List of Todos

AUGMENTATIONS OF KNOT SATELLITES

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Abstract. PARAGRAPH

1. Introduction

Let K be a knot in $\mathbb{R}^3$, let $B \in B_n$ be a braid closing to K, let $R_0 = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$, and let $\mathcal{A}_n$ be the noncommutative unital algebra over $\mathbb{Z}$ freely generated by the $n(n-1)$ elements a_{ij}, $1 \leq i, j \leq n$. From B we define a certain ideal $\mathcal{I} \subset \mathcal{A}_n \otimes R_0$, and the *degree zero homology of the combinatorial knot DGA* is $HC_0(K) = \mathcal{A}_n \otimes R_0/\mathcal{I}$.Since the description of $\mathcal{I}$ is fairly involved, we delay its definition until Section 2.

is this the right kind of slash?

It was shown in [Ng08] that the isomorphism class of $HC_0(K)$ is unchanged under conjugation and by positive and negative stabilization of B, so $HC_0(K)$ is an invariant of K by Markov's theorem.An *augmentation* of K is a homomorphism $\epsilon\colon \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ that descends to $HC_0(K)$, and the *rank* of ϵ is given by the rank of $\epsilon(\mathbf{A})$, where

repeating from background section

$$\mathbf{A_{ij}} = \begin{cases} a_{ij} & i < j \\ -\mu a_{ij} & i > j \\ 1 - \mu & i = j \end{cases}$$

The *augmentation rank* of K, written $\mathrm{ar}(K)$, is the maximum rank among augmentations of K.

Let $\tau_{m,l} \in B_{pk}$ be defined by $\tau_{m,l} = \sigma_m \sigma_{m+1} \cdots \sigma_{m+l-1}$, and let $\Sigma_n^{(p)} \in B_{pk}$ be defined by $\Sigma_n^{(p)} = \tau_{np,p}\tau_{np-1,p} \cdots \tau_{np-p+1,p}$ (see Figure 1). Then if $B \in B_k$ is given by the braid word $\sigma_{n_1}\sigma_{n_2} \cdots \sigma_{n_m}$, we define the *p-copy* $B^{(p)}$ of B to be $B^{(p)} = \Sigma_{n_1}^{(p)}\Sigma_{n_2}^{(p)} \cdots \Sigma_{n_m}^{(p)}$. Our main result shows that certain satellites of knots with augmentation rank equal to braid index also have augmentation rank equal to bridge index.

can we just say $(p, 0)$-cable with the blackboard framing?

Theorem 3.1. *Let $B \in B_k$ have augmentation rank k, and let $B'' \in B_p$ have augmentation rank p. If B' is the braid B'' included into B_{pk}, then $B^{(p)}B'$ has augmentation rank pk.*

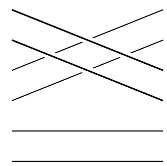

FIGURE 1. $\Sigma_1^{(2)}$

The *meridional rank* of K, written $\mathrm{mr}(K)$, is the minimal size of a meridional generating set of the knot group of K. It is bounded above by the bridge number $b(K)$, and Problem 1.11 of [Kir95] asks whether $\mathrm{mr}(K) = b(K)$ for all knots K. Cornwell has proven that the *augmentation rank* $\mathrm{ar}(K)$ of K (which is defined in Section 2) bounds the meridional rank from below, and that $\mathrm{ar}(K) = \mathrm{mr}(K) = b(K)$ for some families of knots, including torus knots [Cor13b].

Corollary 2.4 ([Cor13b])**.** *Given a knot $K \subset S^3$,*

$$ar(K) \leq mr(K) \leq b(K)$$

Corollary 1.1. *Let K be a knot with augmentation rank equal to its braid index, and let $p, q > 0$ with $\gcd(p, q) = 1$ and $p < q$. Then the (p, q)-cable of K taken with the blackboard framing has augmentation rank equal to its braid index.*

2. Background

We review in Section 2.1 the construction of $HC_0(K)$ from the viewpoint of the combinatorial knot DGA, which was first defined in [Ng08]; our conventions are those given in [Ng12]. In Section 2.2 we discuss augmentations in knot contact homology and their rank, which gives a bound on the meridional rank of the knot group useful for studying the relation between meridional rank and bridge number. Finally, in Section 2.3 is a discussion of techniques from [Cor13a] that we use to calculate the augmentation rank.

Throughout the paper we denote by B_n the n-strand braid group. We orient braids from left to right and label the strands $1, \ldots, n$, with 1 the topmost to n the bottommost strand. We work with the generating set $\{\sigma_i^{\pm}, i = 1, \ldots, n\}$ of B_n, where σ_i has strands i and $i+1$ that cross once in the manner depicted in Figure 2. As usual, a braid may be closed to a link as depicted in Figure 3. The *writhe* (or algebraic sum) of a braid $B \in B_n$, denoted $\omega(B)$, is the sum of the exponents in a factorization of B in terms of the generators.

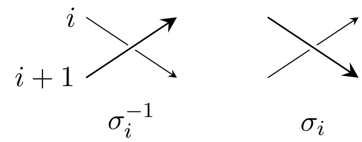

FIGURE 2. Generators of B_n

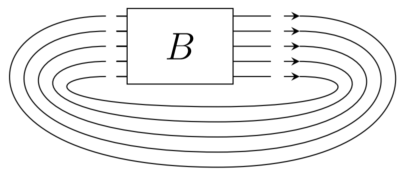

FIGURE 3. The closure of the braid B

2.1. **Knot contact homology.** Here we cover the necessary preliminaries for defining the combinatorial knot DGA of Ng. This DGA was defined in order to be a calculation of knot contact homology and was shown to be so in [EENS13] (see [Ng12] for more details). Let $\mathcal{A}_n$ be the noncommutative unital algebra over $\mathbb{Z}$ freely generated by a_{ij}, $1 \leq i \neq j \leq n$. We define a homomorphism $\phi : B_n \to \operatorname{Aut} \mathcal{A}_n$ by defining it on the generators of B_n:

$$\phi_{\sigma_k} : \begin{cases} a_{ij} \mapsto a_{ij} & i, j \neq k, k+1 \\ a_{k+1,i} \mapsto a_{ki} & i \neq k, k+1 \\ a_{i,k+1} \mapsto a_{ik} & i \neq k, k+1 \\ a_{k,k+1} \mapsto -a_{k+1,k} & \\ a_{k+1,k} \mapsto -a_{k,k+1} & \\ a_{ki} \mapsto a_{k+1,i} - a_{k+1,k}a_{ki} & i \neq k, k+1 \\ a_{ik} \mapsto a_{i,k+1} - a_{ik}a_{k,k+1} & i \neq k, k+1 \end{cases} \tag{1}$$

