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EECS 212
Homework 1

Problem 1.1

0.1

The issue present in this problem is that the direction of the equality was supposed to be swapped after when a division using the log(1/2) (a negative number) was carried out.

$\log(\frac{1}{2}) = -0.30103$

When introduced into the equation the direction of the equality sign should have been made to be "<" instead of ">" greater than.

0.2

$1cent = 0.01\$ = (0.01\$)^2$
This step is erroneous. Since the dollar sign is within the bounds of the parentheses the units have become $\$^2$ which is not equivalent to dollars. The real conversion step should have been:
$= \$(0.1)^2$
equivalently $(10cents)^2$ does not equal 100 cents either.

0.3

By making the assumption that a = b that means a - b = 0. On line 4 you cannot simplofy by dividing both sides by (a - b). Since this is zero and invalid mathematical operation as dividing by zero is undefined.

Problem 1.2

The proof as it stands only shows that given $\frac{a+b}{2} = \sqrt{ab}$ that (a - b)$^2 \geq 0$. Writign the proof in backwards order would have shown that we could take 2 real numbers a and b and (a - b)$^2 \geq 0$ to form the result $\frac{a+b}{2} = \sqrt{ab}$

Problem 1.5

Let us assume 2 cases. One in which a^b =$\sqrt{2}^2$ is rational and one in which it is irrational.
Case 1: $\sqrt{2}^2$ is rational.

Let Let a = b = $\sqrt{2}$, $\sqrt{2}$ is irrational so a and b are irrational. $\sqrt{2}^2$ results in 1.63253 which is a rational number.

Case 2: Assuming $\sqrt{2}^2$ is irrational.

Say a = $\sqrt{2}^2$ and b = $\sqrt{2}$ where we are assuming both a and b are irrational. a^b =$\sqrt{2}^2$ = 2. 2 is rational.

Through case 1 we see that given an irrational number there is a possibility that an irrational power may result in a rational number.

Problem 2

Statement: If a positive integer is the product of two distinct primes, then its square root is irrational.

Contrapositive: If the square root of a positive integer is rational, then it is not the product of two distinct primes.

Let us assume that the integer was a product of two distinct primes. Such that we have two prime numbers m and n a rational number k and a positive integer p. We can represent this statement as such:

p = m*n
and

Since $\sqrt{p}$ is a rational number it can be represented as $\frac{a}{b}$ where a and b are two co-prime numbers meaning they share only a common factor of 1 to represent a ratio.

Rewrite:

$\sqrt{p} = \frac{a}{b}$
p = $\frac{a^2}{b^2}$

Combining these two equations we get:

$\frac{a^2}{b^2}$ = m*n
a^2 = $m * n * b^2$

This implies that a^2 divides b^2 by $m * n$. Which means that since a is a factor of a^2, a would also divide b^2 and subsequently a would divide b.

Here we have reached a contradiction. In establishing that $\frac{a}{b}$ was a rational number that implied that a and b only shared a common factor of 1. However, if the the positive integer p was a product of two distinct primes it would imply that a divides b which violates the definition of a rational number.

Therefore, through contradiction we can say that it cannot be the product of two distinct primes given that the square root of the positive integer p is rational.

So If the square root of a positive integer is rational, then it is not the product of two distinct primes. Equivalently if a positive integer is the product of two distinct primes, then its square root is irrational.

Problem 3

Statement: Any stack of $n \geq 5$ pancakes can be sorted using at most 2n - 5 flips.

Suppose we have some function F(n) which represents the number of flips it takes to flip n pancakes.
There is also a function G() which represents the maximum number of flips that it takes to move the largest pancake in a stack to the bottom of a stack through the bring to the top algorithm. The bring to top algorithm works by taking the biggest pancake in the stack and flipping it to the top by flipping from underneath it. Then flipping the whol stack of pancakes to bring it to the bottom of the stack. If the pancake is already at the top of the stack it would take one flip, if the pancake was elsewhere in the stack it would then take two flips based on the scenario previously described, and if the pancake is already at the bottom it would take 0 flips given that it is already in place.

S(n) : F(n) $\leq$ 2n - 5

Base Case:

F(5) $\leq$ 2(5) - 5 : 5 $\leq$ 5

We have been given that $F(5) = 5$ which is equal to 5. So the base case passes.
Inductive Step:
F(n+1) $\leq$ 2(n+1) - 5

F(n+1) $\leq$ 2n - 3

Based on the problem we know that given that we add another pancake to the stack it will take G(),at most 2, flips in order to move the largest pancake to the bottom of the stack. So it will then in the worst case take F(n) flips in order to reorder the other n pancakes in the stack.

