Classification, Halfspaces, the Perceptron algorithm

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Outline

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Classification tasks

Half-space model

Perceptron algorithm

Example of Classification

Use some model to determine a digit that was (hand)written in an image

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- Convert image to a vector (*in some way*) $\longrightarrow \mathbf{x}$.
- Your model's output: $\hat{y}(\mathbf{x})$ is the (predicted) digit.

Provided with your data, an "observation" $y \in \{0, 1, \ldots, 9\}$ of the digit being written.

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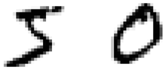
y and $\hat{y}$ are numbers on number line; but, use them like labels (or, separate buckets) to group points $\mathbf{x}$. When $y = 5$, predicting $\hat{y} = 4$ is not any better than $\hat{y} = 0$.



Close only counts in ~~horseshoes~~ …Regression

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values of function $\leftrightarrow$ prediction $\hat{y}$; error term ε, so that $y = \hat{y} + \varepsilon$.

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"Regression"

"Classification" tasks: the value y is a label and might not even be a number. The prediction $\hat{y}$ is simply wrong, or not; close doesn't count. Good model: when $\hat{y} = y$ as often as possible.

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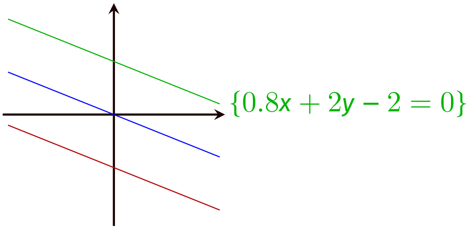

Figure: A few hyperplanes in $\mathbb{R}^2$.

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- $\mathbf{w}$ is a vector that is orthogonal to a $(d-1)$-dimensional subspace of $\mathbb{R}^d$; $|b|$ corresponds to a translation away from the origin.

Half-space model

Using the notation from last slide:
a half-space model in $\mathbb{R}^d$ is determined by $d+1$ parameters $w_1, w_2, \ldots, w_d, b$; the first d parameters grouped into a vector: $\mathbf{w} = (w_1, w_2, \ldots, w_d)$.

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Given $\mathbf{x} \in \mathbb{R}^d$, the side of the hyperplane H it is on is determined by the sign of $\mathbf{w} \cdot \mathbf{x} + b$. Our half-space model: $h : \mathbb{R}^d \setminus H \longrightarrow \{1, -1\}$.

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- (Positive side) Say that $h(\mathbf{x}) = 1$ if $\mathbf{w} \cdot \mathbf{x} + b > 0$.
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If there exists a hyperplane, given by some $\mathbf{w}$, b, so that $\mathbf{x}$ has one of the labels if and only if it is on the positive side, the labeled data are called **linearly separable**.

Linearly separable

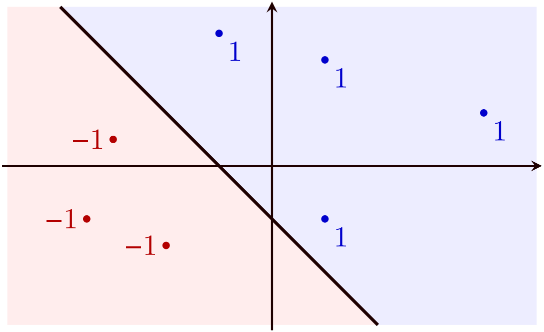

Figure: The hyperplane $H = \{(x, y) \in \mathbb{R}^2 : x + y + 1 = 0\}$, corresponding positive and negative regions, $\mathbf{w} = (1, 1)$, $b = 1$

Not linearly separable

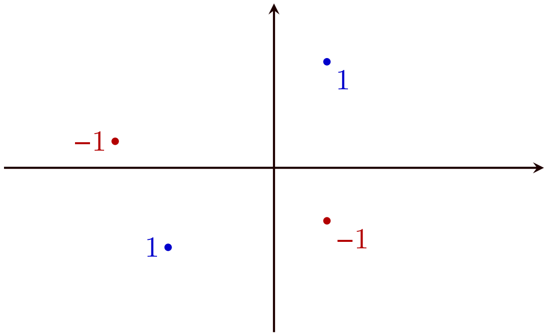

Figure: A data set in $\mathbb{R}^2$ that is not linearly separable.

