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1. Introduction

introduce corollary for iterated cables

Let K be a knot in S^3. The *meridional rank* of K, written $\mathrm{mr}(K)$, is the minimal size of a meridional generating set of the knot group of K. It is bounded above by the bridge number $b(K)$, and Problem 1.11 of [Kir95] asks whether $\mathrm{mr}(K) = b(K)$ for all knots K. Cornwell has proven that the *augmentation rank* $\mathrm{ar}(K)$ of K (which is defined in Section 2) bounds the meridional rank from below, and that $\mathrm{ar}(K) = \mathrm{mr}(K) = b(K)$ for some families of knots, including torus knots [Cor13b]

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The main result of this paper is that if $\mathrm{ar}(K) = b(K)$ and K is the closure of a braid with even writhe and index equal to $b(K)$, then the augmentation rank and bridge number are equal for any (p, q)-cable of K where $\gcd(p, q) = 1$ and $p < q$.

2. Background

We begin in Section 2.1 by reviewing the construction of $HC_0(K)$ from the viewpoint of the combinatorial knot DGA, which was first defined in [Ng08]; our conventions are those given in [Ng12]. In Section 2.2 we discuss augmentations in knot contact homology and their rank, which gives a bound on the meridional rank of the knot group useful for studying the relation between meridional rank and bridge number. Finally, in Section 2.3 is a discussion of techniques from [Cor13a] that we use to calculate the augmentation rank.

2.1. Knot contact homology.

Knot contact homology. We begin with the necessary preliminaries for defining the combinatorial knot DGA of Ng. This DGA was defined to be a calculation of knot contact homology and was shown to be so in [EENS13] (see [Ng12] for more details). Let $\mathcal{A}_n$ be the noncommutative unital algebra over $\mathbb{Z}$ generated by a_{ij}, $1 \leq i \neq j \leq n$. Let B_n be the braid group on n strands, and define $\phi : B_n \to \mathrm{Aut}\,\mathcal{A}_n$ by defining it on the generators of $\mathcal{A}_n$ and extending by linearity

$$\phi_{\sigma_k} : \begin{cases} a_{ij} \mapsto a_{ij} & i, j \neq k, k+1 \\ a_{k+1,i} \mapsto a_{ki} & i \neq k, k+1 \\ a_{i,k+1} \mapsto a_{ik} & i \neq k, k+1 \\ a_{k,k+1} \mapsto -a_{k+1,k} & \\ a_{k+1,k} \mapsto -a_{k,k+1} & \\ a_{ki} \mapsto a_{k+1,i} - a_{k+1,k}a_{ki} & i \neq k, k+1 \\ a_{ik} \mapsto a_{i,k+1} - a_{ik}a_{k,k+1} & i \neq k, k+1 \end{cases} \tag{1}$$

Let $\iota\colon B_n \to B_{n+1}$ be the inclusion that adds in an $(n+1)$th strand that doesn't interact with the others, and define $\phi_B^* \in \operatorname{Aut}\mathcal{A}_{n+1}$ by $\phi_B^* = \phi_B \circ \iota$. We then define the $n \times n$ matrices Φ_B^L and Φ_B^R with entries in $\mathcal{A}_n$ by

$$\phi_B^*(a_{i,n+1}) = \sum_{j=1}^{n} (\Phi_B^L)_{ij} a_{j,n+1}$$

$$\phi_B^*(a_{n+1,i}) = \sum_{j=1}^{n} a_{n+1,j} (\Phi_B^R)_{ji}$$

Let ω be the writhe of B, and define matrices $\mathbf{A}$ and Λ by

$$\mathbf{A_{ij}} = \begin{cases} a_{ij} & i<j \\ -\mu a_{ij} & i>j \\ 1-\mu & i=j \end{cases} \tag{2}$$

$$\Lambda = \operatorname{diag}[\lambda\mu^{\omega}, 1, \ldots, 1]. \tag{3}$$

Definition Suppose that K is the closure of $B \in B_n$ and let R_0 be the Laurent polynomial ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$. Define $\mathcal{I} \subset \mathcal{A}_n \otimes R_0$ to be the ideal generated by the entries of $\mathbf{A} - \boldsymbol{\Lambda} \cdot \boldsymbol{\Phi}_{\mathbf{B}}^{\mathbf{L}} \cdot \mathbf{A}$ and $\mathbf{A} - \mathbf{A} \cdot \boldsymbol{\Phi}_{\mathbf{B}}^{\mathbf{R}} \cdot \boldsymbol{\Lambda}^{-\mathbf{1}}$. The *degree zero homology of the combinatorial knot DGA* is $\mathrm{HC}_0(K) = (\mathcal{A}_n \otimes R_0)/\mathcal{I}$.

It was shown in [Ng08] that the isomorphism class of $HC_0(K)$ is unchanged under conjugation and by positive and negative stabilization of B, hence $HC_0(K)$ is an invariant of the knot K by Markov's theorem. While we only consider $HC_0(K)$ here, it is part of the larger invariant, the combinatorial knot DGA of K, studied in [Ng08] which is a computation of the Legendrian contact homology of a Legendrian lift of K to the cosphere bundle over $\mathbb{R}^3$ ([EENS13]).

The proofs in Section 3 will require a number of computations of $\phi_B(a_{ij})$ for particular braids $B \in B_n$. Such computations are greatly benefited by an alternate description of the map ϕ_B on $\mathcal{A}_n$ (or $\mathcal{A}_{n+1}$) which we will use liberally. In this description we use a punctured disk D and consider B as a mapping class of D.

