# Variations on theme of Linear Regression

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    - Ignores that all are contributing together to `Sales`.
    - Doesn't give predictive ability that matches data.

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where $p_i, i = 0, 1, \ldots, d$ are coefficients to be fit from the data; $\varepsilon$ is random variable with expected value 0.

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- Simple linear regression case, $d = 1$: $p_0$ is the slope, $p_1$ is intercept.
- Advertising data set: independent variables are `TV`, `Radio`, `Newspaper`; $d = 3$.

# Working with multiple independent variables

To find the coefficients, alter procedure a bit.
Matrix $A$ is size $n \times (d+1)$ and has column for each variable (and a column of ones). That is, treating each $\vec{x}_i$ as a column vector (with one entry for each data point),

$$A = \begin{bmatrix} \vec{x}_0, & \vec{x}_1, & \ldots, & \vec{x}_{d-1}, & \vec{1} \end{bmatrix}.$$

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- Larger $d \longrightarrow$ more likely $A^T A$ is poorly conditioned (potential issues from numerically computing its inverse).

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Contrast with result of three separate linear regressions, below.

| Variable | `TV` | `Radio` | `Newspaper` |
|---|---|---|---|
| LSR line | $0.0475x_0 + 7.0326$ | $0.2025x_1 + 9.3116$ | $0.0547x_2 + 12.3514$ |
| $R^2$ | $0.612$ | $0.332$ | $0.052$ |

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The value of $R^2$ with all three predictor (independent) variables is: 0.89721. What conclusion can we draw?

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Hypothesis testing: choose a $p$-value threshold (often $< 0.05$ or $< 0.01$). The $p$-value corresponds to some $t$-statistic – use regression coefficient ($\hat{p}_i$ for $x_i$) and standard error.

- In example, if using simple linear regression on `Newspaper`, would get the variable is significant. However, using multiple regression with `TV`, `Radio`, and `Newspaper`, get very large $p$-value $\longrightarrow$ so, not significant.

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Alternatively: if sample regression coeff. varies a lot (relative to size) compared to coeff.s of the other var's, that variable is not significant.

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So: Take (*many*) random subsamples of data (fraction of whole set); compute $\hat{p}_i$ for those. Standard deviation of them $\approx$ *SE*.

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So: Take (*many*) random subsamples of data (fraction of whole set); compute $\hat{p}_i$ for those. Standard deviation of them $\approx SE$.
Then: use regression coeff. from whole data set, $\approx p_i$. If standard deviation divided by this coeff. is (order(s) of magnitude) larger than the same for other var's $\longrightarrow$ variable is not significant.

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## Powers of $x$ in place of multiple variables

Often, a linear model does not seem like a good fit for our data. What about trying to fit the data to a polynomial?

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i.e., consider the model

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For the procedure, use essentially the same idea for the matrix $A$, but using powers of single variable $x$ instead of using different independent variables[3]. Given data with $x$-coordinates $x_1, x_2, \ldots, x_n$, the matrix $A$ is known as a **Vandermonde matrix**.

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$$A = \begin{bmatrix} x_1^d & \ldots & x_1^2 & x_1 & 1 \\ x_2^d & \ldots & x_2^2 & x_2 & 1 \\ \vdots & & \vdots & \vdots & \\ x_n^d & \ldots & x_n^2 & x_n & 1 \end{bmatrix}$$

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## Example

Taking the `'College.csv'` data set from the DataSets folder. Two of the columns are `'Top10perc'` and `'Top25perc'`. For the schools in the data set, these columns give the percentage of the entering class that were in the top 10% (resp. 25%) of their graduating high school class.[4]

[4] Removed rows that contained schools receiving fewer than 2500 applications.

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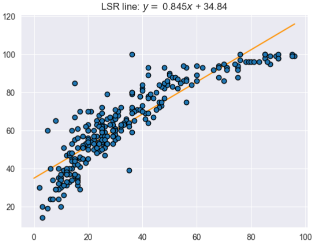


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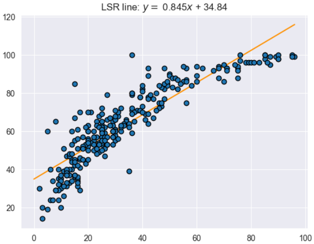

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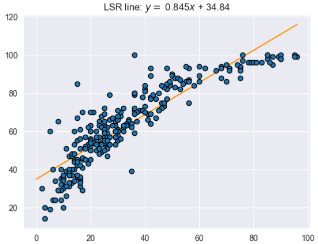
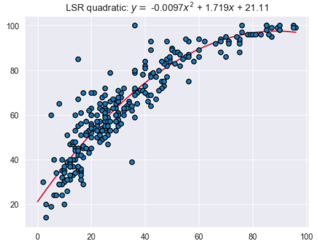



Next, the data set with a least squares quadratic polynomial fit. The $R^2$ value is 0.854.



# Value of $R^2$ as polynomial degree increases

What will happen to the value of $R^2$ if we increase the degree of the polynomial that we fit to the data?

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- Note: Suppose that $n > d$. A Vandermonde matrix for $x$-values $x_1, x_2, \ldots, x_n$, which has $d + 1$ columns (so, highest power is $x_i^d$), will have rank $d + 1$ if and only if there are $d + 1$ of the $x_i$ that are distinct.

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  *If $x_1, x_2, \ldots, x_{d+1}$ are pairwise distinct, say, then the determinant of the $(d+1) \times (d+1)$ submatrix for their corresponding rows is*

$$\prod_{1 \le i < j \le d+1} (x_j - x_i).$$

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set $A_0$: the Vandermonde matrix used to fit polynomial of degree $d$; set $A_1$: the one used for polynomial of degree $d+1$. [5]

From Note, as long as enough of the $x_i$ are distinct, $\mathrm{rank}(A_1) = \mathrm{rank}(A_0) + 1$.

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Meaning: $\mathrm{Col}(A_0)$ is proper subspace of $\mathrm{Col}(A_1)$. So, using $A_1$ makes $|y - \hat{y}|^2$ smaller. Since $\sum (y - \bar{y})^2$ is unchanged, makes $R^2$ closer to 1.

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