Logistic Regression

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Feb 27, 2025

Outline

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Perceptron algorithm might give model h with many data points that are *near* the decision boundary.

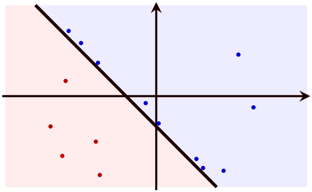

Figure: Many points near the decision boundary

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- Also, the immediate change of label across the boundary (a discontinuity in the model) ...perhaps not “natural”?

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Reconsidering the Half-space Model

Logistic model

Incorporating a probability into half-space model

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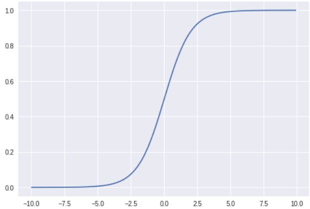

Logistic model, continued

"Logistic regression", used for binary classification, as follows.

Find hyperplane H, as determined by some $\mathbf{w}$ and b, that fits labeled data well; given new $\mathbf{x} \in \mathbb{R}^d$, find $z = \mathbf{w} \cdot \mathbf{x} + b$, then compute $\sigma(z)$.

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- *Logistic model helps decision boundary not be near data* (more on this later).

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Some rationale for the use of the logistic function $\frac{1}{1+e^{-z}}$ in this process, comes from an inverse direction.

- If wanting to use *Maximum Likelihood Estimation* to get binary classification model, then some typical simplifying assumptions on conditional probability P, given model parameters, of observing $\mathbf{x_i}$ $\rightsquigarrow \log\left(\frac{P}{1-P}\right)$ being linear.

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- Relevant: relationship between gradient of a function and its directional derivative (discussed in Calc III).

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Could introduce additional parameter (more flexibility).

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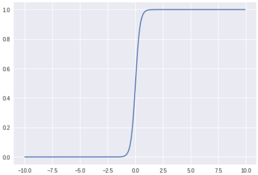
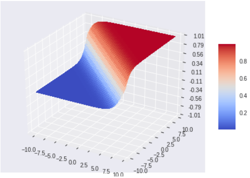
(**Left:** graph with $k = 5$; **Right:** applied to points in $\mathbb{R}^2$)





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