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AUGMENTATION RANK OF SATELLITES WITH BRAID PATTERN

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Abstract. Given a knot K in S^3, a question raised by Cappell and Shaneson asks if the meridional rank of K equals the bridge number of K. Using augmentations in knot contact homology we consider the persistence of equality between these two invariants under satellite operations on K with a braid pattern. In particular, we answer the question in the affirmative for a large class of iterated torus knots.

1. Introduction

Let K be an oriented knot in S^3 and denote by π_K the fundamental group of its complement $\overline{S^3 \setminus n(K)}$, with some basepoint. We call an element of π_K a *meridian* if it is represented by the oriented boundary of a disc, embedded in S^3, whose interior intersects K positively once. The group π_K is generated by meridians; the *meridional rank* of K, written $\mathrm{mr}(K)$, is the minimal size of a generating set containing only meridians.

Choose a height function $h : S^3 \to \mathbb{R}$. The *bridge number* of K, denoted $b(K)$, is the minimum of the number of local maxima of $h|_{\varphi(S^1)}$ among $\varphi : S^1 \to S^3$ which realize K.

By considering Wirtinger's presentation of π_K one can show that $\mathrm{mr}(K) \leq b(K)$ for any $K \subset S^3$. Whether the bound is equality for all knots is an open question attibuted to Cappell and Shaneson [Kir95, Prob. 1.11]. Equality is known to hold for some families of knots due to work of various authors ([BZ85, Cor13b, RZ87]).

Here we study *augmentations* of K, which are maps that arise in the study of knot contact homology. To each augmentation is associated a rank and there is a maximal rank of augmentations of a given K, called the *augmentation rank* $\mathrm{ar}(K)$. For any K the inequality $\mathrm{ar}(K) \leq \mathrm{mr}(K)$ holds (see Section 3.2). We discuss the behavior of $\mathrm{ar}(K)$ under certain satellite operations with a braid pattern.

To be precise, denote the group of braids on n strands by B_n and write $\hat{\beta}$ for the *braid closure* of a braid β (see Section 3, Figure 3). We write $\imath_n$ for the identity in B_n.

Throughout the paper we let $\alpha \in B_k$ and $\gamma \in B_p$ and set $K = \hat{\alpha}$. Note that $\mathrm{ar}(K) \leq k$.

Definition 1.1. Let $\imath_p(\alpha)$ be the braid in B_{kp} obtained by replacing each strand of α by p parallel copies (in the blackboard framing). Let $\bar{\gamma}$ be the inclusion of γ into B_{kp} by the map $\sigma_i \mapsto \sigma_i, 1 \le i \le p-1$. Set $\gamma(\alpha) = \iota_p(\alpha)\bar{\gamma}$. The *braid satellite* of K associated to α, γ is defined as $K(\alpha,\gamma) = \widehat{\gamma(\alpha)}$.

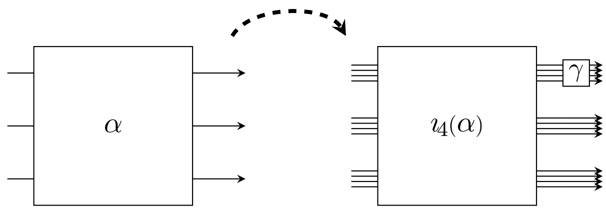

FIGURE 1. Constructing $\gamma(\alpha)$ from α; case $p = 4$.

As defined $K(\alpha,\gamma)$ depends on the choice of α – in general $\widehat{\gamma(\alpha)} \ne \hat{\gamma}(\hat{\alpha})$. However, the construction is more intrinsic if we require the index k of α to be minimal among braid representatives of K (see Section 2).

Note that if $\hat{\alpha}$ and $\hat{\gamma}$ are each a knot, $K(\alpha,\gamma)$ is also. Our principal result is the following.

Theorem 1.2. *If $\alpha \in B_k$ and $\gamma \in B_p$ are such that $\mathrm{ar}(\hat{\alpha}) = k$ and $\mathrm{ar}(\hat{\gamma}) = p$, then $\mathrm{ar}(K(\alpha,\gamma)) = kp$.*

A corollary of Theorem 1.2 involves Cappell and Shaneson's question for iterated torus knots. Let $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$ be integral vectors. We write $T(\mathbf{p},\mathbf{q})$ for the $(\mathbf{p},\mathbf{q})$ *iterated torus knot*, defined as follows. Define $T(\mathbf{p},\mathbf{q})$ inductively so that, if $\hat{\mathbf{p}}, \hat{\mathbf{q}}$ are the truncated lists obtained from $\mathbf{p},\mathbf{q}$ by removing the last integer in each, then $T(\mathbf{p},\mathbf{q})$ is the (p_n, q_n)-cable of $T(\hat{\mathbf{p}},\hat{\mathbf{q}})$. By convention $T(\emptyset,\emptyset)$ is the unknot.

The knot $T(\mathbf{p},\mathbf{q})$ is well-defined when a framing convention is chosen at each stage of cabling. In contrast to the traditional choice of Seifert framing at each stage, we choose a framing so that if α is of minimal braid index such that $T(\hat{\mathbf{p}},\hat{\mathbf{q}}) = \hat{\alpha}$ then $T(\mathbf{p},\mathbf{q}) = K(\alpha, (\sigma_1 \dots \sigma_{p_n-1})^{q_n})$.

Corollary 1.3. *Given integral vectors $\mathbf{p}$ and $\mathbf{q}$, suppose that $|p_i| < |q_i|$ and $gcd(p_i, q_i) = 1$ for each $1 \le i \le n$. Then*

$$\mathrm{ar}(T(\mathbf{p},\mathbf{q})) = \mathrm{mr}(T(\mathbf{p},\mathbf{q})) = b(T(\mathbf{p},\mathbf{q})) = p_1 p_2 \dots p_n.$$

Our assumption that $|p_i| < |q_i|$ in Corollary 1.3 is needed for the hypothesis of Theorem 1.2 that the associated braids have closures with augmentation rank equal to the braid index. This requirement is not a deficiency of our techniques; there are families of cables of $(n, n+1)$ torus knots which do not attain the large augmentation rank in Corollary 1.3.

Theorem 1.4. *For $p > 1$ and $n > 1$, let $K = T((n,p),(n+1,1))$. Then $\mathrm{ar}(K) < np$.*

It is natural to wonder if the augmentation rank is multiplicative under weaker assumptions on α, γ than those in Theorem 1.2. The following is a possible generalization.

Conjecture 1.5. *Suppose $K = \hat{\alpha}$ for $\alpha \in B_k$, and that α has minimal index among braids with the same closure. Let $\gamma \in B_p$. Then* $\mathrm{ar}(K(\alpha,\gamma)) \geq \mathrm{ar}(\hat{\alpha})\,\mathrm{ar}(\hat{\gamma})$.

Remark 1.6. There are examples when the inequality of Conjecture 1.5 is strict (see Section 5).

The paper is organized as follows. In Section 3 we give the needed background in knot contact homology, specifically Ng's cord algebra, and discuss augmentation rank and the relationship to meridional rank. Section 3.3 reviews of techniques to be used in the proof of Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.2, its requisite supporting lemmas, and Corollary 1.3. Finally, Section 5 considers the sharpness of our results. We prove Theorem 1.4 and briefly discuss the more general case, Conjecture 1.5.

2. Satellite operators and the braid satellite

Definition 1.1 of the braid satellite $K(\alpha,\gamma)$ produces a satellite of $\hat{\alpha}$. As defined, the resulting satellite depends on the braid representative of $\hat{\alpha}$. We remark here how to avoid this ambiguity.

A tubular neighborhood of a null-homologous knot K in a 3-manifold has a standard identification with $S^1 \times D^2$ determined by an oriented Seifert surface that K bounds. Given a knot $P \subset S^1 \times D^2$, as per the usual convention, let $P(K)$ be the satellite of K with pattern P obtained with this framing.

Proposition 2.1. *Given a knot K and a braid $\gamma \in B_p$, let ω be the writhe of a minimal index closed braid representing K. Let $P \subset S^1 \times D^2$ be the braid closure of $\Delta^{2\omega}\gamma$, where Δ^2 is the full twist in B_p. Then $K(\alpha,\gamma) = P(K)$ for any minimal index braid α with $K = \hat{\alpha}$.*

Proof. The principal observation is that, since the Jones conjecture holds [DP13, LM13], the writhe of α must be ω. Thus the blackboard framing of the closure of $\imath_p(\alpha)\Delta^{-2\omega}$ agrees with the $(p,0)$-cable of K (with Seifert framing). □

We note, the satellite $T(\mathbf{p},\mathbf{q})$ corresponds to the $(p_n, p_n\omega_n + q_n)$-cable of $T(\hat{\mathbf{p}},\hat{\mathbf{q}})$, where ω_n is defined inductively by $\omega_n = p_{n-1}\omega_{n-1} + (p_{n-1}-1)q_{n-1}$ and $\omega_1 = 0$.

3. Background

We review in Section 3.1 the construction of $HC_0(K)$ from the viewpoint of the combinatorial knot DGA, which was first defined in [Ng08]; our conventions are those given in [Ng12]. In Section 3.2 we discuss augmentations in knot contact homology and their rank, which gives a lower bound on the meridional rank of the knot group. Section 3.3 contains a discussion of techniques from [Cor13a] that we use to calculate the augmentation rank.

Throughout the paper we orient n-braids in B_n from left to right, labeling the strands $1, \dots, n$, with 1 the topmost and n the bottommost strand. We work with Artin's generators $\{\sigma_i^{\pm}, i = 1, \dots, n-1\}$ of B_n, where in σ_i only the i and $i+1$ strands interact, and they cross once in the manner depicted in Figure 2. Given a braid $\beta \in B_n$, the braid closure $\hat{\beta}$ of β is the link obtained

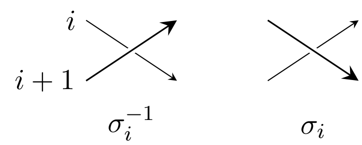

FIGURE 2. Generators of B_n

as shown in Figure 3. The *writhe* (or algebraic length) of β, denoted $\omega(\beta)$, is the sum of exponents of the Artin generators in a word representing β.