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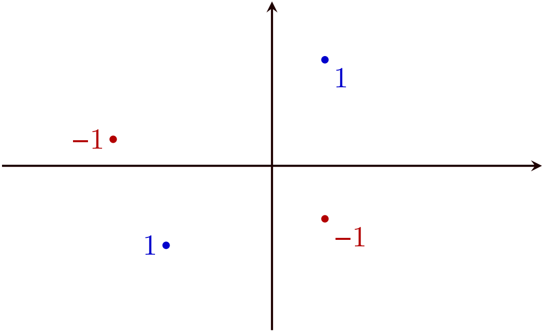

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- A criterion (checkable, in theory) that is equivalent to "not linearly separable"?

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Perceptron algorithm

Setup for Perceptron algorithm

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To describe it: for each $\mathbf{x}_i$, use X_i to denote the $(d+1)$-vector consisting of $\mathbf{x}_i$ with 1 appended at the end;

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Note that $W \cdot X_i = \mathbf{w} \cdot \mathbf{x}_i + b$.

For linearly separable data, our goal is to find $W \in \mathbb{R}^{d+1}$ so that, for all $1 \leq i \leq n$, $W \cdot X_i$ and y_i have the same sign (are both positive or both negative).

- Equivalently, we need $y_i W \cdot X_i > 0$ for all $1 \leq i \leq n$.

Perceptron algorithm

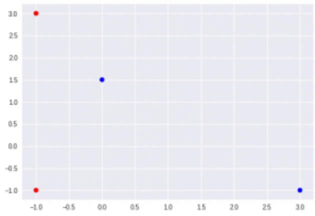
Suppose the data is linearly separable. Also, `x` is an $n \times d$ array of points, with ith row equal to $\mathtt{x}_i$, and `y` is array of the labels. The Perceptron algorithm finds `W` iteratively as follows.

```
input: x, y  ## x is n by d, y is 1d array
X ← append 1 to each row of x
W ← (0,0,...,0)  ## Initial W
while (exists i with y[i]*dot(W, X[i]) ≤ 0){
    W ← W + y[i]*X[i]
}
return W
```

Example

A simple example in $\mathbb{R}^2$, with $n = 4$ points.

$$\text{x:} \begin{bmatrix} -1 & 3 \\ -1 & -1 \\ 3 & -1 \\ 0 & 1.5 \end{bmatrix} \qquad \text{y:} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$



Example, continued

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Beginning with $W^{(1)} = (0, 0, 0)$, next step:
$W^{(2)} = \vec{0} + y_1 X_1 = -1 * (-1, 3, 1) = (1, -3, -1)$.
Next: since $y_1 W^{(2)} \cdot X_1 > 0$, check
$y_2 W^{(2)} \cdot X_2 = -1 * (-1 + 3 - 1) = -1$. So,

$$W^{(3)} = W^{(2)} + y_2 X_2 = (2, -2, -2).$$

Continue in this way – on each step check dot products (in order) with $y_1 X_1, y_2 X_2, y_3 X_3, y_4 X_4$. Eventually you return the vector $W^{(10)} = (4, -0.5, 1)$.
i.e., $H = \{(x_1, x_2) \in \mathbb{R}^2 : \ 4x_1 - 0.5x_2 + 1 = 0\}$ separates the points.

Perceptron algorithm, stopping time

Under our assumptions for Perceptron algorithm, a guarantee on eventually stopping.

Theorem

Define $R = \max_i |X_i|$ *and* $B = \min_i\{|V| : \forall i, y_i V \cdot X_i \geq 1\}$*. Then, the Perceptron algorithm stops after at most* $(RB)^2$ *iterations and, when it stops with output* W*, then* $y_i W \cdot X_i > 0$ *for all* $1 \leq i \leq n$*.*

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Idea of proof: Write W^* for vector that realizes the minimum B. Also, write $W^{(t)}$ for the vector W on the t^{th} step, with $W^{(1)} = (0, 0, \ldots, 0)$.

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Now, by Cauchy-Schwarz inequality, $T \leq BR\sqrt{T}$, which we can rearrange to $T \leq (BR)^2$.

Another example, the Iris data set

First discussed by R.A. Fisher in a 1936 paper, Iris data set commonly used in explanations. It contains 150 points in $\mathbb{R}^4$, each for an individual iris flower from one of 3 species: Iris setosa, Iris virginica, and Iris versicolor. The 4 coordinates are measurements of sepal length, sepal width, petal length, and petal width (in cm).

Iris setosa points are linearly separable from the other two.

Labels: *Iris setosa* $\longleftarrow$ 1; *Other species* $\longleftarrow$ -1.

Begin by opening the notebook `'perceptron-iris-notebook.ipynb'` ...After completing the algorithm, should get $\mathbf{w} = (1.3, 4.1, -5.2, -2.2)$ and $b = 1$.

Figure: Images by G. Robertson, E. Hunt, Radomil ©CC BY-SA 3.0