Let D be a flat disk, to the right of B, with n points (punctures) where it intersects $K = \widehat{B}$ (see Figure 1). We assume the n punctures of D to be collinear, on a line that separates D into upper and lower half-disks. Denote by c_{ij} the isotopy class (fixing endpoints) of a path that is contained in the upper half-disk of D, with initial endpoint on the i^{th} strand and terminal endpoint on the j^{th} strand.

Considering B as a mapping class element, let $B \cdot c_{ij}$ denote the isotopy class of the path to which c_{ij} is sent. Viewing D from the left (as pictured), σ_k acts by rotating the k- and $(k+1)$-punctures an angle of π about their midpoint in counter-clockwise fashion. Consider the *algebra of paths* over $\mathbb{Z}$ generated by isotopy classes of paths in D with endpoints on punctures, modulo the relation in Figure 2. Let $\psi(c_{ij}) = a_{ij}$ if $i<j$, and $\psi(c_{ij}) = -a_{ij}$

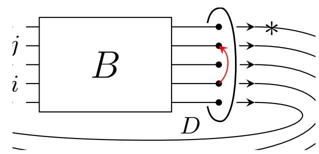

FIGURE 1. Cord c_{ij} of $K = \widehat{B}$

if $i > j$. This defines an algebra map to $\mathcal{A}_n$ satisfying $\psi(B \cdot c_{ij}) = \phi_B(\psi(c_{ij}))$ (see [Ng05]).

FIGURE 2. Relation in the algebra of paths

Let perm : $B_n \to S_n$ denote the standard homomorphism from B_n to the symmetric group S_n. We will make use of the following property of the automorphism ϕ_B.

Lemma 2.1. *For some $B \in B_n$ and $1 \leq i \neq j \leq n$, consider the element $\phi_B(a_{ij}) \in \mathcal{A}_n$ as a polynomial expression in the (non-commuting) variables $\{a_{ij}, 1 \leq i \neq j \leq n\}$. Writing $i' = perm(B)(i)$ and $j' = perm(B)(j)$, every non-constant monomial in $\phi_B(a_{ij})$ is a constant times a product $\prod_{k=0}^{m} a_{i_k, i_{k+1}}$ where $m \geq 1$ and $i_0 = i'$, $i_m = j'$, and $i_k \neq i_{k+1}$ for each $0 \leq k \leq m-1$.*

Proof. Suppose a path c in D starts at puncture k and ends at puncture k'. The relation in Figure 2 equates c with a sum (or difference) of another path with the same endpoints and a product of two paths, one beginning at k and the other ending at k'. A finite number of applications of this relation expresses c in the generators c_{ij}. The result follows since the class $B \cdot c_{ij}$ is represented by a path beginning at the i' puncture and ending at the j' puncture.

Alternatively, the statement follows from noting that (1) defining ϕ_{σ_k} has the desired property and that $\phi : B_n \to \mathrm{Aut}(\mathcal{A}_n)$ is a homomorphism. $\square$

2.2. **Augmentations and augmentation rank.** Let S be a ring with 1, and consider it a differential graded algebra with grading 0 and trivial differential. An augmentation of a DGA $(\mathcal{A}, \partial)$ to $(S, 0)$ is a graded homomorphism $\epsilon : \mathcal{A} \to S$ that intertwines the differential. In the case of knot contact homology, the combinatorial knot DGA is supported in non-negative grading, implying that augmentations correspond to ring homomorphisms $HC_0(K) \to S$. We will consider only when $S = \mathbb{C}$.

Definition An *a*ugmentation of a cord algebra $\mathcal{C}_K$ is a homomorphism $\epsilon \colon \mathcal{C}_K \to \mathbb{C}$

A correspondence between augmentations and particular representations of the knot group were studied in [Cor13a]. Let π_K be the fundamental group of the complement of a knot $K \subset S^3$. Recall that, if we call any $g \in \pi_K$ a *meridian* if it may be represented by the boundary of an embedded disk in S^3 that intersects K in exactly one point, then π_K is generated by meridians. We may pick any one meridian m and generate π_K by conjugates of m.

Definition For any integer $r \geq 1$ we call a homomorphism $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ a *KCH representation* if a meridian m of K such that $\rho(m)$ is diagonalizable and has eigenvalue 1 with multiplicity $r-1$. We call ρ a *KCH irrep* if it is irreducible.

In [Ng08], Ng describes an isomorphism between $HC_0(K)$ and an algebra constructed from elements of π_K. As discussed in [Ng12], by utilizing this isomorphism a KCH representation $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ induces an augmentation $\epsilon_\rho : HC_0(K) \to \mathbb{C}$. It was shown in [Cor13a] that (essentially) all augmentations arise in this fashion, and that the dimension of an inducing KCH irrep is invariant of the augmentation that can be described from the matrix $\mathbf{A}$. Specifically, if we write $\epsilon(\mathbf{A})$ for the matrix of values $(\epsilon(\mathbf{A}_{ij}))$, then we have the following theorem.

Theorem 2.2 ([Cor13a]). *For every augmentation $\epsilon : HC_0(K) \to \mathbb{C}$ such that $\epsilon(\mu) \neq 1$, there is a KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$, and r is the rank of $\epsilon(\mathbf{A})$.*

Considering Theorem 2.2 we make the following definition.

Definition The *rank* of an augmentation $\epsilon : HC_0(K) \to \mathbb{C}$ with $\epsilon(\mu) \neq 1$ equals the rank of $\epsilon(\mathbf{A})$. Given a knot K, the *augmentation rank of* K, denoted $\mathrm{ar}(K)$, is the maximum of all ranks of augmentations $\epsilon : HC_0(K) \to \mathbb{C}$.