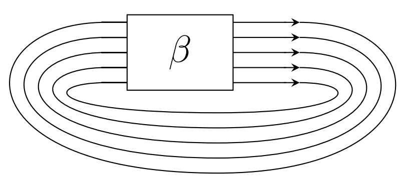

FIGURE 3. The braid closure of β

3.1. **Knot contact homology.** We review the construction of the combinatorial knot DGA of Ng (in fact, we discuss only the degree zero part as this will suffice for our purposes). This DGA was defined in order to be a calculation of knot contact homology and was shown to be so in [EENS13] (see [Ng12] for more details). Let $\mathcal{A}_n$ be the noncommutative unital algebra over $\mathbb{Z}$ freely generated by a_{ij}, $1 \le i \ne j \le n$. We define a homomorphism $\phi : B_n \to \operatorname{Aut} \mathcal{A}_n$ by defining it on the generators of B_n:

$$\phi_{\sigma_k} : \begin{cases} a_{ij} \mapsto a_{ij} & i, j \ne k, k+1 \\ a_{k+1,i} \mapsto a_{ki} & i \ne k, k+1 \\ a_{i,k+1} \mapsto a_{ik} & i \ne k, k+1 \\ a_{k,k+1} \mapsto -a_{k+1,k} & \\ a_{k+1,k} \mapsto -a_{k,k+1} & \\ a_{ki} \mapsto a_{k+1,i} - a_{k+1,k}a_{ki} & i \ne k, k+1 \\ a_{ik} \mapsto a_{i,k+1} - a_{ik}a_{k,k+1} & i \ne k, k+1 \end{cases} \tag{1}$$

Let $\iota \colon B_n \to B_{n+1}$ be the inclusion $\sigma_i \mapsto \sigma_i$ so that the $(n+1)$ strand does not interact with those from $\beta \in B_n$, and define $\phi_\beta^* \in \operatorname{Aut} \mathcal{A}_{n+1}$ by $\phi_\beta^* = \phi_\beta \circ \iota$. We then define the $n \times n$ matrices Φ_β^L and Φ_β^R with entries in $\mathcal{A}_n$ by

$$\phi_\beta^*(a_{i,n+1}) = \sum_{j=1}^{n} (\Phi_\beta^L)_{ij} a_{j,n+1}$$

$$\phi_\beta^*(a_{n+1,i}) = \sum_{j=1}^{n} a_{n+1,j} (\Phi_\beta^R)_{ji}$$

Finally, let R_0 be the Laurent polynomial ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ and define matrices $\mathbf{A}$ and $\mathbf{\Lambda}$ over R_0 by

$$\mathbf{A}_{ij} = \begin{cases} a_{ij} & i < j \\ -\mu a_{ij} & i > j \\ 1 - \mu & i = j \end{cases} \tag{2}$$

$$\mathbf{\Lambda} = \mathrm{diag}[\lambda \mu^{\omega(\beta)}, 1, \ldots, 1]. \tag{3}$$

Definition 3.1. Suppose that K is the closure of $\beta \in B_n$. Define $\mathcal{I} \subset \mathcal{A}_n \otimes R_0$ to be the ideal generated by the entries of $\mathbf{A} - \mathbf{\Lambda} \cdot \mathbf{\Phi}_\beta^{\mathbf{L}} \cdot \mathbf{A}$ and $\mathbf{A} - \mathbf{A} \cdot \mathbf{\Phi}_\beta^{\mathbf{R}} \cdot \mathbf{\Lambda}^{\mathbf{-1}}$. The *degree zero homology of the combinatorial knot DGA* is $\mathrm{HC}_0(K) = (\mathcal{A}_n \otimes R_0)/\mathcal{I}$.

It was shown in [Ng08] that the isomorphism class of $HC_0(K)$ is unchanged by the Markov moves on β, hence $HC_0(K)$ is an invariant of the knot K. We only consider $HC_0(K)$ here, but there is a larger invariant, the differential graded algebra discussed in [Ng12].

The proofs in Section 4 require a number of computations of ϕ_β for particular braids β. Such computations are benefited by an alternate description of the map ϕ_β, which we now describe and will use in Section 4.

Let D be a flat disk, to the right of β, with n points (punctures) where it intersects $K = \hat{\beta}$ (see Figure 4). We assume the n punctures of D to be collinear, on a line that separates D into upper and lower half-disks. Denote by c_{ij} the isotopy class (fixing endpoints) of a path that is contained in the upper half-disk of D, with initial endpoint on the i^{th} strand and terminal endpoint on the j^{th} strand.

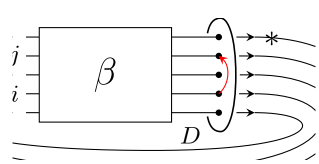

FIGURE 4. Cord c_{ij} of $K = \hat{\beta}$

Consider β as a mapping class and let $\beta \cdot c_{ij}$ denote the isotopy class of the path to which c_{ij} is sent. If D, as viewed from the left (as pictured), is oriented as the plane then σ_k acts by rotating the k- and $(k+1)$-punctures an angle of π about their midpoint in counter-clockwise fashion. Consider the *algebra of paths* over $\mathbb{Z}$ generated by isotopy classes of paths in D with endpoints on punctures, modulo the relation in Figure 5 (paths depicted there are understood to agree outside the neighborhood of the puncture shown). It was shown in [Ng05] that the algebra map to $\mathcal{A}_n$ defined by $F(c_{ij}) = a_{ij}$ if $i < j$, and $F(c_{ij}) = -a_{ij}$ if $i > j$ satisfies $F(\beta \cdot c_{ij}) = \phi_\beta(F(c_{ij}))$.

FIGURE 5. Relation in the algebra of paths

Let perm : $B_n \to S_n$ denote the homomorphism from B_n to the symmetric group sending σ_k to the simple transposition interchanging $k, k+1$. We will make use of the following.

Lemma 3.2. *For some $\beta \in B_n$ and $1 \leq i \neq j \leq n$, consider $(\Phi_\beta^L)_{ij} \in \mathcal{A}_n$ as a polynomial expression in the (non-commuting) variables $\{a_{kl}, 1 \leq k \neq l \leq n\}$. Writing $i' = perm(\beta)(i)$, every monomial in $(\Phi_\beta^L)_{ij}$ is a constant times $a_{i'i_1}a_{i_1i_2}\dots a_{i_{l-1},j}$ for some $l \geq 0$, the monomial being a constant if $l = 0$ and only if $i' = j$.*

Proof. Include β in B_{n+1} and consider the isotopy class of the path $\beta \cdot c_{i,n+1}$ which begins at i' and ends at $n+1$ (as β does not interact with the $(n+1)$ strand. Applying the relation in Figure 5 to the path equates it with a sum (or difference) of another path with the same endpoints and a product of two paths, the first beginning at i' and the other ending at $n+1$. A finite number of applications of this relation allows one to express the path as a polynomial in the $c_{kl}, 1 \leq k \neq l \leq n$ where each monomial has the form $c_{i'i_1}\dots c_{i_{l-1},j}$. The result then follows from the fact that $\phi_\beta(a_{i,n+1}) = \phi_\beta(F(c_{i,n+1})) = F(\beta \cdot c_{i,n+1})$. □

3.2. **Augmentations and augmentation rank.** Let S be a ring with 1, and consider it a differential graded algebra supported in grading 0, with trivial differential. Augmentations of $(\mathcal{A}, \partial)$ are DGA maps $(\mathcal{A}, \partial) \to (S, 0)$. For our setting, if $\beta \in B_n$ is a braid representative of K, such a map corresponds precisely to a homomorphism $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ such that ϵ sends each generator of $\mathcal{I}$ to zero (see Definition 3.1).

Definition 3.3. Suppose that K is the closure of $\beta \in B_n$. An *augmentation* of K is a homomorphism $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ such that each element of $\mathcal{I}$ is sent by ϵ to zero.

A correspondence between augmentations and certain representations of the knot group π_K were studied in [Cor13a]. Recall that π_K is generated by meridians. Fix some meridian m and generate π_K by conjugates of m.

Definition 3.4. For any integer $r \geq 1$, a homomorphism $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ is a *KCH representation* if $\rho(m)$ is diagonalizable and has an eigenvalue of 1 with multiplicity $r-1$. We call ρ a *KCH irrep* if it is irreducible.

In [Ng08], Ng describes an isomorphism between $HC_0(K)$ and an algebra constructed from elements of π_K. As discussed in [Ng12] a KCH representation $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ induces an augmentation ϵ_ρ of K. Given an augementation, the first author showed how to construct a KCH representation that induces it. In fact, we have the following rephrasing of results from [Cor13a].

Theorem 3.5 ([Cor13a]). *Let $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ be an augmentation with $\epsilon(\mu) \neq 1$. There is a KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$. Furthermore, for any KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$, the rank of $\epsilon(\mathbf{A})$ equals r.*

Considering Theorem 3.5 we make the following definition.

Definition 3.6. The *rank* of an augmentation $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ with $\epsilon(\mu) \neq 1$ is the rank of $\epsilon(\mathbf{A})$. Given a knot K, the *augmentation rank of K*, denoted $\mathrm{ar}(K)$, is the maximum rank among augmentations of K.

Remark 3.7. The augmentation rank can be defined for target rings other than $\mathbb{C}$, but this paper only considers augmentations as in Definition 3.3.

It is the case that $\mathrm{ar}(K)$ is well-defined. That is, given K there is a bound on the maximal rank of an augmentation of K.

Theorem 3.8 ([Cor13b]). *Given a knot $K \subset S^3$, if $g_1, \ldots, g_d$ are meridians that generate π_K and $\rho : \pi_K \to GL_r\mathbb{C}$ is a KCH irrep then $r \leq d$.*

As in the introduction, if we denote the meridional rank of π_K by $\mathrm{mr}(K)$, then Theorem 3.8 implies that $\mathrm{ar}(K) \leq \mathrm{mr}(K)$. In addition, the geometric quantity $b(K)$ called the bridge index of K is never less than $\mathrm{mr}(K)$. Thus we have the following corollary.

