List of Todos

AUGMENTATION RANK OF SATELLITES WITH BRAID PATTERN

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Abstract. Given a knot K in S^3, a question raised by Cappell and Shaneson asks if the meridional rank of K equals the bridge number of K. Using augmentations in knot contact homology we consider the persistence of equality between these two invariants under satellite operations on K with a braid pattern. In particular, we answer the question in the affirmative for a large class of iterated torus knots.

1. Introduction

Let K be an oriented knot in S^3 and denote by π_K the fundamental group of its complement $\overline{S^3 \setminus n(K)}$, with some basepoint. We call an element of π_K a *meridian* if it is represented by the oriented boundary of a disc, embedded in S^3, whose interior intersects K positively once. The group π_K is generated by meridians; the *meridional rank* of K, written $\mathrm{mr}(K)$, is the minimal size of a generating set containing only meridians.

Choose a height function $h : S^3 \to \mathbb{R}$. The *bridge number* of K, denoted $b(K)$, is the minimum of the number of local maxima of $h|_{\varphi(S^1)}$ among $\varphi : S^1 \to S^3$ which realize K.

A basic argument shows that $\mathrm{mr}(K) \leq b(K)$ for any $K \subset S^3$. Whether the bound is equality for all knots is an open question attibuted to Cappell and Shaneson [Kir95, Prob. 1.11]. Equality is known to hold for some families of knots due to work of various authors.

Here we study the behavior of *augmentations*, which are maps that arise in the study of knot contact homology. To each augmentation is associated a rank, and there is a maximal rank of augmentations of a given K, called the *augmentation rank* $\mathrm{ar}(K)$. For any K the inequality $\mathrm{ar}(K) \leq \mathrm{mr}(K)$ holds (see Section 2.2 below).

Our main result indicates how $\mathrm{ar}(K)$ behaves under satellite operations with a braid pattern, provided the companion and pattern have sufficiently large augmentation rank. To be precise let $B \in B_k$, and $B' \in B_p$, be braids on k, and p, strands respectively. Write $\widehat{B}$ for the *braid closure* of B (see Section 2, Figure 2) and set $K = \widehat{B}$. Throughout the paper only braid closures that are knots are considered.

Definition 1.1. Denote by $_{\bar{p}}B$ the braid in B_{kp} which consists of each strand of B being replaced by p parallel copies of that strand (in the blackboard framing). Upon including B' into B_{kp} by juxtaposing it with an identity braid on $(k-1)p$ strands, define the *braid satellite* of K associated to B, B' to be $K(B, B') = \widehat{{}_{\bar{\bar{p}}}B \cdot B'}$.

make a figure

When $\widehat{B}$ and $\widehat{B'}$ are each a knot, $K(B, B')$ is also. We show the following.

remark on difference from $B'(K)$ in satellite literature

Theorem 1.2. *If $B \in B_k$ is such that $\text{ar}(\widehat{B}) = k$ and $B' \in B_p$ such that $\text{ar}(\widehat{B'}) = p$, then $\text{ar}(K(B, B')) = kp$.*

Remark 1.3. It is natural to ask if the augmentation rank is multiplicative under weaker assumptions on B, B'. Braid stabilization can show it is submultiplicative at best. In Section 4 we show that it is not submultiplicative in general.

We obtain a corollary involving Cappell and Shaneson's question for iterated torus knots. Let $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$ be integral vectors. We write $T(\mathbf{p}, \mathbf{q})$ for the $(\mathbf{p}, \mathbf{q})$ *iterated torus knot*, defined as follows. Define $T(\mathbf{p}, \mathbf{q})$ inductively so that, if $\widehat{\mathbf{p}}, \widehat{\mathbf{q}}$ are the truncated lists obtained from $\mathbf{p}, \mathbf{q}$ by removing the last integer in each, then $T(\mathbf{p}, \mathbf{q})$ is the (p_n, q_n)-cable of $T(\widehat{\mathbf{p}}, \widehat{\mathbf{q}})$. By convention $T(\emptyset, \emptyset)$ is the unknot.

The knot $T(\mathbf{p}, \mathbf{q})$ is well-defined only if a framing convention is chosen at each stage of cabling. In contrast to the traditional choice of Seifert framing at each stage, we choose a framing so that if B is of minimal braid index such that $T(\widehat{\mathbf{p}}, \widehat{\mathbf{q}}) = \widehat{B}$ then $T(\mathbf{p}, \mathbf{q}) = K(B, (\sigma_1 \dots \sigma_{p_n - 1})^{q_n})$.

Corollary 1.4. *Given integral vectors $\mathbf{p}$ and $\mathbf{q}$, suppose that $|p_i| < |q_i|$ and $gcd(p_i, q_i) = 1$ for each $1 \le i \le n$. Then $\text{ar}(T(\mathbf{p}, \mathbf{q})) = p_1 p_2 \dots p_n$. Moreover, $\text{mr}(T(\mathbf{p}, \mathbf{q})) = b(T(\mathbf{p}, \mathbf{q}))$ for all such $\mathbf{p}$ and $\mathbf{q}$.*

The paper is organized as follows. In Section 2 we give the needed background in knot contact homology and discuss the rank of augmentations and the relationship to the meridional rank. In Section 2.3 we review some techniques that will be used in the proof of Theorem 1.2. Section 3 is mainly devoted to the proof of Theorem 1.2, and in Section 4 we consider the more general case.

2. Background

We review in Section 2.1 the construction of $HC_0(K)$ from the viewpoint of the combinatorial knot DGA, which was first defined in [Ng08]; our conventions are those given in [Ng12]. In Section 2.2 we discuss augmentations in knot contact homology and their rank, which gives a lower bound on the meridional rank of the knot group useful for studying the relation between meridional rank and bridge number. Finally, in Section 2.3 is a discussion of techniques from [Cor13a] that we use to calculate the augmentation rank.

Throughout the paper we denote by B_n the group of n-braids. We orient n-braids from left to right, labeling the strands $1, \dots, n$, with 1 the topmost and n the bottommost strand. We work with Artin's generators $\{\sigma_i^{\pm}, i = 1, \ldots, n-1\}$ of B_n, where in σ_i only the i and $i+1$ strands interact, and they cross once in the manner depicted in Figure 1. Given a braid B the

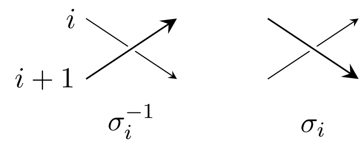

FIGURE 1. Generators of B_n

braid closure $\widehat{B}$ of B is the link obtained as shown in Figure 2. The *writhe* (or algebraic length) of B, denoted $\omega(B)$, is the sum of exponents of the Artin generators in a word representing B.