Let $\iota\colon B_n \to B_{n+1}$ be the inclusion $\sigma_i \mapsto \sigma_i$ so that strand $(n+1)$ does not interact with those from $B \in B_n$, and define $\phi_B^* \in \operatorname{Aut} \mathcal{A}_{n+1}$ by $\phi_B^* = \phi_B \circ \iota$. We then define the $n \times n$ matrices Φ_B^L and Φ_B^R with entries in $\mathcal{A}_n$ by

$$\phi_B^*(a_{i,n+1}) = \sum_{j=1}^{n} (\Phi_B^L)_{ij} a_{j,n+1}$$

$$\phi_B^*(a_{n+1,i}) = \sum_{j=1}^{n} a_{n+1,j} (\Phi_B^R)_{ji}$$

Letting $\omega(B)$ be the writhe of B, define matrices $\mathbf{A}$ and $\mathbf{\Lambda}$ by

$$\mathbf{A_{ij}} = \begin{cases} a_{ij} & i < j \\ -\mu a_{ij} & i > j \\ 1 - \mu & i = j \end{cases} \tag{2}$$

$$\mathbf{\Lambda} = \mathrm{diag}[\lambda\mu^{\omega(\mathbf{B})}, \mathbf{1}, \ldots, \mathbf{1}]. \tag{3}$$

Definition Suppose that K is the closure of $B \in B_n$ and let R_0 be the Laurent polynomial ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$. Define $\mathcal{I} \subset \mathcal{A}_n \otimes R_0$ to be the ideal generated by the entries of $\mathbf{A} - \mathbf{\Lambda} \cdot \mathbf{\Phi}^{\mathbf{L}}_{\mathbf{B}} \cdot \mathbf{A}$ and $\mathbf{A} - \mathbf{A} \cdot \mathbf{\Phi}^{\mathbf{R}}_{\mathbf{B}} \cdot \mathbf{\Lambda}^{\mathbf{-1}}$. The *degree zero homology of the combinatorial knot DGA* is $\mathrm{HC}_0(K) = (\mathcal{A}_n \otimes R_0)/\mathcal{I}$.

It was shown in [Ng08] that the isomorphism class of $HC_0(K)$ is unchanged under conjugation and by positive and negative stabilization of B, hence $HC_0(K)$ is an invariant of the knot K by Markov's theorem. We only consider $HC_0(K)$ here, but there is a larger invariant, the differential graded algebra discussed in [Ng12], where the image of the differential may be generated by the same elements as $\mathcal{I}$.

The proofs in Section 3 require a number of computations of $\phi_B(a_{ij})$ for particular braids $B \in B_n$. Such computations are greatly benefited by an alternate description of the map ϕ_B, which follows, that we will use liberally.

Let D be a flat disk, to the right of B, with n points (punctures) where it intersects $K = \widehat{B}$ (see Figure 4). We assume the n punctures of D to be collinear, on a line that separates D into upper and lower half-disks. Denote by c_{ij} the isotopy class (fixing endpoints) of a path that is contained in the upper half-disk of D, with initial endpoint on the i^{th} strand and terminal endpoint on the j^{th} strand.

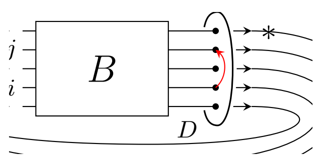

FIGURE 4. Cord c_{ij} of $K = \widehat{B}$

Considering B as a mapping class element of the punctured disk, let $B \cdot c_{ij}$ denote the isotopy class of the path to which c_{ij} is sent. Viewing D from the left (as pictured), σ_k acts by rotating the k- and $(k+1)$-punctures an angle of π about their midpoint in counter-clockwise fashion. Consider the *algebra of paths* over $\mathbb{Z}$ generated by isotopy classes of paths in D with endpoints on punctures, modulo the relation in Figure 5 (paths depicted there are understood to agree outside the neighborhood of the puncture shown). Let $F(c_{ij}) = a_{ij}$ if $i < j$, and $F(c_{ij}) = -a_{ij}$ if $i > j$. This was shown in [Ng05] to define an algebra map to $\mathcal{A}_n$ satisfying $F(B \cdot c_{ij}) = \phi_B(F(c_{ij}))$.

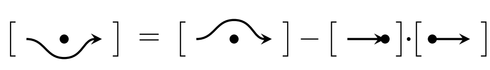

FIGURE 5. Relation in the algebra of paths

Let perm : $B_n \to S_n$ denote the homomorphism from B_n to the symmetric group sending σ_k to the transposition interchanging $k, k+1$. We will make use of the following property of ϕ_B.

Lemma 2.1. *For some $B \in B_n$ and $1 \le i \ne j \le n$, consider the element $\phi_B(a_{ij}) \in \mathcal{A}_n$ as a polynomial expression in the (non-commuting) variables $\{a_{ij}, 1 \le i \ne j \le n\}$. Writing $i' = perm(B)(i)$ and $j' = perm(B)(j)$, every non-constant monomial in $\phi_B(a_{ij})$ is a constant times $\prod_{k=0}^{l-1} a_{i_k,i_{k+1}}$, where $l \ge 1$ and $i_0 = i'$, $i_l = j'$, and $i_k \ne i_{k+1}$ for each $0 \le k \le l-1$.*

Proof. Suppose a path c in D starts at puncture p and ends at puncture q. The relation in Figure 5 equates c with a sum (or difference) of another path with the same endpoints and a product of two paths, one beginning at p and the other ending at q. A finite number of applications of this relation allows one to express c as a polynomial in the $c_{pq}, 1 \le p \ne q \le n$. The result follows since the class $B \cdot c_{ij}$ is represented by a path with endpoints the i' and j' punctures.