Corollary 3.9 ([Cor13b]). *Given a knot $K \subset S^3$,*

$$ar(K) \leq mr(K) \leq b(K)$$

As a result, to verify for K that $\mathrm{mr}(K) = b(K)$ it suffices to find an augmentation of K with rank equal to $b(K)$. As we discuss in the next section, we will concern ourselves in this paper with a setting where $\mathrm{ar}(K) = n$ and there is a braid $\beta \in B_n$ which closes to K. This is a special situation, since $b(K)$ is strictly less than the braid index for many knots.

3.3. **Finding augmentations.** The following theorem concerns the behavior of the matrices Φ^L_β and Φ^R_β under the product in B_n. It is an essential tool for studying $HC_0(K)$ and will be central to our arguments.

Theorem 3.10 ([Ng05], Chain Rule)**.** *Let β_1, β_2 be braids in B_n. Then $\Phi^L_{\beta_1\beta_2} = \phi_{\beta_1}(\Phi^L_{\beta_2}) \cdot \Phi^L_{\beta_1}$ and $\Phi^R_{\beta_1\beta_2} = \Phi^R_{\beta_1} \cdot \phi_{\beta_1}(\Phi^R_{\beta_2})$.*

The main result of this paper concerns augmentations with rank equal to the braid index of the knot K. Define the diagonal matrix $\Delta(\beta) = \mathrm{diag}[(-1)^{w(\beta)}, 1, \ldots, 1]$. The following statement follows from results in [Cor13a, Section 5].

Theorem 3.11 ([Cor13a])**.** *If K is the closure of $\beta \in B_n$ and has a rank n augmentation $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$, then*

$$\epsilon(\Phi^L_\beta) = \Delta(\beta) = \epsilon(\Phi^R_\beta). \tag{4}$$

Furthermore, any homomorphism $\epsilon : \mathcal{A}_n \to \mathbb{C}$ which satisfies (4) can be extended to $\mathcal{A}_n \otimes R_0$ to produce a rank n augmentation of K.

The proof of Theorem 1.2 relies on this characterization of rank n augmentations. That is, given $\alpha \in B_k$ with a rank k augmentation $\epsilon : \mathcal{A}_k \to \mathbb{C}$, and $\gamma \in B_p$ with a rank p augmentation $\epsilon' : \mathcal{A}_p \to \mathbb{C}$ we show that $K(\alpha, \gamma)$ has a rank kp augmentation by defining a homomorphism $\psi : \mathcal{A}_{kp} \to \mathcal{A}_k \otimes \mathcal{A}_p$ so that $(\epsilon \otimes \epsilon') \circ \psi$ satisfies (4) for the braid $\gamma(\alpha)$.

There is a symmetry on the matrices Φ^L_β and Φ^R_β that is relevant to the study of augmentations in this setting. Define an involution $x \mapsto \overline{x}$ on $\mathcal{A}_n$ (termed *conjugation*) as follows: first set $\overline{a_{ij}} = a_{ji}$; then, for any $x, y \in \mathcal{A}_n$, define $\overline{xy} = \overline{y}\,\overline{x}$ and extend the operation linearly to $\mathcal{A}_n$. We have the following symmetry.

Theorem 3.12 ([Ng05], Prop. 6.2)**.** *For a matrix of elements in $\mathcal{A}_n$, let $\overline{M}$ be the matrix such that $\left(\overline{M}\right)_{ij} = \overline{M_{ij}}$. Then for $\beta \in B_n$, Φ^R_β is the transpose of $\overline{\Phi^L_\beta}$.*

4. Main Result

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proof of Corollary 1.3

We begin in Section 4.1 with an intermediate result, Proposition 4.1, followed by our main theorem. In Section 4.2 we prove Lemma 4.2, which is used to prove Proposition 4.1, as well as Lemma 4.3, which is used to prove Lemma 4.2.

4.1. **Proof of main result.**

In this section we prove our main result, which we now recall. The notation of Theorem 1.2 will be used throughout Section 4.

Theorem 1.2. If $\alpha \in B_k$ and $\gamma \in B_p$ are such that $\mathrm{ar}(\hat{\alpha}) = k$ and $\mathrm{ar}(\hat{\gamma}) = p$, then $\mathrm{ar}(K(\alpha, \gamma)) = kp$.

Theorem 1.2 is proved by defining an algebra map $\psi\colon \mathcal{A}_{pk} \to \mathcal{A}_k \otimes \mathcal{A}_p$ such that $\psi(\Phi^L_{\alpha\bar{\gamma}})$ factors suitably across the tensor product. This is the content of Proposition 4.1, which follows from Lemma 4.2, which in turn relies on a calculation (Lemma 4.3) of $\phi^{\pm 1}_{\imath_p(\sigma_n)}(a_{ij})$.

For $1 \leq i \leq kp$, write $i = q_i p + r_i$, where $1 \leq r_i \leq p$ and $0 \leq q_i \leq k-1$. For each generator $a_{ij}, 1 \leq i \neq j \leq kp$, define

$$\psi(a_{ij}) = \begin{cases} 1 \otimes a_{r_i r_j} & : q_i = q_j \\ a_{q_i+1, q_j+1} \otimes 1 & : r_i = r_j \\ 0 & : (q_i - q_j)(r_i - r_j) < 0 \\ a_{q_i+1, q_j+1} \otimes a_{r_i r_j} & : (q_i - q_j)(r_i - r_j) > 0 \end{cases}, \tag{5}$$

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which determines an algebra map $\psi \colon \mathcal{A}_{pk} \to \mathcal{A}_k \otimes \mathcal{A}_p$. Note that if $\psi(a_{ij}) \in 1 \otimes \mathcal{A}_p$ then $q_i = q_j$ or $a_{ij} \in \ker \psi$. Also, if we extend conjugation to $\mathcal{A}_k \otimes \mathcal{A}_p$ by applying it to each factor, then $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$. We have the following proposition.

Proposition 4.1. *For any braid α, $\psi\left(\Phi^L_{\imath_p(\alpha)}\right) = \Phi^L_\alpha \otimes I_p$ and $\psi\left(\Phi^R_{\imath_p(\alpha)}\right) = \Phi^R_\alpha \otimes I_p$*

say more here

Note that by $\Phi^L_\alpha \otimes I_p$ we mean the Kronecker product of Φ^L_α and I_p, or equivalently their tensor product as $\mathcal{A}_k$- and $\mathcal{A}_p$-linear automorphisms of $\mathcal{A}_{k+1}$ and $\mathcal{A}_{p+1}$, respectively.

The proof of Proposition 4.1 relies heavily on the following technical lemma, whose proof we delay until Section 4.2.

Lemma 4.2. $\psi(\phi_{\imath_p(\alpha)}(a_{ij})) = (\phi_\alpha \otimes \mathrm{id})(\psi(a_{ij}))$ *for any $\alpha \in B_k$*

Proof of Proposition 4.1. Fix $\alpha \in B_{kp}$ and $1 \leq i \leq kp$. By the definitions of Φ^L_α and $\Phi^L_{\imath_p(\alpha)}$ and Lemma 4.2 we have that

$$\begin{aligned} \left(\sum_{l=1}^{k} \left(\Phi^L_\alpha\right)_{q_i+1,l} a_{l*}\right) \otimes a_{r_i *} &= (\phi_\alpha \otimes \mathrm{id})\, \Psi\left(a_{i*}\right) \\ &= \Psi\left(\phi_{\imath_p(\alpha)}\left(a_{i*}\right)\right) \\ &= \sum_{j=1}^{kp} \Psi\left(\left(\Phi^L_{\imath_p(\alpha)}\right)_{ij}\right)\left(a_{q_j+1,*} \otimes a_{r_j *}\right) \end{aligned}$$

Thus we have that $\Psi\left(\left(\Phi^L_{\imath_p(\alpha)}\right)_{ij}\right) = 0$ if $r_i \neq r_j$ and $\Psi\left(\left(\Phi^L_{\imath_p(\alpha)}\right)_{ij}\right) = \left(\Phi^L_\alpha\right)_{q_i+1,q_j+1}$ if $r_i = r_j$, so $\Psi\left(\Phi^L_{\imath_p(\alpha)}\right) = \Phi^L_\alpha \otimes I_p$. Since $\Phi^R_\alpha = \overline{\Phi^L_\alpha}^t$ and $\Psi(\overline{a_{ij}}) = \overline{\Psi(a_{ij})}$, we then have that $\Psi\left(\Phi^R_{\imath_p(\alpha)}\right) = \Phi^R_\alpha \otimes I_p$ as well. $\square$

Proof of Theorem 1.2. By Theorem 3.11 there exist augmentations $\epsilon_k \colon \mathcal{A}_k \otimes R_0 \to \mathbb{C}$ and $\epsilon_p \colon \mathcal{A}_p \otimes R_0 \to \mathbb{C}$, for the closures of α, γ respectively, such that $\epsilon_k\left(\Phi^L_\alpha\right) = \epsilon_k\left(\Phi^R_\alpha\right) = \Delta(\alpha)$ and $\epsilon_p\left(\Phi^L_\gamma\right) = \epsilon_p\left(\Phi^R_\gamma\right) = \Delta(\gamma)$. Theorem 3.11 also implies that it suffices to prove that there exists an augmentation $\epsilon\colon \mathcal{A}_{pk}\otimes R_0 \to \mathbb{C}$ such that $\epsilon\left(\Phi^L_{\gamma(\alpha)}\right) = \epsilon\left(\Phi^R_{\gamma(\alpha)}\right) = \Delta(\gamma(\alpha))$.

Below we will define a homomorphism $\delta\colon \mathcal{A}_p \to \mathbb{C}$ such that for each generator a_{ij} we have $\delta(a_{ij}) = \pm\epsilon_p(a_{ij})$, the sign depending on the parity of $w(\alpha)$ and p. Let $\pi\colon \mathbb{C}\otimes\mathbb{C}\to\mathbb{C}$ be the multiplication $a\otimes b\mapsto ab$. Our desired map is defined by $\epsilon = \pi\circ(\epsilon_k\otimes\delta)\circ\psi$.