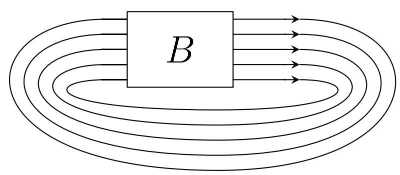

FIGURE 2. The braid closure of B

2.1. **Knot contact homology.** We review the construction of the combinatorial knot DGA of Ng (in fact, we discuss only the degree zero part as this will suffice for our purposes). This DGA was defined in order to be a calculation of knot contact homology and was shown to be so in [EENS13] (see [Ng12] for more details). Let $\mathcal{A}_n$ be the noncommutative unital algebra over $\mathbb{Z}$ freely generated by a_{ij}, $1 \leq i \neq j \leq n$. We define a homomorphism $\phi : B_n \to \operatorname{Aut} \mathcal{A}_n$ by defining it on the generators of B_n:

$$\phi_{\sigma_k} : \begin{cases} a_{ij} \mapsto a_{ij} & i,j \neq k, k+1 \\ a_{k+1,i} \mapsto a_{ki} & i \neq k, k+1 \\ a_{i,k+1} \mapsto a_{ik} & i \neq k, k+1 \\ a_{k,k+1} \mapsto -a_{k+1,k} & \\ a_{k+1,k} \mapsto -a_{k,k+1} & \\ a_{ki} \mapsto a_{k+1,i} - a_{k+1,k}a_{ki} & i \neq k, k+1 \\ a_{ik} \mapsto a_{i,k+1} - a_{ik}a_{k,k+1} & i \neq k, k+1 \end{cases} \tag{1}$$

Let $\iota \colon B_n \to B_{n+1}$ be the inclusion $\sigma_i \mapsto \sigma_i$ so that the $(n+1)$ strand does not interact with those from $B \in B_n$, and define $\phi_B^* \in \operatorname{Aut} \mathcal{A}_{n+1}$ by $\phi_B^* = \phi_B \circ \iota$. We then define the $n \times n$ matrices Φ_B^L and Φ_B^R with entries in $\mathcal{A}_n$ by

$$\phi_B^*(a_{i,n+1}) = \sum_{j=1}^{n} (\Phi_B^L)_{ij} a_{j,n+1}$$

$$\phi_B^*(a_{n+1,i}) = \sum_{j=1}^{n} a_{n+1,j} (\Phi_B^R)_{ji}$$

Finally, let R_0 be the Laurent polynomial ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ and define matrices $\mathbf{A}$ and $\mathbf{\Lambda}$ over R_0 by

$$\mathbf{A}_{ij} = \begin{cases} a_{ij} & i < j \\ -\mu a_{ij} & i > j \\ 1 - \mu & i = j \end{cases} \tag{2}$$

$$\mathbf{\Lambda} = \mathrm{diag}[\lambda \mu^{\omega(B)}, 1, \ldots, 1]. \tag{3}$$

Definition 2.1. Suppose that K is the closure of $B \in B_n$. Define $\mathcal{I} \subset \mathcal{A}_n \otimes R_0$ to be the ideal generated by the entries of $\mathbf{A} - \mathbf{\Lambda} \cdot \mathbf{\Phi_B^L} \cdot \mathbf{A}$ and $\mathbf{A} - \mathbf{A} \cdot \mathbf{\Phi_B^R} \cdot \mathbf{\Lambda^{-1}}$. The *degree zero homology of the combinatorial knot DGA* is $\mathrm{HC}_0(K) = (\mathcal{A}_n \otimes R_0)/\mathcal{I}$.

It was shown in [Ng08] that the isomorphism class of $HC_0(K)$ is unchanged by the Markov moves on B, hence $HC_0(K)$ is an invariant of the knot K. We only consider $HC_0(K)$ here, but there is a larger invariant, the differential graded algebra discussed in [Ng12].

The proofs in Section 3 require a number of computations of ϕ_B for particular braids B. Such computations are greatly benefited by an alternate description of the map ϕ_B, which we now give and will use without comment in Section 3.

Let D be a flat disk, to the right of B, with n points (punctures) where it intersects $K = \widehat{B}$ (see Figure 3). We assume the n punctures of D to be collinear, on a line that separates D into upper and lower half-disks. Denote by c_{ij} the isotopy class (fixing endpoints) of a path that is contained in the upper half-disk of D, with initial endpoint on the i^{th} strand and terminal endpoint on the j^{th} strand.

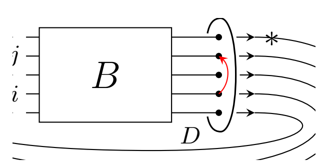

FIGURE 3. Cord c_{ij} of $K = \widehat{B}$

Consider B as a mapping class and let $B \cdot c_{ij}$ denote the isotopy class of the path to which c_{ij} is sent. If D, as viewed from the left (as pictured), is oriented as the plane then σ_k acts by rotating the k- and $(k+1)$-punctures an angle of π about their midpoint in counter-clockwise fashion. Consider the *algebra of paths* over $\mathbb{Z}$ generated by isotopy classes of paths in D with endpoints on punctures, modulo the relation in Figure 4 (paths depicted there are understood to agree outside the neighborhood of the puncture shown). It was shown in [Ng05] that the algebra map to $\mathcal{A}_n$ defined by $F(c_{ij}) = a_{ij}$ if $i < j$, and $F(c_{ij}) = -a_{ij}$ if $i > j$ satisfies $F(B \cdot c_{ij}) = \phi_B(F(c_{ij}))$.

FIGURE 4. Relation in the algebra of paths

Let perm : $B_n \to S_n$ denote the homomorphism from B_n to the symmetric group sending σ_k to the simple transposition interchanging $k, k+1$. We will make use of the following.

Lemma 2.2. *For some $B \in B_n$ and $1 \leq i \neq j \leq n$, consider $(\Phi^L_B)_{ij} \in \mathcal{A}_n$ as a polynomial expression in the (non-commuting) variables $\{a_{kl}, 1 \leq k \neq l \leq n\}$. Writing $i' = perm(B)(i)$, every monomial in $(\Phi^L_B)_{ij}$ is a constant times $a_{i'i_1}a_{i_1i_2}\ldots a_{i_{l-1},j}$ for some $l \geq 0$, the monomial being a constant if $l = 0$ and only if $i' = j$.*

Proof. Include B in B_{n+1} and consider the isotopy class of the path $B \cdot c_{i,n+1}$ which begins at i' and ends at $n+1$ (as B does not interact with the $(n+1)$ strand. Applying the relation in Figure 4 to the path equates it with a sum (or difference) of another path with the same endpoints and a product of two paths, the first beginning at i' and the other ending at $n+1$. A finite number of applications of this relation allows one to express the path as a polynomial in the $c_{kl}, 1 \leq k \neq l \leq n$ where each monomial has the form $c_{i'i_1}\ldots c_{i_{l-1},j}$. The result then follows from the fact that $\phi_B(a_{i,n+1}) = \phi_B(F(c_{i,n+1})) = F(B \cdot c_{i,n+1})$.

Alternatively, the statement follows from noting that (1) defining ϕ_{σ_k} has the desired property and that $\phi : B_n \to \mathrm{Aut}(\mathcal{A}_n)$ is a homomorphism. $\square$

2.2. **Augmentations and augmentation rank.** Let S be a ring with 1, and consider it a differential graded algebra supported in grading 0, with trivial differential. Augmentations of $(\mathcal{A}, \partial)$ are DGA maps $(\mathcal{A}, \partial) \to (S, 0)$. For our setting, if $B \in B_n$ is a braid representative of K, such a map corresponds precisely to a homomorphism $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ such that ϵ sends each generator of $\mathcal{I}$ to zero (see Definition 2.1).