Remark The augmentation rank of a knot could be defined for augmentations into other rings, but we deal in this paper with augmentations to $\mathbb{C}$.

It is the case that $\mathrm{ar}(K)$ is well-defined. That is, given a knot K there is a bound on the maximal rank of an augmentation $\epsilon : HC_0(K) \to \mathbb{C}$ that is provided by through the correspondence $\rho \leftrightarrow \epsilon_\rho$ and fact that π_K is generated by meridians.

Theorem 2.3 ([Cor13b]). *Given a knot $K \subset S^3$, if $g_1, \ldots, g_d$ are meridians that generate π_K and $\rho : \pi_K \to GL_r\mathbb{C}$ is a KCH irrep then $r \leq d$.*

As in the introduction, if we denote the meridional rank of π_K by $\mathrm{mr}(K)$, then Theorem 2.3 implies that $\mathrm{ar}(K) \leq \mathrm{mr}(K)$. In addition, the geometric quantity $b(K)$ called the bridge index of K is never less than $\mathrm{mr}(K)$. Thus we have the inequality

make it a corollary to refer to later

$$\mathrm{ar}(K) \leq \mathrm{mr}(K) \leq b(K).$$

As a result, to verify for K that $\mathrm{mr}(K) = b(K)$ it suffices to find an augmentation of K with rank equal to $b(K)$. As we discuss in the next section, we will concern ourselves in this paper with a setting where $\mathrm{ar}(K) = n$ and there is a braid $B \in B_n$ which closes to K. This is a special situation, since $b(K)$ is strictly less than the braid index for many knots.

2.3. **Finding augmentations.** Throughout the paper we denote by B_n the n-strand braid group, where our braids are oriented from left to right. We will often label the strands of a braid $1, \ldots, n$, with 1 the topmost to n the bottommost strand. The group B_n has standard generators $\{\sigma_i^{\pm}, i = 1, \ldots, n\}$ which have only the i and $i+1$ strands crossing once, and in the manner depicted in the projections of Figure 3. As usual, a braid may

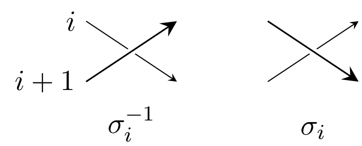

FIGURE 3. Generators of B_n

be closed to a link as depicted in Figure 4. The *w*rithe (or algebraic sum) of a braid B, denoted $w(B)$, is the sum of the exponents in a factorization of B in terms of the standard generators.

The following theorem concerns the behavior of the matrices Φ^L_B and Φ^R_B under the product in B_n. It is an essential tool for studying augmentations from the perspective of the combinatorial DGA, and it will be central to our arguments. Following language of that paper, we refer to the result as the Chain Rule.

Theorem 2.4 ([Ng05], Chain Rule)**.** *Let B, B' be braids in B_n. Then $\Phi^L_{BB'} = \phi_B(\Phi^L_{B'}) \cdot \Phi^L_B$ and $\Phi^R_{BB'} = \Phi^R_B \cdot \phi_B(\Phi^R_{B'})$.*

This paper concerns augmentations with rank equal to the braid index of the knot K. Suppose that K is the closure of $B \in B_n$ and define the diagonal matrix $\Delta(B) = \mathrm{diag}[(-1)^{w(B)}, 1, \ldots, 1]$. By considering the generators of the ideal $\mathcal{I}$ from Definition 2.1 the following statement follows from results in [Cor13a, Section 5].

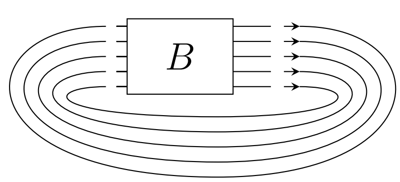

FIGURE 4. The closure of the braid B

Theorem 2.5 ([Cor13a])**.** *If K is the closure of $B \in B_n$ and has a rank n augmentation $\epsilon : \mathcal{A}_n \to \mathbb{C}$, then*

$$\epsilon(\Phi^L_B) = \Delta(B) = \epsilon(\Phi^R_B). \tag{4}$$

Furthermore, any homomorphism $\epsilon : \mathcal{A}_n \to \mathbb{C}$ which satisfies (4) determines a rank n augmentation of K.

There is a symmetry on the matrices Φ^L_B and Φ^R_B that is relevant to the study of augmentations in this setting. Define an involution $x \mapsto \overline{x}$ on $\mathcal{A}_n$ (termed *conjugation*) as follows: first set $\overline{a_{ij}} = a_{ji}$; then, for any $x, y \in \mathcal{A}_n$, define $\overline{xy} = \overline{y}\,\overline{x}$ and extend the operation linearly to $\mathcal{A}_n$. We have the following symmetry.

Theorem 2.6 ([Ng05], Prop. 6.2)**.** *For a matrix of elements in $\mathcal{A}_n$, let $\overline{M}$ be the matrix such that $\left(\overline{M}\right)_{ij} = \overline{M_{ij}}$. Then for $B \in B_n$, Φ^R_B is the transpose of $\overline{\Phi^L_B}$.*

Define $\mathcal{A}^{ab}_n$ to be the quotient of $\mathcal{A}_n$ by the ideal generated by $\{xy - yx | x, y \in \mathcal{A}_n\} \cup \{a_{ij} - a_{ji} | i \neq j\}$. Any homomorphism $\epsilon : \mathcal{A}_n \to \mathbb{C}$ such that $\epsilon(a_{ij} - a_{ji}) = 0$ for all $i \neq j$ determines a homomorphism $\epsilon : \mathcal{A}^{ab}_n \to \mathbb{C}$. As a consequence, Theorems 2.5 and 2.6 imply that finding $\epsilon : \mathcal{A}_n \to \mathbb{C}$ such that $\epsilon(a_{ij} - a_{ji}) = 0$ and satisfying $\epsilon(\Phi^L_B) = \Delta(B)$ suffices to determine a rank n augmentation. There do exist braids $B \in B_n$ and homomorphisms $\epsilon : \mathcal{A}_n \to \mathbb{C}$ satisfying (4), such that ϵ does not descend to $\mathcal{A}^{ab}_n$. However, to date every such braid $B \in B_n$ that has been found, also admits a homomorphism satisfying (4) that does descend to $\mathcal{A}^{ab}_n$.

