A Mathematical Interpretation of Feynman Diagrams and Dimensional Regularization via Hopf Albegras

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Abstract: A Feynman diagram is a pictorial representation of a specific quantum process. However, this representation corresponds to a perturbative approach of the phenomenon. Due to this appears some kind of divergences, called *ultraviolet divergences*, which are not desirable in the description of an observable phenomenon. In physics this problem is fully described by the *Renormalization Group* method. Nevertheless, there are a lot of problems when we want to understand this method formally. For instances, the formal definition of the *Feynman path-integral* or the *regularization of Feynman integrals*. In this work, we give a abstract definition of Feynman diagrams using Hopf algebras in order to avoid this problems in the formal definition of a quantum field theory. Besides, we also give an own interpretation of the works of Alain Connes and Dirk Kreimer[CK00, CK01, CK98] about the abstract view of the dimensional regularization process in a quantum field theory.

Keywords: Quantum Field Theory, Feynman Diagrams, Hopf Algebra, ϕ-four Theory, Quantum Electrodynamics

Dedicated to

All people in my daily life who give me force to carry on with my dreams. I'm really grateful with you. THANKS YOU!!

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Introduction

The **Renormalization** is some of the more important concepts in physics. The principal idea of a renormalization method is describe a physical system in the more simple way, i.e with least degrees of freedom possible[Cos11, Has99]. There are several methods of renormalization. For instances, the iterative reduction of the size of the Hamiltonian matrix associated to a *quantum many-body system*[Whi92] or the integration over certain degrees of freedom in a field theory. The latter corresponds to the usual renormalization method in quantum field theory called *Wilson's Rernomalization Group*[Wil71a, Wil71b] where the fields with high energy are integrated. With respect to the perturbative theory Wilson's renormalization method corresponds effectively to eliminate the *ultraviolet divergences* due to the quantization of the theory.

A quantization process corresponds to an assignment of an operator for each field. Likewise to a renormalization process, there are several methods of quantization like the functional quantization develop by *Richard Feynman* in 1948 where the propagator of a field depends on an integration over all possible classical paths called *Feynman path-integral*. However, there is not a good definition of the path-integral. The usual quantization of a classical field theory corresponds effectively to a certain set of assignments, called *Feynman rules*, of analytical expressions into a pictorial representation called *Feynman diagram*.

A interesting mathematical problem is understand formally the definition of the Feynman diagrams associated to a quantum field theory or describe the regularization of that Feynman diagrams. In this thesis, we study abstractly the Feynman diagrams as some kind of graphs. Besides, inspired by the work of *A. Connes* and *D. Kreimer*[CK98, CK00, CK01] about the *Hopf algebra* associated to a quantum field theory, we study this Hopf algebra structure and its relation with the regularization problem of Feynman diagrams in the particular case of the *dimensional regularization*.

This document is organized as follows. In Chapter 1, we give a some basic theory about quantum field theory and the regularization problem giving the usual physical definition of a Feynman diagram. In Chapter 2, we show an alternative and abstract definition of Feynman diagrams. We also give the basic definition about the Hopf algebra structure. In Chapter 3, we present the specific Hopf algebra of Feynman diagrams associated to a quantum field theory and show the principal result of Connes and Kreimer about *Hopf algebra renormalization*[CK00].

CHAPTER 1

Functional Quantization and Feynman Diagrams

In this chapter, we introduce the concept of *renormalization* in the particular case of a Quantum Field Theory (QFT). We define a QFT using the functional quantization without care about the formal details in the definition of the Feynman path-integral, besides we try to understand the way of how we can obtain from a *renormalized theory* some elemental information about of the processes associated to a particular QFT (like the renormalization of the electric charge in the Quantum Electrodynamics (QED).)

1.1 Physics and Renormalization

Let us consider a physical phenomena, like the motion of an electron in an electromagnetic field or the refraction of the light inside of a material. In most cases, the phenomena can be described through a functional, $S : \Gamma(E)^g \to \mathbb{R}$, from the sections of some bundle E, where g is a positive integer number called *number of degrees of freedom*[Pol04]. The functional S is called *action* and it can be written in terms of a function $\mathcal{L} : \Gamma(E)^g \to C^\infty(E, \mathbb{R})$ called *Lagrangian* as

$$S(\sigma_1, \sigma_2, \cdots, \sigma_g) = \int_E \mathcal{L}(\sigma_1(x), \sigma_2(x), \cdots, \sigma_g(x))dx, \tag{1.1}$$

where dx is a measure over E that depends of the phenomena itself. The following examples show this fact.