Alternatively, the statement follows from noting that (1) defining ϕ_{σ_k} has the desired property and that $\phi : B_n \to \mathrm{Aut}(\mathcal{A}_n)$ is a homomorphism. □

2.2. **Augmentations and augmentation rank.** Let S be a ring with 1, and consider it a differential graded algebra supported in grading 0, with trivial differential. Augmentations of $(\mathcal{A}, \partial)$ are DGA maps $(\mathcal{A}, \partial) \to (S, 0)$. For our setting, if $B \in B_n$ is a braid representative of K, such a map corresponds precisely to a homomorphism $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ such that ϵ sends each generator (mentioned in 2.1) of $\mathcal{I}$ to zero.

Definition Suppose that K is the closure of $B \in B_n$. An *augmentation* of K is a homomorphism $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ such that each element of $\mathcal{I}$ is sent by ϵ to zero.

A correspondence between augmentations and particular representations of the knot group of K were studied in [Cor13a]. Let π_K be the fundamental group of the complement of $K \subset S^3$. An element $g \in \pi_K$ is called a *meridian* if it may be represented by the boundary of an embedded disk in S^3 that intersects K in exactly one point. Recall that π_K is generated by meridians. We may fix a meridian m and generate π_K by conjugates of m.

Definition For any integer $r \ge 1$, a homomorphism $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ is a *KCH representation* if there is a meridian m of K such that $\rho(m)$ is diagonalizable and has eigenvalue 1 with multiplicity $r-1$. We call ρ a *KCH irrep* if it is irreducible.

In [Ng08], Ng describes an isomorphism between $HC_0(K)$ and an algebra constructed from elements of π_K. As discussed in [Ng12]a KCH representation $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ induces an augmentation ϵ_ρ of K. Given an augementation, the first author showed how to construct a KCH representation that induces it. In fact, we have the following rephrasing of results from [Cor13a].

Theorem 2.2 ([Cor13a])**.** *Let $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ be an augmentation with $\epsilon(\mu) \ne 1$. There is a KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$. Furthermore, for any KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$, r equals the rank of $\epsilon(\mathbf{A})$.*

Considering Theorem 2.2 we make the following definition.

Definition The *rank* of an augmentation $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ with $\epsilon(\mu) \neq 1$ is the rank of $\epsilon(\mathbf{A})$. Given a knot K, the *augmentation rank of* K, denoted $\mathrm{ar}(K)$, is the maximum rank among augmentations of K.

Remark The augmentation rank can be defined for target rings other than $\mathbb{C}$, but this paper only considers augmentations as in 2.2.

It is the case that $\mathrm{ar}(K)$ is well-defined. That is, given K there is a bound on the maximal rank of an augmentation of K.

Theorem 2.3 ([Cor13b])**.** *Given a knot $K \subset S^3$, if $g_1, \ldots, g_d$ are meridians that generate π_K and $\rho : \pi_K \to GL_r\mathbb{C}$ is a KCH irrep then $r \leq d$.*

As in the introduction, if we denote the meridional rank of π_K by $\mathrm{mr}(K)$, then Theorem 2.3 implies that $\mathrm{ar}(K) \leq \mathrm{mr}(K)$. In addition, the geometric quantity $b(K)$ called the bridge index of K is never less than $\mathrm{mr}(K)$. Thus we have the following corollary:

I made the inequality a corollary here

Corollary 2.4 ([Cor13b])**.** *Given a knot $K \subset S^3$,*

$$ar(K) \leq mr(K) \leq b(K)$$

As a result, to verify for K that $\mathrm{mr}(K) = b(K)$ it suffices to find an augmentation of K with rank equal to $b(K)$. As we discuss in the next section, we will concern ourselves in this paper with a setting where $\mathrm{ar}(K) = n$ and there is a braid $B \in B_n$ which closes to K. This is a special situation, since $b(K)$ is strictly less than the braid index for many knots.

2.3. **Finding augmentations.** The following theorem concerns the behavior of the matrices Φ^L_B and Φ^R_B under the product in B_n. It is an essential tool for studying $HC_0(K)$ and will be central to our arguments.

Theorem 2.5 ([Ng05], Chain Rule)**.** *Let B, B' be braids in B_n. Then $\Phi^L_{BB'} = \phi_B(\Phi^L_{B'}) \cdot \Phi^L_B$ and $\Phi^R_{BB'} = \Phi^R_B \cdot \phi_B(\Phi^R_{B'})$.*

The main result of this paper concerns augmentations with rank equal to the braid index of the knot K. Suppose that K is the closure of $B \in B_n$ and define the diagonal matrix $\Delta(B) = \mathrm{diag}[(-1)^{w(B)}, 1, \ldots, 1]$. The following statement follows from results in [Cor13b, Section 5].

Theorem 2.6 ([Cor13a])**.** *If K is the closure of $B \in B_n$ and has a rank n augmentation $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$, then*

$$\epsilon(\Phi^L_B) = \Delta(B) = \epsilon(\Phi^R_B). \tag{4}$$

Furthermore, any homomorphism $\epsilon : \mathcal{A}_n \to \mathbb{C}$ which satisfies (4) can be extended to $\mathcal{A}_n \otimes R_0$ to produce a rank n augmentation of K.

Our proof of Theorem 3.1 relies on this characterization of rank n augmentations. Suppose the knot K is the closure of $B \in B_k$ and has a rank k augmentation ϵ_k. In Section 3 we consider $B' \in B_p$ which has closure admitting a rank p augmentation ϵ_p. Applying the braid satellite construction to B, B' we obtain a satellite of K. We prove the theorem in Section 3 by describing a map from ϵ_k and ϵ_p that satisfies (4) for the braid satellite. By Theorem 4 this determines the desired rank pk augmentation.

described in the introduction...

There is a symmetry on the matrices Φ_B^L and Φ_B^R that is relevant to the study of augmentations in this setting. Define an involution $x \mapsto \overline{x}$ on $\mathcal{A}_n$ (termed *conjugation*) as follows: first set $\overline{a_{ij}} = a_{ji}$; then, for any $x, y \in \mathcal{A}_n$, define $\overline{xy} = \overline{y}\,\overline{x}$ and extend the operation linearly to $\mathcal{A}_n$. We have the following symmetry.

Theorem 2.7 ([Ng05], Prop. 6.2)**.** *For a matrix of elements in $\mathcal{A}_n$, let $\overline{M}$ be the matrix such that $\left(\overline{M}\right)_{ij} = \overline{M_{ij}}$. Then for $B \in B_n$, Φ_B^R is the transpose of $\overline{\Phi_B^L}$.*

It may be appropriate here to indicate that $\mathrm{ar}(K) < \mathrm{mr}(K)$ sometimes (maybe in previous subsection), and talk about the 2-cable of the trefoil that does not have $\mathrm{ar}(K, \mathbb{C}) = 4$

3. Main Result

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bring in equations to fit margins

introduce what will be done in the section: first section on notation, main result will follow from Lemma ____ and the Chain Rule

Let K be a knot and let B be a braid with closure K.
We then have the following result.