The Chain Rule theorem gives that

$$\pi\circ(\epsilon_k\otimes\delta)\circ\psi\left(\Phi^L_{\gamma(\alpha)}\right) = \pi\circ(\epsilon_k\otimes\delta)\psi\left(\phi_{\imath_p(\alpha)}\left(\Phi^L_{\bar\gamma}\right)\right)\psi\left(\Phi^L_{\imath_p(\alpha)}\right) \tag{6}$$

Consider the homomorphism $\phi_{\imath_p(\alpha)}$ through the description given in Section 3.1. As each $a_{ij}, 1\le i\ne j\le p$, is represented up to sign by the isotopy class c_{ij}, and the leftmost p punctures are moved as one block by the action of $\imath_p(\alpha)$ on D, there is an $0\le m<k$ so that $\phi_{\imath_p(\alpha)}(a_{ij}) = a_{i+mp,j+mp}$ for each i,j in this range. As $\psi(a_{i+mp,j+mp}) = 1\otimes a_{ij}$,

specify this description better somehow

$$\psi\left(\phi_{\imath_p(\alpha)}\left(\Phi^L_{\bar\gamma}\right)\right) = \left(1\otimes\left(\Phi^L_{\bar\gamma}\right)_{ij}\right)$$

Note that while the entries of $\Phi^L_{\bar\gamma}$ are elements of $\mathcal{A}_{kp}$, all of them lie in the image of the natural inclusion of $\mathcal{A}_p$ into $\mathcal{A}_{kp}$, so we regard the entries of the matrix on the right hand side as elements of $\mathcal{A}_k\otimes\mathcal{A}_p$. By Proposition 4.1, we have that

$$\psi\left(\Phi^L_{\imath_p(\alpha)}\right) = \Phi^L_\alpha\otimes I_p$$

Returning to the right hand side of (6)

$$\begin{aligned}\pi\circ(\epsilon_k\otimes\delta)\left(\psi\left(\phi_{\imath_p(\alpha)}\left(\Phi^L_{\bar\gamma}\right)\right)\psi\left(\Phi^L_{\imath_p(\alpha)}\right)\right) &= \pi\circ(\epsilon_k\otimes\delta)\left(\left(1\otimes\left(\Phi^L_{\bar\gamma}\right)_{ij}\right)\left(\Phi^L_\alpha\otimes I_p\right)\right)\\ &= \delta\left(\Phi^L_{\bar\gamma}\right)\left(\Delta(\alpha)\otimes I_p\right).\end{aligned}$$

We are done if δ may be defined so that $\delta\left(\Phi^L_{\bar\gamma}\right)\left(\Delta(\alpha)\otimes I_p\right) = \Delta(\gamma(\alpha))$. When $w(\alpha)$ is even $w(\imath_p(\alpha))$ is also, and further $\Delta(\alpha) = I_k$. Letting $\delta = \epsilon_p$ makes

$$\delta\left(\Phi^L_{\bar\gamma}\right)\left(\Delta(\alpha)\otimes I_p\right) = \epsilon_p\left(\Phi^L_{\bar\gamma}\right) = \Delta(\bar\gamma) = \Delta(\gamma(\alpha)).$$

Suppose that $w(\alpha)$ is odd. We define $g\colon\{1,\dots,p\}\to\{\pm1\}$ as follows. Let $x_1 = 1$, and $x_l = \mathrm{perm}(\bar\gamma)(x_{l-1})$ for $1<l\le p$. Since the first p strands of $\bar\gamma$ close to a knot, $\mathrm{perm}(\bar\gamma)$ is given by the p-cycle $(x_1x_2\dots x_p)$. If p is even, we let $g(x_1) = 1$, and $g(x_l) = -g(x_{l-1})$ for $1<l\le p$. If p is odd, let $g(x_1) = g(x_2) = 1$ and $g(x_l) = -g(x_{l-1})$ for $2<l\le p$.

Define $\delta:\mathcal{A}_p\to\mathbb{C}$ by setting $\delta(a_{ij}) = g(i)g(j)\epsilon_p(a_{ij})$ for $1\le i\ne j\le p$. Fix i,j and consider a monomial M in $\left(\Phi^L_{\bar\gamma}\right)_{ij}$, which is constant if $i>p$ or $j>p$ so that $\delta(M)$ is defined. If $i,j\le p$, then writing $i' = \mathrm{perm}(\bar\gamma)(i)$, Proposition 3.2 implies $M = c_{ij}a_{i',j_1}a_{j_1,j_2}\dots a_{j_m,j}$ for some $j_1,\dots j_m\in\{1,\dots,p\}$, possibly being constant only if $i' = j$, implying that

$$\delta(M) = g(i')g(j)\left(\prod_{k=1}^{m} g(j_k)^2\right)\epsilon_p(M) = g(i')g(j)\epsilon_p(M).$$

Note, when M is constant then $\delta(M) = M = g(i')g(j)\epsilon_p(M)$ since $i' = j$. Since this holds for each monomial, we have that

$$\delta\left(\left(\Phi^L_{\bar\gamma}\right)_{ij}\right) = g(i')g(j)\epsilon_p\left(\left(\Phi^L_{\bar\gamma}\right)_{ij}\right).$$

When p is even, $w(\imath_p(\alpha))$ is also even and so the opposite parity of $w(\alpha)$. Our definition of g gives $\delta\left(\left(\Phi^L_{\bar\gamma}\right)_{ii}\right) = -\epsilon\left(\left(\Phi^L_{\bar\gamma}\right)_{ii}\right)$ for $i \le p$. Thus

$$\delta\left(\Phi^L_{\bar\gamma}\right) = \begin{pmatrix} (-1)^{w(\bar\gamma)+1} & 0 & 0 \\ 0 & -I_{p-1} & 0 \\ 0 & 0 & I_{(k-1)p} \end{pmatrix}$$

and therefore

$$\delta\left(\Phi^L_{\bar\gamma}\right)(\Delta(\alpha)\otimes I_p) = \mathrm{diag}[(-1)^{w(\alpha)+w(\bar\gamma)+1}, 1\dots 1] = \Delta(\gamma(\alpha))$$

as desired.

When p is odd, $w(\imath_p(\alpha))$ is odd and therefore the same parity of $w(\alpha)$. Our definition of g gives that $\delta\left(\left(\Phi^L_{\bar\gamma}\right)_{11}\right) = \epsilon\left(\left(\Phi^L_{\bar\gamma}\right)_{11}\right)$ and $\delta\left(\left(\Phi^L_{\bar\gamma}\right)_{ii}\right) = -\epsilon\left(\left(\Phi^L_{\bar\gamma}\right)_{ii}\right)$ for $1 < i \le p$, so

$$\delta\left(\Phi^L_{\bar\gamma}\right) = \begin{pmatrix} (-1)^{w(\bar\gamma)} & 0 & 0 \\ 0 & -I_{p-1} & 0 \\ 0 & 0 & I_{(k-1)p} \end{pmatrix}$$

and therefore

$$\delta\left(\Phi^L_{\bar\gamma}\right)(\Delta(\alpha)\otimes I_p) = \mathrm{diag}[(-1)^{w(\alpha)+w(\bar\gamma)}, 1\dots 1] = \Delta(\gamma(\alpha))$$

as desired.

There is little difference in the proof that $\epsilon(\Phi^R_{\gamma(\alpha)}) = \Delta(\gamma(\alpha))$, except that monomials in $(\Phi^R_{\bar\gamma})_{ij}$ are of the form $c_{ij}a_{i,j_1}a_{j_1,j_2}\cdots a_{j_k,j'}$ where $j' = \mathrm{perm}(\bar\gamma)(j)$. Applying Theorem 3.11 now completes the proof. □

4.2. **Supporting Lemmas.** We now prove Lemma 4.2. Figure 6 demonstrates an example for Lemma 4.2, showing that $\psi(\phi_{\imath_2(\sigma_2)}(a_{24})) = \phi_{\sigma_2}\otimes \mathrm{id}(\psi(a_{24}))$. Note that in the figure we condense elements such as $a_{13}\otimes 1$ to a_{13} and include products of algebra elements on a single set of points in order to make the notation cleaner.

In the proof of Lemma 4.2, we will make use of some calculations of $\phi_\alpha(a_{ij})$ for simple braids α. Define $\tau_{m,l} = \sigma_m\sigma_{m+1}\cdots\sigma_{m+l-1}$. We leave it

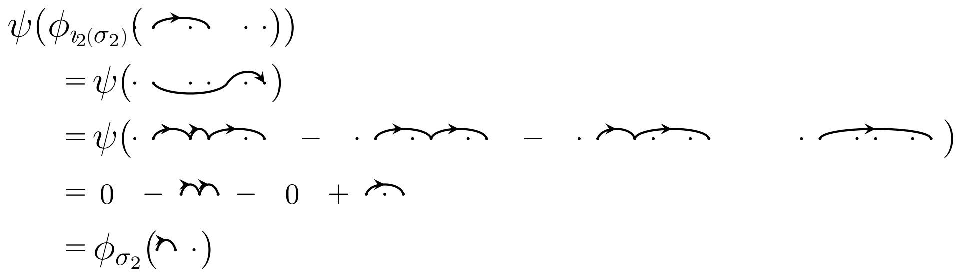


FIGURE 6. Computing $\psi(\phi_{\iota_p(\sigma_2)}(a_{24}))$

to the reader to check that for all $1 \leq m < n$, $1 \leq l \leq n-m$, $i < j$

$$\phi_{\tau_{m,l}}(a_{ij}) = \begin{cases} a_{i+1,j+1} & : m \leq i < j < m+l \\ a_{i-l,j} & : m < m+l = i < j \\ a_{i,j-l} & : i < m < m+l = j \\ -a_{i+1,j-l} & : m \leq i < j = m+l \\ a_{i,j+1} - a_{i,m}a_{m,j+1} & : i < m \leq j < m+l \\ a_{i+1,j} - a_{i+1,m}a_{m,j} & : m \leq i < m+l < j \\ a_{ij} & : \text{otherwise} \end{cases} \tag{7}$$

