Consider the braid ${}^pB$ in B_{pk}. Not bad ...
Let's try $\Phi^L_{{}^p\sigma_n}$ or $\Phi^L_{{}^pB}$. And how about $\Phi_{{}^p\sigma_n^{\pm 1}}$ or ${}^2B$, ${}^2\sigma_n$, or even $\Phi_{{}^2\sigma_n}$
Closure?? $\widehat{{}^pBB'}$ vs. $\widehat{\alpha_p\gamma}$. what about $\mathrm{ar}(\widehat{\alpha})$ vs. $\mathrm{ar}(\hat{\alpha})$
Just say that $\alpha_p \in B_{pk}$ is the p-copy of $\alpha \in B_k$
$\Phi^L_{\imath_p(\alpha)}$ and $\Phi^L_{\beta(\alpha)}$ and $\widehat{\beta(\alpha)}$ (or $\hat{\beta(\alpha)}$)
γ vs $\bar{\gamma}$
$\imath_p(\alpha)$
$\Phi^L_{\imath_p(\alpha)}$ and $\Phi^L_{\beta(\alpha)}$ or... $\Phi^L_{\gamma(\alpha)}$ and $\widehat{\beta(\alpha)}$ (or $\hat{\beta(\alpha)}$)
$\mathscr{S}^L_n$