In Section 3 we demonstrate that a satellite of the closure of $B \in B_k$ with a braid pattern $B' \in B_p$ admits an augmentation with rank pk, provided that the companion has augmentation rank k and the pattern augmentation rank p. If each possesses an augmentation that descends to $\mathcal{A}^{ab}_k$ (or A^{ab}_p respectively) then our methods show that the braid satellite does also.

It may be appropriate here to indicate that $\mathrm{ar}(K) < \mathrm{mr}(K)$ sometimes (maybe in previous subsection), and talk about the 2-cable of the trefoil that does not have $\mathrm{ar}(K, \mathbb{C}) = 4$

3. Main Result

bring in equations to fit margins

talk about how we're working in $\mathcal{A}^{ab}$ the whole time

introduce what will be done in the section: first section on notation, main result will follow from Lemma ____ and the Chain Rule

Let K be a knot and let B be a braid with closure K. Let $\tau_{m,l} \in B_{pk}$ be defined by $\tau_{m,l} = \sigma_m \sigma_{m+1} \cdots \sigma_{m+l-1}$, and let $\Sigma^{(p)}_n \in B_{pk}$ be defined by $\Sigma^{(p)}_n = \tau_{np,p} \tau_{np-1,p} \cdots \tau_{np-p+1,p}$ (see Figure 5).

make clear what's on top of what?

Then if $B \in B_k$ is given by the braid word $\sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_m}$, we define the ***p*-copy** $B^{(p)}$ of B to be

$$B^{(p)} = \Sigma^{(p)}_{n_1} \Sigma^{(p)}_{n_2} \cdots \Sigma^{(p)}_{n_m}$$

We then have the following result.

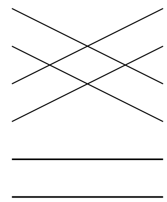

FIGURE 5. $\Sigma_1^{(2)}$

do I want the abelianized algebra here?

Theorem 3.1. *Let $B \in B_k$, and let $B' \in B_{pk}$ be a braid in B_p included into B_{pk} such that the first p strands of B' close to a knot. Suppose that there exists an augmentation $\epsilon_k\colon \mathcal{A}_k^{ab} \to \mathbb{C}$ such that $\epsilon_k\left(\Phi_B^L\right) = \Delta(B)$ and an augmentation $\epsilon_p\colon \mathcal{A}_p^{ab} \to \mathbb{C}$ such that $\epsilon_p\left(\Phi_{B'}^L\right) = \Delta(B')$. Then there exists an augmentation $\epsilon\colon \mathcal{A}_{pk}^{ab} \to \mathbb{C}$such that $\epsilon\left(\Phi_{B^{(p)}B'}^L\right) = \Delta(B^{(p)}B')$.*

braid rep've info needed to make well-defined

cite cornwell

finish this

Among other things, this theorem implies that iterated cables of torus knots have meridional rank equal to their bridge number. Consider a (r,s)-torus knot T with $\gcd(r,s) = 1$ and $r < s$. T has bridge number r and is the closure of a braid B on r strands, and since all torus knots have bridge number equal to their augmentation rank (), we have that there exists an augmentation $\epsilon_T\colon \mathcal{A}_r \to \mathbb{C}$. . given by the braid sum of $T^{(p)}$ with a braid who's first p strands form a torus knot with bridge number (and therefore augmentation rank) equal to p and such that $w(T)$ is even (i.e. a (p,q) torus knot, where $\gcd(p,q) = 1$, $p < q$, and $pq - q$ is even). Theorem 3.1 then says that this cable has augmentation rank equal to its braid index, implying that its meridional rank is equal to its bridge number. Furthermore, we can iterate this process, taking cables of the resulting knots with augmentation rank, bridge number, and braid index all equal.

prob from Kirby list should be mentioned here

in $\mathcal{A}^{ab}$ we are always making $i < j$? let's state so in background

Fix $p > 0$ and let B be a braid on k strands. For each $1 \le i \le pk$ define integers q_i, r_i such that $i = q_i p + r_i$, where $0 < r_i \le p$. Instrumental to the proof of Theorem 3.1 will be the map $\psi\colon \mathcal{A}_{pk}^{ab} \to \mathcal{A}_k^{ab} \otimes \mathcal{A}_p^{ab}$, defined as follows (note that since $a_{ij} \in \mathcal{A}_{pk}^{ab}$, $i < j$, so we must have $q_i \le q_j$):

$$\psi(a_{ij}) = \begin{cases} 1 \otimes a_{r_i r_j} & : q_i = q_j \\ a_{q_i+1, q_j+1} \otimes 1 & : q_i < q_j, r_i = r_j \\ 0 & : q_i < q_j, r_i > r_j \\ a_{q_i+1, q_j+1} \otimes a_{r_i r_j} & : q_i < q_j, r_i < r_j \end{cases}$$

Note that $\psi(a_{ij}) \in 1 \otimes \mathcal{A}_p^{ab}$ or $\psi(a_{ij}) = 0$ if and only if $q_i = q_j$. This homomorphism gives us a way of relating $\Phi_{B^{(p)}}^L$ to Φ_B^L via the following proposition.