We also make the following definitions. Let $W, X \subseteq \{1, \ldots, n\}$, and write the elements of a subset $Y \subseteq X$ as $y_1 < \ldots < y_k$. Define

$$A(i,j,X) = \sum_{Y \subseteq X} (-1)^{|Y|} a_{iy_1} a_{y_1y_2} \cdots a_{y_kj}$$

and

$$A'(i,j,X) = \sum_{Y \subseteq X} (-1)^{|Y|} a_{iy_k} a_{y_ky_{k-1}} \cdots a_{y_1j}$$

and lastly

$$B'(i,j,X,W) = \sum_{Y \subseteq X} c_Y a_{iy_k} a_{y_ky_{k-1}} \cdots a_{y_1j}$$

where $c_Y = (-1)^{|Y|+1}$ if $Y \cap (W \cup \{\max W + 1\}) = \emptyset$, $c_Y = (-1)^{|Y|}$ if $Y \cap W \neq \emptyset$, and $c_Y = 0$ otherwise.

this is perhaps a bad way of putting this, since when $c_Y = 0$, we have the undefined $a_{\max W+1, \max W+1}$ cord appearing in the sum

We have the following lemma

Lemma 4.3. *Let $X_n^{(p)} = \{(n-1)p+1, \dots, np\}$, let $X_{m,l} = \{m, \dots, m+l-1\}$, and let $Y = X_{(n-1)p+1, j-np-1}$ We have*

$$\phi_{\imath_p(\sigma_n)}(a_{ij}) = \begin{cases} a_{i-p,j-p} & : i,j \in X_{n+1}^{(p)} \\ a_{i-p,j} & : j > (n+1)p, i \in X_{n+1}^{(p)} \\ a_{i,j-p} & : i \leq (n-1)p, j \in X_{n+1}^{(p)} \\ a_{i+p,j+p} & : i,j \in X_n^{(p)} \\ B'(i+p, j-p, X_n^{(p)}, Y) & : i \in X_n^{(p)}, j \in X_{n+1}^{(p)} \\ A(i, j+p, X_n^{(p)}) & : i \leq (n-1)p, j \in X_n^{(p)} \\ A'(i+p, j, X_n^{(p)}) & : j > (n+1)p, i \in X_n^{(p)} \\ a_{ij} & : \text{otherwise} \end{cases}$$

Proof of Lemma 4.2. We will prove that for all $1 \leq n < k$

$$\psi \circ \phi_{\imath_p(\sigma_n)} = \left(\phi_{\sigma_n}^{\pm 1} \otimes \mathrm{id}\right) \circ \psi$$

As the map $B_k \to \mathrm{Aut}(\mathcal{A}_k \otimes \mathcal{A}_p)$ given by $\alpha \mapsto \phi_\alpha \otimes \mathrm{id}$ is a homomorphism, this suffices. Note that if $\psi(\phi_{\imath_p(\sigma_n)}(a_{ij})) = (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))$, then

$$\psi(a_{ij}) = \psi\left(\phi_{\imath_p(\sigma_n)}\left(\phi_{\imath_p(\sigma_n)^{-1}}\left(\alpha_{ij}\right)\right)\right) = \left(\phi_{\sigma_n} \otimes \mathrm{id}\right)\left(\psi\left(\phi_{\imath_p(\sigma_n)^{-1}}\left(a_{ij}\right)\right)\right)$$

And applying $\left(\phi_{\sigma_n^{-1}} \otimes \mathrm{id}\right)$ to both sides gives

$$\psi\left(\phi_{\imath_p(\sigma_n)^{-1}}\left(a_{ij}\right)\right) = \left(\phi_{\sigma_n^{-1}} \otimes \mathrm{id}\right)\psi(a_{ij})$$

Furthermore, $\phi_\beta(\overline{a_{ij}}) = \overline{\phi_\beta(a_{ij})}$ for any braid β and $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$, so it suffices to prove the lemma for $\imath_p(\sigma_n)$ in the case where $i < j$. With these restrictions, we then break the statement up into the cases from Lemma 4.3. The first four of these as well as the last case can be checked easily. Consider the sixth case. Lemma 4.3 gives that

$$\psi\left(\phi_{\imath_p(\sigma_n)}(a_{ij})\right) = \sum_{Y \subseteq \{np-p+1, \dots, np\}} (-1)^{|Y|} \psi\left(a_{iy_1} a_{y_1 y_2} \cdots a_{y_k, j+p}\right)$$

Let $\alpha_i = np - p + r_i$. Note that if $np - p + 1 \leq y_1 < \alpha_i$ then $\psi(a_{iy_1}) = 0$, and if $np \geq y_k > \alpha_j$ then $\psi(a_{y_k j}) = 0$, so the sum on the right hand side can be taken over $Y \subseteq \{\alpha_i, \alpha_i + 1, \dots, \alpha_j\}$. Then we manipulate the sum to get

$$\begin{aligned}
&\sum_{Y\subseteq\{\alpha_i,\ldots,\alpha_j\}} (-1)^{|Y|}\psi\left(a_{iy_1}a_{y_1y_2}\cdots a_{y_k,j+p}\right)\\
&=\psi\left(a_{i,j+p}-a_{i,\alpha_i}a_{\alpha_i,j+p}\right)\\
&\quad+\sum_{y=\alpha_i+1}^{\alpha_j}\sum_{Y\subseteq\{y+1,\ldots,\alpha_j\}}(-1)^{|Y|+1}\psi\left(a_{iy}a_{yy_1}\cdots a_{y_k,j+p}\right)+(-1)^{|Y|}\psi\left(a_{i,\alpha_i}a_{\alpha_i,y}a_{yy_1}\cdots a_{y_k,j+p}\right)\\
&=\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right)\\
&\quad+\sum_{y=\alpha_i+1}^{\alpha_j}\sum_{Y\subseteq\{y+1,\ldots,\alpha_j\}}(-1)^{|Y|}\psi\left(a_{i,\alpha_i}a_{\alpha_i,y}-a_{iy}\right)\psi\left(a_{yy_1}\cdots a_{y_k,j+p}\right)
\end{aligned}$$

Note that $r_i = r_{\alpha_i}$ and since we're in the sixth case we have $(n-1)p < j \leq np$, so $q_{\alpha_i} = q_y$. Thus $\psi(a_{i,\alpha_i}) = a_{q_i+1,q_{\alpha_i}+1}\otimes 1 = a_{q_i+1,q_y+1}\otimes 1$ and $\psi(a_{\alpha_i,y}) = 1\otimes a_{r_{\alpha_i},r_y} = 1\otimes a_{r_i,r_y}$, so we have

$$\psi(a_{i,\alpha_i}a_{\alpha_i,y}-a_{iy}) = \left(a_{q_i+1,q_y+1}\otimes 1\right)\left(1\otimes a_{r_i,r_y}\right) - a_{q_i+1,q_y+1}\otimes a_{r_i,r_y} = 0$$

Thus the right hand side reduces to

$$\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right)$$

Remark 4.4. The fact that $\psi(a_{i,\alpha_i}a_{\alpha_i,y}-a_{iy})=0$ and ψ behaves similarly for the analogous terms in cases 5 and 7 is the key to this proof working, and ψ is defined the way it is mainly so that this will be true. As we hinted at earlier, the homomorphism $\rho\colon \mathcal{A}_{pk}\to\mathcal{A}_k$ defined to send a_{ij} to a_{q_i+1,q_j+1} if $r_i=r_j$ and to 0 otherwise would also send these terms to 0, so Proposition 4.1 would still be true with ρ used in the place of ψ. We will need ψ for the proof of the main result, however.

Note that, since we're in the sixth case, $q_j+1=n$. If $r_i=r_j$, then

$$\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right) = \left(a_{q_i+1,n+1}-a_{q_i+1,n}a_{n,n+1}\right)\otimes 1 = (\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))$$

If $r_i<r_j$, then

$$\begin{aligned}
\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right) &= \left(a_{q_i+1,n+1}\otimes a_{r_ir_j}-a_{q_i+1,n}a_{n,n+1}\otimes a_{r_ir_j}\right)\\
&=\left(a_{q_i+1,n+1}-a_{q_i+1,n}a_{n,n+1}\right)\otimes a_{r_ir_j}\\
&=(\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))
\end{aligned}$$

Finally, if $r_i>r_j$, then

$$\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right) = 0 = (\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))$$

Consider the seventh case. Lemma 4.3 gives that

$$\psi\left(\phi_{\imath_p(\sigma_n)}(a_{ij})\right) = \sum_{Y\subseteq\{np-p+1,\ldots,np\}}(-1)^{|Y|}\psi\left(a_{i+p,y_k}a_{y_{k-1}y_{k-2}}\cdots a_{y_1,j}\right)$$

Note that if $\alpha_i < y_k \leq np$ then $\psi(a_{i+p,y_k}) = 0$, and if $np-p+1 \leq y_1 < \alpha_j$ then $\psi(a_{y_1 j}) = 0$, so the sum on the right hand side can be taken over $Y \subseteq \{\alpha_j, \alpha_j+1, \ldots, \alpha_i\}$. Then we manipulate the sum to get

$$\begin{aligned}
&\sum_{Y\subseteq\{\alpha_j,\ldots,\alpha_i\}} (-1)^{|Y|}\psi\left(a_{i+p,y_k}a_{y_k y_{k-1}}\cdots a_{y_1,j}\right)\\
&=\psi\left(a_{i+p,j}-a_{i+p,\alpha_i}a_{\alpha_i,j}\right)\\
&\quad+\sum_{y=\alpha_j}^{\alpha_i-1}\sum_{Y\subseteq\{\alpha_j,\ldots,y-1\}}(-1)^{|Y|+1}\psi\left(a_{i+p,y}a_{y,y_k}\cdots a_{y_1,j}\right)+(-1)^{|Y|}\psi\left(a_{i+p,\alpha_i}a_{\alpha_i,y}a_{yy_k}\cdots a_{y_1,j}\right)\\
&=\psi\left(a_{i+p,j}-a_{i+p,\alpha_i}a_{\alpha_i j}\right)\\
&\quad+\sum_{y=\alpha_j}^{\alpha_i-1}\sum_{Y\subseteq\{\alpha_j,\ldots,y-1\}}(-1)^{|Y|}\psi\left(a_{i+p,\alpha_i}a_{\alpha_i,y}-a_{i+p,y}\right)\psi\left(a_{yy_k}\cdots a_{y_1,j}\right)
\end{aligned}$$