Theorem 3.1. *Let $B \in B_k$, and let $B' \in B_{pk}$ be a braid in B_p included into B_{pk} such that the first p strands of B' close to a knot. Suppose that there exists an augmentation $\epsilon_k \colon \mathcal{A}_k \to \mathbb{C}$ such that $\epsilon_k\left(\Phi_B^L\right) = \epsilon_k\left(\Phi_B^R\right) = \Delta(B)$ and an augmentation $\epsilon_p \colon \mathcal{A}_p \to \mathbb{C}$ such that $\epsilon_p\left(\Phi_{B'}^L\right) = \epsilon_p\left(\Phi_{B'}^R\right) = \Delta(B')$. Then there exists an augmentation $\epsilon \colon \mathcal{A}_{pk} \to \mathbb{C}$ such that $\epsilon\left(\Phi_{B^{(p)}B'}^L\right) = \epsilon\left(\Phi_{B^{(p)}B'}^R\right) = \Delta(B^{(p)}B')$.*

Among other things, this theorem implies that iterated cables of torus knots have meridional rank equal to their bridge number. Consider a (r, s)-torus knot T with $\gcd(r, s) = 1$ and $r < s$. T has bridge number r and is the closure of a braid B on r strands, and since all torus knots have bridge number equal to their augmentation rank (), we have that there exists an augmentation $\epsilon_T \colon \mathcal{A}_r \to \mathbb{C}$. . given by the braid sum of $T^{(p)}$ with a braid who's first p strands form a torus knot with bridge number (and therefore augmentation rank) equal to p and such that $w(T)$ is even (i.e. a (p, q) torus knot, where $\gcd(p, q) = 1$, $p < q$, and $pq - q$ is even). Theorem 3.1 then says that this cable has augmentation rank equal to its braid index, implying

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that its meridional rank is equal to its bridge number. Furthermore, we can iterate this process, taking cables of the resulting knots with augmentation rank, bridge number, and braid index all equal.

prob from Kirby list should be mentioned here

Fix $p > 0$ and let B be a braid on k strands. For each $1 \leq i \leq pk$ define integers q_i, r_i such that $i = q_i p + r_i$, where $0 < r_i \leq p$. Instrumental to the proof of Theorem 3.1 will be the map $\psi\colon \mathcal{A}_{pk} \to \mathcal{A}_k \otimes \mathcal{A}_p$, defined as follows

$$\psi(a_{ij}) = \begin{cases} 1 \otimes a_{r_i r_j} & : q_i = q_j \\ a_{q_i+1, q_j+1} \otimes 1 & : r_i = r_j \\ 0 & : (q_i - q_j)(r_i - r_j) < 0 \\ a_{q_i+1, q_j+1} \otimes a_{r_i r_j} & : (q_i - q_j)(r_i - r_j) > 0 \end{cases}$$

Note that $\psi(a_{ij}) \in 1 \otimes \mathcal{A}_p$ or $\psi(a_{ij}) = 0$ if and only if $q_i = q_j$, and that $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$. This homomorphism gives us a way of relating $\Phi^L_{B^{(p)}}$ to Φ^L_B via the following proposition.

Proposition 3.2. *For any braid B, $\psi\left(\Phi^L_{B^{(p)}}\right) = \left(\left(\Phi^L_B\right)_{ij} \otimes 1\right) \otimes I_p$ and $\psi\left(\Phi^R_{B^{(p)}}\right) = \left(\left(\Phi^R_B\right)_{ij} \otimes 1\right) \otimes I_p$*

make consistent throughout paper

Note that here we mean the tensor product of Φ^L_B and I_p as matrices, not as linear maps, while the tensor product of $\left(\Phi^L_B \otimes I_p\right)_{ij}$ and 1 is a tensor product of algebra elements, so that if we divide the matrix $\psi(\Phi^L_{B^{(p)}}$ into k^2 $p \times p$ blocks, the ijth block is $\left(\Phi^L_B\right)_{ij} I_p$.

It turns out that instead of ψ we could have defined a simpler homomorphism $\rho\colon \mathcal{A}_{pk} \to \mathcal{A}_k$ that would take a_{ij} to $a_{q_{i+1}, q_{j+1}}$ if $r_i = r_j$ and 0 otherwise, and Proposition 3.2 would still be true (this follows from the same ideas used in the proof of Proposition 3.2). The advantage of ψ is that it doesn't send a_{ij} to 0 if $q_i = q_j$, a fact which will be important in the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $\delta\colon \mathcal{A}_p \to \mathbb{C}$ be a homomorphism, let $\pi\colon \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ be a homomorphism defined by $\pi(a \otimes b) = ab$, and set $\epsilon = \pi \circ (\epsilon_k \otimes \delta) \circ \psi$. We will later break the theorem up into three cases depending on the parity of $w(B)$ and p and in each case define δ such that $\delta(a_{ij})$ is one of $\pm\epsilon_p(a_{ij})$ in such a way that ϵ is an augmentation of $B^{(p)}B'$. The Chain Rule theorem gives that

$$\pi \circ (\epsilon_k \otimes \delta) \circ \psi\left(\Phi^L_{B^{(p)}B'}\right) = \pi \circ (\epsilon_k \otimes \delta) \psi\left(\phi_{B^{(p)}}\left(\Phi^L_{B'}\right)\right) \psi\left(\Phi^L_{B^{(p)}}\right) \tag{5}$$

Note that since the non zero or one entries of $\Phi^L_{B'}$ are products of a_{ij} where $i < j \leq p$, $\phi_{B^{(p)}}$ takes each of the a_{ij}'s in these products to $a_{i+mp, j+mp}$ for some $0 \leq m < k$. We have that ψ takes $a_{i+mp, j+mp}$ to $1 \otimes a_{ij}$, however, so

$$\psi\left(\phi_{B^{(p)}}\left(\Phi^L_{B'}\right)\right) = \left(1 \otimes \left(\Phi^L_{B'}\right)_{ij}\right)$$