Proposition 3.2. $\psi\left(\Phi_{B^{(p)}}^L\right) = \Phi_B^L \otimes I_p$

Note that here we mean the tensor product of Φ_B^L and I_p as matrices, not as linear maps. We are also abusing notation here, as what we actually mean is that

$$\psi\left(\Phi^L_{B^{(p)}}\right) = \left(\left(\Phi^L_B \otimes I_p\right)_{ij} \otimes 1\right)$$

where the elements of the matrix on the right hand side are the tensor products of the elements of $\Phi_B^L \otimes I_p$ with 1.

It turns out that instead of ψ we could have defined a simpler homomorphism $\rho\colon \mathcal{A}_{pk} \to \mathcal{A}_k$ that would take a_{ij} to $a_{q_{i+1},q_{j+1}}$ if $r_i = r_j$ and 0 otherwise, and Proposition 3.2 would still be true (this follows from the same ideas used in the proof of Proposition 3.2). The advantage of ψ is that it doesn't send a_{ij} to 0 if $q_i = q_j$, a fact which will be important in the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $\delta\colon \mathcal{A}_p \to \mathbb{C}$ be a homomorphism, and set $\epsilon = (\epsilon_k \otimes \delta) \circ \psi$. We will later break the theorem up into three cases depending on the parity of $w(B)$ and p and in each case define δ such that $\delta(a_{ij})$ is one of $\pm\epsilon_p(a_{ij})$ in such a way that ϵ is an augmentation of $B^{(p)}B'$. The Chain Rule theorem gives that

$$(\epsilon_k \otimes \delta) \circ \psi\left(\Phi^L_{B^{(p)}B'}\right) = (\epsilon_k \otimes \delta)\psi\left(\phi_{B^{(p)}}\left(\Phi^L_{B'}\right)\right)\psi\left(\Phi^L_{B^{(p)}}\right) \tag{5}$$

Note that since the non zero or one entries of $\Phi^L_{B'}$ are products of a_{ij} where $i < j \leq p$, $\phi_{B^{(p)}}$ takes each of the a_{ij}'s in these products to $a_{i+mp,j+mp}$ for some $0 \leq m < k$. We have that ψ takes $a_{i+mp,j+mp}$ to $1 \otimes a_{ij}$, however, so

$$\psi\left(\phi_{B^{(p)}}\left(\Phi^L_{B'}\right)\right) = \left(1 \otimes \left(\Phi^L_{B'}\right)_{ij}\right)$$

By Proposition 3.2, we have that

$$\psi\left(\Phi^L_{B^{(p)}}\right) = \left(\left(\Phi^L_B \otimes I_p\right)_{ij} \otimes 1\right)$$

make this consistent throughout prop 3.2

So returning to the right hand side of (5) we get

need to include homomorphism h taking $a \otimes b$ to ab

using two different tensor products here (and in other places throughout paper)

$$\begin{aligned}(\epsilon_k \otimes \delta)\left(\psi\left(\phi_{B^{(p)}}\left(\Phi^L_{B'}\right)\right)\psi\left(\Phi^L_{B^{(p)}}\right)\right) &= (\epsilon_k \otimes \delta)\left(\left(1 \otimes \left(\Phi^L_{B'}\right)_{ij}\right)\left(\left(\Phi^L_B \otimes I_p\right)_{ij} \otimes 1\right)\right)\\ &= \left(\Delta(B) \otimes I_p\right)\delta\left(\Phi^L_{B'}\right)\end{aligned}$$

So it suffices to find an augmentation δ such that the right hand side is equal to $\Delta(B^{(p)}B')$. If $w(B)$ is even, then we simply let $\delta = \epsilon_p$. Since $w(B)$ is even we know that $w(B^{(p)})$ is also even and that $\Delta(B) = I_k$. Since $\epsilon\left(\Phi^L_{B'}\right) = \Delta(B')$, it follows that the right hand side is equal to $\Delta(B^{(p)}B')$.

Now suppose that $w(B)$ is odd. In a moment we will define $g\colon \{1,\ldots,p\} \to \{\pm 1\}$ for each of the cases for when p is even or odd, but for now let just introducing this notation

$\delta(a_{ij}) = g(i)g(j)e_k(a_{ij})$. Fix i,j and consider a monomial M in $\left(\Phi^L_{B'}\right)_{ij}$. Since B' is a braid on p strands included into B_{pk}, if $i > p$ or $j > p$ then M is 0 or 1 and $\delta(M) = M$. If $i, j \le p$, such a monomial must arise from a product in the algebra of paths in D that begins at $i' = \mathrm{perm}(B')(i)$ and ends at j, so $M = c_{ij}a_{i',j_1}a_{j_1,j_2}\dots a_{j_m,j}$ for some $j_1,\dots j_m \in \{1,\dots,p\}$, unless $i' = j$, in which case it is possible that $M = c_{ij}$. We then see that

$$\delta(M) = g(i')g(j)\left(\prod_{k=1}^{m} g(j_k)^2\right)\epsilon_p(M) = g(i')g(j)\epsilon_p(M)$$

Or $\delta(M) = M = g(i')g(j)\epsilon_p(M)$ in the case that $i' = j$ and $M = c_{ij}$. Since this is true for each monomial M chosen in $\left(\Phi^L_{B'}\right)_{ij}$, we have that

$$\delta\left(\left(\Phi^L_{B'}\right)_{ij}\right) = g(i')g(j)\epsilon_p\left(\left(\Phi^L_{B'}\right)_{ij}\right)$$

Now let $x_1 = 1$, and $x_l = \mathrm{perm}(B')(x_{l-1})$ for $1 < l \le p$. Since the first p strands of B' close to a knot, $\mathrm{perm}(B')$ is given by the p-cycle $(x_1x_2\dots x_p)$.