Note that $r_i = r_{\alpha_i}$ and since we're in the seventh case, $q_{\alpha_i} = q_i$. Thus $\psi(a_{i+p,\alpha_i}) = a_{q_i+2,q_i+1}\otimes 1$ and $\psi(a_{\alpha_i,y}) = 1\otimes a_{r_i,r_y}$, so we have

$$\psi(a_{i+p,\alpha_i}a_{\alpha_i,y}-a_{i+p,y}) = \left(a_{q_i+2,q_i+1}\otimes 1\right)\left(1\otimes a_{r_i,r_y}\right)-a_{q_i+2,q_i+1}\otimes a_{r_i,r_y} = 0$$

Thus the right hand side reduces to

$$\psi\left(a_{i+p,j}-a_{i+p,\alpha_i}a_{\alpha_i,j}\right)$$

Note that, since we're in the seventh case, $q_i+1 = n$ and $q_j > q_i+2$. If $r_i = r_j$, then

$$\psi\left(a_{i+p,j}-a_{i+p,\alpha_i}a_{\alpha_i,j}\right) = \left(a_{n+1,q_j+1}-a_{n+1,n}a_{n,q_j+1}\right)\otimes 1 = (\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))$$

If $r_i < r_j$, then

$$\begin{aligned}
\psi\left(a_{i+p,j}-a_{i+p,\alpha_i}a_{\alpha_i,j}\right) &= \left(a_{n+1,q_j+1}\otimes a_{r_i r_j}-a_{n+1,n}a_{n,q_j+1}\otimes a_{r_i r_j}\right)\\
&=(\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))
\end{aligned}$$

Finally, if $r_i > r_j$, then

$$\psi\left(a_{i+p,j}-a_{i+p,\alpha_i}a_{\alpha_i,j}\right) = 0 = (\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))$$

Consider the fifth case. Lemma 4.3 gives that

$$\begin{aligned}
\psi\left(\phi_{\bar{\bar{p}}\sigma_n}(a_{ij})\right) = &\sum_{Y\subseteq\{np-p+1,\ldots,np\},Y\cap X_{np-p+1,j-np-1}\neq\emptyset}(-1)^{|Y|}\psi(a_{i+p,y_k}a_{y_k y_{k-1}}\cdots a_{y_1,j-p})\\
&-\sum_{Y\subseteq\{np-p+1,\ldots,np\},Y\cap X_{np-p+1,j-np}=\emptyset}(-1)^{|Y|}\psi(a_{i+p,y_k}a_{y_k y_{k-1}}\cdots a_{y_1,j-p})
\end{aligned}$$

Note that if $\alpha_i < y_k \leq np$ then $\psi(a_{i+p,y_k}) = 0$, so if $r_i < r_j$, then all terms in the the second sum on the right hand side are sent to zero, and manipulating the first sum gives

$$\begin{aligned}
\psi\left(\phi_{\bar{\bar{p}}\sigma_n}(a_{ij})\right) &= \sum_{Y\subseteq\{np-p+1,\ldots,\alpha_i\}} (-1)^{|Y|}\psi(a_{i+p,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j-p}) \\
&= \psi\left(-a_{i+p,\alpha_i}a_{\alpha_i,j-p}\right) \\
&\quad+\sum_{y=np-p+1}^{\alpha_i-1}\sum_{Y\subseteq\{np-p+1,\ldots,y-1\}}(-1)^{|Y|}\psi\left(a_{i+p,\alpha_i}a_{\alpha_i,y}-a_{i+p,y}\right)\psi\left(a_{yy_k}\cdots a_{y_1,j-p}\right) \\
&= \psi\left(-a_{i+p,\alpha_i}a_{\alpha_i,j-p}\right) \\
&= -a_{q_i+2,q_i+1}\otimes a_{r_i,r_j} \\
&= (\phi_{\sigma_n}\otimes \mathrm{id})(\psi(a_{ij}))
\end{aligned}$$

The third equality holds because $\psi\left(a_{i+p,\alpha_i}a_{\alpha_i,y}-a_{i+p,y}\right)=0$. If $r_i=r_j$, then all of the terms in the second sum are sent to zero except for $-a_{i+p,j-p}$, giving

$$\begin{aligned}
\psi\left(\phi_{\bar{\bar{p}}\sigma_n}(a_{ij})\right) &= \sum_{Y\subseteq\{np-p+1,\ldots,\alpha_i\}} (-1)^{|Y|}\psi(a_{i+p,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j-p}) \\
&= \psi\left(-a_{i+p,j-p}\right) \\
&\quad+\sum_{y=np-p+1}^{\alpha_i-1}\sum_{Y\subseteq\{np-p+1,\ldots,y-1\}}(-1)^{|Y|}\psi\left(a_{i+p,\alpha_i}a_{\alpha_i,y}-a_{i+p,y}\right)\psi\left(a_{yy_k}\cdots a_{y_1,j-p}\right) \\
&= \psi\left(-a_{i+p,j-p}\right) \\
&= -a_{q_i+2,q_i+1}\otimes 1 \\
&= (\phi_{\sigma_n}\otimes \mathrm{id})(\psi(a_{ij}))
\end{aligned}$$

Finally, if $r_i>r_j$, then using the notation above for $B'(i{+}p,j{-}p,X_n^{(p)},X_{(n-1)p+1,j-np-1})$ we see that for a given $Y\subseteq X_n^{(p)}$ that if $\alpha_i\notin Y$, then $c_Y=-c_{Y\cup\{\alpha_i\}}$ since $\alpha_i\notin X_{(n-1)p+1,j-np-1}$. We then have that

$$\begin{aligned}
\psi\left(\phi_{\bar{\bar{p}}\sigma_n}(a_{ij})\right) &= \sum_{Y\subseteq\{np-p+1,\ldots,\alpha_i\}} c_Y\psi(a_{i+p,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j-p}) \\
&= \sum_{y=np-p+1}^{\alpha_i-1}\sum_{Y\subseteq\{np-p+1,\ldots,y-1\}}c_Y\psi\left(a_{i+p,\alpha_i}a_{\alpha_i,y}-a_{i+p,y}\right)\psi\left(a_{yy_k}\cdots a_{y_1,j-p}\right) \\
&= 0 \\
&= (\phi_{\sigma_n}\otimes \mathrm{id})(\psi(a_{ij}))
\end{aligned}$$

□

Proof of Lemma 4.3. We will prove a more general statement than the one presented in Lemma 4.3. Let $\kappa_{m,l}=\tau_{m+l-1,p}\tau_{m+l-2,p}\cdots\tau_{m,p}$. We will prove that if $i<j$ and $l\leq p$ then

$$\phi_{\kappa_{m,l}}(a_{ij}) = \begin{cases} a_{i-p,j-p} & : i,j \in X_{m+p,l} \\ a_{i-p,j} & : j \geq m+l+p, i \in X_{m+p,l} \\ a_{i,j-p} & : i < m, j \in X_{m+p,l} \\ a_{i+l,j+l} & : i,j \in X_{m,l} \\ B'(i+l, j-p, X_{m,l}, X_{m,j-m-p}) & : i \in X_{m,l}, j \in X_{m+p,l} \\ A(i, j+l, X_{m,l}) & : i < m, j \in X_{m,l} \\ A'(i+l, j, X_{m,l}) & : j \geq m+l+pi \in X_{m,l} \\ a_{ij} & : \text{otherwise} \end{cases}$$

Letting $l = p$ and $m = (n-1)p+1$ then gives us Lemma 4.3 as a special case. The first four cases as well as the eighth can be easily checked. We will prove the remaining cases by induction on l. Consider the sixth case. The base case is covered by (7). For the inductive step, we have that

$$\begin{aligned}
\phi_{\kappa_{m,l}}(a_{ij}) &= \phi_{\tau_{m+l-1,p}}\left(\phi_{\kappa_{m,l-1}}(a_{ij})\right) \\
&= \sum_{Y \subseteq \{m,\ldots,m+l-2\}} (-1)^{|Y|} \phi_{\tau_{m+l-1,p}}\left(a_{i,y_1} a_{y_1 y_2} \cdots a_{y_k,j+l-1}\right) \\
&= \sum_{Y \subseteq \{m,\ldots,m+l-2\}} (-1)^{|Y|} a_{iy_1} a_{y_1 y_2} \cdots a_{y_{k-1} y_k}\left(a_{y_k,j+l} - a_{y_k,m+l-1} a_{m+l-1,j+l}\right) \\
&= \sum_{Y \subseteq \{m,\ldots,m+l-1\}} (-1)^{|Y|} a_{i,y_1} a_{y_1 y_2} \cdots a_{y_k,j+l} \\
&= A(i, j+l, X_{m,l})
\end{aligned}$$

Where the second equality holds because $l \leq p$.

Consider the seventh case. The base case is covered by (7). For the inductive step, we have that

$$\begin{aligned}
\phi_{\kappa_{m,l}}(a_{ij}) &= \phi_{\tau_{m+l-1,p}}\left(\phi_{\kappa_{m,l-1}}(a_{ij})\right) \\
&= \sum_{Y \subseteq \{m,\ldots,m+l-2\}} (-1)^{|Y|} \phi_{\tau_{m+l-1,p}}\left(a_{i+l-1,y_k} a_{y_k y_{k-1}} \cdots a_{y_1,j}\right) \\
&= \sum_{Y \subseteq \{m,\ldots,m+l-2\}} (-1)^{|Y|} \left(a_{i+l,y_k} - a_{i+l,m+l-1} a_{m+l-1,y_k}\right) a_{y_k y_{k-1}} \cdots a_{y_1 j} \\
&= \sum_{Y \subseteq \{m,\ldots,m+l-1\}} (-1)^{|Y|} a_{i+l,y_k} a_{y_k y_{k-1}} \cdots a_{y_1,j} \\
&= A'(i+l, j, X_{m,l})
\end{aligned}$$

Where in the second equality we make use of the facts that $l \leq p$, $j \geq m+l+p$, and that $\phi_{\tau_{m+l-1,p}}(a_{i,y_k}) = \overline{\phi_{\tau_{m+l-1,p}}(a_{y_k,i})}$.

