Optimization from Calculus

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Outline

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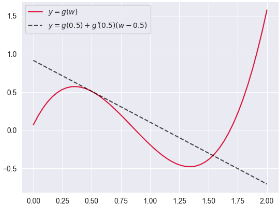
Linear approximations

$g : \mathbb{R} \to \mathbb{R}$, a twice-differentiable function (at least). Write w for input to g, so $g(w)$.

Linear approximation to $g(w)$. At a point $(v, g(v))$ on its graph, function whose graph is the tangent line:

$$h(w) = g(v) + g'(v)(w - v).$$

- Also called first order Taylor series approximation (or Taylor polynomial) of g.
- Approximates values of g, for inputs near v.



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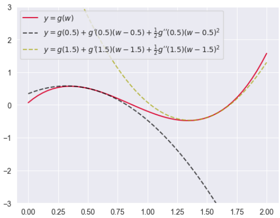
- If $g'(v) = 0$ and $g''(v) < 0$, then opposite of the last item is true.



First order approximation in multiple variables

Extending to the setting of multiple variables.

- Use the gradient where, if $\mathbf{w} = \left[w_1\, w_2\ \ldots\ w_N\right]^T$, then

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- The graph of $h(\mathbf{w})$ is the tangent (hyper)plane to graph of $g(\mathbf{w})$ at the point $(\mathbf{v}, g(\mathbf{v}))$.

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- There is something like the second derivative test in this general case; is more involved to describe. In practice, techniques using first order approximations are the most commonly used.

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- Means that $\mathbf{v}$ is point where all partial derivatives of g are zero.
- A minimum of g can only occur at a stationary point. However, other things can happen at a stationary point too – maximum of g or a "saddle" point.