What we have (when $w(B)$ is odd) is a solution that gives us a matrix

$$\begin{pmatrix} -\mathrm{Id}_p & 0 \\ 0 & \mathrm{Id}_{p(k-1)} \end{pmatrix}$$

(the "1" on diagonals here being $1 \otimes 1 \in \mathcal{A}_k \otimes \mathcal{A}_p$), which comes about from the matrix $\psi(\Phi^L_{B^{(p)}B'})$ being the product of: a $k \times k$ matrix of $p \times p$ matrices with each entry being $(\Phi^L_B)_{ij}\Phi^L_{B'}$ times the $pk \times pk$ matrix with entries $1 \otimes \Phi^L_{B'}$, where we consider $B' \in B_{pk}$ (sitting in the subgroup generated by $\sigma_1, \dots, \sigma_p$) rather than in $B_p$.

We run with the fact that we can get the homomorphism sending $\Phi^L_{B^{(p)}}$ to the matrix described above.

**Claim:** It would suffice to show that if $p < q$ are coprime and $B' = \tau_p^q$, where $\tau_p = \sigma_1 \dots \sigma_{p-1}$, then there is a homomorphism $f : \mathcal{A}_p^{ab} \to \mathbb{C}$ so that (including $B'$ into $B_{pk}$),

$$f\left(\Phi^L_{B'}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mathrm{Id}_{p-1} & 0 \\ 0 & 0 & \mathrm{Id}_{p(k-1)} \end{pmatrix}.$$

If there is such an $f$ then note that

$$\begin{pmatrix} -\mathrm{Id}_p & 0 \\ 0 & \mathrm{Id}_{p(k-1)} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mathrm{Id}_{p-1} & 0 \\ 0 & 0 & \mathrm{Id}_{p(k-1)} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \mathrm{Id}_{pk-1} \end{pmatrix},$$

and this matrix is $\Delta(B^{(p)}B')$ since $B' = \tau_p^q$ (with $p, q$ coprime) means that $w(B')$ and $p$ have opposite parity, so in the case that $w(B)$ is odd, $w(B^{(p)}) + w(B')$ is odd. So it remains to prove that there is such an $f$.

Let $p$ be even and let $\epsilon : \mathcal{A}_p^{ab} \to \mathbb{C}$ be such that $\epsilon(\Phi^L_{B'}) = \Delta(B')$. Define $f(a_{ij}) = (-1)^{j-i}\epsilon(a_{ij})$ and let $i' = \mathrm{perm}(B')(i)$ (this is the puncture that the monomial must start on). Fix $i, j$ and consider a monomial $M = c_{ij}a_{i',j_1}a_{j_1,j_2}\dots a_{j_m,j}$ in $\left(\Phi^L_{B'}\right)_{ij}$. We have used that such a monomial must arise from a product in the algebra of paths in $D$ that begins at $i' = \mathrm{perm}(B')(i)$ and ends at $j$.

Now we see that $f(M) = (-1)^{\sum_{n=0}^m (j_{n+1}-j_n)}\epsilon(M) = (-1)^{j-i'}\epsilon(M)$ where $j_0 = i'$ and $j_{m+1} = j$. The power of $-1$ here is independent of the particular monomial chosen in $\left(\Phi^L_{B'}\right)_{ij}$ and so $f((\Phi^L_{B'})_{ij}) = \pm\epsilon((\Phi^L_{B'})_{ij})$. When $i = j$, the sign is, in fact, negative since the difference $\mathrm{perm}(B')(i) - i \bmod p$ must be invertible in $\mathbb{Z}/p$ since $B'$ closes to a knot (here we used the particular cyclic form of $\mathrm{perm}(\tau_p^q)$). When $p$ is even this means that the difference $i' - j$ is odd.

Now suppose that $p$ is odd and let's just consider the case that $q = p + 1$. Here we must define $f$ in a slightly different manner. Again let $\epsilon(\Phi^L_{B'}) = \Delta(B')$ as before (now $\Delta(B') = \mathrm{Id}_p$ since $w(B')$ is even). We would like $f$ to change the sign on some of the $\epsilon((\Phi^L_{B'})_{ij})$ again, but we want

$f((\Phi^L_{B'})_{ii}) = -\epsilon((\Phi^L_{B'})_{ii})$ only if $i > 1$. Define $f$ by $f(a_{ij}) = (-1)^{j-i+1}\epsilon(a_{ij})$ if $\mathrm{perm}(B')(1) = i$ and $f(a_{ij}) = (-1)^{j-i}\epsilon(a_{ij})$ otherwise.

As before we get $f(M) = \pm\epsilon(M)$ for each monomial, the sign only depending on $i, j$. Here is why it only depends on $i, j$: suppose that $i$ is such that $\mathrm{perm}(B')(1) = i$. Then there must be an odd number of generators $a_{ik} = a_{ki}$ in any monomial $M = c_{ij}a_{i,j_1}a_{j_1,j_2}\ldots a_{j_m,j}$ appearing in the first row of $\Phi^L_{B'}$ since (1) the first generator $a_{i,j_1}$ is such a generator and (2) since $j \neq i$, any time that a path corresp. to $a_{ki}$ comes into the puncture $i$, it must then leave it by some $a_{i,k'}$. Thus for some $M$ appearing in the first row, $f(M) = (-1)(-1)^{j-i}\epsilon(M)$. On the other hand, if $M$ appears in some other row $k$ (and column $j$), then $\mathrm{perm}(B')(k) = k' \neq i$, so the path corresponding to $M$ cannot begin at puncture $i$. We get $f(M) = (-1)^{\delta_{ij}}(-1)^{j-k'}\epsilon(M)$ which, since $i$ is fixed as $\mathrm{perm}(B')(1)$, has sign that only depends on $k, j$.