Definition 2.3. Suppose that K is the closure of $B \in B_n$. An *augmentation* of K is a homomorphism $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ such that each element of $\mathcal{I}$ is sent by ϵ to zero.

A correspondence between augmentations and particular representations of the knot group π_K were studied in [Cor13a]. Recall that π_K is generated by meridians. We may fix a meridian m and generate π_K by conjugates of m.

Definition 2.4. For any integer $r \geq 1$, a homomorphism $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ is a *KCH representation* if there is a meridian m of K such that $\rho(m)$ is diagonalizable and has an eigenvalue of 1 with multiplicity $r-1$. We call ρ a *KCH irrep* if it is irreducible.

In [Ng08], Ng describes an isomorphism between $HC_0(K)$ and an algebra constructed from elements of π_K. As discussed in [Ng12] a KCH representation $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ induces an augmentation ϵ_ρ of K. Given an augementation, the first author showed how to construct a KCH representation that induces it. In fact, we have the following rephrasing of results from [Cor13a].

Theorem 2.5 ([Cor13a])**.** *Let $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ be an augmentation with $\epsilon(\mu) \neq 1$. There is a KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$. Furthermore, for any KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$, r equals the rank of $\epsilon(\mathbf{A})$.*

Considering Theorem 2.5 we make the following definition.

Definition 2.6. The *rank* of an augmentation $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ with $\epsilon(\mu) \neq 1$ is the rank of $\epsilon(\mathbf{A})$. Given a knot K, the *augmentation rank of* K, denoted $\mathrm{ar}(K)$, is the maximum rank among augmentations of K.

Remark 2.7. The augmentation rank can be defined for target rings other than $\mathbb{C}$, but this paper only considers augmentations as in Definition 2.3.

It is the case that $\mathrm{ar}(K)$ is well-defined. That is, given K there is a bound on the maximal rank of an augmentation of K.

Theorem 2.8 ([Cor13b])**.** *Given a knot $K \subset S^3$, if $g_1, \ldots, g_d$ are meridians that generate π_K and $\rho : \pi_K \to GL_r\mathbb{C}$ is a KCH irrep then $r \leq d$.*

As in the introduction, if we denote the meridional rank of π_K by $\mathrm{mr}(K)$, then Theorem 2.8 implies that $\mathrm{ar}(K) \leq \mathrm{mr}(K)$. In addition, the geometric quantity $b(K)$ called the bridge index of K is never less than $\mathrm{mr}(K)$. Thus we have the following corollary: I made the inequality a corollary here

Corollary 2.9 ([Cor13b])**.** *Given a knot $K \subset S^3$,*

$$ar(K) \leq mr(K) \leq b(K)$$

As a result, to verify for K that $\mathrm{mr}(K) = b(K)$ it suffices to find an augmentation of K with rank equal to $b(K)$. As we discuss in the next section, we will concern ourselves in this paper with a setting where $\mathrm{ar}(K) = n$ and there is a braid $B \in B_n$ which closes to K. This is a special situation, since $b(K)$ is strictly less than the braid index for many knots.

2.3. **Finding augmentations.** The following theorem concerns the behavior of the matrices Φ^L_B and Φ^R_B under the product in B_n. It is an essential tool for studying $HC_0(K)$ and will be central to our arguments.

Theorem 2.10 ([Ng05], Chain Rule)**.** *Let B, B' be braids in B_n. Then $\Phi^L_{BB'} = \phi_B(\Phi^L_{B'}) \cdot \Phi^L_B$ and $\Phi^R_{BB'} = \Phi^R_B \cdot \phi_B(\Phi^R_{B'})$.*

The main result of this paper concerns augmentations with rank equal to the braid index of the knot K. Define the diagonal matrix $\Delta(B) = \mathrm{diag}[(-1)^{w(B)}, 1, \dots, 1]$. The following statement follows from results in [Cor13b, Section 5].

but the theorem is marked Cor13a?

Theorem 2.11 ([Cor13a])**.** *If K is the closure of $B \in B_n$ and has a rank n augmentation $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$, then*

$$\epsilon(\Phi^L_B) = \Delta(B) = \epsilon(\Phi^R_B). \tag{4}$$

Furthermore, any homomorphism $\epsilon : \mathcal{A}_n \to \mathbb{C}$ which satisfies (4) can be extended to $\mathcal{A}_n \otimes R_0$ to produce a rank n augmentation of K.

The proof of Theorem 1.2 relies on this characterization of rank n augmentations. That is, given a braid $B \in B_k$ with a rank k augmentation $\epsilon : \mathcal{A}_k \to \mathbb{C}$, and $B' \in B_p$ with a rank p augmentation $\epsilon' : \mathcal{A}_p \to \mathbb{C}$ we show that $K(B, B')$ has a rank kp augmentation by defining a homomorphism $\psi : \mathcal{A}_{kp} \to \mathcal{A}_k \otimes \mathcal{A}_p$ so that $(\epsilon \otimes \epsilon') \circ \psi$ satisfies (4) for the braid $\bar{\bar{p}}BB'$.

There is a symmetry on the matrices Φ^L_B and Φ^R_B that is relevant to the study of augmentations in this setting. Define an involution $x \mapsto \overline{x}$ on $\mathcal{A}_n$ (termed *conjugation*) as follows: first set $\overline{a_{ij}} = a_{ji}$; then, for any $x, y \in \mathcal{A}_n$, define $\overline{xy} = \overline{y}\,\overline{x}$ and extend the operation linearly to $\mathcal{A}_n$. We have the following symmetry.

Theorem 2.12 ([Ng05], Prop. 6.2)**.** *For a matrix of elements in $\mathcal{A}_n$, let $\overline{M}$ be the matrix such that $\left(\overline{M}\right)_{ij} = \overline{M_{ij}}$. Then for $B \in B_n$, Φ^R_B is the transpose of $\overline{\Phi^L_B}$.*

It may be appropriate here to indicate that $\mathrm{ar}(K) < \mathrm{mr}(K)$ sometimes (maybe in previous subsection), and talk about the 2-cable of the trefoil that does not have $\mathrm{ar}(K, \mathbb{C}) = 4$

3. Main Result

figure out this two tensor products nonsense

how do I bring in equations to fit margins?

We begin in Section 3.1 with our main theorem. The proof relies upon an intermediate result, Proposition 3.1, which is shown in the subsequent Section 3.2 along with some supporting lemmas.

3.1. **Proof of main result.** In this section we prove our main result, which we now recall. The notation of Theorem 1.2 will be used throughout Section 3.

Theorem 1.2. Let $B \in B_k$ have augmentation rank k, and let $B'' \in B_p$ have augmentation rank p. If B' is the braid B'' included into B_{pk}, then $\bar{\bar{p}}BB'$ has augmentation rank pk.

Theorem 1.2 is proved by defining an algebra map $\psi\colon \mathcal{A}_{pk} \to \mathcal{A}_k \otimes \mathcal{A}_p$ such that $\psi(\Phi^L_{\bar{\bar{p}}BB'})$ factors suitably across the tensor product. This is the content of Proposition 3.1, which follows from Lemmas 3.2 and 3.4, the former following from a calculation (Lemma 3.5) and the latter depending on the former.

