p$ this tells us that all the off-diagonal entries go to zero.

Also, if we send $Q^k_{pp}(\mathbf{a})$ to 1 then this tells us that the diagonal elements Q^k_{ii}, $i > 1$ all go to 1.

The polynomials just along bottom row are simple enough, one should be able to show they have the solutions we need once $k > p$, maybe using a triangular set type argument.