By Proposition 3.2, we have that

$$\psi\left(\Phi^L_{B^{(p)}}\right) = \left(\left(\Phi^L_B\right)_{ij} \otimes 1\right) \otimes I_p = \left(\left(\Phi^L_B \otimes I_p\right)_{ij} \otimes 1\right)$$

So returning to the right hand side of (5) we get

$$\begin{aligned}\pi \circ (\epsilon_k \otimes \delta)\left(\psi\left(\phi_{B^{(p)}}\left(\Phi^L_{B'}\right)\right)\psi\left(\Phi^L_{B^{(p)}}\right)\right) &= \pi \circ (\epsilon_k \otimes \delta)\left(\left(1 \otimes \left(\Phi^L_{B'}\right)_{ij}\right)\left(\left(\Phi^L_B \otimes I_p\right)_{ij} \otimes 1\right)\right) \\ &= \pi\left(\left(\Delta(B) \otimes I_p\right)\delta\left(\Phi^L_{B'}\right)\right)\end{aligned}$$

So it suffices to find an augmentation δ such that the right hand side is equal to $\Delta(B^{(p)}B')$. If $w(B)$ is even, then we simply let $\delta = \epsilon_p$. Since $w(B)$ is even we know that $w(B^{(p)})$ is also even and that $\Delta(B) = I_k$. Since $\epsilon_p\left(\Phi^L_{B'}\right) = \Delta(B')$, it follows that the right hand side is equal to $\Delta(B^{(p)}B')$.

Now suppose that $w(B)$ is odd. In a moment we will define $g\colon \{1, \ldots, p\} \to \{\pm 1\}$ for each of the cases for when p is even or odd, but for now let $\delta(a_{ij}) = g(i)g(j)e_k(a_{ij})$. Fix i, j and consider a monomial M in $\left(\Phi^L_{B'}\right)_{ij}$. Since B' is a braid on p strands included into B_{pk}, if $i > p$ or $j > p$ then M is 0 or 1 and $\delta(M) = M$. If $i, j \leq p$, such a monomial must arise from a product in the algebra of paths in D that begins at $i' = \mathrm{perm}(B')(i)$ and ends at j, so $M = c_{ij}a_{i',j_1}a_{j_1,j_2} \ldots a_{j_m,j}$ for some $j_1, \ldots j_m \in \{1, \ldots, p\}$, unless $i' = j$, in which case it is possible that $M = c_{ij}$. We then see that

just introducing this notation

$$\delta(M) = g(i')g(j)\left(\prod_{k=1}^{m} g(j_k)^2\right)\epsilon_p(M) = g(i')g(j)\epsilon_p(M)$$

Or $\delta(M) = M = g(i')g(j)\epsilon_p(M)$ in the case that $i' = j$ and $M = c_{ij}$. Since this is true for each monomial M chosen in $\left(\Phi^L_{B'}\right)_{ij}$, we have that

$$\delta\left(\left(\Phi^L_{B'}\right)_{ij}\right) = g(i')g(j)\epsilon_p\left(\left(\Phi^L_{B'}\right)_{ij}\right)$$

Now let $x_1 = 1$, and $x_l = \mathrm{perm}(B')(x_{l-1})$ for $1 < l \leq p$. Since the first p strands of B' close to a knot, $\mathrm{perm}(B')$ is given by the p-cycle $(x_1x_2 \ldots x_p)$.

Suppose p is even. Then we let $g(x_1) = 1$, and $g(x_l) = -g(x_{l-1})$ for $1 < l \leq p$. Since p is even, $w(B^{(p)})$ is even and therefore the opposite parity of $w(B)$. Our definition of g gives that $\delta\left(\left(\Phi^L_{B'}\right)_{ii}\right) = -\epsilon\left(\left(\Phi^L_{B'}\right)_{ii}\right)$ for $i \leq p$, so

$$\delta\left(\Phi^L_{B'}\right) = \begin{pmatrix} (-1)^{w(B')+1} & 0 & 0 \\ 0 & -I_{p-1} & 0 \\ 0 & 0 & I_{(k-1)p} \end{pmatrix}$$

and therefore

$$\pi\left(\left(\Delta(B) \otimes I_p\right)\delta\left(\Phi^L_{B'}\right)\right) = \mathrm{diag}[(-1)^{w(B)+w(B')+1}, 1 \ldots 1] = \Delta(B^{(p)}B')$$

as desired.

Next suppose that p is odd. Then we let $g(x_1) = g(x_2) = 1$ and $g(x_l) = -g(x_{l-1})$ for $2 < l \leq p$. Since p is odd, $w(B^{(p)})$ is odd and therefore the same parity of $w(B)$. Our definition of g gives that $\delta\left(\left(\Phi^L_{B'}\right)_{11}\right) = \epsilon\left(\left(\Phi^L_{B'}\right)_{11}\right)$ and $\delta\left(\left(\Phi^L_{B'}\right)_{ii}\right) = -\epsilon\left(\left(\Phi^L_{B'}\right)_{ii}\right)$ for $1 < i \leq p$, so

$$\delta\left(\Phi^L_{B'}\right) = \begin{pmatrix} (-1)^{w(B')} & 0 & 0 \\ 0 & -I_{p-1} & 0 \\ 0 & 0 & I_{(k-1)p} \end{pmatrix}$$

and therefore

$$\pi\left((\Delta(B) \otimes I_p)\, \delta\left(\Phi^L_{B'}\right)\right) = \operatorname{diag}[(-1)^{w(B)+w(B')}, 1 \dots 1] = \Delta(B^{(p)}B')$$

as desired. Similarly, we have that

$$\pi \circ (\epsilon_k \otimes \delta) \circ \psi\left(\Phi^R_{B^{(p)}B'}\right) = \pi \circ (\epsilon_k \otimes \delta)\left(\left(\left(\Phi^R_B \otimes I_p\right)_{ij} \otimes 1\right)\left(1 \otimes \left(\Phi^R_{B'}\right)_{ij}\right)\right)$$

but since $\epsilon_k\left(\Phi^L_B\right) = \epsilon_k\left(\Phi^R_B\right)$ and $\epsilon_p\left(\Phi^L_{B'}\right) = \epsilon_p\left(\Phi^R_{B'}\right)$, in each case above we have

$$\begin{aligned}\pi \circ (\epsilon_k \otimes \delta)\left(\left(\left(\Phi^R_B \otimes I_p\right)_{ij} \otimes 1\right)\left(1 \otimes \left(\Phi^R_{B'}\right)_{ij}\right)\right) &= \pi \circ (\epsilon_k \otimes \delta)\left(\left(\left(\Phi^L_B \otimes I_p\right)_{ij} \otimes 1\right)\left(1 \otimes \left(\Phi^L_{B'}\right)_{ij}\right)\right) \\ &= \Delta\left(B^{(p)}B'\right)\end{aligned}$$

Which completes the proof. $\square$

We will use the following two lemmas in our proof of Proposition 3.2.