For each generator $a_{ij}, 1 \leq i \neq j \leq kp$, define

$$\psi(a_{ij}) = \begin{cases} 1 \otimes a_{r_i r_j} & : q_i = q_j \\ a_{q_i+1,q_j+1} \otimes 1 & : r_i = r_j \\ 0 & : (q_i - q_j)(r_i - r_j) < 0 \\ a_{q_i+1,q_j+1} \otimes a_{r_i r_j} & : (q_i - q_j)(r_i - r_j) > 0 \end{cases}, \tag{5}$$

which determines an algebra map $\psi\colon \mathcal{A}_{pk} \to \mathcal{A}_k \otimes \mathcal{A}_p$. Note that if $\psi(a_{ij}) \in 1 \otimes \mathcal{A}_p$ then $q_i = q_j$ or $a_{ij} \in \ker \psi$. Also, if we extend conjugation to $\mathcal{A}_k \otimes \mathcal{A}_p$ by applying it to each factor, then $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$. We have the following proposition.

Proposition 3.1. *For any braid B, $\psi\left(\Phi^L_{\bar{\bar{p}}B}\right) = \left(\left(\Phi^L_B\right)_{ij} \otimes 1\right) \otimes I_p$ and $\psi\left(\Phi^R_{\bar{\bar{p}}B}\right) = \left(\left(\Phi^R_B\right)_{ij} \otimes 1\right) \otimes I_p$* make consistent throughout paper

Note that here we mean the tensor product of Φ^L_B and I_p as matrices, not as linear maps, while the tensor product of $\left(\Phi^L_B \otimes I_p\right)_{ij}$ and 1 is a tensor product of algebra elements, so that if we divide the matrix $\psi\left(\Phi^L_{\bar{\bar{p}}B}\right)$ into k^2 $p \times p$blocks, the ijth block is $\left(\Phi^L_B\right)_{ij} I_p$. awkward

Proof of Theorem 1.2. By Theorem 2.11 there exist augmentations $\epsilon_k\colon \mathcal{A}_k \otimes R_0 \to \mathbb{C}$ and $\epsilon_p\colon \mathcal{A}_p \otimes R_0 \to \mathbb{C}$, for the closures of B, B' respectively, such that $\epsilon_k\left(\Phi^L_B\right) = \epsilon_k\left(\Phi^R_B\right) = \Delta(B)$ and $\epsilon_p\left(\Phi^L_{B'}\right) = \epsilon_p\left(\Phi^R_{B'}\right) = \Delta(B')$. Theorem 2.11 also implies that it suffices to prove that there exists an augmentation $\epsilon\colon \mathcal{A}_{pk} \otimes R_0 \to \mathbb{C}$ such that $\epsilon\left(\Phi^L_{\bar{\bar{p}}BB'}\right) = \epsilon\left(\Phi^R_{\bar{\bar{p}}BB'}\right) = \Delta(\bar{\bar{p}}BB')$.

Below we will define a homomorphism $\delta\colon \mathcal{A}_p \to \mathbb{C}$ such that $\delta = \pm\epsilon_p$, misleading? the sign depending on the parity of $w(B)$ and p. Let $\pi\colon \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ be the multiplication isomorphism $a \otimes b \mapsto ab$. Our desired map is defined by $\epsilon = \pi \circ (\epsilon_k \otimes \delta) \circ \psi$.

The Chain Rule theorem gives that

$$\pi \circ (\epsilon_k \otimes \delta) \circ \psi\left(\Phi^L_{\bar{\bar{p}}BB'}\right) = \pi \circ (\epsilon_k \otimes \delta)\psi\left(\phi_{\bar{\bar{p}}B}\left(\Phi^L_{B'}\right)\right)\psi\left(\Phi^L_{\bar{\bar{p}}B}\right) \tag{6}$$

Consider the homomorphism $\phi_{\bar{\bar{p}}B}$ through the description given in Section 2.1. As each $a_{ij}, 1 \leq i \neq j \leq p$, is represented by the isotopy class c_{ij}, and the leftmost p punctures are moved as one block by the action of $\bar{\bar{p}}B$ on D, there is an $0 \leq m < k$ so that $\phi_{\bar{\bar{p}}B}(a_{ij}) = a_{i+mp,j+mp}$ for each i,j in this range. As $\psi(a_{i+mp,j+mp}) = 1 \otimes a_{ij}$,

$$\psi\left(\phi_{\bar{\bar{p}}B}\left(\Phi^L_{B'}\right)\right) = \left(1 \otimes \left(\Phi^L_{B'}\right)_{ij}\right)$$

By Proposition 3.1, we have that

$$\psi\left(\Phi^L_{\bar{\bar{p}}B}\right) = \left(\left(\Phi^L_B\right)_{ij} \otimes 1\right) \otimes I_p = \left(\left(\Phi^L_B \otimes I_p\right)_{ij} \otimes 1\right)$$

this is maybe confusing, since i,j in the middle part range over different values than i,j on the RHS

Returning to the right hand side of (6)

$$\begin{aligned}\pi \circ (\epsilon_k \otimes \delta)\left(\psi\left(\phi_{\bar{\bar{p}}B}\left(\Phi^L_{B'}\right)\right)\psi\left(\Phi^L_{\bar{\bar{p}}B}\right)\right) &= \pi \circ (\epsilon_k \otimes \delta)\left(\left(1 \otimes \left(\Phi^L_{B'}\right)_{ij}\right)\left(\left(\Phi^L_B \otimes I_p\right)_{ij} \otimes 1\right)\right)\\ &= \delta\left(\Phi^L_{B'}\right)\left(\Delta(B) \otimes I_p\right).\end{aligned}$$

We are done if δ may be defined so that $\delta\left(\Phi^L_{B'}\right)\left(\Delta(B) \otimes I_p\right) = \Delta(\bar{\bar{p}}BB')$. When $w(B)$ is even $w(\bar{\bar{p}}B)$ is also, and further $\Delta(B) = I_k$. Letting $\delta = \epsilon_p$ makes

$$\delta\left(\Phi^L_{B'}\right)\left(\Delta(B) \otimes I_p\right) = \epsilon_p\left(\Phi^L_{B'}\right) = \Delta(B') = \Delta(\bar{\bar{p}}BB').$$

Suppose that $w(B)$ is odd. We define $g\colon \{1,\ldots,p\} \to \{\pm 1\}$ as follows. Let $x_1 = 1$, and $x_l = \mathrm{perm}(B')(x_{l-1})$ for $1 < l \leq p$. Since the first p strands of B' close to a knot, $\mathrm{perm}(B')$ is given by the p-cycle $(x_1x_2\ldots x_p)$. If p is even, we let $g(x_1) = 1$, and $g(x_l) = -g(x_{l-1})$ for $1 < l \leq p$. If p is odd, let $g(x_1) = g(x_2) = 1$ and $g(x_l) = -g(x_{l-1})$ for $2 < l \leq p$.

