# Gradient descent on Logistic model

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# Outline

# Setup

As usual, sample data $\mathcal{S}$; write $(\mathbf{x}_i, y_i)$ for point in $\mathcal{S}$, with $y_i \in \{1, -1\}$. Parameter space: $\Omega = \mathbb{R}^{d+1} = \{(\mathbf{w}, b) \mid \mathbf{w} \in \mathbb{R}^d, b \in \mathbb{R}\}$. For each $\omega = (\mathbf{w}, b) \in \Omega$, have function $f_\omega : \mathbb{R}^d \to (0, 1)$ defined as

$$f_\omega(\mathbf{x}) = \sigma(\mathbf{w} \cdot \mathbf{x} + b),$$

where $\sigma$ is the logistic function. (So, our function output is our probability that the correct label for $\mathbf{x}$ is $+1$.)

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What do we use for the empirical loss function?

- *Considering how model should work*: when correct $y$ is $+1$ (and $\mathbf{x}$ is far from the hyperplane), good situation is when $f_\omega(\mathbf{x})$ very close to $1$. Alternatively, if the correct $y$ is $-1$, then we would like $f_\omega(\mathbf{x})$ to be close to $0$.

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- *From considerations about model*: It helps to alter numerical values of our labels: for $i$ with $y_i = 1$, define $\tilde{y}_i = 1$; for $i$ with $y_i = -1$, define $\tilde{y}_i = 0$.[1]

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We then use the **log-loss** function (a.k.a. **binary cross-entropy**). It is defined by

$$\mathcal{L}_{\mathcal{S}}(\omega) = \frac{1}{n} \sum_{i=1}^{n} -\tilde{y}_i \log(f_\omega(\mathbf{x}_i)) - (1 - \tilde{y}_i) \log(1 - f_\omega(\mathbf{x}_i)).$$

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# Gradient descent with logistic regression

Let's compute the gradient of the log-loss. Pick $1 \leq i \leq n$, and write the coordinates (also called *features*) of the vector $\mathbf{x}_i \in \mathbb{R}^d$ as $(x_{i,1}, x_{i,2}, \ldots, x_{i,d})$.

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$$\frac{d}{dw_j}(-\log(f_\omega(\mathbf{x}_i))) = -\frac{x_{i,j} * f_\omega(\mathbf{x}_i) * (1 - f_\omega(\mathbf{x}_i))}{f_\omega(\mathbf{x}_i)} = -x_{i,j} * (1 - f_\omega(\mathbf{x}_i)).$$

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The partial with respect to $b$ is simply $-(1 - f_\omega(\mathbf{x}_i))$.

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On the other hand, if $\tilde{y}_i = 0$ then the per-example log-loss at $\mathbf{x}_i$ is $-\log(1 - f_\omega(\mathbf{x}_i))$.

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and the partial with respect to $b$ is $f_\omega(\mathbf{x}_i)$.

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and the partial with respect to $b$ is $f_\omega(\mathbf{x}_i)$.
Hence, we have, for each $1 \leq j \leq d$,

$$\frac{d}{dw_j}\mathcal{L}_{\mathcal{S}} = \frac{1}{n}\sum_{i=1}^{n} -\tilde{y}_i x_{i,j} * (1 - f_\omega(\mathbf{x}_i)) + (1 - \tilde{y}_i) x_{i,j} f_\omega(\mathbf{x}_i),$$

and

$$\frac{d}{db}\mathcal{L}_{\mathcal{S}} = \frac{1}{n}\sum_{i=1}^{n} -\tilde{y}_i * (1 - f_\omega(\mathbf{x}_i)) + (1 - \tilde{y}_i) f_\omega(\mathbf{x}_i).$$

# Similarity to Perceptron algorithm updates

Consider the **per-example loss**: given one "example" $(\mathbf{x}_i, y_i) \in \mathcal{S}$, have the term $-\tilde{y}_i \log(f_\omega(\mathbf{x}_i)) - (1 - \tilde{y}_i)\log(1 - f_\omega(\mathbf{x}))$.
Its partial w.r.t. $w_j$ is

$$-\tilde{y}_i x_{i,j} * (1 - f_\omega(\mathbf{x}_i)) + (1 - \tilde{y}_i) x_{i,j} f_\omega(\mathbf{x}_i).$$