Consider the fifth case. We have from the seventh case that

$$\phi_{\kappa_{m,j-m-p}}(a_{ij}) = A'(i+j-m-p, j, X_{m,j-m-p})$$

We then have that

$$\begin{aligned}
&\phi_{\tau_{m+(j-m-p),p}}(A'(i+j-m-p,j,X_{m,j-m-p}))\\
&= \sum_{Y\subseteq\{m,\ldots,j-p-1\}} (-1)^{|Y|}\phi_{\tau_{j-p,p}}\left(a_{i+j-m-p,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j}\right)\\
&= -a_{i+j-m-p+1,j-p}\\
&+ \sum_{Y\subseteq\{m,\ldots,j-p-1\},Y\neq\emptyset} (-1)^{|Y|}\left(a_{i+j-m-p+1,y_k} - a_{i+j+m-p+1,j-p}a_{j-p,y_k}\right)a_{y_ky_{k-1}}\cdots a_{y_2y_1}a_{y_1,j-p}\\
&= B'(i+j-m-p+1,j-p,X_{m,j-m-p+1},X_{m,j-m-p})
\end{aligned}$$

For $l > j-m-p+1$, $l \leq p$, we have

$$\begin{aligned}
&\tau_{m+l-1,p}(B'(i+l-1,j-p,X_{m,l-1},X_{m,j-m-p}))\\
&= \sum_{Y\subseteq\{m,\ldots,m+l-2\},Y\cap X_{m,j-m-p}\neq\emptyset} (-1)^{|Y|}\tau_{m+l-1,p}(a_{i+l-1,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j-p})\\
&- \sum_{Y\subseteq\{m,\ldots,m+l-2\},Y\cap X_{m,j-m-p+1}=\emptyset} (-1)^{|Y|}\tau_{m+l-1,p}(a_{i+l-1,y_k}a_{y_ky_{k-1}}\cdots a_{y_1,j-p})\\
&= \sum_{Y\subseteq\{m,\ldots,m+l-2\},Y\cap X_{m,j-m-p}\neq\emptyset} (-1)^{|Y|}(a_{i+l,y_k} - a_{i+l,m+l-1}a_{m+l-1,y_k})a_{y_ky_{k-1}}\cdots a_{y_1,j-p}\\
&- \sum_{Y\subseteq\{m,\ldots,m+l-2\},Y\cap X_{m,j-m-p+1}=\emptyset} (-1)^{|Y|}(a_{i+l,y_k} - a_{i+l,m+l-1}a_{m+l-1,y_k})a_{y_ky_{k-1}}\cdots a_{y_1,j-p}\\
&= B'(i+l,j-p,X_{m,l},X_{m,j-m-p})
\end{aligned}$$

giving the desired result. □

5. Comments on augmentation rank and multiplicativity

We split this section in two parts. In the first we prove Theorem 1.4, showing that some cables of torus knots have augmentation rank less than their bridge number. In the second part we discuss how this result, and some computational evidence may fit into a generalization of Theorem 1.2, given by Conjecture 1.5.

5.1. Cables of $(n, n+1)$ torus knots.

Theorem 1.4. For $p > 1$ and $n > 1$, let $K = T((p,n),(1,n+1))$. Then $\mathrm{ar}(K) < pn$.

Proof. For notational convenience we write the proof for the case that $p = 2$, remarking that the same techniques also prove the result for general $p > 1$. The special role that the element $a_{32} \in \mathcal{A}_{2n}$ plays in our proof would be replaced by $a_{p+1,p} \in \mathcal{A}_{pn}$ in the general case.

Let $\tau = \sigma_1 \ldots \sigma_{n-1} \in B_n$ and set $\alpha = \tau^{n+1}$, which has the $(n, n+1)$ torus knot as its braid closure. Then $K = K(\alpha, \sigma_1) = \widehat{\iota_2(\alpha)\sigma_1}$.

Let $k \geq 1$. Every entry of $\Phi^L_{\mathfrak{v}_2(\tau)^k}$ is a polynomial in the (non-commuting) variables $\mathbf{a} = (a_{12}, a_{13}, \ldots, a_{2n-1,2n})$. For each $1 \leq i, j \leq n$ write $A^k_{i,j}(\mathbf{a})$ for entry $\left(\Phi^L_{\mathfrak{v}_2(\tau)^k}\right)_{2i-1,2j-1}$ and similarly write $B^k_{i,j}(\mathbf{a}), C^k_{i,j}(\mathbf{a})$, and $D^k_{i,j}(\mathbf{a})$ for the entries in the $(2i-1, 2j), (2i, 2j-1)$, and $(2i, 2j)$ spots, respectively.

Note for $1 \leq i \leq n-1$, that $\begin{pmatrix} A^1_{ij} & B^1_{ij} \\ C^1_{ij} & D^1_{ij} \end{pmatrix}$ is the 2×2 identity if $j = i+1$ and is the zero matrix otherwise (if $j \neq 1$). Further, when $i = n$ this is the identity matrix if $j = 1$ and is zero otherwise.

Since $\Phi^L_{\mathfrak{v}_2(\tau)^{k+1}} = \phi_{\mathfrak{v}_2(\tau)}(\Phi^L_{\mathfrak{v}_2(\tau)^k})\Phi^L_{\mathfrak{v}_2(\tau)}$ we see that for $2 \leq j \leq n$

$$\begin{aligned} B^{k+1}_{i,j}(\mathbf{a}) &= B^k_{i,j-1}(\phi_{\mathfrak{v}_2(\tau)}(\mathbf{a})) \\ D^{k+1}_{i,j}(\mathbf{a}) &= D^k_{i,j-1}(\phi_{\mathfrak{v}_2(\tau)}(\mathbf{a})). \end{aligned}$$

Also, since $\Phi^L_{\mathfrak{v}_2(\tau)^{k+1}} = \phi_{\mathfrak{v}_2(\tau)^k}(\Phi^L_{\mathfrak{v}_2(\tau)})\Phi^L_{\mathfrak{v}_2(\tau)^k}$ we have that

$$\begin{aligned} B^{k+1}_{n,j}(\mathbf{a}) &= B^k_{1,j}(\mathbf{a}) \\ D^{k+1}_{n,j}(\mathbf{a}) &= D^k_{1,j}(\mathbf{a}). \end{aligned}$$

Combining these two you get that for $j \geq 2$, $X^{k+1}_{1,j}(\mathbf{a}) = X^k_{1,j-1}(\phi_{\mathfrak{v}_2(\tau)}(\mathbf{a})) = X^{k+1}_{n,j-1}(\phi_{\mathfrak{v}_2(\tau)}(\mathbf{a}))$ for $X \in \{B, D\}$. Also, for $1 \leq i \leq n-1$,

$$B^{k+1}_{i,j}(\mathbf{a}) = B^k_{i+1,j}(\mathbf{a}) + A^1_{i,1}(\phi_{\mathfrak{v}_2(\tau)^k}(\mathbf{a}))B^k_{1,j}(\mathbf{a}) + B^1_{i,1}(\phi_{\mathfrak{v}_2(\tau)^k}(\mathbf{a}))D^k_{1,j}(\mathbf{a}). \tag{8}$$

Starting at (8) with $i \leq n-1$, and $k = n$ we get

$$\begin{aligned} B^{n+1}_{i,j}(\mathbf{a}) &= B^n_{i+1,j}(\mathbf{a}) + A^1_{i,1}(\phi_{\mathfrak{v}_2(\tau)^n}(\mathbf{a}))B^n_{1,j}(\mathbf{a}) + B^1_{i,1}(\phi_{\mathfrak{v}_2(\tau)^n}(\mathbf{a}))D^n_{1,j}(\mathbf{a}) \\ &= B^n_{i+1,1}(\mathbf{a}) + A^1_{i,1}(\phi_{\mathfrak{v}_2(\tau)^n}(\mathbf{a}))B^{n+1}_{n,j}(\mathbf{a}) + B^1_{i,1}(\phi_{\mathfrak{v}_2(\tau)^n}(\mathbf{a}))D^{n+1}_{n,j}(\mathbf{a}). \end{aligned}$$

Since $\varphi(B^{n+1}_{n,j}(\mathbf{a})) = 0$ and $\varphi(D^{n+1}_{n,j}(\mathbf{a})) = 0$ for $1 \leq j \leq n$ by assumption, this says that $B^{n+1}_{i,j}$ and $B^n_{i+1,j}$ have the same image under φ. Using the first set of equations, this means that $\varphi(B^{n+1}_{i,j}(\mathbf{a})) = \varphi(B^{n+1}_{i+1,j+1}(\phi^{-1}_{\mathfrak{v}_2(\tau)}(\mathbf{a})))$.

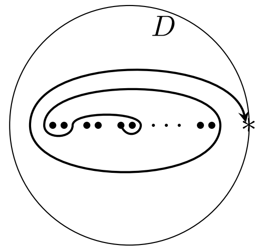
Hence we have that $\varphi(B^{n+1}_{1,1}(\mathbf{a})) = \varphi(B^{n+1}_{n,n}(\phi^{-n+1}_{\mathfrak{v}_2(\tau)}(\mathbf{a}))) = \varphi(B^1_{1,1}(\mathbf{a})) = \varphi(-a_{32})$. We will show that $\varphi(a_{32}) = 0$.

Write $(M)_i$ for the i^{th} row of a matrix M and $\mathbf{e}_i$ for the vector $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^{2n}$, where the 1 is in the i^{th} position.

Lemma 5.1. *Suppose $\varphi : \mathcal{A}_{2n} \to \mathbb{C}$ is a homomorphism with the property $\varphi((\Phi^L_{\mathfrak{v}_2(\alpha)})_{2i-1}) = \mathbf{e}_{2i-1}$ and $\varphi((\Phi^L_{\mathfrak{v}_2(\alpha)})_{2i}) = \mathbf{e}_{2i}$ for $2 \leq i \leq n$. Then $\varphi(a_{32}) = 0$.*

Proof. We arrive at the result by considering a computation of the row $(\Phi^L_{\mathfrak{v}_2(\alpha)})_3$, which is obtained from the equation $\phi^*_{\mathfrak{v}_2(\alpha)}(a_{3,2n+1}) = \sum_{j=1}^{2n}(\Phi^L_{\mathfrak{v}_2(\alpha)})_{3,j}a_{j,2n+1}$. Described as an element in the algebra of paths (see Section 3.1), the image $\phi^*_{\mathfrak{v}_2(\alpha)}(a_{3,2n+1})$ is given up to isotopy by the curve in Figure 7.