Define $\delta : \mathcal{A}_p \to \mathbb{C}$ by setting $\delta(a_{ij}) = g(i)g(j)\epsilon_k(a_{ij})$ for $1 \leq i \neq j \leq p$. Fix i,j and consider a monomial M in $\left(\Phi^L_{B'}\right)_{ij}$, which is constant if $i > p$ or $j > p$ so that $\delta(M)$ is defined. If $i,j \leq p$, then writing $i' = \mathrm{perm}(B')(i)$, Proposition 2.2 implies $M = c_{ij}a_{i',j_1}a_{j_1,j_2}\ldots a_{j_m,j}$ for some $j_1,\ldots j_m \in \{1,\ldots,p\}$, possibly being constant only if $i' = j$, implying that

$$\delta(M) = g(i')g(j)\left(\prod_{k=1}^m g(j_k)^2\right)\epsilon_p(M) = g(i')g(j)\epsilon_p(M).$$

Note, when M is constant then $\delta(M) = M = g(i')g(j)\epsilon_p(M)$ since $i' = j$. Since this holds for each monomial, we have that

$$\delta\left(\left(\Phi^L_{B'}\right)_{ij}\right) = g(i')g(j)\epsilon_p\left(\left(\Phi^L_{B'}\right)_{ij}\right).$$

When p is even, $w(\bar{\bar{p}}B)$ is also even and so the opposite parity of $w(B)$. Our definition of g gives $\delta\left(\left(\Phi^L_{B'}\right)_{ii}\right) = -\epsilon\left(\left(\Phi^L_{B'}\right)_{ii}\right)$ for $i \leq p$. Thus

$$\delta\left(\Phi^L_{B'}\right) = \begin{pmatrix} (-1)^{w(B')+1} & 0 & 0 \\ 0 & -I_{p-1} & 0 \\ 0 & 0 & I_{(k-1)p}\end{pmatrix}$$

and therefore

$$\delta\left(\Phi^L_{B'}\right)\left(\Delta(B) \otimes I_p\right) = \mathrm{diag}[(-1)^{w(B)+w(B')+1}, 1\ldots 1] = \Delta(\bar{\bar{p}}BB')$$

as desired.

When p is odd, $w(\bar{\bar{p}}B)$ is odd and therefore the same parity of $w(B)$. Our definition of g gives that $\delta\left(\left(\Phi^L_{B'}\right)_{11}\right) = \epsilon\left(\left(\Phi^L_{B'}\right)_{11}\right)$ and $\delta\left(\left(\Phi^L_{B'}\right)_{ii}\right) = -\epsilon\left(\left(\Phi^L_{B'}\right)_{ii}\right)$ for $1 < i \leq p$, so

$$\delta\left(\Phi^L_{B'}\right) = \begin{pmatrix} (-1)^{w(B')} & 0 & 0 \\ 0 & -I_{p-1} & 0 \\ 0 & 0 & I_{(k-1)p} \end{pmatrix}$$

and therefore

$$\delta\left(\Phi^L_{B'}\right)\left(\Delta(B) \otimes I_p\right) = \mathrm{diag}[(-1)^{w(B)+w(B')}, 1 \ldots 1] = \Delta(\bar{\bar{p}}BB')$$

as desired.

There is little difference in the proof that $\epsilon(\Phi^R_{\bar{\bar{p}}B}) = \Delta(\bar{\bar{p}}BB')$, except that monomials in $(\Phi^R_{B'})_{ij}$ are of the form $c_{ij}a_{i,j_1}a_{j_1,j_2}\cdots a_{j_k,j'}$ where $j' = \mathrm{perm}(B')(j)$. Applying Theorem 2.11 now completes the proof. $\square$

3.2. **Proposition 3.1 and supporting lemmas.** We use the following two lemmas to prove Proposition 3.1. Figure 5 demonstrates an example for Lemma 3.2, showing that $\psi(\phi_{\bar{\bar{2}}\sigma_2}(a_{24})) = \phi_{\sigma_2} \otimes \mathrm{id}(\psi(a_{24}))$. Note that in the figure we condense elements such as $a_{13} \otimes 1$ to a_{13} and include products of algebra elements on a single set of points in order to make the notation cleaner.

Lemma 3.2. $\psi \circ \phi^{\pm 1}_{\bar{\bar{p}}\sigma_n} = \left(\phi^{\pm 1}_{\sigma_n} \otimes \mathrm{id}\right) \circ \psi$ *for all* $1 \leq n < k$.

Remark 3.3. As the map $B_k \to \mathrm{Aut}(\mathcal{A}_k \otimes \mathcal{A}_p)$ given by $B \mapsto \phi_B \otimes \mathrm{id}$ is a homomorphism, Lemma 3.2 immediately implies that $\psi(\phi_{\bar{\bar{p}}B}(a_{ij})) = (\phi_B \otimes \mathrm{id})(\psi(a_{ij}))$ for any $B \in B_k$.

Lemma 3.4 (Base Case). $\psi\left(\Phi^L_{\bar{\bar{p}}\sigma_n^{\pm 1}}\right) = \left(\left(\Phi^L_{\sigma_n^{\pm 1}}\right)_{ij} \otimes 1\right) \otimes I_p$

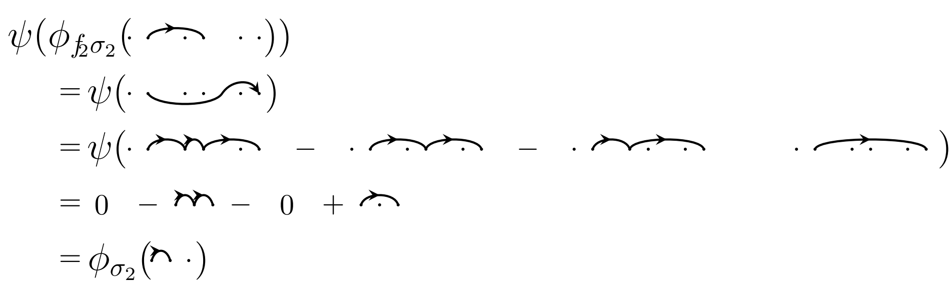

Figure 5. Computing $\psi(\phi_{\bar{\bar{p}}\sigma_2}(a_{24}))$

Proof of Proposition 3.1. Let $B = \sigma_{n_1}^{q_1} \cdots \sigma_{n_r}^{q_r}$, where $1 \leq n_i < k$ and $q_i = \pm 1$. We will prove the proposition by induction on r. The base case is already taken care of by Lemma 3.4. Suppose that the proposition holds for braids of length $r-1$. Let $B' = \sigma_{n_1}^{q_1} \cdots \sigma_{n_{r-1}}^{q_{r-1}}$ Then by the Chain Rule and Lemmas 3.2 and 3.4, we have that

should I pick something other than B'?

