Classification, Halfspaces, the Perceptron algorithm

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Outline

Classification tasks

Half-space model

Perceptron algorithm

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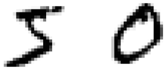
Example of Classification

Use some model to determine a digit that was (hand)written in an image

$$\texttt{0, 1, 2, 3, 4, 5, 6, 7, 8,} \text{ or } 9.$$

- Convert image to a vector (*in some way*) $\longrightarrow \mathbf{x}$.
- Your model's output: $\hat{y}(\mathbf{x})$ is the (predicted) digit.

Provided with your data, an "observation" $y \in \{0, 1, \ldots, 9\}$ of the digit being written.
y and $\hat{y}$ are numbers on number line; but, use them like labels (or, separate buckets) to group points $\mathbf{x}$. When $y = 5$, predicting $\hat{y} = 4$ is not any better than $\hat{y} = 0$.



Close only counts in ~~horseshoes~~ ...Regression

In linear regression, on indpt. variables $x_0, x_1, \ldots, x_{d-1}$, had (affine) linear function $\hat{y} = p_0 x_0 + p_1 x_1 + \ldots + p_{d-1} x_{d-1} + p_d$; values of function $\leftrightarrow$ prediction $\hat{y}$; error term ε, so that $y = \hat{y} + \varepsilon$. An observation y for each data point $\mathbf{x} = (x_0, x_1, \ldots, x_{d-1}) \in \mathbb{R}^d$. The **linear model** $\mathbf{x} \mapsto \hat{y}$ approximates $\mathbf{x} \mapsto y$.[1]

We expect that $|y - \hat{y}|$ is almost never exactly 0; a good model: one where $|y - \hat{y}|$ is small on average (but, still positive).

"Regression"

"Classification" tasks: the value y is a label and might not even be a number. The prediction $\hat{y}$ is simply wrong, or not; close doesn't count. Good model: when $\hat{y} = y$ as often as possible.

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[1] Should consider the output y here to be a random variable, with distribution that depends on $\mathbf{x}$. In simple linear regression, ε is a random variable and $y = p_0 x_0 + p_1 + \varepsilon$.

Outline

A linear model for classification

Binary classification: Data is from $\mathbb{R}^d$ for some $d > 0$ and we only have two labels, $\{1, -1\}$.

A hyperplane in $\mathbb{R}^d$ is an (affine) linear subspace that has dimension $d - 1$ (and separates $\mathbb{R}^d$ in two components).

Using coordinates $(x_1, x_2, \ldots, x_d)$ in $\mathbb{R}^d$, a hyperplane H may be determined from $d + 1$ numbers $w_1, w_2, \ldots, w_d$, and b. It consists of solutions to

$$w_1 x_1 + w_2 x_2 \ldots + w_d x_d + b = 0.$$

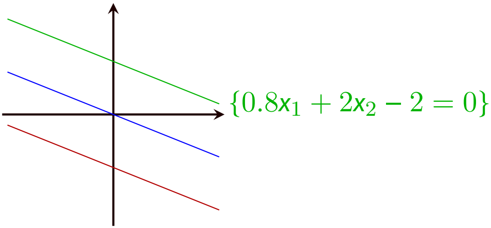

Figure: A few hyperplanes in $\mathbb{R}^2$.

A linear model for classification

Binary classification: Data is from $\mathbb{R}^d$ for some $d > 0$ and we only have two labels, $\{1, -1\}$.
A hyperplane in $\mathbb{R}^d$ is an (affine) linear subspace that separates $\mathbb{R}^d$ in two. Using coordinates $(x_1, x_2, \ldots, x_d)$ in $\mathbb{R}^d$, a hyperplane H may be determined from $d + 1$ numbers $w_1, w_2, \ldots, w_d$, and b. It consists of solutions to

$$w_1 x_1 + w_2 x_2 \ldots + w_d x_d + b = 0.$$

- Rewriting in vector form: $\mathbf{w} = (w_1, w_2, \ldots, w_d)$, look for solutions $\mathbf{x} \in \mathbb{R}^d$ to the equation $\mathbf{w} \cdot \mathbf{x} + b = 0$.
- $\mathbf{w}$ is a vector that is orthogonal to a $(d - 1)$-dimensional subspace of $\mathbb{R}^d$; $|b|$ corresponds to a translation away from the origin.