Suppose p is even. Then we let $g(x_1) = 1$, and $g(x_l) = -g(x_{l-1})$ for $1 < l \le p$. Since p is even, $w(B^{(p)})$ is even and therefore the opposite parity of $w(B)$. Our definition of g gives that $\delta\left(\left(\Phi^L_{B'}\right)_{ii}\right) = -\epsilon\left(\left(\Phi^L_{B'}\right)_{ii}\right)$ for $i \le p$, so

$$\delta\left(\Phi^L_{B'}\right) = \begin{pmatrix} (-1)^{w(B')+1} & 0 & 0 \\ 0 & -I_{p-1} & 0 \\ 0 & 0 & I_{(k-1)p} \end{pmatrix}$$

and therefore

$$(\Delta(B)\otimes I_p)\,\delta\left(\Phi^L_{B'}\right) = \mathrm{diag}[(-1)^{w(B)+w(B')+1}, 1\dots 1] = \Delta(B^{(p)}B')$$

as desired.

Lastly suppose that p is odd. Then we let $g(x_1) = g(x_2) = 1$ and $g(x_l) = -g(x_{l-1})$ for $2 < l \le p$. Since p is odd, $w(B^{(p)})$ is odd and therefore the same parity of $w(B)$. Our definition of g gives that $\delta\left(\left(\Phi^L_{B'}\right)_{11}\right) = \epsilon\left(\left(\Phi^L_{B'}\right)_{11}\right)$ and $\delta\left(\left(\Phi^L_{B'}\right)_{ii}\right) = -\epsilon\left(\left(\Phi^L_{B'}\right)_{ii}\right)$ for $1 < i \le p$, so

$$\delta\left(\Phi^L_{B'}\right) = \begin{pmatrix} (-1)^{w(B')} & 0 & 0 \\ 0 & -I_{p-1} & 0 \\ 0 & 0 & I_{(k-1)p} \end{pmatrix}$$

and therefore

$$(\Delta(B)\otimes I_p)\,\delta\left(\Phi^L_{B'}\right) = \mathrm{diag}[(-1)^{w(B)+w(B')}, 1\dots 1] = \Delta(B^{(p)}B')$$

as desired. $\square$

We will use the following two lemmas in our proof of Proposition 3.2.

Lemma 3.3. *$\psi(\phi_{\Sigma_n^{(p)}}(a_{ij})) = (\phi_{\sigma_n}\otimes \mathrm{id})(\psi(a_{ij}))$ for all $1 \le n < k$, $1 \le i, j \le pk$.*

Lemma 3.4. $\psi\left(\Phi^L_{\Sigma_n^{(p)}}\right) = \Phi^L_{\sigma_n} \otimes I_p$

Proof of Proposition 3.2. Let $B = \sigma_{n_1} \cdots \sigma_{n_l}$, $1 \leq n < k$. We will prove the proposition by inducting on l. The base case is already taken care of by Lemma 3.4. Suppose that the proposition is true for braids of length $l-1$. Let $B' = \sigma_{n_1} \cdots \sigma_{n_{l-1}}$ Then by the Chain Rule and Lemmas 3.3 and 3.4, we have that

$$\begin{aligned}
\psi\left(\Phi^L_{B^{(p)}}\right) &= \psi\left(\phi_{B'^{(p)}}\left(\Phi^L_{\Sigma_{n_l}^{(p)}}\right) \cdot \Phi^L_{B'^{(p)}}\right) \\
&= (\phi_{B'} \otimes \mathrm{id})\left(\psi\left(\Phi^L_{\Sigma_{n_l}^{(p)}}\right)\right) \cdot \left(\Phi^L_{B'} \otimes I_p\right) \\
&= (\phi_{B'} \otimes \mathrm{id})\left(\Phi^L_{\sigma_{n_l}} \otimes I_p\right) \cdot \left(\Phi^L_{B'} \otimes I_p\right) \\
&= \Phi^L_B \otimes I_p
\end{aligned}$$

□

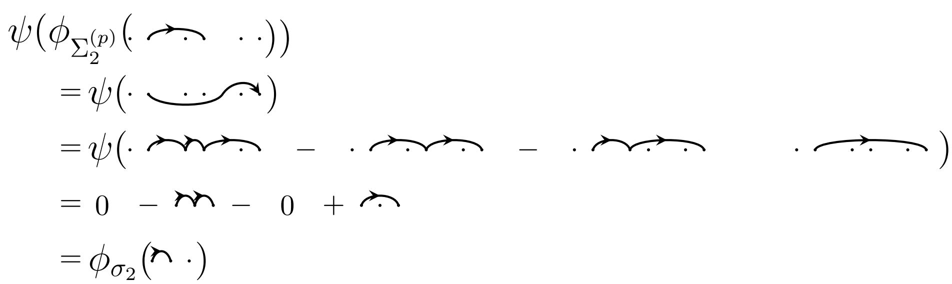

Figure 6. Computing $\psi(\phi_{\Sigma_2^{(p)}}(a_{24}))$

In the proof of Lemmas 3.3 and 3.4, we will make use of some calculations of $\phi_B(a_{ij})$ for simple braids B. It can easily be checked that for all $1 \leq m < n$, $1 \leq l \leq n-m$, $i < j$:

recall definition of $\tau_{m,l}$

$$\phi_{\tau_{m,l}}(a_{ij}) = \begin{cases} a_{i+1,j+1} & : m \leq i < j < m+l \\ a_{i-l,j} & : m < m+l = i < j \\ a_{i,j-l} & : i < m < m+l = j \\ a_{i+1,j-l} & : m \leq i < j = m+l \\ a_{i,j+1} - a_{i,m}a_{m,j+1} & : i < m \leq j < m+l \\ a_{i+1,j} - a_{i+1,m}a_{m,j} & : m \leq i < m+l < j \\ a_{ij} & : \text{otherwise} \end{cases} \tag{6}$$