Example 1.1.1. (Classical Mechanics*)* Consider the motion of a particle of mass m in a central potential, $V(r)$. The trajectory of the particle is described by a set of generalized coordinates[1], $q(t) = (q_1(t), \cdots, q_s(t))$, and its mechanical state is completely determined by the set $\{q(t), \dot{q}(t)\} = \{(q_1(t), \cdots, q_s(t)), (\dot{q}_1(t), \cdots, \dot{q}_s(t))\}$[2] . The Lagrangian of this system is given by[LDL69, Section 5]

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2}m|\dot{q}(t)|^2 - V(|q(t)|). \tag{1.2}$$

[1]In this case, $2s$ is the number of degrees of freedom and its number depends of the constrains of the motion.

[2]The notation $\dot{q}(t)$ means the total derivative of the vector q, $dq(t)/dt$.

Thus, the action $S : C^\infty(\mathbb{R}, \mathbb{R}^{2s}) \to \mathbb{R}$ for the particle in a central potential is

$$S(q(t), \dot{q}(t)) = \int_{t=t_0}^{t=t_1} \mathcal{L}(q(t), \dot{q}(t))dt. \tag{1.3}$$

Example 1.1.2. *(*Quantum Mechanics*)* In quantum mechanics, the state of a single spinless particle of mass m in a given potential V, is described by the Hamiltonian

$$H = -\frac{1}{2m}\nabla + V, \tag{1.4}$$

where H is an operator acting over $L^2(\mathbb{R}^3, \mathbb{C}) =: \mathcal{H}$. The stationary state[3] of the particle with an energy E is completely determined by its wavefunction, $\psi \in \mathcal{H}$, and its action $S : \mathcal{H} \to \mathbb{R}$ is given by[LDL77, Section 20]

$$S(\psi, \bar{\psi}) = \int_{\mathbb{R}^3} \bar{\psi}(x)(H - E)\psi(x)dx. \tag{1.5}$$

The description of the phenomena associated to the action S, as the motion of a classical or quantum particle, is giving by the *principle of least action*[LDL69, Section 2].

Principle of Least Action 1.1.1. *The physical phenomena associated to S is characterized by the sections $\sigma_1, \cdots, \sigma_g \in \Gamma(E)$ if S satisfies the following equation* [4]

$$\delta S|_{\sigma_1,\cdots,\sigma_g} = 0, \tag{1.6}$$

where δS is the functional variation[Has99] of S.

Let us see two examples of how to apply this principle.

Example 1.1.3. Example 1, the principle of least action gives the following equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0. \tag{1.7}$$

This equation is called *Euler-Lagrange* equation and for the case of a particle in a central field, we have that

$$\frac{1}{2}m\ddot{q} = -\frac{\partial V}{\partial q}. \tag{1.8}$$

This is the usual *Newton's second law* that describes the motion of a particle under the effect of a central force.

[3]Corresponding to the solution of the time-independent Schödinger equation.

[4]Equation (1.6) means that the physical phenomena occurs when S is an extremum. However, this doesn't mean that S is a minima as the name of this principle suggests.

Example 1.1.4. In the case of quantum mechanics, the variation of the action can be done over the wavefunction ψ or its complex conjugate. Taking the variation over $\bar{\psi}$, the Euler-Lagrange equation for this case is

$$(H - E)\psi = 0. \tag{1.9}$$

This is the time-independent Schrödinger equation that describes the stationary state of a spinless particle.

The above examples show that the sections $\sigma_1, \cdots, \sigma_g$ which satisfy the principle of least action (1.6) describe the most important characteristics of the considered phenomena like the generalized coordinates and velocities of a classical particle or the wavefunction of a quantum particle. If we can find the action S and its corresponding sections $\sigma_1, \cdots, \sigma_g$ which describe the phenomena, we say that S and the sections $(\sigma_1, \cdots, \sigma_g)$ give a *theory* of the phenomena. Moreover, since the sections of a given theory depend on the coordinates of the physical space-time, M, we say that S corresponds to a *non-relativistic theory* (resp. *relativistic theory*) if the sections belong to some representation of the Galilean group (resp. Lorentz group)[5].

The last definition of the non-relativistic (relativistic) theory corresponds to the notion of *covariance* of the theory due to the relative description of the considered phenomena by an observer.

1.1.1 Renormalization

We have said that given a physical phenomena one can describe the phenomena through an action $S : \Gamma(E)^g \to \mathbb{R}$ with g sections which satisfy the principle of least action. A natural question at this point is if the number of degrees of freedom is unique for a given phenomena. The answer to this question is **NO** and this is the main idea behind the method of renormalization of a theory[Cos11].

