Logistic Regression

Chris Cornwell

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Outline

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Decision boundaries

Suppose function $h : \mathbb{R}^d \longrightarrow \{y_1, y_2, \ldots, y_m\}$ gives predictions for a classification task (data points $\mathbf{x} \in \mathbb{R}^d$, with m labels).

For $i \in \{1, 2, \ldots, m\}$, define C_{y_i} to be the set of points with label y_i, i.e.,

$$C_{y_i} = h^{-1}(y_i) = \{\mathbf{x} \in \mathbb{R}^d \mid h(\mathbf{x}) = y_i\}.$$

For $i \neq j$, a point is in **decision boundary for** h when it's in the boundary of both C_{y_i} and C_{y_j}. For half-space model (last lecture) determined by $\mathbf{w}$ and b, decision boundary is the hyperplane – i.e., points $\mathbf{x}$ such that $\mathbf{w} \cdot \mathbf{x} + b = 0$.

Perceptron algorithm might give model h with many data points that are *near* the decision boundary.

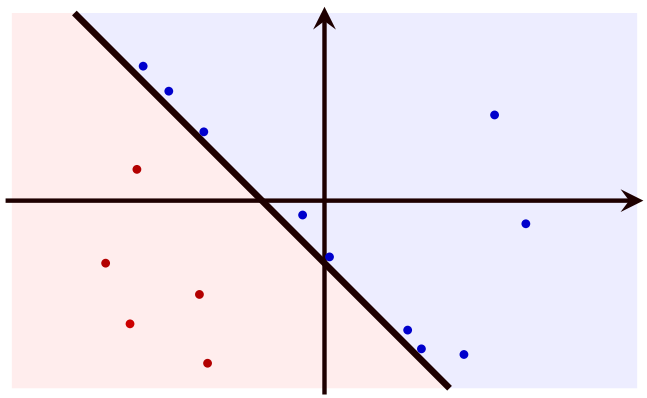


Figure: Many points near the decision boundary

But...data is messy

Many points "close" to the decision boundary $\Longrightarrow$ likely for newly observed data to appear on "wrong side" of decision boundary.
Perceptron algorithm has nothing to correct for this.
Some thoughts:

- When $\mathbf{x}$ close to decision boundary, we feel less *confident* in giving the label $h(\mathbf{x})$. In contrast, when $\mathbf{x}$ farther from boundary, and only one label is observed nearby, more confidence is warranted.
- Also, the immediate change of label across the boundary (a discontinuity in the model) ...perhaps not "natural"?

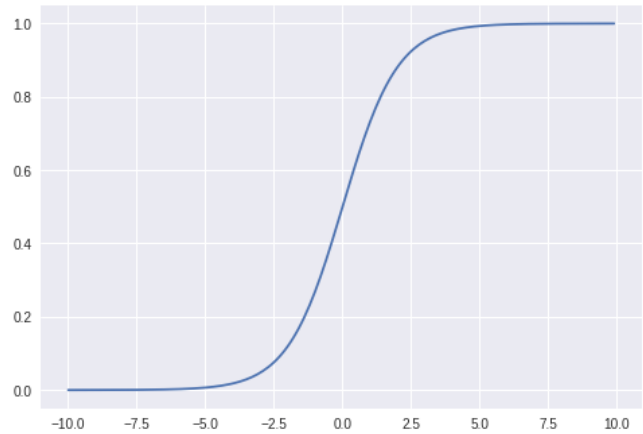
Outline

Incorporating a probability into half-space model

Don't only capture the sign of $\mathbf{w} \cdot \mathbf{x} + b$; instead, compose $\mathbf{w} \cdot \mathbf{x} + b$ with the **logistic function**.

$$\sigma(z) = \frac{1}{1 + e^{-z}}.$$

- $0 < \sigma(z) < 1$ for all $z \in \mathbb{R}$;
- $\lim_{z \to \infty} \sigma(z) = 1$ and $\lim_{z \to -\infty} \sigma(z) = 0$;
- $\sigma(0) = 1/2$.



Logistic model, continued

"Logistic regression", used for binary classification, as follows.
Find hyperplane H, as determined by some $\mathbf{w}$ and b, that fits labeled data well; given new $\mathbf{x} \in \mathbb{R}^d$, find $z = \mathbf{w} \cdot \mathbf{x} + b$, then compute $\sigma(z)$.