We also make the following definition
Let $X \subseteq \{1, \ldots, n\}$, and write the elements of a subset $Y \subseteq X$ as $y_1 < \ldots < y_k$ Define

$$A(i,j,X) = \sum_{Y \subseteq X} (-1)^{|Y|} a_{iy_1} a_{y_1y_2} \cdots a_{y_kj}$$

and

$$A'(i,j,X) = \sum_{Y \subseteq X} (-1)^{|Y|} a_{iy_k} a_{y_k y_{k-1}} \cdots a_{y_1 j}$$

and have the following lemma

Changed $X_{m,l}$ by adding p to everything, need to make sure still works with proof

check

Lemma 3.5. *Let* $\kappa_{m,l} = \tau_{m+l-1,p}\tau_{m+l-2,p}\cdots\tau_{m,p}$*, and let* $X_{m,l} = \{m,\ldots,m+l-1\}$*. Then*

$$\phi_{\kappa_{m,l}}(a_{ij}) = \begin{cases} a_{i-p,j-p} & : m+p \leq i < j < m+l+p \\ a_{i-p,j} & : m+p \leq i < m+l+p \leq j \\ a_{i,j-p} & : i < m < m+p \leq j < m+l+p \\ a_{i+l,j+l} & : m \leq i < j < m+p \\ A'(i+l, j-p, X_{m,l} \setminus (j-p)) & : m \leq i < m+p \leq j < m+l+p \\ A(i, j+l, X_{m,l}) & : i < m \leq j < m+p < m+l+p \\ A'(i+l, j, X_{m,l}) & : m \leq i < m+p < m+l+p \leq j \\ a_{ij} & : \text{otherwise} \end{cases}$$

Note that letting $l = p$ and $m = (n-1)p+1$ gives us $\phi_{\Sigma_n^{(p)}}(a_{ij})$ as a special case. Letting $X_n^{(p)} = \{(n-1)p+1,\ldots,np\}$, we have

$$\phi_{\Sigma_n^{(p)}}(a_{ij}) = \begin{cases} a_{i-p,j-p} & : np < i < j \leq (n+1)p \\ a_{i-p,j} & : np < i \leq (n+1)p < j \\ a_{i,j-p} & : i \leq (n-1)p < np < j \leq (n+1)p \\ a_{i+p,j+p} & : (n-1)p < i < j \leq np \\ A'(i+p, j-p, X_n^{(p)} \setminus (j-p)) & : (n-1)p < i \leq np < j \leq (n+1)p \\ A(i, j+p, X_n^{(p)}) & : i \leq (n-1)p < j \leq np < (n+1)p \\ A'(i+p, j, X_n^{(p)}) & : (n-1)p < i \leq np < (n+1)p < j \\ a_{ij} & : \text{otherwise} \end{cases}$$

Proof of Lemma 3.3. The first four cases as well as the last case from Lemma 3.5 can be checked easily. Consider the sixth case. Lemma 3.5 gives that

$$\psi\left(\phi_{\Sigma_n^{(p)}}(a_{ij})\right) = \sum_{Y \subseteq \{np-p+1,\ldots,np\}} (-1)^{|Y|} \psi\left(a_{iy_1} a_{y_1 y_2} \cdots a_{y_k, j+p}\right)$$

Let $\alpha_i = np - p + r_i$. Note that if $y_1 < \alpha_i$ then $\psi(a_{iy_1}) = 0$, and if $y_k > \alpha_j$ then $\psi(a_{y_k j}) = 0$, so the sum on the right hand side can be taken over $Y \subseteq \{\alpha_i, \alpha_i + 1, \ldots, \alpha_j\}$. Then we manipulate the sum to get

do I need to explain what I'm doing here?

$$
\begin{aligned}
&\sum_{Y\subseteq\{\alpha_i,\dots,\alpha_j\}} (-1)^{|Y|}\psi\left(a_{iy_1}a_{y_1y_2}\cdots a_{y_k,j+p}\right)\\
&=\psi\left(a_{i,j+p}-a_{i,\alpha_i}a_{\alpha_i,j+p}\right)\\
&\quad+\sum_{y=\alpha_i+1}^{\alpha_j}\sum_{Y\subseteq\{y+1,\dots,\alpha_j\}} (-1)^{|Y|+1}\psi\left(a_{iy}a_{yy_1}\cdots a_{y_k,j+p}\right)+(-1)^{|Y|}\psi\left(a_{i,\alpha_i}a_{\alpha_i,y}a_{yy_1}\cdots a_{y_k,j+p}\right)\\
&=\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right)\\
&\quad+\sum_{y=\alpha_i+1}^{\alpha_j}\sum_{Y\subseteq\{y+1,\dots,\alpha_j\}} (-1)^{|Y|}\psi\left(a_{i,\alpha_i}a_{\alpha_i,y}-a_{iy}\right)\psi\left(a_{yy_1}\cdots a_{y_k,j+p}\right)
\end{aligned}
$$