Half-space model

Using the notation from last slide:
a half-space model in $\mathbb{R}^d$ is determined by $d+1$ parameters $w_1, w_2, \ldots, w_d, b$, which determine a hyperplane H; the first d parameters grouped into a vector: $\mathbf{w} = (w_1, w_2, \ldots, w_d)$.
Given $\mathbf{x} \in \mathbb{R}^d$, the side of the hyperplane H it is on is determined by the sign of $\mathbf{w} \cdot \mathbf{x} + b$. Our half-space model: $h : \mathbb{R}^d \setminus H \longrightarrow \{1, -1\}$.

- (Positive side) Say that $h(\mathbf{x}) = 1$ if $\mathbf{w} \cdot \mathbf{x} + b > 0$.
- (Negative side) Say that $h(\mathbf{x}) = -1$ if $\mathbf{x} \cdot \mathbf{x} + b < 0$.

Given data with $\{\pm 1\}$ labels, if there exists a hyperplane H so that $\mathbf{x}$ has label 1 if and only if it is on the positive side, the labeled data are called **linearly separable**.

Linearly separable

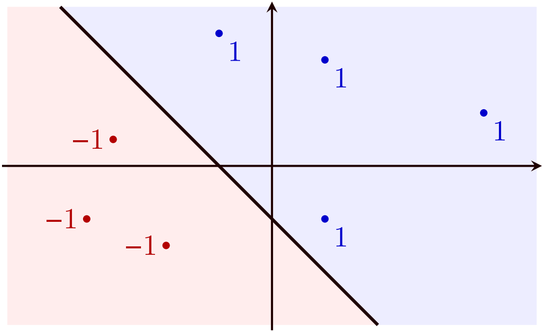

Figure: The hyperplane $H = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 + 1 = 0\}$, corresponding positive and negative regions, $\mathbf{w} = (1, 1)$, $b = 1$

Not linearly separable

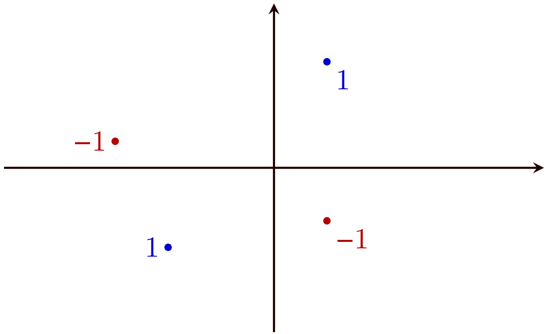

Figure: A data set in $\mathbb{R}^2$ that is not linearly separable.

- A criterion (checkable, in theory) that is equivalent to “not linearly separable”?

Outline

Setup for Perceptron algorithm

Labeled data: $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)$, with $\mathbf{x}_i \in \mathbb{R}^d$ and $y_i \in \{\pm 1\}$ for all i.
Assuming labeled data is linearly separable, the Perceptron algorithm is a procedure that is guaranteed to find a hyperplane that separates the data.[2]
To describe it: for each $\mathbf{x}_i$, use X_i to denote the $(d+1)$-vector consisting of $\mathbf{x}_i$ with 1 appended at the end;
Additionally, use W to denote the vector $\mathbf{w}$ with b appended at the end.
Note that $W \cdot X_i = \mathbf{w} \cdot \mathbf{x}_i + b$.
For linearly separable data, our goal is to find $W \in \mathbb{R}^{d+1}$ so that $W \cdot X_i$ and y_i have the same sign (both positive or both negative), for all $1 \le i \le n$.

- Equivalently, we need $y_i W \cdot X_i > 0$ for all $1 \le i \le n$.

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[2] Introduced in *The perceptron: A probabilistic model for information storage and organization in the brain*, F. Rosenblatt, Psychological Review **65** (1958), 386–407.