$$\begin{aligned}
\psi\left(\Phi^L_{\bar{\bar{p}}B}\right) &= \psi\left(\phi_{\bar{\bar{p}}B'}\left(\Phi^L_{\bar{\bar{p}}\sigma_{n_r}^{q_r}}\right) \cdot \Phi^L_{\bar{\bar{p}}B'}\right) \\
&= (\phi_{B'} \otimes \mathrm{id})\left(\psi\left(\Phi^L_{\bar{\bar{p}}\sigma_{n_r}^{q_r}}\right)\right) \cdot \left(\left(\Phi^L_{B'}\right)_{ij} \otimes 1\right) \otimes I_p \\
&= (\phi_{B'} \otimes \mathrm{id})\left(\left(\left(\Phi^L_{\sigma_{n_r}^{q_r}}\right)_{ij} \otimes 1\right) \otimes I_p\right) \cdot \left(\left(\Phi^L_{B'}\right)_{ij} \otimes 1\right) \otimes I_p \\
&= \left(\left(\Phi^L_B\right)_{ij} \otimes 1\right) \otimes I_p
\end{aligned}$$

We also have then that $\psi\left(\Phi^R_{\bar{\bar{p}}B}\right) = \left(\left(\Phi^R_B\right)_{ij} \otimes 1\right) \otimes I_p$ as well, since $\Phi^R_B = \overline{\Phi^L_B}^t$ and $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$. $\square$

In the proof of Lemmas 3.2 and 3.4, we will make use of some calculations of $\phi_B(a_{ij})$ for simple braids B. Recall that $\tau_{m,l} = \sigma_m \sigma_{m+1} \cdots \sigma_{m+l-1}$. It can easily be checked that for all $1 \leq m < n$, $1 \leq l \leq n-m$, $i < j$

$$\phi_{\tau_{m,l}}(a_{ij}) = \begin{cases} a_{i+1,j+1} & : m \leq i < j < m+l \\ a_{i-l,j} & : m < m+l = i < j \\ a_{i,j-l} & : i < m < m+l = j \\ a_{i+1,j-l} & : m \leq i < j = m+l \\ a_{i,j+1} - a_{i,m}a_{m,j+1} & : i < m \leq j < m+l \\ a_{i+1,j} - a_{i+1,m}a_{m,j} & : m \leq i < m+l < j \\ a_{ij} & : \text{otherwise} \end{cases} \tag{7}$$

We also make the following definition
Let $X \subseteq \{1, \ldots, n\}$, and write the elements of a subset $Y \subseteq X$ as $y_1 < \ldots < y_k$. Define

$$A(i,j,X) = \sum_{Y \subseteq X} (-1)^{|Y|} a_{iy_1} a_{y_1 y_2} \cdots a_{y_k j}$$

and

$$A'(i,j,X) = \sum_{Y \subseteq X} (-1)^{|Y|} a_{iy_k} a_{y_k y_{k-1}} \cdots a_{y_1 j}$$

and have the following lemma

Lemma 3.5. *Let* $X_n^{(p)} = \{(n-1)p+1, \dots, np\}$*. We have*

$$\phi_{\bar{\bar{p}}\sigma_n}(a_{ij}) = \begin{cases} a_{i-p,j-p} & : np < i < j \le (n+1)p \\ a_{i-p,j} & : np < i \le (n+1)p < j \\ a_{i,j-p} & : i \le (n-1)p < np < j \le (n+1)p \\ a_{i+p,j+p} & : (n-1)p < i < j \le np \\ A'(i+p, j-p, X_n^{(p)}) & : (n-1)p < i \le np < j \le (n+1)p \\ A(i, j+p, X_n^{(p)}) & : i \le (n-1)p < j \le np < (n+1)p \\ A'(i+p, j, X_n^{(p)}) & : (n-1)p < i \le np < (n+1)p < j \\ a_{ij} & : \text{otherwise} \end{cases}$$

Proof of Lemma 3.2. Note that if $\psi(\phi_{\bar{\bar{p}}\sigma_n}(a_{ij})) = (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))$, then

$$\psi(a_{ij}) = \psi\left(\phi_{\bar{\bar{p}}\sigma_n}\left(\phi_{\bar{\bar{p}}\sigma_n^{-1}}(\alpha_{ij})\right)\right) = (\phi_{\sigma_n} \otimes \mathrm{id})\left(\psi\left(\phi_{\bar{\bar{p}}\sigma_n^{-1}}(a_{ij})\right)\right)$$

And applying $\left(\phi_{\sigma_n^{-1}} \otimes \mathrm{id}\right)$ to both sides gives

$$\psi\left(\phi_{\bar{\bar{p}}\sigma_n^{-1}}(a_{ij})\right) = \left(\phi_{\sigma_n^{-1}} \otimes \mathrm{id}\right)\psi(a_{ij})$$

Furthermore, $\phi_B(\overline{a_{ij}}) = \overline{\phi_B(a_{ij})}$and $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$, so it suffices to prove the lemma for $\bar{\bar{p}}\sigma_n$ in the case where $i < j$. does this need justification?

With these restrictions, we then break the statement up into the cases from Lemma 3.5, from which the first four cases as well as the last case can be checked easily. Consider the sixth case. Lemma 3.5 gives that

$$\psi\left(\phi_{\bar{\bar{p}}\sigma_n}(a_{ij})\right) = \sum_{Y \subseteq \{np-p+1, \dots, np\}} (-1)^{|Y|}\psi\left(a_{iy_1}a_{y_1y_2}\cdots a_{y_k,j+p}\right)$$

Let $\alpha_i = np - p + r_i$. Note that if $y_1 < \alpha_i$ then $\psi(a_{iy_1}) = 0$, and if $y_k > \alpha_j$ then $\psi(a_{y_kj}) = 0$, so the sum on the right hand side can be taken over $Y \subseteq \{\alpha_i, \alpha_i + 1, \dots, \alpha_j\}$. Then we manipulate the sum to get do I need to explain what I'm doing here?

$$\begin{aligned} &\sum_{Y \subseteq \{\alpha_i, \dots, \alpha_j\}} (-1)^{|Y|}\psi\left(a_{iy_1}a_{y_1y_2}\cdots a_{y_k,j+p}\right) \\ &= \psi\left(a_{i,j+p} - a_{i,\alpha_i}a_{\alpha_i,j+p}\right) \\ &\quad + \sum_{y=\alpha_i+1}^{\alpha_j} \sum_{Y \subseteq \{y+1, \dots, \alpha_j\}} (-1)^{|Y|+1}\psi\left(a_{iy}a_{yy_1}\cdots a_{y_k,j+p}\right) + (-1)^{|Y|}\psi\left(a_{i,\alpha_i}a_{\alpha_i,y}a_{yy_1}\cdots a_{y_k,j+p}\right) \\ &= \psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right) \\ &\quad + \sum_{y=\alpha_i+1}^{\alpha_j} \sum_{Y \subseteq \{y+1, \dots, \alpha_j\}} (-1)^{|Y|}\psi\left(a_{i,\alpha_i}a_{\alpha_i,y} - a_{iy}\right)\psi\left(a_{yy_1}\cdots a_{y_k,j+p}\right) \end{aligned}$$

Note that $r_i = r_{\alpha_i}$ and since we're in the sixth case we have $(n-1)p < j \le np$, so $q_{\alpha_i} = q_y$. Thus $\psi(a_{i,\alpha_i}) = a_{q_i+1,q_{\alpha_i}+1} \otimes 1 = a_{q_i+1,q_y+1} \otimes 1$ and

