Variations on theme of Linear Regression

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Outline

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Multiple variables

Polynomial fitting

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  - Ignores that all are contributing together to `Sales`.
  - Doesn't give predictive ability that matches data.

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where $p_i, i = 0, 1, \ldots, d$ are coefficients to be fit from the data; ε is random variable with expected value 0.

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- Advertising data set: independent variables are `TV`, `Radio`, `Newspaper`; $d = 3$.

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To find the coefficients, alter procedure a bit.
Matrix A is size $n \times (d+1)$ and has column for each variable (and a column of ones). That is, treating each $\vec{x}_i$ as a column vector (with one entry for each data point),

$$A = \left[\vec{x}_0, \quad \vec{x}_1, \quad \ldots, \quad \vec{x}_{d-1}, \quad \vec{1}\right].$$

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- Larger $d \longrightarrow$ more likely $A^T A$ is poorly conditioned (potential issues from numerically computing its inverse).

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Contrast with result of three separate linear regressions, below.

Variable	TV	Radio	Newspaper
LSR line	$0.0475x_0 + 7.0326$	$0.2025x_1 + 9.3116$	$0.0547x_2 + 12.3514$
R^2	0.612	0.332	0.052

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The value of R^2 with all three predictor (independent) variables is: 0.89721. What conclusion can we draw?

How small to decide not significant?

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Hypothesis testing: choose a p-value threshold (often < 0.05 or < 0.01). The p-value corresponds to some t-statistic – use regression coefficient ($\hat{p}_i$ for x_i) and standard error.

- In example, if using simple linear regression on `Newspaper`, would get the variable is significant. However, using multiple regression with `TV`, `Radio`, and `Newspaper`, get very large p-value $\longrightarrow$ so, not significant.

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- p-value large when t-statistic is small, which is when SE is large *relative to size of* $\hat{p}_i$. [2]

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Intuitive estimate of significance

Checking whether fluctuation of regression coefficient for an independent variable, relative to coeff.'s size, is large.

[3]*Some evidence in literature (Goodhue-Lewis, 2012) that not much precision is to be gained with more than 100 samples, for bootstrapping standard errors.

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2. Use regression coeff. from whole data set, $\approx p_i$. If standard dev. found in 1., divided by this coeff., is larger than about 0.5, variable is not significant.
   - Since we are *estimating some things* here, don't use as a hard cutoff. Getting 0.48, versus 0.59, would perhaps both be *weakly* significant. However, if larger than 1.5, say, definitely not significant.

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Powers of x in place of multiple variables

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i.e., consider the model

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For the procedure, use essentially the same idea for the matrix A, but using powers of single variable x instead of using different independent variables[5]. Given data with x-coordinates $x_1, x_2, \ldots, x_n$, the matrix A is known as a **Vandermonde matrix**.

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Example

Taking the `'College.csv'` data set from the DataSets folder. Two of the columns are `'Top10perc'` and `'Top25perc'`. For the schools in the data set, these columns give the percentage of the entering class that were in the top 10% (resp. 25%) of their graduating high school class.[6]

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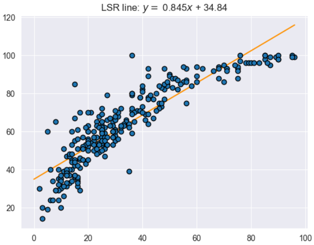

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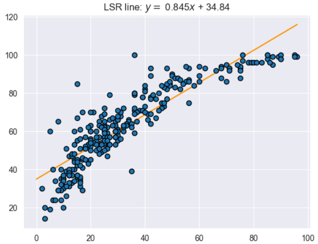

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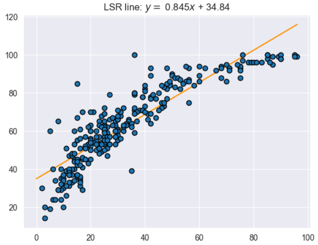
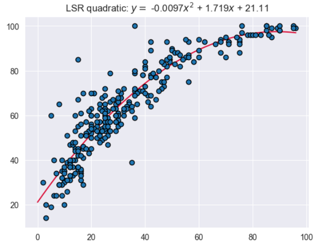


Next, the data set with a least squares quadratic polynomial fit. The R^2 value is 0.854.



Value of R^2 as polynomial degree increases

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- Note: Suppose that $n > d$. A Vandermonde matrix for x-values $x_1, x_2, \ldots, x_n$, which has $d + 1$ columns (so, highest power is x_i^d), will have rank $d + 1$ if and only if there are $d + 1$ of the x_i that are distinct.

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Meaning: $\mathrm{Col}(A_0)$ is proper subspace of $\mathrm{Col}(A_1)$. So, using A_1 makes $|y - \hat{y}|^2$ smaller. Since $\sum(y - \bar{y})^2$ is unchanged, makes R^2 closer to 1.

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