- (Logistic regression) $\sigma(z)$, the *probability* that the label is $+1$.

1. If $\mathbf{w} \cdot \mathbf{x} + b > 0$ is very large ($\mathbf{x}$ far away from H and on positive side), then $\sigma(z)$ is very close to 1.
2. If $\mathbf{w} \cdot \mathbf{x} + b < 0$ has abs. value very large ($\mathbf{x}$ far away from H on negative side), then $\sigma(z)$ very close to 0.[1]
3. If $\mathbf{x}$ is contained in H itself, $z = 0$ and $\sigma(z) = 0.5$.

Logistic model (a binary classifier): $h(\mathbf{x}) = 1$ if $\sigma(\mathbf{w} \cdot \mathbf{x} + b) \geq 0.5$, and $h(\mathbf{x}) = -1$ otherwise.

- *Remember that probability (certainty) in the prediction.*
- *Logistic model helps decision boundary not be near data* (more on this later).

[1] Since -1 is only other label, high probability of having label -1.

Logistic model, continued

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Some rationale for the use of the logistic function $\frac{1}{1+e^{-z}}$ in this process, comes from an inverse direction.

- If wanting to use *Maximum Likelihood Estimation* to get binary classification model, then some typical simplifying assumptions on conditional probability P, given model parameters, of observing $\mathbf{x_i}$ $\rightsquigarrow \log\left(\frac{P}{1-P}\right)$ being linear.

How to find $\mathbf{w}$ and b

Given $\{\pm 1\}$ labeled data, how do we go about finding $\mathbf{w}$ and b to use in the logistic (regression) model?

- Even if data is linearly separable, Perceptron algorithm does not try to make hyperplane be positioned "away from" data (Disadvantage).
- If data is not linearly separable, what should be done?

Future lectures: Will discuss using optimization (calculus-based) to find best parameters $w_1, w_2, \ldots, w_d, b$; process called Gradient Descent.

- Relevant: relationship between gradient of a function and its directional derivative (discussed in Calc III).

More messy versus less messy data

Could introduce additional parameter (more flexibility).

For $k > 0$, define

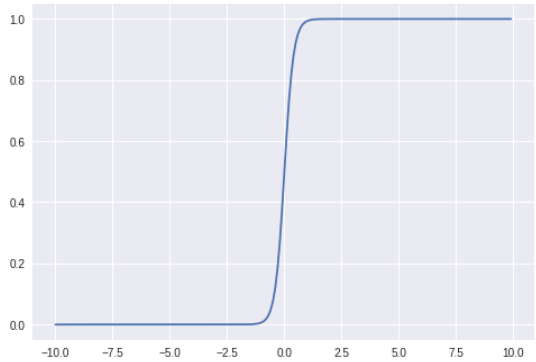
$$\sigma_k(z) = \frac{1}{1 + e^{-kz}}.$$

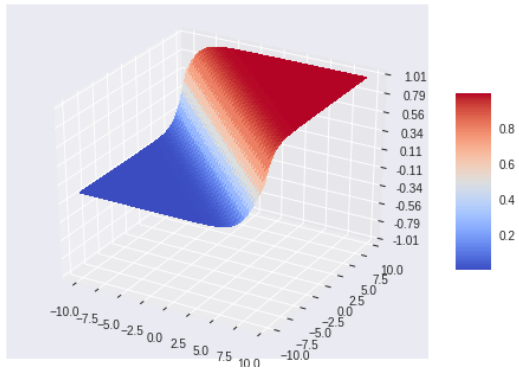
$0 < k < 1$: values of $\sigma_k(z)$ transition from 0 to 1 more slowly.

$k > 1$: values of $\sigma_k(z)$ transition from 0 to 1 more quickly. (think about derivative)

Hence, if know data has more noise, might use $0 < k < 1$ to decrease measure of confidence in prediction. In contrast, very "clean" data, interpretation of the model might benefit from $k > 1$.[2]

(**Left:** graph with $k = 5$; **Right:** applied to points in $\mathbb{R}^2$)





[2] I first saw the idea to introduce k, and interpret the model this way, from Eli Grigsby at Boston College