$\psi(a_{\alpha_i,y}) = 1 \otimes a_{r_{\alpha_i},r_y} = 1 \otimes a_{r_i,r_y}$, so we have

$$\psi(a_{i,\alpha_i}a_{\alpha_i,y} - a_{iy}) = \left(a_{q_i+1,q_y+1} \otimes 1\right)\left(1 \otimes a_{r_i,r_y}\right) - a_{q_i+1,q_y+1} \otimes a_{r_i,r_y} = 0$$

Thus the right hand side reduces to

$$\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right)$$

Remark 3.6. The fact that $\psi(a_{i,\alpha_i}a_{\alpha_i,y} - a_{iy}) = 0$ and ψ behaves similarly for the analogous terms in cases 5 and 7 is the key to this proof working, and ψ is defined the way it is mainly so that this will be true. As we hinted at earlier, the homomorphism $\rho\colon \mathcal{A}_{pk} \to \mathcal{A}_k$ defined to send a_{ij} to a_{q_i+1,q_j+1} if $r_i = r_j$ and to 0 otherwise would also send these terms to 0, so Proposition 3.1 would still be true with ρ used in the place of ψ. We will need ψ for the proof of the main result, however.

Note that, since we're in the sixth case, $q_j + 1 = n$. If $r_i = r_j$, then

$$\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right) = \left(a_{q_i+1,n+1} - a_{q_i+1,n}a_{n,n+1}\right)\otimes 1 = (\phi_{\sigma_n}\otimes \mathrm{id})(\psi(a_{ij}))$$

If $r_i < r_j$, then

$$\begin{aligned}\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right) &= \left(a_{q_i+1,n+1} \otimes a_{r_ir_j} - a_{q_i+1,n}a_{n,n+1} \otimes a_{r_ir_j}\right)\\ &= \left(a_{q_i+1,n+1} - a_{q_i+1,n}a_{n,n+1}\right) \otimes a_{r_ir_j}\\ &= (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))\end{aligned}$$

Finally, if $r_i > r_j$, then

$$\psi\left(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}\right) = 0 = (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij}))$$

The proof for the seventh case goes exactly as the proof for the sixth case except with all i's replaced with $i + p$, all $(j + p)$'s replaced with j, all y_i's replaced with y_{k+1-i}, and with α_i and α_j swapped. The proof for the fifth case goes exactly as the proof for the seventh. □

check this

check

Proof of Lemma 3.4. First we will prove the lemma for $\bar{\bar{p}}\sigma_n$. We can extend the definition of ψ to be an algebra morphism from the free module over $\mathcal{A}_{pk}$ generated by the symbols $\{a_{i*} | 1 \le i \le pk\}$ to the free module over $\mathcal{A}_k \otimes \mathcal{A}_p$ generated by $\{a_{i*} | 1 \le i \le k\}$ by defining $\psi(a_{i*}) = a_{i*}$ and extending it to an algebra morphism. Then the statement of the lemma is equivalent to saying that for all $1 \le i \le pk$, the coefficient of a_{j*} in $\psi\left(\phi_{\bar{\bar{p}}\sigma_n}(a_{i*})\right)$ is equal to 0 unless $r_j = r_i$, in which case it is equal to the coefficient of a_{q_j*} in $\phi_{\sigma_n}(a_{q_i*})$. If $q_i + 1 \ne n$, this fact can be easily checked. In the case that $q_i + 1 = n$, we have $i = (n-1)p + r_i = \alpha_i$, so

$$\psi\left(\phi_{\bar{\bar{p}}\sigma_n}(a_{i*})\right) = \psi\left(A(i+p, *, \{np - p + 1, \ldots, np\}\right)$$

which is equal to

$$\psi(a_{i+p,*} - a_{i+p,\alpha_i}a_{\alpha_i,*}) = a_{i+p,*} - (a_{n+1,n} \otimes 1)a_{\alpha_i,*}$$

by the same argument that was used in Lemma 3.2. The coefficients of the a_{j*} are equal to the coefficients of the a_{q_j*} in $\phi_{\sigma_n}(a_{q_i*})$, so we have $\psi\left(\Phi^L_{\bar{\bar{p}}\sigma_n}\right) = \left(\left(\Phi^L_{\sigma_n}\right)_{ij} \otimes 1\right) \otimes I_p$.

Using this fact, the Chain Rule, and Lemma 3.2, we have

$$\begin{aligned}\left((I_{pk})_{ij}\otimes 1\right) &= \psi\left(\Phi^L_{\bar{\bar{p}}\sigma_n^{-1}\bar{\bar{p}}\sigma_n}\right)\\ &= \psi\left(\phi_{\bar{\bar{p}}\sigma_n^{-1}}\left(\Phi^L_{\bar{\bar{p}}\sigma_n}\right)\right)\psi\left(\Phi^L_{\bar{\bar{p}}\sigma_n^{-1}}\right)\\ &= \left(\phi_{\sigma_n^{-1}}\otimes \mathrm{id}\right)\left(\left((\Phi_{\sigma_n})_{ij}\otimes 1\right)\otimes I_p\right)\psi\left(\Phi^L_{\bar{\bar{p}}\sigma_n^{-1}}\right)\end{aligned}$$

But note that the Chain Rule also gives that $\left(\left(\Phi^L_{\sigma_n^{-1}}\right)_{ij}\otimes 1\right)\otimes I_p$ is the inverse of $\left(\phi_{\sigma_n^{-1}}\otimes \mathrm{id}\right)\left(\left((\Phi_{\sigma_n})_{ij}\otimes 1\right)\otimes I_p\right)$, so

$$\psi\left(\Phi^L_{\bar{\bar{p}}\sigma_n^{-1}}\right) = \left(\left(\Phi^L_{\sigma_n^{-1}}\right)_{ij}\otimes 1\right)\otimes I_p$$

which completes the proof. $\square$

Proof of Lemma 3.5. check

We will prove a more general statement than the one presented in Lemma 3.5. Let $\kappa_{m,l} = \tau_{m+l-1,p}\tau_{m+l-2,p}\cdots\tau_{m,p}$, and let $X_{m,l} = \{m,\dots,m+l-1\}$. We will prove that if $i<j$, then check

$$\phi_{\kappa_{m,l}}(a_{ij}) = \begin{cases} a_{i-p,j-p} & : m+p\le i<j<m+l+p\\ a_{i-p,j} & : m+p\le i<m+l+p\le j\\ a_{i,j-p} & : i<m<m+p\le j<m+l+p\\ a_{i+l,j+l} & : m\le i<j<m+p\\ A'(i+l,j-p,X_{m,l}) & : m\le i<m+p\le j<m+l+p\\ A(i,j+l,X_{m,l}) & : i<m\le j<m+p<m+l+p\\ A'(i+l,j,X_{m,l}) & : m\le i<m+p<m+l+p\le j\\ a_{ij} & : \text{otherwise}\end{cases}$$

Letting $l=p$ and $m=(n-1)p+1$ then gives us Lemma **??** as a special case. The first four cases as well as the eighth can be easily checked. We will prove the remaining cases by induction on l. Consider the sixth case. The base case is covered by (7). For the inductive step, we have that

$$\begin{aligned}\phi_{\kappa_{m,l}}(a_{ij}) &= \phi_{\tau_{m+l-1,p}}\left(\phi_{\kappa_{m,l-1}}(a_{ij})\right)\\ &= \sum_{Y\subseteq\{m,\dots,m+l-2\}}(-1)^{|Y|}\phi_{\tau_{m+l-1,p}}\left(a_{i,y_1}a_{y_1y_2}\cdots a_{y_k,j+l-1}\right)\\ &= \sum_{Y\subseteq\{m,\dots,m+l-2\}}(-1)^{|Y|}a_{iy_1}a_{y_1y_2}\cdots a_{y_{k-1}y_k}\left(a_{y_k,j+l}-a_{y_k,m+l-1}a_{m+l-1,j+l}\right)\\ &= \sum_{Y\subseteq\{m,\dots,m+l-1\}}(-1)^{|Y|}a_{i,y_1}a_{y_1y_2}\cdots a_{y_k,j+l}\\ &= A(i,j+l,X_{m,l})\end{aligned}$$

is this clear/can it be shortened?

