Linear Regression, Method 1

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Feb 11, 2025

Outline

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The goal

- Setting: have points in the plane, say n of them. Say the points are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.
- **Goal:** *Model* them as being “noisy” points from a line, finding “best fit” line (the closest linear model). This line is also called the **least squares regression** (LSR) line.

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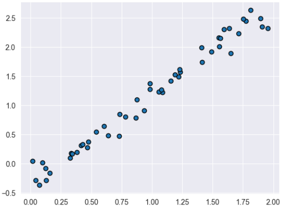
- **Running example:** A data set, `'Example1.csv'`, with 50 points, is available here; these points are displayed in the plot.



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Finding the LSR line

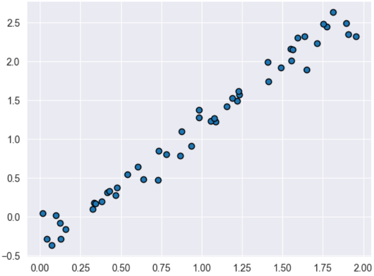

Figure: Our running example

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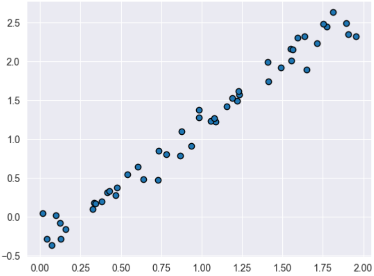

Figure: Our running example

How do we find the LSR line?
Can get the slope *m*, intercept *b* simply from using the `polyfit` function in NumPy. If `x`, `y` are the arrays with *x*- and *y*-coordinates:

```
np.polyfit(x,y,1)
```

Finding the LSR line

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- If a slope m and intercept b existed so that $(x_1, y_1), \ldots, (x_{50}, y_{50})$ *were* points on $y = mx + b$, then

$$y_i = mx_i + b$$

would hold for all $1 \leq i \leq 50$.

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1. Write those 50 equations as a matrix equation. Setting:

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_{50} & 1 \end{bmatrix}; \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{50} \end{bmatrix},$$

and writing[1] $\mathbf{p} = \begin{bmatrix} m \\ b \end{bmatrix}$, the matrix equation is $A\mathbf{p} = \mathbf{y}$.

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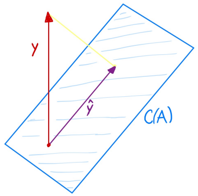
- Get vector $\hat{\mathbf{y}}$ that is *as close to* $\mathbf{y}$ *as possible*, so that $A\mathbf{p} = \hat{\mathbf{y}}$ has a solution.



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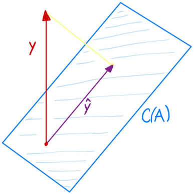
- For each i, we either increase or decrease y_i by a (hopefully small) amount, $y_i \rightsquigarrow \hat{y}_i$. We make $|\mathbf{y} - \hat{\mathbf{y}}|$ as small as possible.



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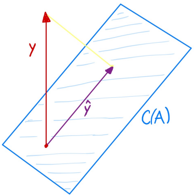
2. Done by solving
   $A^T A\mathbf{p} = A^T\mathbf{y}$ (normal equations).
   If $\mathbf{p} = \begin{bmatrix} \hat{m} \\ \hat{b} \end{bmatrix}$ is the solution,
   then $\hat{\mathbf{y}}$ is given by $A \begin{bmatrix} \hat{m} \\ \hat{b} \end{bmatrix}$.



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- $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1$ in null space of A^T and $\mathbf{z}_2$ in column space of A.
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  (*Note: Null space of A^T orthogonal to column space of A.*)
- As $\mathbf{z}_2$ in column space, $\exists\, \hat{\mathbf{p}}$ so that $A\hat{\mathbf{p}} = \mathbf{z}_2$. But then,

$$A^T(A\hat{\mathbf{p}}) = A^T\mathbf{z}_2 = A^T(\mathbf{y} - \mathbf{z}_1) = A^T\mathbf{y}.$$

And $\mathbf{z}_2$ is closest, since subtracted $\mathbf{z}_1$ from $\mathbf{y}$, orthogonal to column space:

$$\mathbf{z}_2 = \hat{\mathbf{y}}.$$

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So, three steps:

1. Write the n equations in matrix form. (get matrix A, vector $\mathbf{y}$)
2. Get matrix $A^T A$ and vector $A^T \mathbf{y}$ for normal equations: $A^T A\mathbf{p} = A^T \mathbf{y}$.
3. Use a method to solve normal equations for $\mathbf{p}$.

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Solving normal equation, in pseudocode

Procedure to carry out the steps:

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Given $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, as a NumPy array (call it `D`, with shape `(n,2)`):

```
A ← [x coordinates, all 1s] # 2-column matrix
y ← y coordinates
# next, get 2x2 matrix and 2-vector
Compute A.T times A; compute A.T times y
Solve normal eq'ns (numpy solve, or use inverse)
return solution
```

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Result on running example

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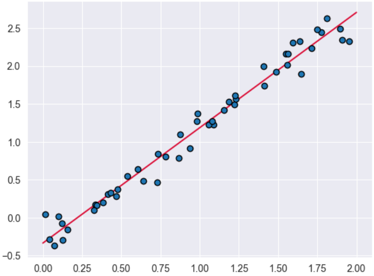
A plot of the line (in red), alongside the points, looks as follows.



Another example, Advertising data

In the DataSets folder, the `'Advertising.csv'` file contains data on amounts spent (in thousands of dollars) on TV, Radio, and Newspaper advertising in 200 different markets, as well as the amounts sold in each market (in thousands of units).

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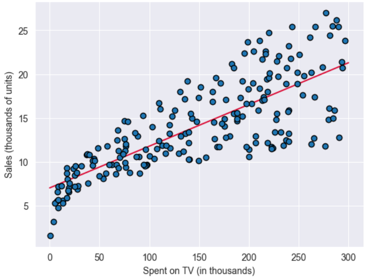
We will look more at this data later. For now, plotted here are the columns (`'TV'`, `'Sales'`).



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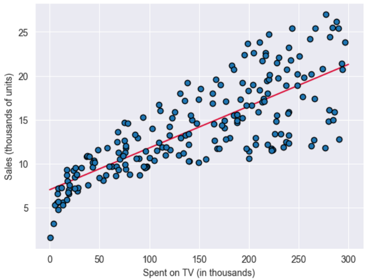

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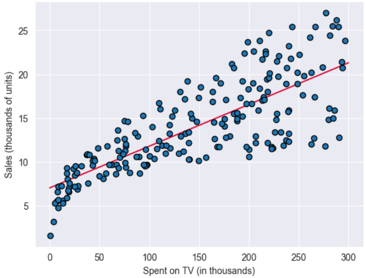
The LSR line for the data is then **not** the same line, if you switch the roles of TV and Sales in the algorithm to get $\hat{\mathbf{p}}$.



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