In other words: F(n+1) = F(n) + G() flips.

In the base case we showed that F(n) = 2n - 5
F(n+1) = 2n - 5 + 2 = 2n - 3, in the worst case.

2n - 3 = 2n - 3

Therefore, any stack of $n \geq 5$ pancakes can be sorted using at most 2n - 5 flips.

Problem 4

$\sum_{k=0}^{n} k^3 + 2k^2 - k + 1 = an^4 + bn^3 + cn^2 + e$

We must first solve for a, b c, d , and e. We can do this by setting up a system of equations for the unknowns. Given by case k = 0 we can solve for the coefficient e which is 1. Proceed to solve the summation for k = 1 to k =4.

k(1)=4 = a + b + c + d + 1

k(2)=19 = 16a + 8b + 4c + d + 1

k(3)=62 = 81a + 27b + 9c + 3d + 1

k(4)=155 = 256a + 64b + 16c + 4d + 1

We can then solve this system of linear equations and find that the values for a, b, c, d ,and e are $\frac{1}{4}$, $\frac{7}{6}$, $\frac{3}{4}$, $\frac{5}{6}$, and 1 respectively.

Given these coefficents we must now show through Induction that these coefficients solve for all values of n.

Base Case (n = 0):

$0^3 + 2(0)^2 - 0 + 1 = \frac{1}{4}(0)^4 + \frac{7}{6}(0)^3 + \frac{3}{4}(0)^2 + \frac{5}{6}(0) + 1$
1 = 1 : Base Case is true.

Inductive Hyopthesis:

For all values of n $\sum_{k=0}^{n} k^3 + 2k^2 - k + 1 = an^4 + bn^3 + cn^2 + e$ holds.
Inductive step:

$\sum_{k=0}^{n+1} k^3 + 2k^2 - k + 1 = a(n+1)^4 + b(n+1)^3 + c(n+1)^2 + e$

We now assume that k(n) is true. We must now show that k(n + 1) is true.

We know that the following represents the next value in the sequence, k(n +1):

k(n) + $(n+1)^3 + 2(n+1)^2 - (n+1) + 1$

We can use the inductive hypothesis to replace k(n) with $an^4 + bn^3 + cn^2 + e$:

k(n+1) = $an^4 + bn^3 + cn^2 + e + (n+1)^3 + 2(n+1)^2 - (n+1) + 1$

Through the base case we know that $\sum_{k=0}^{n} k^3 + 2k^2 - k + 1 = an^4 + bn^3 + cn^2 + e = \frac{1}{4}n^4 + \frac{7}{6}n^3 + \frac{3}{4}n^2 + \frac{5}{6}n + 1$

So equivalently k(n+1) = $\frac{1}{4}n^4 + \frac{7}{6}n^3 + \frac{3}{4}n^2 + \frac{5}{6}n + 1 + (n+1)^3 + 2(n+1)^2 - (n+1) + 1$

In order to show that the following function holds for all n it must be true that the successor to n will also hold true. So the following expression below and the previous function must be equivalent:

$an^4 + bn^3 + cn^2 + e$

k(n + 1) = $\frac{1}{4}(n+1)^4 + \frac{7}{6}(n+1)^3 + \frac{3}{4}(n+1)^2 + \frac{5}{6}(n+1) + 1$
Check for equality:

$\frac{1}{4}n^4 + \frac{7}{6}n^3 + \frac{3}{4}n^2 + \frac{5}{6}n + 1 + (n+1)^3 + 2(n+1)^2 - (n+1) + 1 = \frac{1}{4}(n+1)^4 + \frac{7}{6}(n+1)^3 + \frac{3}{4}(n+1)^2 + \frac{5}{6}(n+1) + 1$

Simplify:
$\frac{n^4}{4} + \frac{13n^3}{6} + \frac{23n^2}{4} + \frac{41n}{6} + 4 = \frac{n^4}{4} + \frac{13n^3}{6} + \frac{23n^2}{4} + \frac{41n}{6} + 4$

So they are equivalent. Therefore there are some coefficients for a, b, c, d, and e that hold for all possible values of n.

