Logistic Regression

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Feb 27, 2025

Outline

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Reconsidering the Half-space Model

Logistic model

Decision boundaries

For model h, made for classification task (with data points $\mathbf{x} \in \mathbb{R}^d$), write $C_y \subset \mathbb{R}^d$ for the set of points with label y, i.e.,

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Say $y \neq y'$ and a point is in the boundary of both C_y and $C_{y'}$. We say that point is on a **decision boundary** of the model. In a half-space model (last lecture), the hyperplane determined by $\mathbf{w}$ and b is the decision boundary.

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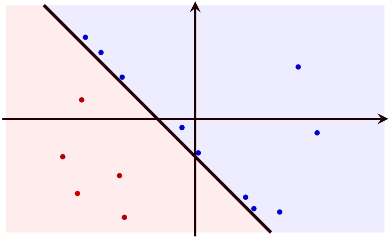

Figure: Many points near the decision boundary

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Also...the immediate change of label across the boundary (a discontinuity in the model) ...perhaps not "natural"?

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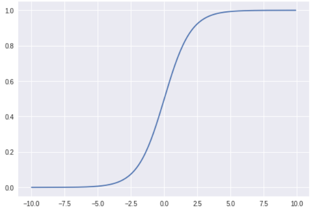
Incorporating a probability into half-space model

Instead of only capturing the sign of $\mathbf{w} \cdot \mathbf{x} + b$, compose it with the **logistic function**.

$$\sigma(z) = \frac{1}{1 + e^{-z}}.$$

- $0 < \sigma(z) < 1$ for all $z \in \mathbb{R}$;
- $\lim_{z \to \infty} \sigma(z) = 1$ and $\lim_{z \to -\infty} \sigma(z) = 0$;
- $\sigma(0) = 1/2$.



Logistic model, continued

"Logistic regression", used for binary classification, as follows.

Find hyperplane H, as determined by some $\mathbf{w}$ and b, that fits labeled data well; given new $\mathbf{x} \in \mathbb{R}^d$, find $z = \mathbf{w} \cdot \mathbf{x} + b$, then compute $\sigma(z)$.

- (Logistic regression) $\sigma(z)$, the *probability* that the label is $+1$.

[1]Since -1 is only other label, high probability of having label -1.

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1. If $\mathbf{w} \cdot \mathbf{x} + b > 0$ is very large ($\mathbf{x}$ far away from H and on positive side), then $\sigma(z)$ is very close to 1.
2. If $\mathbf{w} \cdot \mathbf{x} + b < 0$ has abs. value very large ($\mathbf{x}$ far away from H on negative side), then $\sigma(z)$ very close to 0.[1]
3. If $\mathbf{x}$ is contained in H itself, $z = 0$ and $\sigma(z) = 0.5$.

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Binary classifier model from logistic regression: $h(\mathbf{x}) = 1$ if $\sigma(\mathbf{w} \cdot \mathbf{x} + b) \geq 0.5$, and $h(\mathbf{x}) = -1$ otherwise.

- *Remember that probability (certainty) in the prediction.*

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Some rationale for the use of the logistic function $\frac{1}{1+e^{-z}}$ in this process, comes from an inverse direction.

- If wanting to use *Maximum Likelihood Estimation* to get a binary model, and making some typical simplifying assumptions on conditional probability (P), given model parameters, of observing some $\mathbf{x_i} \rightsquigarrow \log\left(\frac{P}{1-P}\right)$ being linear.

How to find $\mathbf{w}$ and b

Given $\{\pm 1\}$ labeled data, how do we go about finding $\mathbf{w}$ and b to use in the logistic (regression) model?

- Even if data is linearly separable, Perceptron algorithm does not try to make hyperplane be positioned "away from" data (Disadvantage).
- If data is not linearly separable, what should be done?

Future lectures: Will discuss using optimization (calculus-based) to find best parameters $w_1, w_2, \ldots, w_d, b$; process called Gradient Descent.

- Relevant: relationship between gradient of a function and its directional derivative.

More messy versus less messy data

Could introduce additional parameter (more flexibility).

For $k > 0$, define

$$\sigma_k(z) = \frac{1}{1 + e^{-kz}}.$$

$0 < k < 1$: values of $\sigma_k(z)$ transition from 0 to 1 more slowly.

$k > 1$: values of $\sigma_k(z)$ transition from 0 to 1 more quickly. (think about derivative)

Hence, if know data has more noise, might use $0 < k < 1$ to decrease measure of confidence in prediction. In contrast, very "clean" data, interpretation of the model might benefit from $k > 1$.

(**Left:** graph with $k = 5$; **Right:** applied to points in $\mathbb{R}^2$)