Definition 1.1.1. Let $S : \Gamma(E)^g \to \mathbb{R}$ be the action associated to some theory. If there is an EFFECTIVE Lagrangian $\mathcal{L}_{eff} : \Gamma(E)^{g'} \to C^\infty(E, \mathbb{R})$ with $g' < g$ such that

$$S(\sigma_1, \cdots, \sigma_g) = \int_E \mathcal{L}_{eff}(\sigma'_1(x), \cdots, \sigma'_{g'}(x))dx, \tag{1.10}$$

the theory is called *renormalizable.*

The latter definition says that a theory is renormalizable if we can describe the same theory as a theory with less degrees of freedom such that the action keeps invariant. The fact that the action doesn't change implies that the sections $\sigma_1, \cdots, \sigma_g$ as well as $\sigma'_1, \cdots, \sigma'_{g'}$ describe the same phenomena. In fact, since the action is invariant the principle of least action gives us a set of g' sections associated to the effective Lagrangian which describe the same phenomena that the original theory. The process of finding an effective Lagrangian that keeps the action invariant is called *renormalization.*

[5] The Galilean group is related to the isometry group of a space-time with an absolute time, whereas the Lorentz group corresponds to the isometry group of the complete space-time.

Usually the term "renormalization" is related only with quantum field theories. However, this process is one of the most important and useful methods in all branches of physics as we can see in the following example[CK00].

Example 1.1.5. (Classical Renormalization) Let us consider a spherical rigid balloon moving in a fluid. We are interested in the motion of the balloon between a interval of time $[0, t]$. Obviously, we can consider the motion of the surface of the balloon like a rigid body and describe the motion of the surface only with the coordinates of the center of the balloon $r_c = (x_c, y_c, z_c)$. Thus the theory that describes the motion of the balloon has at least 6 degrees of freedom[6]. Nevertheless, we also need to describe the motion of the fluid which is given by five quantities, 3 components of the fluid velocity u, the presion P and the density of the fluid ρ. Then, the action S that describes the motion of the balloon needs a total of 11 sections and can be written as follows

$$S(r_c, \dot{r}_c, u, P, \rho) = \int_0^t \mathcal{L}(r_c(s), \dot{r}_c(s), u(s), P(s), \rho(s))ds. \tag{1.11}$$

Solving the Euler-Lagrange equations for this theory is very hard or even impossible if we change the shape of the balloon. However, we can describe the motion of the balloon only with the coordinates of its center of mass through the following effective Lagrangian

$$\mathcal{L}_{eff}(r_c(t), \dot{r}_c(t)) = \frac{1}{2}m\dot{r}_c^2(t), \tag{1.12}$$

where the parameter m is called inertial mass. Here the inertial mass is not only the mass m_0 of the balloon but is given by

$$m = m_0 + \frac{1}{2}M, \tag{1.13}$$

where M is the mass of the volume of the fluid occupied by the balloon (from the Archimedean law). The theory defined from $\mathcal{L}_{eff}$ describes the motion of the balloon with only six degrees of freedom.

We must note that the process of renormalization in the latter example gives us a shift of the mass of the balloon, $\delta m = m - m_0$, which comes from the integration of the degrees of freedom of the fluid. Since, the shift δm comes from a renormalization process the parameter $m_0 + \delta m := m$ is called *renormalized parameter* and the fixed parameter m_0 is called *bare parameter* and this phenomena is characteristic of a renormalization process.

1.2 Quantum Field Theory

Now, let us introduce a particular set of physical theories called quantum field theories which will be the main object of study in this work. This kind of theories pretend to explain the fundamental interaction between the particles that compounds our universe, like the electromagnetic interaction or the strong nuclear force which keeps the nucleus of

[6] 3 position coordinates and 3 velocity coordinates of the center of the balloon.

an atom bounded. Due to the small range and the usual velocities of those interactions, we need to consider a quantum and relativistic nature of this phenomena.

Classical Electrodynamics **Scalar Fields** Fluid Mechanics
Classical Mechanics

Vector Fields Field Theory **Tensor Fields**

QED
QCD **Spinorial Fields**

FIGURE 1.1. Kind of field in some typical classical field theories.

The theory described in the latter section is called *classical theory of fields*. In this theory the sections that describe the physical phenomena are called *fields*. Due to the covariance of the theories, there are different kinds of fields to consider (see figure [1.1]). For example in a field theory for the electromagnetic interaction, the electric and magnetic fields are described by using a scalar potential ϕ, and a vector potential $\mathbf{A}$, by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{1.14}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1.15}$$

Thus, the theory for the electromagnetic interaction is described by one scalar field, one vector field[7]. However, in order to describe the motion of a charge in an electromagnetic field we also need to include another vector field that describes the position of the charge.

