Variations on theme of Linear Regression

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Outline

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Multiple variables

Polynomial fitting

Working with multiple independent variables

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  - Ignores that all are contributing together to `Sales`.
  - Doesn't give predictive ability that matches data.

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- Advertising data set: independent variables are `TV`, `Radio`, `Newspaper`; $d = 3$.

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To find the coefficients, alter procedure a bit.
Matrix A is size $n \times (d+1)$ and has column for each variable (and a column of ones). That is, treating each $\vec{x}_i$ as a column vector (with one entry for each data point),

$$A = \begin{bmatrix} \vec{x}_0, & \vec{x}_1, & \dots, & \vec{x}_{d-1}, & \vec{1} \end{bmatrix}.$$

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- Larger $d \longrightarrow$ more likely $A^T A$ is poorly conditioned (potential issues from numerically computing its inverse).

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Contrast with result of three separate linear regressions, below.

Variable	`TV`	`Radio`	`Newspaper`
LSR line	$0.0475x_0 + 7.0326$	$0.2025x_1 + 9.3116$	$0.0547x_2 + 12.3514$
R^2	0.612	0.332	0.052

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The value of R^2 with all three predictor (independent) variables is: 0.89721. What conclusion can we draw?

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Can take (many) random subsamples of your data (fraction of whole data set); compute $\hat{p}_i$ for those. Find standard deviation of them $\longrightarrow$ approximates *SE*.
Next, approximate p_i with regression coefficient from your whole data set. If the standard deviation above, divided by this coeff., is (order(s) of magnitude) larger than the same thing for other variables, this variable is not significant.

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For the procedure, use essentially the same idea for the matrix A, but using powers of your variable x (or, variables) instead of using different independent variables. Given data with x-coordinates $x_1, x_2, \ldots, x_m$, the matrix A is known as a **Vandermonde matrix**.

Example

Taking the `'College.csv'` data set from the DataSets folder. Two of the columns are `'Top10perc'` and `'Top25perc'`. For the schools in the data set, these columns give the percentage of the entering class that were in the top 10% (resp. 25%) of their graduating high school class.[2]

[2] Removed rows that contained schools receiving fewer than 2500 applications.

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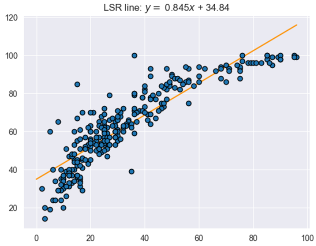

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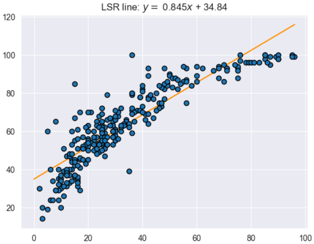

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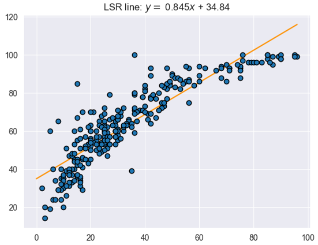
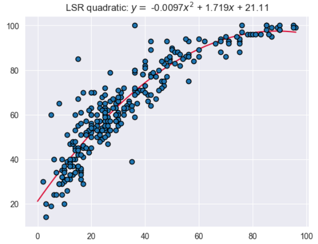


Next, the data set with a least squares quadratic polynomial fit. The R^2 value is 0.854.



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- Note: Suppose that $n > d$. A Vandermonde matrix for x-values $x_1, x_2, \ldots, x_n$, which has $d + 1$ columns (so, highest power is x_i^d), will have rank $d + 1$ if and only if there are $d + 1$ of the x_i that are distinct.

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