Lemma 3.3. *$\psi(\phi_{\Sigma_n^{\pm(p)}}(a_{ij})) = (\phi_{\sigma_n^{\pm 1}} \otimes \mathrm{id})(\psi(a_{ij}))$ for all $1 \leq n < k$, $1 \leq i, j \leq pk$.*

Lemma 3.4. $\psi\left(\Phi^L_{\Sigma_n^{\pm(p)}}\right) = \left(\left(\Phi^L_{\sigma_n^{\pm 1}}\right)_{ij} \otimes 1\right) \otimes I_p$

Proof of Proposition 3.2. Let $B = \sigma_{n_1}^{q_1} \cdots \sigma_{n_r}^{q_r}$, where $1 \leq n_i < k$ and $q_i = \pm 1$. We will prove the proposition by inducting on r. The base case is already taken care of by Lemma 3.4. Suppose that the proposition holds for braids of length $r-1$. Let $B' = \sigma_{n_1}^{q_1} \cdots \sigma_{n_{r-1}}^{q_{r-1}}$ Then by the Chain Rule and Lemmas 3.3 and 3.4, we have that

$$\begin{aligned}\psi\left(\Phi^L_{B^{(p)}}\right) &= \psi\left(\phi_{B'^{(p)}}\left(\Phi^L_{\Sigma_{n_r}^{q_r(p)}}\right) \cdot \Phi^L_{B'^{(p)}}\right) \\ &= (\phi_{B'} \otimes \mathrm{id})\left(\psi\left(\Phi^L_{\Sigma_{n_r}^{q_r(p)}}\right)\right) \cdot \left(\left(\Phi^L_{B'}\right)_{ij} \otimes 1\right) \otimes I_p \\ &= (\phi_{B'} \otimes \mathrm{id})\left(\left(\left(\Phi^L_{\sigma_{n_r}^{q_r}}\right)_{ij} \otimes 1\right) \otimes I_p\right) \cdot \left(\left(\Phi^L_{B'}\right)_{ij} \otimes 1\right) \otimes I_p \\ &= \left(\left(\Phi^L_B\right)_{ij} \otimes 1\right) \otimes I_p\end{aligned}$$

Which implies that $\psi\left(\Phi^R_{B^{(p)}}\right) = \left(\left(\Phi^R_B\right)_{ij} \otimes 1\right) \otimes I_p$ as well, since $\Phi^R_B = \overline{\Phi^L_B}^t$ and $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$. $\square$

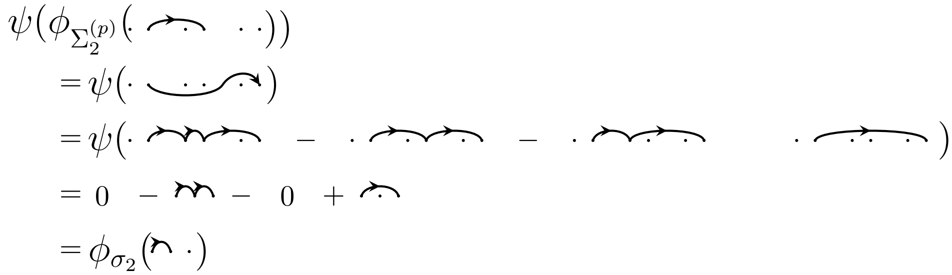

FIGURE 6. Computing $\psi(\phi_{\Sigma_2^{(p)}}(a_{24}))$

give setting for figure 6

recall definition of $\tau_{m,l}$

In the proof of Lemmas 3.3 and 3.4, we will make use of some calculations of $\phi_B(a_{ij})$ for simple braids B. It can easily be checked that for all $1 \leq m < n$, $1 \leq l \leq n-m$, $i < j$:

$$\phi_{\tau_{m,l}}(a_{ij}) = \begin{cases} a_{i+1,j+1} & : m \leq i < j < m+l \\ a_{i-l,j} & : m < m+l = i < j \\ a_{i,j-l} & : i < m < m+l = j \\ a_{i+1,j-l} & : m \leq i < j = m+l \\ a_{i,j+1} - a_{i,m}a_{m,j+1} & : i < m \leq j < m+l \\ a_{i+1,j} - a_{i+1,m}a_{m,j} & : m \leq i < m+l < j \\ a_{ij} & : \text{otherwise} \end{cases} \tag{6}$$

We also make the following definition
Let $X \subseteq \{1, \ldots, n\}$, and write the elements of a subset $Y \subseteq X$ as $y_1 < \ldots < y_k$. Define

$$A(i,j,X) = \sum_{Y \subseteq X} (-1)^{|Y|} a_{iy_1} a_{y_1y_2} \cdots a_{y_kj}$$

and

$$A'(i,j,X) = \sum_{Y \subseteq X} (-1)^{|Y|} a_{iy_k} a_{y_ky_{k-1}} \cdots a_{y_1j}$$

and have the following lemma

Changed $X_{m,l}$ by adding p to everything, need to make sure still works with proof

check

Lemma 3.5. *Suppose $i < j$. Let $\kappa_{m,l} = \tau_{m+l-1,p}\tau_{m+l-2,p} \cdots \tau_{m,p}$, and let $X_{m,l} = \{m, \ldots, m+l-1\}$. Then*

$$\phi_{\kappa_{m,l}}(a_{ij}) = \begin{cases} a_{i-p,j-p} & : m+p \leq i < j < m+l+p \\ a_{i-p,j} & : m+p \leq i < m+l+p \leq j \\ a_{i,j-p} & : i < m < m+p \leq j < m+l+p \\ a_{i+l,j+l} & : m \leq i < j < m+p \\ A'(i+l, j-p, X_{m,l} \setminus (j-p)) & : m \leq i < m+p \leq j < m+l+p \\ A(i, j+l, X_{m,l}) & : i < m \leq j < m+p < m+l+p \\ A'(i+l, j, X_{m,l}) & : m \leq i < m+p < m+l+p \leq j \\ a_{ij} & : \text{otherwise} \end{cases}$$