The proof of the seventh case goes exactly as the proof of the sixth, with all i's replaced with $i+l$, j's replaced with $j-l$, and y_i's replaced with y_{k-i+1}. The proof of the fifth case goes exactly as the proof of the seventh. $\square$

4. Comments on augmentation rank and multiplicativity

In this section we address two questions arising from Theorem 1.2. We first consider, in Section 4.1, the possibility of weaker assumptions for which the multiplicative property of the theorem still holds - that is, what is necessary for $\mathrm{ar}(K(B,B')) = \mathrm{ar}(\widehat{B})\,\mathrm{ar}(\widehat{B'})$? We will provide some examples showing that augmentation rank is not multiplicative (nor sub-multiplicative) in the general case.

Second, we consider in Section 4.2 how atypical is the equality $\mathrm{ar}(B) = n$ (for $B \in B_n$). The comments of this section are somewhat speculative. We recall the Dehornoy order on the braid group and remark on the possibility of a relationship to the augmentation rank.

4.1. **Augmentation rank does not multiply.** The knot $K(B,B')$ obtained by the braid satellite construction depends on the braid B chosen to represent $\widehat{B}$ (as we are using B to determine the framing). It is thus not surprising that stabilization can be used to find B, B' such that $\mathrm{ar}(K(B,B')) < \mathrm{ar}(B)\,\mathrm{ar}(B')$.

In fact, for $B \in B_k$ define $S^{\pm}(B)$ as the stabilization $\iota(B)\sigma_1^{\pm} \in B_{k+1}$ where $\iota : B_k \to B_{k+1}$ is the map given on generators by $\iota(\sigma_i) = \sigma_{i+1}$ for $1 \le i \le k-1$. Then for any $B' \in B_p$, a knot equal to $K(S^{\pm}(B), B')$ is also equal to the braid satellite $K(B, \Delta^{\pm 2}B')$, where Δ^2 is the full twist in B_p. This can be seen by "destabilizing" the companion torus in the satellite (see Figure **??**).

Setting $k = 1$ and taking B' so that $\mathrm{ar}(\widehat{B'}) = p$ and $\mathrm{ar}(\widehat{\Delta^{\pm}B'}) < p$ gives that $\mathrm{ar}(K(S^{\pm}(B), B')) = \mathrm{ar}(\widehat{\Delta^{\pm}B'}) < p = \mathrm{ar}(\widehat{B})\,\mathrm{ar}(\widehat{B'})$. As a less trivial example, we remark that (taking $\sigma_1 \in B_2$) we have $\mathrm{ar}(K(\sigma_1^3, \sigma_1)) < 4$, however $K(\sigma_1^3, \sigma_1) = K(S^{-1}(\sigma_1^3), \sigma_1^3)$ and $\mathrm{ar}(\widehat{S^{-1}(\sigma_1^3)}) = \mathrm{ar}(\widehat{\sigma_1^3}) = 2$. (Note, in more conventional notation, $K(\sigma_1^3, \sigma_1)$ is the $(2,7)$-cable of the right-handed trefoil.

It should be observed that every example thus far where the rank is submultiplicative (strictly) has involved taking a B which is not minimal braid index. It can occur that the rank is super-multiplicative. By finding a solution to (4) for $B = \bar{\bar{2}}\sigma_1^5 \cdot \sigma_1 \in B_4$ we can show that $\mathrm{ar}(K(\sigma_1^5, \sigma_1)) = 4$, even though $\mathrm{ar}(\widehat{\sigma_1^5}) = 2$ and $\mathrm{ar}(\widehat{\sigma_1}) = 1$.

insert observations on exchange moves. conjecture should be that supermultiplicative holds for minimal braids?, maybe even multiplicative if unique conjugacy class in closure?

4.2. **Braid closures with full rank.** We recall the Dehornoy order on B_n, which is a total, right invariant order $<_D$ on the braid group. A braid $B \in B_n$ is *positive* in B_n if $e <_D B$ and *negative* if $B <_D e$, where e denotes the identity element of B_n. Right invariance of the ordering immediately implies that B is positive if and only if B^{-1} is negative.

Let $\Delta = (\sigma_1 \dots \sigma_{n-1})(\sigma_1 \dots \sigma_{n-2}) \dots (\sigma_1)$ be the half twist in B_n. Recall that $\Delta^2 = (\sigma_1 \dots \sigma_{n-1})^n$ generates the center of B_n []. As a consequence, if $\Delta^2 <_D B$ then $\Delta^{-2}B$ is positive which implies that $\Delta^{2m} <_D B^m$ for any $m > 0$.

Early calculations seem to indicate that the lack of a solution to equation (4) can only occur for braids that are in a sufficiently small interval about e in the Dehornoy order. It is a remarkable fact, resulting from Birman and Menasco's Markov Theorem without Stabilization [**?**], that for each braid index n there is a number m_n such that if $\Delta^{2m_n} <_D B$ in B_n then n is the minimal braid index of the closure of B, and every braid in B_n with the same closure as B is conjugate to B. By results in [] it must be that $m_n \geq n-1$.

Note that if $\Delta^{2m_n} <_D B$ then $\Delta^{-2m_n}B$ is positive and so $B = \Delta^{2m_n}\beta$, where β is positive.

If the closure of $B \in B_n$ is K and $b(K) < n$ then there cannot be rank n augmentations. For each 2-bridge knot K, up to crossing number 10, which has minimal braid index 3 we have taken a braid B which has closure K and calculated that $\mathrm{ar}(\Delta^2 B) = 3$ or $\mathrm{ar}(\Delta^4 B)$ (here Δ is the half twist in B_3. In addition, it appears that if $\mathrm{ar}(\Delta^{2l}B) = n$ then $\mathrm{ar}(\Delta^{2m}B) = n$ for $m > l$. Such calculations can sometimes be done for braids in B_4, though they can quickly become prohibitively complicated as braid length increases.

We remark that it is possible to find strictly increasing sequences of braids (which remain less than a fixed power of Δ^2) where the augmentation rank remains constant, and less than n.

Question: For a given braid index n, does there exist a number m_n such that if $\Delta^{2m_n} <_D B$ then $\mathrm{ar}(B) = n$?

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