Optimization from Calculus

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Outline

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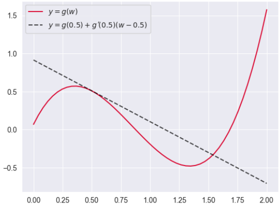
Linear approximations

$g : \mathbb{R} \to \mathbb{R}$, a twice-differentiable function (at least). Write w for input to g, so $g(w)$.

Linear approximation to $g(w)$. At a point $(v, g(v))$ on its graph, function whose graph is the tangent line:

$$h(w) = g(v) + g'(v)(w - v).$$

- Also called first order Taylor series approximation (or Taylor polynomial) of g.
- Approximates values of g, for inputs near v.



Second order (quadratic) approximations

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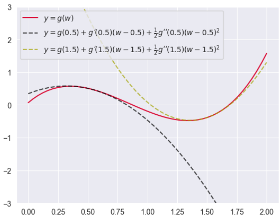
- If $g'(v) = 0$ and $g''(v) < 0$, then opposite of the last item is true.



First order approximation in multiple variables

Extending to the setting of multiple variables.

- Use the gradient where, if $\mathbf{w} = [w_1 \, w_2 \; \ldots \; w_N]^T$, then

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- The graph of $h(\mathbf{w})$ is the tangent (hyper)plane to graph of $g(\mathbf{w})$ at the point $(\mathbf{v}, g(\mathbf{v}))$.

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- There is something like the second derivative test in this general case; is more involved to describe. In practice, techniques using first order approximations are the most commonly used.

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A **stationary point** (or, **critical point**) of g is a point $\mathbf{v}$ where $\nabla g(\mathbf{v})$ is the zero vector. (Called the "first order condition for optimality.")

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- A minimum of g can only occur at a stationary point. However, other things can happen at a stationary point too – maximum of g or a "saddle" point.

The Chain rule

The **chain rule** is very useful for understanding derivatives and gradients. The more general version of it ("suped up" from Calculus I):

[1]"Df" and "Dg" mean, take the appropriate version of the derivative for the function. If the function is from $\mathbb{R}$ to $\mathbb{R}$, this is the f' from Calc I; if it is multi-variable (from $\mathbb{R}^N$ to $\mathbb{R}$, for some $N > 1$), then Df means ∇f. If f has vector output, there is a matrix of partial derivatives for Df.

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Example: given a fixed vector $\mathbf{a}$, set $h(\mathbf{w}) = \frac{1}{1+e^{\mathbf{a}^T\mathbf{w}}}$. This is a composition of $f(x) = \frac{1}{1+e^x}$ and $g(\mathbf{w}) = \mathbf{a}^T\mathbf{w}$. Thus,

$$\nabla h(\mathbf{w}) = f'(g(\mathbf{w})) \cdot \nabla g(\mathbf{w}) = \left(\frac{-e^{\mathbf{a}^T\mathbf{w}}}{1+e^{\mathbf{a}^T\mathbf{w}}}\right)\mathbf{a}.$$

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Example of stationary points

Here we look at the stationary points of the function $f(w_1, w_2) = w_1^3 + 3w_1w_2 + w_2^3$.

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  $3w_1^2 + 3w_2 = 0$ and
  $3w_1 + 3w_2^2 = 0$; so, $w_2 = -w_1^2$;
  using this and simplifying,
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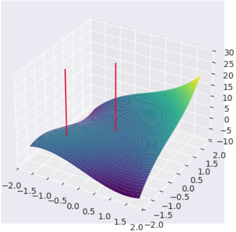
- We get: either $w_1 = 0$ (with $w_2 = 0$ also), or $w_1 = -1$ (with $w_2 = -1$ also). See Figure on right.



Vertical lines placed over stationary points $(0, 0)$ and $(-1, -1)$

Convex functions

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[2]A different definition (which is equivalent): g is **convex at** v if the linear approximation at v has a graph that is below the graph of g, near v; it is then **convex** if it is convex at v for all v in $\mathbb{R}$. This extends to multiple variables.

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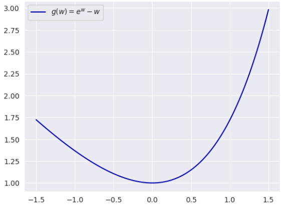
Example.
The function $g(w) = e^w - w$ is a convex function, since $g''(w) = e^w$, and e^v is bigger than zero for all v.



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