Note that letting $l = p$ and $m = (n-1)p + 1$ gives us $\phi_{\Sigma_n^{(p)}}(a_{ij})$ when $i < j$ as a special case. Letting $X_n^{(p)} = \{(n-1)p+1, \ldots, np\}$, we have

$$\phi_{\Sigma_n^{(p)}}(a_{ij}) = \begin{cases} a_{i-p,j-p} & : np < i < j \leq (n+1)p \\ a_{i-p,j} & : np < i \leq (n+1)p < j \\ a_{i,j-p} & : i \leq (n-1)p < np < j \leq (n+1)p \\ a_{i+p,j+p} & : (n-1)p < i < j \leq np \\ A'(i+p, j-p, X_n^{(p)} \setminus (j-p)) & : (n-1)p < i \leq np < j \leq (n+1)p \\ A(i, j+p, X_n^{(p)}) & : i \leq (n-1)p < j \leq np < (n+1)p \\ A'(i+p, j, X_n^{(p)}) & : (n-1)p < i \leq np < (n+1)p < j \\ a_{ij} & : \text{otherwise} \end{cases}$$

Proof of Lemma 3.3. Note that if $\psi(\phi_{\Sigma_n^{(p)}}(a_{ij})) = (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))$, then

$$(\phi_{\sigma_n} \otimes \mathrm{id}) \left(\psi \left(\phi_{\Sigma_n^{-(p)}} (a_{ij}) \right) \right) = \psi(a_{ij})$$

So $\psi \circ \phi_{\Sigma_n^{-(p)}}$ is the inverse function of $(\phi_{\sigma_n} \otimes \mathrm{id})$, and therefore

$$\psi \left(\phi_{\Sigma_n^{-(p)}} (a_{ij}) \right) = \left(\phi_{\sigma_n^{-1}} \otimes \mathrm{id} \right) (a_{ij})$$

Furthermore, $\phi_B(\overline{a_{ij}}) = \overline{\phi_B(a_{ij})}$and $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$, so it suffices to prove the lemma for $\Sigma_n^{(p)}$ in the case where $i < j$.

does this need justification?

With these restrictions, we then break the statement up into the cases from Lemma 3.5, from which the first four cases as well as the last case can be checked easily. Consider the sixth case. Lemma 3.5 gives that

$$\psi \left(\phi_{\Sigma_n^{(p)}}(a_{ij}) \right) = \sum_{Y \subseteq \{np-p+1, \ldots, np\}} (-1)^{|Y|} \psi \left(a_{iy_1} a_{y_1 y_2} \cdots a_{y_k, j+p} \right)$$

Let $\alpha_i = np - p + r_i$. Note that if $y_1 < \alpha_i$ then $\psi(a_{iy_1}) = 0$, and if $y_k > \alpha_j$ then $\psi(a_{y_k j}) = 0$, so the sum on the right hand side can be taken over $Y \subseteq \{\alpha_i, \alpha_i + 1, \ldots, \alpha_j\}$. Then we manipulate the sum to get

do I need to explain what I'm doing here?

$$\begin{aligned} &\sum_{Y \subseteq \{\alpha_i, \ldots, \alpha_j\}} (-1)^{|Y|} \psi \left(a_{iy_1} a_{y_1 y_2} \cdots a_{y_k, j+p} \right) \\ &= \psi \left(a_{i,j+p} - a_{i,\alpha_i} a_{\alpha_i, j+p} \right) \\ &\quad + \sum_{y=\alpha_i+1}^{\alpha_j} \sum_{Y \subseteq \{y+1, \ldots, \alpha_j\}} (-1)^{|Y|+1} \psi \left(a_{iy} a_{yy_1} \cdots a_{y_k, j+p} \right) + (-1)^{|Y|} \psi \left(a_{i,\alpha_i} a_{\alpha_i, y} a_{yy_1} \cdots a_{y_k, j+p} \right) \\ &= \psi \left(a_{i,j+p} - a_{i\alpha_i} a_{\alpha_i, j+p} \right) \\ &\quad + \sum_{y=\alpha_i+1}^{\alpha_j} \sum_{Y \subseteq \{y+1, \ldots, \alpha_j\}} (-1)^{|Y|} \psi \left(a_{i,\alpha_i} a_{\alpha_i, y} - a_{iy} \right) \psi \left(a_{yy_1} \cdots a_{y_k, j+p} \right) \end{aligned}$$

Note that $r_i = r_{\alpha_i}$ and since we're in the sixth case we have $(n-1)p < j \leq np$, so $q_{\alpha_i} = q_y$. Thus $\psi(a_{i,\alpha_i}) = a_{q_i+1,q_{\alpha_i}+1} \otimes 1 = a_{q_i+1,q_y+1} \otimes 1$ and $\psi(a_{\alpha_i,y}) = 1 \otimes a_{r_{\alpha_i},r_y} = 1 \otimes a_{r_i,r_y}$, so we have

$$\psi(a_{i,\alpha_i}a_{\alpha_i,y} - a_{iy}) = \left(a_{q_i+1,q_y+1} \otimes 1\right)\left(1 \otimes a_{r_i,r_y}\right) - a_{q_i+1,q_y+1} \otimes a_{r_i,r_y} = 0$$

Thus the right hand side reduces to

$$\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right)$$

Remark The fact that $\psi(a_{i,\alpha_i}a_{\alpha_i,y} - a_{iy}) = 0$ and ψ behaves similarly for the analogous terms in the other cases is the key to this proof working, and ψ is defined the way it is mainly so that this will be true. As we hinted at earlier, the homomorphism $\rho\colon \mathcal{A}_{pk} \to \mathcal{A}_k$ defined to send a_{ij} to a_{q_i+1,q_j+1} if $r_i = r_j$ and to 0 otherwise would also send these terms to 0, so Proposition 3.2 would still be true with ρ used in the place of ψ. We will need ψ for the proof of the main result, however.