Perceptron algorithm

Suppose the data is linearly separable. Also, `x` is an $n \times d$ array of points, with i[th] row equal to $\mathtt{x}_i$, and `y` is array of the labels. The Perceptron algorithm finds `W` iteratively as follows.[3]

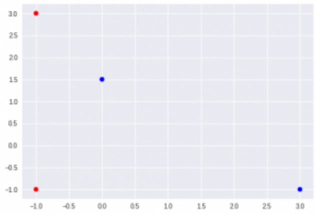
```
input: x, y  ## x is n by d, y is 1d array
X ← append 1 to each row of x
W ← (0,0,...,0)  ## Initial W
while (exists i with y[i]*dot(W, X[i]) ≤ 0){
    W ← W + y[i]*X[i]
}
return W
```

[3] Recall, the left-facing arrow below means *assign* to variable on left.

Example

A simple example in $\mathbb{R}^2$, with $n = 4$ points.

$$\text{x:} \begin{bmatrix} -1 & 3 \\ -1 & -1 \\ 3 & -1 \\ 0 & 1.5 \end{bmatrix} \qquad \text{y:} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$



Example, continued

A simple example in $\mathbb{R}^2$, with $n = 4$ points.

$$\text{x:}\begin{bmatrix} -1 & 3 \\ -1 & -1 \\ 3 & -1 \\ 0 & 1.5 \end{bmatrix} \qquad \text{y:}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

Use $W^{(t)}$ for value of W on step t. Start: $W^{(1)} = (0, 0, 0)$.
Next step: $W^{(2)} = \vec{0} + y_1 X_1 = -1 * (-1, 3, 1) = (1, -3, -1)$.
Next: since $y_1 W^{(2)} \cdot X_1 > 0$, check
$y_2 W^{(2)} \cdot X_2 = -1 * (-1 + 3 - 1) = -1$. So,

$$W^{(3)} = W^{(2)} + y_2 X_2 = (2, -2, -2).$$

Continue in this way – on each step check dot products (in order) with
$y_1 X_1, y_2 X_2, y_3 X_3, y_4 X_4$. Eventually you return the vector
$W^{(10)} = (4, -0.5, 1)$.
i.e., $H = \{(x_1, x_2) \in \mathbb{R}^2 : \; 4x_1 - 0.5x_2 + 1 = 0\}$ separates the points.

Perceptron algorithm, stopping time

Under our assumptions for Perceptron algorithm, a guarantee on eventually stopping.

Theorem

Define $R = \max_i |X_i|$ *and* $B = \min_i\{|V| : \forall i, y_i V \cdot X_i \geq 1\}$. *Then, the Perceptron algorithm stops after at most* $(RB)^2$ *iterations and, when it stops with output* W*, then* $y_i W \cdot X_i > 0$ *for all* $1 \leq i \leq n$.

Idea of proof: Write W^* for vector that realizes the minimum B. Also, write $W^{(t)}$ for the vector W on the t^{th} step, with $W^{(1)} = (0, 0, \ldots, 0)$. Using how $W^{(t+1)}$ is obtained from $W^{(t)}$, can show that $W^* \cdot W^{(T+1)} \geq T$ after $T + 1$ iterations. Also, using the condition on $W^{(T)}$ that necessitates an update, can show that $|W^{(T+1)}| \leq \sqrt{T}R$. (Both statements, use induction.)

Now, by Cauchy-Schwarz inequality, $T \leq BR\sqrt{T}$, which we can rearrange to $T \leq (BR)^2$.

Another example, the Iris data set

First discussed by R.A. Fisher in a 1936 paper, Iris data set commonly used in explanations. It contains 150 points in $\mathbb{R}^4$, each for an individual iris flower from one of 3 species: Iris setosa, Iris virginica, and Iris versicolor. The 4 coordinates are measurements of sepal length, sepal width, petal length, and petal width (in cm).

Iris setosa points are linearly separable from the other two.

Labels: *Iris setosa* $\longleftarrow$ 1; *Other species* $\longleftarrow$ -1.

Begin by opening the notebook `'perceptron-iris-notebook.ipynb'` ...After completing the algorithm, should get final $W = (\mathbf{w}, b)$, where $\mathbf{w} = (1.3, 4.1, -5.2, -2.2)$ and $b = 1$.

Figure: Images by G. Robertson, E. Hunt, Radomil