Why might $\epsilon(\sum_{r=1}^{p} v_r a_{rp}) = 0$?

(1) It suffices to show that $\epsilon(\sum_{r=1}^{p} v_r a_{rp}) = \epsilon(w_{2p+1,p})$.

(2) It suffices to show that $\epsilon(\phi^n_{\imath_p(\tau)}(-a_{2p+1,p})) = 0$.

Note that $-a_{2p+1,p} = \phi_{\imath_p(\tau)}(\sum_{S\subset X_n^{(p)}}(-1)^{|S|}a_{p+1,s_1}a_{s_1,s_2}\dots a_{s_k,np})$

Well, since $\phi_{\imath_p(\alpha)}(\mathbf{A}) = \Phi^L_{\imath_p(\alpha)}\mathbf{A}\Phi^R_{\imath_p(\alpha)}$ and so for $i > p$ and $j > p$, $\epsilon(a_{ij}) = \epsilon(\phi_{\imath_p(\alpha)}(a_{ij}))$ and so

$$\begin{aligned}\sum_{S\subset X_n^{(p)}}(-1)^{|S|}\epsilon(a_{p+1,s_1}a_{s_1,s_2}\dots a_{s_k,np}) &= \sum_{S\subset X_n^{(p)}}(-1)^{|S|}\epsilon(\phi_{\imath_p(\alpha)}(a_{p+1,s_1}a_{s_1,s_2}\dots a_{s_k,np}))\\ &= \epsilon(\phi^n_{\imath_p(\tau)}(-a_{2p+1,p}))\end{aligned}$$

(3) It also suffices to show that $\epsilon(\sum_{S\subset X_3^{(p)}}(-1)^{|S|}a_{2p+1,s_1}a_{s_1,s_2}\dots a_{s_k,np}) = 0$. This was verified for $T((4,2),(5,1))$.

(4) Along another line of thought...Each of the entries in Φ^{n+1}_{22} are sent to same place as entries in Φ^2_{12} and the first row of this looks like $w_{2p+1,j}$ for $p+1 \le j \le 2p$. Because of this, $\epsilon(w_{2p+1,j}) = 0$ for $p+1 < j \le 2p$ and $\epsilon(w_{2p+1,p+1}) = 1$. Hence, $\epsilon(w_{2p+1,p}) = \epsilon(-a_{2p+1,p}) + \epsilon(a_{2p+1,p+1}a_{p+1,p})$. Note that $1 = \epsilon(w_{2p+1,p+1}) = \epsilon(-a_{2p+1,p+1})$ and so

$$\epsilon(a_{2p+1,p}) = -\epsilon(a_{p+1,p}).$$

Now, an argument like you give in the paper so far would show that $\epsilon(a_{2p+1,p}) = 0$ if $n = 3+1$. That is, you may be able to just pass the buck up to the next row... and have an argument that $\epsilon(a_{(n-1)p+1,p}) = 0$.

Let's just make sure:

Claim: For any $1 \le i < n-1$ we have $\epsilon(a_{ip+1,p}) = \pm\epsilon(a_{(i+1)p+1,p})$.

Proof. Note that $\epsilon(w_{ip+1,p}) = \epsilon((\Psi^i_{11})_{1p})$ which agrees with (ϵ of) the $((i-1)p+1,p)$ entry of $\Phi^L_{\imath_p(\alpha)}$. (more general statment replacing p with any j such that $j \le ip$).

Case $i = 1$. For any $j \le 2p$ we have that $\epsilon(w_{2p+1,j})$ agrees with the $(p+1,j)$ entry. Since these are zero for $j > p+1$ we can use the relation in $\mathscr{S}_{np}$ to get that $1 = \epsilon(w_{2p+1,p+1}) = \epsilon(-a_{2p+1,p+1})$ ($w_{2p+1,p+1}$ is a sum of $-a_{2p+1,p+1}$ and terms with $w_{2p+1,j}$, $j > p+1$ in them). Now, again using the spanning arc relation,

$$0 = \epsilon(w_{2p+1,p}) = \epsilon(-a_{2p+1,p}) - \epsilon(w_{p+1,p}) = -\epsilon(a_{2p+1,p}) + \epsilon(a_{p+1,p}).$$

What happens for $i = 2$? First, the $\epsilon(w_{3p+1,j}) = 0$ for $2p+1 < j \le 3p$. Also, $\epsilon(w_{3p+1,2p+1}) = -\epsilon(a_{3p+1,2p+1}) = 1$ and hence

$$0 = \epsilon(w_{3p+1,2p}) = -\epsilon(a_{2p+1,2p}) - \epsilon(a_{3p+1,2p}).$$

Also $0 = \epsilon(w_{3p+1,2p-1}) = -\epsilon(a_{2p+1,2p-1}) - \epsilon(a_{3p+1,2p-1})$, and so on for $\epsilon(w_{3p+1,j})$ with $j = 2p-2,\dots,p$ (to p in particular) so we have $\epsilon(a_{3p+1,p}) = -\epsilon(a_{2p+1,p}) = \epsilon(a_{p+1,p})$. The induction argument should be apparent. $\square$