Gradient Descent

Chris Cornwell

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Outline

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$$\frac{|\omega_j^{(t)} - \omega_j^{(t-1)}|}{|\omega_j^{(t-1)}|} \leq \varepsilon, \qquad \forall 1 \leq j \leq p.$$

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Important: expected value of gradient used should be $\nabla\mathcal{L}_{\mathcal{S}}$ (which is, hopefully, close to gradient of population loss).

Example with Linear regression

Last time: given sample data $\mathcal{S}$ for simple linear regression, and using MSE as empirical loss, $\mathcal{L}_{\mathcal{S}}(m,b) = \frac{1}{n}\sum_{i=1}^{n}(mx_i + b - y_i)^2$, we found

$$\nabla\mathcal{L}_{\mathcal{S}}(m,b) = \left(\frac{2}{n}\sum_{i=1}^{n}(mx_i + b - y_i)x_i, \quad \frac{2}{n}\sum_{i=1}^{n}(mx_i + b - y_i)\right).$$

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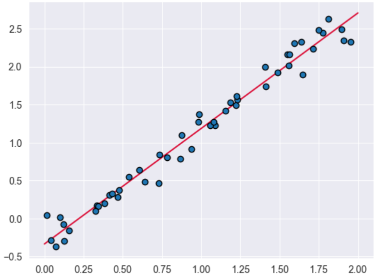
Example: batch gradient descent working on the `'Example1.csv'` data.

The LSR line, using closed form.

- $\hat{m} \approx 1.520, \hat{b} = -0.3346$:



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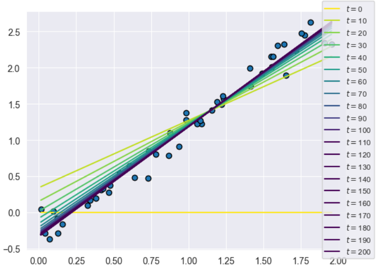

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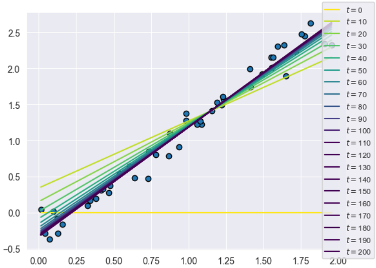
Example: batch gradient descent working on the `'Example1.csv'` data.

Plot of selected lines found during batch GD updates; starting parameters $m = 0, b = 0$; learning rate set to 0.1. Parameter values on iteration 208: $m = 1.519$, $b = -0.334$.



Implementing Gradient Descent

Some notes on implementation of the gradient descent updates.

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Presuming `x` and `y` are arrays of coordinates for sample data, and that `n = len(x)`, the following computes the partial derivatives:

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partial_m = (2/n)*np.sum( (m*x + b - y)*x )
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```
## lr is learning rate; threshhold is for stopping;
input: X, y, lr, threshhold
p ← initial array of parameters
while (max of last_update > threshhold){
    grad ← compute_grad(p, X, y)
    last_update ← | grad / p | ## entrywise array division
    # handle p[i] near 0
    p ← p − lr*grad
}
return p
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Convergence

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Short answer: No, not necessarily.

...so, *in what cases* can we guarantee such a thing?

Issues with convergence

To demonstrate the difficulty, imagine a "toy" loss function: $\boldsymbol{\ell} : \mathbb{R} \to \mathbb{R}$ with $\boldsymbol{\ell}(w) = w^2$. At each w we have $\nabla \boldsymbol{\ell} = \left(\frac{d\ell}{dw}\right) = (2w)$.

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Say learning rate: $\eta > 1$. Then, at any $w^{(t)} > 0$, we get

$$w^{(t+1)} = w^{(t)} - 2\eta w^{(t)} < w^{(t)} - 2w^{(t)} = -w^{(t)},$$

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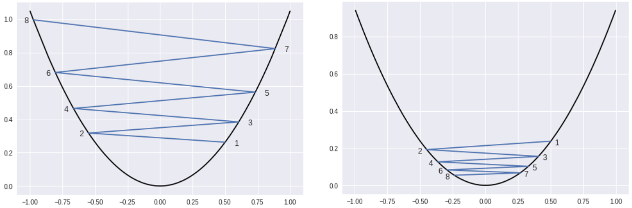

Figure: Gradient descent on $\ell(w) = w^2$. Left: $\eta = 1.05$; right: $\eta = 0.95$.

Guarantees of convergence

If your loss function is differentiable and a **convex function**, and if have some “control” on size of the gradient then, by choosing η small enough, can guarantee convergence.

[2] Meaning: $\exists$ a constant C s.t. for all ω_1, ω_2, $|\nabla\mathcal{L}(\omega_1) - \nabla\mathcal{L}(\omega_2)| \leq C|\omega_1 - \omega_2|$.

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Theorem

Suppose that $\mathcal{L} : \mathbb{R}^p \to \mathbb{R}$ is differentiable and convex, and suppose that $\nabla \mathcal{L}$ is Lipschitz continuous[2] *with some constant $C > 0$ and that $\eta \leq 1/C$. Then, for a minimizer ω^* of $\mathcal{L}$,*

$$\mathcal{L}(\omega^{(t)}) - \mathcal{L}(\omega^*) \leq \frac{|\omega^{(0)} - \omega^*|^2}{2\eta t}.$$

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- The difference between $\mathcal{L}(\omega^{(t)})$ and the minimum of $\mathcal{L}$ is bounded by a constant times $1/t$.

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Previous theorem requires using actual gradient of $\mathcal{L}$ in each update step. Here is a convergence guarantee that allows for a random vector $\mathbf{D}_t$, in place of $\nabla\mathcal{L}$, as long as $\mathbb{E}[\mathbf{D}_t|\omega^{(t)}] = \nabla\mathcal{L}(\omega^{(t)})$.

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Theorem

Suppose that $\mathcal{L}$ is differentiable and convex, that $\omega^{(0)} = \mathbf{0}$, and that $\eta = \frac{1}{\sqrt{K}}$ for an integer $K > 0$. Finally, suppose that $|\mathbf{D}_t| \leq 1$ for all $1 \leq t \leq K$. Then, for a minimizer ω^* of $\mathcal{L}$,

$$\mathbb{E}[\mathcal{L}(\bar{\omega})] - \mathcal{L}(\omega^*) \leq \frac{1}{\sqrt{K}}$$

where $\bar{\omega}$ is the average of $\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(K)}$.