Note that $r_i = r_{\alpha_i}$ and since we're in the sixth case we have $(n-1)p < j \leq np$, so $q_{\alpha_i} = q_y$. Thus $\psi(a_{i,\alpha_i}) = a_{q_i+1,q_{\alpha_i}+1}\otimes 1 = a_{q_i+1,q_y+1}\otimes 1$ and $\psi(a_{\alpha_i,y}) = 1\otimes a_{r_{\alpha_i},r_y} = 1\otimes a_{r_i,r_y}$, so we have

$$
\psi(a_{i,\alpha_i}a_{\alpha_i,y}-a_{iy}) = \left(a_{q_i+1,q_y+1}\otimes 1\right)\left(1\otimes a_{r_i,r_y}\right) - a_{q_i+1,q_y+1}\otimes a_{r_i,r_y} = 0
$$

Thus the right hand side reduces to

$$
\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right)
$$

Remark The fact that $\psi(a_{i,\alpha_i}a_{\alpha_i,y}-a_{iy}) = 0$ and ψ behaves similarly for the analogous terms in the other cases is the key to this proof working, and ψ is defined the way it is mainly so that this will be true. As we hinted at earlier, the homomorphism $\rho\colon \mathcal{A}_{pk}\to\mathcal{A}_k$ defined to send a_{ij} to a_{q_i+1,q_j+1} if $r_i = r_j$ and to 0 otherwise would also send these terms to 0, so Proposition 3.2 would still be true with ρ used in the place of ψ. We will need ψ for the proof of the main result, however.

Note that, since we're in the sixth case, $q_j+1 = n$. If $r_i = r_j$, then

$$
\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right) = \left(a_{q_i+1,n+1}-a_{q_i+1,n}a_{n,n+1}\right)\otimes 1 = (\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))
$$

If $r_i < r_j$, then

$$
\begin{aligned}
\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right) &= \left(a_{q_i+1,n+1}\otimes a_{r_ir_j} - a_{q_i+1,n}a_{n,n+1}\otimes a_{r_ir_j}\right)\\
&= \left(a_{q_i+1,n+1}-a_{q_i+1,n}a_{n,n+1}\right)\otimes a_{r_ir_j}\\
&= (\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))
\end{aligned}
$$

Finally, if $r_i > r_j$, then

$$
\psi\left(a_{i,j+p}-a_{i\alpha_i}a_{\alpha_i,j+p}\right) = 0 = (\phi_{\sigma_n}\otimes\mathrm{id})(\psi(a_{ij}))
$$

The proof for the seventh case goes exactly as the proof for the sixth case except with all i's replaced with $i+p$, all $(j+p)$'s replaced with j, all y_i's replaced with y_{k+1-i}, and with α_i and α_j swapped. The proof for the fifth case goes exactly as the proof for the seventh, except that $j-p$ is removed from the set that Y is a subset of in all the sums. check this

□

check

Proof of Lemma 3.4. We can extend the definition of ψ to be from the free module over $\mathcal{A}_{pk}$ generated by $\{a_{i*} | 1 \leq i \leq pk\}$ to the free module over $\mathcal{A}_k \otimes \mathcal{A}_p$ generated by $\{a_{i*} | 1 \leq i \leq k\}$ by defining $\psi(a_{i*}) = a_{i*}$ and extending by linearity. Then the statement of the lemma is equivalent to saying that for all $1 \leq i \leq pk$, the coefficient of a_{j*} in $\psi\left(\phi_{\Sigma_n^{(p)}}(a_{i*})\right)$ is equal to 0 unless $r_j = r_i$, in which case it is equal to the coefficient of a_{q_j*} in $\phi_{\sigma_n}(a_{q_i*})$. If $q_i + 1 \neq n$, this fact can be easily checked. In the case that $q_i + 1 = n$, we have that

$$\psi\left(\phi_{\Sigma_n^{(p)}}(a_{i*})\right) = \psi\left(A(i+p, *, \{np - p + 1, \ldots, np\}\right)$$

which is equal to

$$\psi(a_{i+p,*} - a_{i+p,\alpha_i} a_{\alpha_i,*}) = a_{i+p,*} - a_{q_i,q_i+1} a_{\alpha_i,*}$$

by the same argument that was used in Lemma 3.3. The coefficients of the a_{j*} are equal to the coefficients of the a_{q_j*} in $\phi_{\sigma_n}(a_{q_i*})$, so we're done. □

Proof of Lemma 3.5. add other cases

The first four cases as well as the eighth can be easily checked. We will prove the remaining cases by induction on l. Consider the sixth case. The base case is covered by (6). For the inductive step, we have that

$$\begin{aligned}
\phi_{\kappa_{m,l}}(a_{ij}) &= \phi_{\tau_{m,p}}\left(\phi_{\kappa_{m+1,l-1}}(a_{ij})\right) \\
&= \sum_{Y \subseteq \{m+p+1,\ldots,m+l+p-1\}} (-1)^{|Y|} \phi_{\tau_{m,p}}\left(a_{i-l+1,y_1} a_{y_1 y_2} \cdots a_{y_k,j}\right) \\
&= \sum_{Y \subseteq \{m+p+1,\ldots,m+l+p-1\}} (-1)^{|Y|} \left(a_{i-l,y_1} - a_{i-l,m+p} a_{m+p,y_1}\right) a_{y_1 y_2} \cdots a_{y_k,j} \\
&= \sum_{Y \subseteq \{m+p,\ldots,m+l+p-1\}} (-1)^{|Y|} a_{i-l,y_1} a_{y_1 y_2} \cdots a_{y_k,j} \\
&= A(i-l, j, X_{m,l})
\end{aligned}$$

is this clear/can it be shortened?

□

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