Denote the curve on the left in Figure 8 by x_i. Denote by y_{ij} the curve on the right in the same figure, and let $y_{ij} = 1$ if $i = j$. Using the relation 5 to

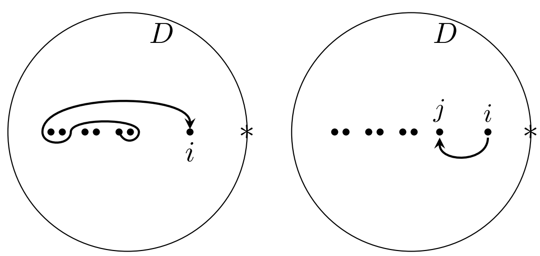


$\iota_2(\alpha) \cdot c_{3,2n+1}$

FIGURE 7. Computing third row of $\Phi^L_{\iota_2(\alpha)}$

pull the curve in Figure 7 across the rightmost $2n-6$ punctures, we consider how to get a coefficient of $a_{j,2n+1}$. For each $j \geq 7$ we get a coefficient of $a_{j,2n+1}$ as below.

$$\langle \phi^*_{\iota_2(\alpha)}(a_{3,2n+1}), a_{j,2n+1} \rangle = \sum_{i=j}^{2n} x_i y_{i,j} \tag{9}$$



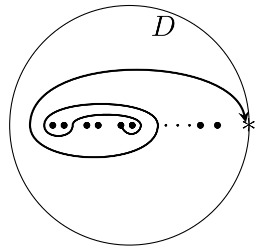
x_i $\qquad$ y_{ij}

FIGURE 8. Coefficients that are sent to zero.

As x_{2n} is the coefficient of $a_{2n,2n+1}$, our assumption makes $\varphi(x_{2n}) = 0$. This in turn implies that $\varphi(x_{2n-1}) = 0$ by considering the case $j = 2n-1$. Continuing in this way, we see that $\varphi(x_j) = 0$ for all $j > 6$. Letting w denote the curve of Figure 9, we see that (applying φ to the coefficients of the $a_{j,2n+1}$) $\varphi(\phi^*_{\iota_2(\alpha)}(a_{3,2n+1})) = \varphi(w)$. We remark, the curve w is the image of $c_{3,2n+1}$ under the braid $\phi_{\iota_2(\sigma_1\sigma_2)^4}$ in $B_6 \subset B_{2n}$.

A similar argument to the above shows that the (coefficients of the) image under φ of $\iota_2(\alpha) \cdot c_{4,2n+1}$, $\iota_2(\alpha) \cdot c_{5,2n+1}$, and $\iota_2(\alpha) \cdot c_{6,2n+1}$ also correspond to the image of $\iota_2(\sigma_1\sigma_2)^4 \cdot c_{i,7}$. This implies that φ, when applied to the 6×6 matrix $\Phi^L_{\iota_2(\sigma_1\sigma_2)^4}$, sends the last four rows to $\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6$. It can be directly calculated that any such map on $\mathcal{A}_6$ satisfies $\varphi(-a_{32}) = 0$, concluding the proof of the lemma. $\square$

To complete the theorem, recall that $K = \widehat{\sigma_1(\alpha)}$. Any augmentation with rank $2n$ would require a homomorphism $\epsilon : \mathcal{A}_{2n} \to \mathbb{C}$ such that $\epsilon\left(\Phi^L_{\sigma_1(\alpha)}\right) =$



$w = \iota_2(\sigma_1\sigma_2)^4 \cdot c_{3,2n+1}$

FIGURE 9. Reduction of computation

$\Delta(\sigma_1(\alpha))$. By the Chain rule $\Phi^L_{\sigma_1(\alpha)} = \phi_{\iota_2(\alpha)}(\Phi^L_{\sigma_1})\Phi^L_{\iota_2(\alpha)}$ which, by the form of $\Phi^L_{\sigma_1}$, has (except for the first two rows) the same rows as $\Phi^L_{\iota_2(\alpha)}$.

As a consequence, ϵ must send the i^{th} row of $\Phi^L_{\iota_2(\alpha)}$ to $\mathbf{e_i}$ for $i \geq 3$. By Lemma 5.1 we see that $\epsilon\left((\Phi^L_{\iota_2(\alpha)})_{12}\right) = 0$. As $\Phi^L_{\sigma_1(\alpha)} = \phi_{\iota_2(\alpha)}(\Phi^L_{\sigma_1})\Phi^L_{\iota_2(\alpha)}$ indicates that $\Phi^L_{\sigma_1(\alpha)}$ has a second row equal to the first row of $\Phi^L_{\iota_2(\alpha)}$, there is a diagonal entry in $\epsilon\left(\Phi^L_{\sigma_1(\alpha)}\right)$ equal to zero, so it cannot equal $\Delta(\sigma_1(\alpha))$, a contradiction. □

5.2. **Augmentation rank does not multiply.** As discussed in Section 2 the braid satellite $K(\alpha, \gamma)$ depends only on γ and the closure $\hat{\alpha}$ if α has minimal braid index. We give some motivation for Conjecture 1.5, that $\mathrm{ar}(K(\alpha,\gamma)) \geq \mathrm{ar}(\hat{\alpha})\,\mathrm{ar}(\hat{\gamma})$.

Theorem 5.2. *For any braid α with $K = \hat{\alpha}$ and any braid γ, we have $\mathrm{ar}(K(\alpha,\gamma)) \geq \mathrm{ar}(K)$.*

Proof. The knot group $\pi_{K(\alpha,\gamma)}$ is isomorphic to a product of π_K and the group for the closure of $\Delta^{2\omega}\gamma$ amalgamated along a $\mathbb{Z}^2$ coming from the torus boundary T of the neighborhood $n(K)$. Choose the basepoint of $\pi_{K(\alpha,\gamma)}$ on T and include π_K into the amalgamated product via the inclusion of the complement.

Let m_1 be the meridian of K determined by a based loop contained in T that is contractible in $n(K)$. Suppose that $\rho : \pi_K \to \mathrm{GL}_n\mathbb{C}$ is an irreducible KCH representation with $\widetilde{M} = \rho(m_1) = \mathrm{diag}[\widetilde{\mu}_0, 1, \ldots, 1]$ for some $\widetilde{\mu}_0 \in \mathbb{C} \setminus \{0\}$. Choose any p^{th} root μ_0 of $\widetilde{\mu}_0$.

Consider a collection of meridians $m_1, \ldots, m_r$ of K that generate π_K. For each $1 \leq i \leq r$ there are p meridians $m_{i1}, \ldots, m_{ip}$ of $K(\alpha,\gamma)$ such that $m_{i1}m_{i2}\ldots m_{ip} = m_i$. Set $\sigma(m_{1j}) = \mathrm{diag}[\mu_0, 1, \ldots, 1] = M$ for $1 \leq j \leq p$. Then, for each $2 \leq i \leq r$ find $w_i \in \pi_K$ so that $m_i = w_i m_1 w_i^{-1}$ and set $\sigma(m_{ij}) = \rho(w_i)M\rho(w_i)^{-1}$ for $1 \leq j \leq p$.

As $\pi_{K(\alpha,\gamma)}$ has a presentation with every relation of the form $m_{i,j} = w_i m_{1,k} w_i^{-1}$ for some $1 \leq j, k \leq p$ and $1 \leq i \leq r$ we see that $\sigma : \pi_{K(\alpha,\gamma)} \to \mathrm{GL}_n\mathbb{C}$ is a well-defined KCH representation. Moreover, σ has the same image as ρ, implying it is irreducible. This shows that $\mathrm{ar}(K(\alpha,\gamma)) \geq \mathrm{ar}(K)$. □

It should also be noted that, for $P = \widehat{\Delta^{2\omega}\gamma}$, we have $\mathrm{ar}(K(\alpha, \gamma)) \geq \mathrm{ar}(P)$ also. This follows from Proposition 2.1 and the fact that there is a surjection $\pi_{K(\alpha,\gamma)} \to \pi_P$, preserving peripheral structures (see Proposition 3.4 in [SW06], for example).

Oddly, the product $\mathrm{ar}(K)\,\mathrm{ar}(P)$ does not relate well to $\mathrm{ar}(K(\alpha, \gamma))$, even when α has minimal index: from Theorem 1.4 we find examples where $\mathrm{ar}(K(\alpha, \gamma)) < \mathrm{ar}(K)\,\mathrm{ar}(P)$ and from Theorem 1.2 there are examples with $\mathrm{ar}(K(\alpha, \gamma)) > \mathrm{ar}(K)\,\mathrm{ar}(P)$. However, to our knowledge the statement of Conjecture 1.5 could hold.

There are cases where $\mathrm{ar}(K(\alpha, \gamma))$ is strictly larger than $\mathrm{ar}(K)\,\mathrm{ar}(\hat{\gamma})$. One example can be found from the $(2, 11)$-cable of the $(2, 5)$ torus knot. By finding a solution to (4) for $\alpha = \sigma_1^5 \in B_2$ and $\gamma = \sigma_1 \in B_2$, we can compute that $\mathrm{ar}(K(\sigma_1^5, \sigma_1)) = 4$, even though $\mathrm{ar}(\hat{\sigma_1^5}) = 2$ and $\mathrm{ar}(\hat{\sigma_1}) = 1$. Unfortunately, any more examples of cables of torus knots (not covered by Theorems 1.2 and 1.4) seem outside of our computational abilities.

We end with a simple remark on computational observations. By the inequalities in (3.9) if a knot has bridge number less than its minimal braid index n, it cannot have augmentation rank equal to n. Take a minimal index braid representative of such a knot, and multiply that braid by successively higher powers of $\Delta^2 \in B_n$, testing in each instance if the closure has augmentation rank equal to n. In examples, the power of Δ^2 need not be very high, compared to n, before a braid with augmentation rank n appears. The existence of such an augmentation also seems to persist. Note that braids which are sufficiently far in Dehornoy's order from the identity were shown in [MN04] to not admit a Birman-Menasco template, and thus are minimal index representatives by the MTWS. For a given braid index n, is there a number m_n such that whenever Δ^{2m_n} is less than $\alpha \in B_n$ in the Dehornoy order, then $\mathrm{ar}(\hat{\alpha}) = n$?

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