Note that, since we're in the sixth case, $q_j + 1 = n$. If $r_i = r_j$, then

$$\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right) = \left(a_{q_i+1,n+1} - a_{q_i+1,n}a_{n,n+1}\right) \otimes 1 = (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))$$

If $r_i < r_j$, then

$$\begin{aligned}\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right) &= \left(a_{q_i+1,n+1} \otimes a_{r_ir_j} - a_{q_i+1,n}a_{n,n+1} \otimes a_{r_ir_j}\right)\\ &= \left(a_{q_i+1,n+1} - a_{q_i+1,n}a_{n,n+1}\right) \otimes a_{r_ir_j}\\ &= (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))\end{aligned}$$

Finally, if $r_i > r_j$, then

$$\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right) = 0 = (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))$$

The proof for the seventh case goes exactly as the proof for the sixth case except with all i's replaced with $i+p$, all $(j+p)$'s replaced with j, all y_i's replaced with y_{k+1-i}, and with α_i and α_j swapped. The proof for the fifth case goes exactly as the proof for the seventh, except that $j-p$ is removed from the set that Y is a subset of in all the sums. $\square$

check this

check

Proof of Lemma 3.4. First we will prove the lemma for $\Sigma_n^{(p)}$. We can extend the definition of ψ to be from the free module over $\mathcal{A}_{pk}$ generated by $\{a_{i*} | 1 \leq i \leq pk\}$ to the free module over $\mathcal{A}_k \otimes \mathcal{A}_p$ generated by $\{a_{i*} | 1 \leq i \leq k\}$ by defining $\psi(a_{i*}) = a_{i*}$ and extending by linearity. Then the statement of the lemma is equivalent to saying that for all $1 \leq i \leq pk$, the coefficient of a_{j*} in $\psi\left(\phi_{\Sigma_n^{(p)}}(a_{i*})\right)$ is equal to 0 unless $r_j = r_i$, in which case it is equal to the coefficient of a_{q_j*} in $\phi_{\sigma_n}(a_{q_i*})$. If $q_i + 1 \neq n$, this fact can be easily checked. In the case that $q_i + 1 = n$, we have that

$$\psi\left(\phi_{\Sigma_n^{(p)}}(a_{i*})\right) = \psi\left(A(i+p, *, \{np-p+1, \ldots, np\}\right)$$

which is equal to

$$\psi(a_{i+p,*} - a_{i+p,\alpha_i}a_{\alpha_i,*}) = a_{i+p,*} - (a_{q_i,q_i+1} \otimes 1)a_{\alpha_i,*}$$

by the same argument that was used in Lemma 3.3. The coefficients of the a_{j*} are equal to the coefficients of the a_{q_j*} in $\phi_{\sigma_n}(a_{q_i*})$, so we have $\psi\left(\Phi^L_{\Sigma_n^{(p)}}\right) = \left(\left(\Phi^L_{\sigma_n}\right)_{ij} \otimes 1\right) \otimes I_p$.

Using this fact, the Chain Rule, and Lemma 3.3, we have

$$\begin{aligned}\left((I_{pk})_{ij} \otimes 1\right) &= \psi\left(\Phi^L_{\Sigma_n^{(p)}\Sigma_n^{-(p)}}\right)\\ &= \psi\left(\phi_{\Sigma_n^{-(p)}}\left(\Phi^L_{\Sigma_n^{(p)}}\right)\right)\psi\left(\Phi^L_{\Sigma_n^{-(p)}}\right)\\ &= \left(\phi_{\sigma_n^{-1}} \otimes \mathrm{id}\right)\left(\left((\Phi_{\sigma_n})_{ij} \otimes 1\right) \otimes I_p\right)\psi\left(\Phi^L_{\Sigma_n^{-(p)}}\right)\end{aligned}$$

But note that the Chain Rule also gives that $\left(\left(\Phi^L_{\sigma_n^{-1}}\right)_{ij} \otimes 1\right) \otimes I_p$ is the inverse of $\left(\phi_{\sigma_n^{-1}} \otimes \mathrm{id}\right)\left(\left((\Phi_{\sigma_n})_{ij} \otimes 1\right) \otimes I_p\right)$, so

$$\psi\left(\Phi^L_{\Sigma_n^{-(p)}}\right) = \left(\left(\Phi^L_{\sigma_n^{-1}}\right)_{ij} \otimes 1\right) \otimes I_p$$

which completes the proof. □

Proof of Lemma 3.5. add other cases

The first four cases as well as the eighth can be easily checked. We will prove the remaining cases by induction on l. Consider the sixth case. The base case is covered by (6). For the inductive step, we have that

$$\begin{aligned}\phi_{\kappa_{m,l}}(a_{ij}) &= \phi_{\tau_{m,p}}\left(\phi_{\kappa_{m+1,l-1}}(a_{ij})\right)\\ &= \sum_{Y \subseteq \{m+p+1,\dots,m+l+p-1\}} (-1)^{|Y|}\phi_{\tau_{m,p}}\left(a_{i-l+1,y_1}a_{y_1y_2}\cdots a_{y_k,j}\right)\\ &= \sum_{Y \subseteq \{m+p+1,\dots,m+l+p-1\}} (-1)^{|Y|}\left(a_{i-l,y_1} - a_{i-l,m+p}a_{m+p,y_1}\right)a_{y_1y_2}\cdots a_{y_k,j}\\ &= \sum_{Y \subseteq \{m+p,\dots,m+l+p-1\}} (-1)^{|Y|}a_{i-l,y_1}a_{y_1y_2}\cdots a_{y_k,j}\\ &= A(i-l,j,X_{m,l})\end{aligned}$$

□ is this clear/can it be shortened?

References

[Cor13a] C. Cornwell. KCH representations, augmentations, and A-polynomials, 2013. arXiv: 1310.7526.

[Cor13b] C. Cornwell. Knot contact homology and representations of knot groups. arXiv: 1303.4943, 2013.

[EENS13] T. Ekholm, J. Etnyre, L. Ng, and M. Sullivan. Knot contact homology. *Geom. Topol.*, 17:975–1112, 2013.

[Kir95] R. (Ed.) Kirby. Problems in low-dimensional topology. In *Proceedings of Georgia Topology Conference, Part 2*, pages 35–473. Press, 1995.

[Ng05] L. Ng. Knot and braid invariants from contact homology I. *Geom. Topol.*, 9:247–297, 2005.

[Ng08] L. Ng. Framed knot contact homology. *Duke Math. J.*, 141(2):365–406, 2008.

[Ng12] L. Ng. A topological introduction to knot contact homology, 2012. arXiv: 1210.4803.