If $\tilde{y}_i = 1$, but $f_\omega(\mathbf{x}_i)$ is close to 0, then this is *almost* $-y_i x_{i,j}$. Were we to do an update with $\eta = 1$ then we would have $w_j^{(t)} \approx w_j^{(t-1)} + y_i x_{i,j}$. This is almost the Perceptron update (in the $j^{th}$ coordinate).
Likewise, if $\tilde{y}_i = 0$, but $f_\omega(\mathbf{x}_i)$ is close to $1$, then the partial is approximately $x_{i,j}$; so, using $\eta = 1$, we would get $w_j^{(t)} \approx w_j^{(t-1)} - x_{i,j} = w_j^{(t-1)} + y_i x_{i,j}$. Again, this is almost the Perceptron algorithm update.

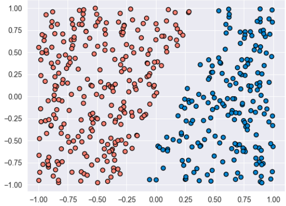
# Example with Logistic model

Given $\pm 1$-labeled sample data $\mathcal{S} = \{(\mathbf{x}_i, y_i)\}_{i=1}^n$, we found an expression for each partial derivative of $\mathcal{L}_\mathcal{S}$. In fact, if we call $w_{d+1} = b$ and $x_{i,d+1} = 1$ for every $1 \le i \le n$, then

$$\frac{d}{dw_j}\mathcal{L}_\mathcal{S} = \frac{1}{n}\sum_{i=1}^{n} -\tilde{y}_i x_{i,j}(1 - f_\omega(\mathbf{x}_i)) + (1-\tilde{y}_i)x_{i,j}f_\omega(\mathbf{x}_i)$$

for every $1 \le j \le d+1$.

**Example:** consider simulated sample data below, drawn uniformly from subset of $[-1,1]^2$ that is at least a fixed distance from $H = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1 - \frac{2}{3}x_2 - \frac{1}{5} = 0\}$. Positively labeled points are in blue.



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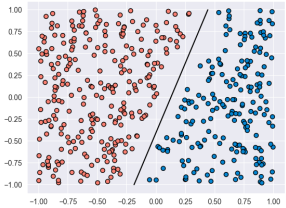
for every $1 \le j \le d + 1$.

**Example (cont'd):** Batch gradient descent, with learning rate $0.5$, stopping with threshhold $0.0005$, gives (approximately) parameters for:

$$\hat{H} = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2.01x_1 - 0.63x_2 - 0.26 = 0\}.$$

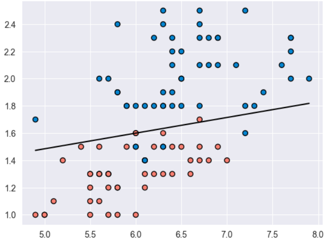


# Logistic regression on the Iris data

Recall the Iris data set: 150 points, 50 from each of three species of Iris flower. Two of the species in the data set, Iris versicolor and Iris virginica, are not linearly separable.

We can use gradient descent on the logistic model to find a hyperplane that does *well* in classifying the versicolor versus virginica – it correctly classifies 97 out of the 100 points.

For visualization, I first show just the first and fourth coordinates, and the results for logistic model in 2D.



# Logistic regression on the Iris data

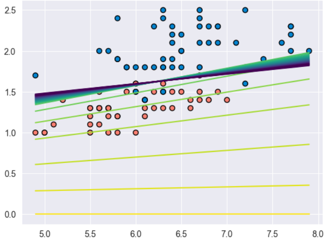
Pictured below are selected lines found during the batch gradient descent.

As before, yellow-to-purple is progression through the procedure.

Consecutive lines that are shown have $200$ updates between them; $\approx 4000$ updates in total.

The accuracy in this 2D projection is 92% (the final hyperplane correctly labeled 92 out of 100).[2]

The model on the points in $\mathbb{R}^4$ took less updates (just under $3000$). It had 97% accuracy on the data.




[2]Recall, the model labels the point with $+1$ when $f_\omega(\mathbf{x}) \geq 0.5$.