Problem 5

Theorem: Given a natural number n > 0 and a set of n + 1 positive integers, none of them exceeding 2n, there is at least one integer in the set that divides another integer in the set.

Base case: n = 1. Set is of size 2 with no numbers greater than 2(1) = 2.

Only one set [1,2] that represents this. The base case passes.

Inductive Hypothesis: We assume that for a set of size n + 1 with elements no large than 2n there exists two integers p and q such that p divides q.

Now lets prove for n + 1.

We now have a set that contains (n + 1) + 1 = n + 2 numbers in the set and values that are no greater than 2(n + 1) = 2n + 2.

There are now two new possible numbers that may be present in the set. 2n + 1 and 2n + 2.

That now brings about the presense of a few cases that we will evaluate:
Case 1: (Neither 2n + 1 or 2n + 2 are in the set):

Since we have some Set, lets call it A, that includes n + 2 elements with all elements less than or equal to 2n in value. We can create a subset B which is within the space of A that has n + 1 elements that are all values less than or equal to 2n. Since it holds for the subset B it then follows that it is true for the whole set. Therefore by the inductive hypothesis we can conclude that there are some integers p and q such that p divides q.

Case 2 (2n + 1 is in the set but not 2n +2):

Given that only 2n + 1 is in the set but not 2n + 2. We have a set of n + 2 elements, A. If we seperate 2n + 1 into its own subset, lets call it C, we now have a subset of the data of n + 1 elements that have values less than or equal to 2n. By the inductive hypothesis we know that we can conclude that there are some integers p and q such that p divides q given that we have a set of n + 1 elements with all elements less than 2n. Since it holds for a subset of A it follows that it holds for the whole combined set A. So it is true.

Case 3 (2n + 2 is in the set but not 2n + 1):

Given that only 2n + 2 is in the set but not 2n + 1 . We have a set of n + 2 elements, A. If we seperate 2n + 1 into its own subset, lets call it C, we now have a subset of the data, B, of n + 1 elements that have values less than or equal to 2n. By the inductive hypothesis we know that we can conclude that there are some integers p and q such that p divides q given that we have a set of n + 1 elements with all elements less than 2n. Since it holds for a subset of A it follows that it holds for the whole combined set A. So it is true.

Case 4: (Both 2n + 1 and 2n+2 are in the set):

If both 2n +1 and 2n + 2 are in the set we can come to a conclusion examining these two subcases.

n +1 is in the set:

If 2 or n + 1 is in the set you can rewrite 2n + 2 as 2(n + 1) which shows that n + 1 divides 2n + 2. Therefore, therefore it would be true.

n + 1 is not in the set:

The relationship between n + 1 and 2n + 2 takes the form (k , 2k) where k = n + 1. One thing to note is that the only two elements in the set that are greater than 2n are 2n + 1 and 2n + 2. You can use the k in the pair(k,2k) to replace 2k in the set. This can be done because within the range of 1 to 2n there are no numbers that are multiples of (n +1) other than itself as the next highest multiple is 2(n + 1) = 2n + 2. Since 2n + 2 is in the set it can effectively replace n +1 in the set as it will not affect the integrity of the set. 2n + 2 can be reduced to n + 1 as everything that is a factor of n + 1 is also a factor of 2n + 2. So in other words anything that divides n + 1 also divides 2n + 2 .

Following this replacement of 2n + 2 with n + 1 in the set we can now extrapolate that we can form a set of A, where we have removed 2n + 1, n + 1 elements where the values are less than or equal to 2n. Therefore, through the inductive hypothesis we can conclude that there exists a p and q such that p divides q given a set of n + 1 elements with all elements values less than 2n. Since this is a subset of the whole set it holds true for the whole set.

Problem 6

Statement: Given an array A of n distinct integers, the algorithm QuickSort sorts the numbers in ascending order.

Base Case (Array of Size 1):

QuickSort (A(1)) = 1, which is true.

Inductive Hypothesis:

Given a Array of size $\leq$ n QuickSort will be able to sort the list.

Inductive Step (Array of n + 1):

Given an array of size n + 1. The nature of the QuickSort algorithm takes out one integer in the Array A to be used as a pivot. And then uses this pivot to form a comparison and seperate the rest of the integers into two lists. Given that these two subsets will each be of size $\leq$ n , the Quick Sort algorithm can sort the list as given by the inductive hypothesis. Supporting that an array of n + 1 will be sorted by quick sort.