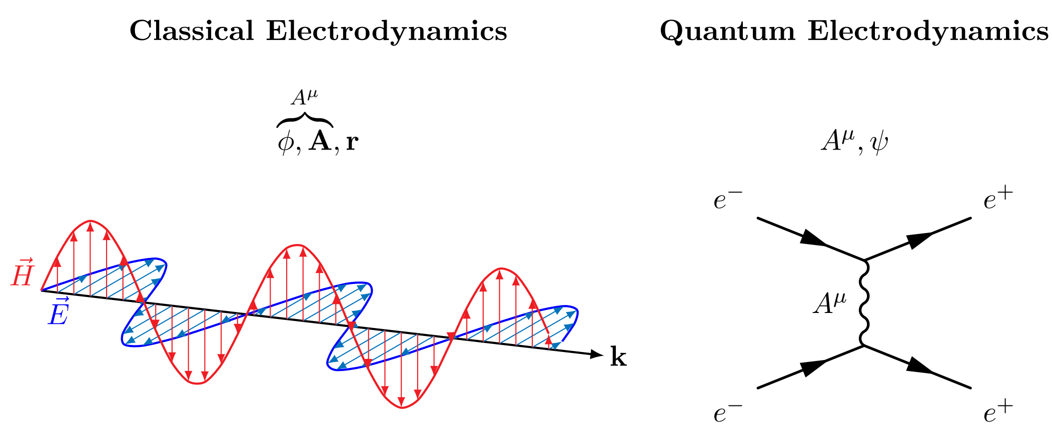


FIGURE 1.2. Classic vs Quantum Electrodynamics

With this considerations, the electromagnetic theory describes the motion of an electron e^- in a electromagnetic field and the dynamics of the electromagnetic field itself. Nevertheless, there are other interactions between charged particles and an electromagnetic field that are not considered in this *classical theory of the electromagnetic interaction*. For example, the degree of freedom of spin of an electron is not considered in the classical theory. In order to consider the interaction between the spin of an

[7]Since the electromagnetic theory must be a relativistic theory the two fields in this theory are interpreted as only one object called four-potential $A^\mu = (\phi, \mathbf{A})$.

electron and the electromagnetic field, a more complete theory can be obtain if we describe the electron with a spinorial field, ψ[VBBP82]. But even this theory does not include all the phenomena associated to the electromagnetic interaction because phenomena like the creation and annihilation of an electron with an antielectron (positron, e^+) is not predicted by the classical theory [see figure 1.2]. In order to include this kind of phenomena, which are due to quantum effects, we need to consider a process called *quantization*. The process of quantization can be understood as a process where we assign a quantum field $\hat{\psi}$(operator over some Hilbert space) to a classical field ψ.

When we can define this assignment we have a *quantum field theory*. A problem with the process of quantization is that there is not a unique way to do it, but in this work we consider only the process called *functional quantization*.

Functional Quantization

Now let us give more details about the process of functional quantization. This method was completely developed by *Richard Feynman* in 1948. In order to give a simple explanation of the method, we will consider just a simple scalar theory called ϕ^4-theory[MD95].

Definition 1.2.1. (ϕ^4-THEORY) Let M be the Minkowski space-time[8] and $\phi \in C^\infty(M)$. The ϕ^4-theory is defined by the following Lagrangian,

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4, \tag{1.16}$$

where ∂_μ means the partial derivative with respect to the coordinate x^μ and $m, \lambda \in \mathbb{R}$. This theory can be interpreted as a particle of mass m described by the scalar field ϕ in presence of an interacting potential $V(\phi) = \frac{1}{4!}\lambda\phi^4$.

The ϕ^4-theory is a toy model to describe several phenomena in physics; in particular, it is a model to describe the second order phase transitions in statistical physics[MD95, EML80a, EML80b]. However, in the present work we use this theory to understand different concepts in the formulation of a quantum field theory. The classical theory given by the Lagrangian (1.16) describes the dynamics (or propagation) of the scalar field ϕ in the space-time. When we solve the Euler-Lagrange equation of this theory we obtain a particular dynamics of the field restricted by the principle of least action. However, in a quantum theory it is not enough to consider this particular propagation.

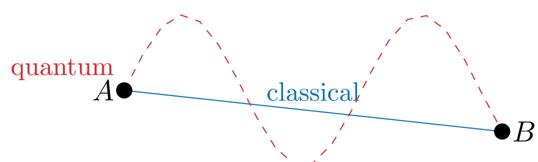


FIGURE 1.3. Classical vs Quantum Path

[8] M is a manifold of dimension 4 with a flat pseudo metric defined by $g = dt^2 - dx^2 - dy^2 - dz^2$.

Let us consider a particle moving in the space-time. In a classical theory, the particle follows a particular trajectory determined by the principle of least action (solid blue line in Figure [1.3]). However, in a quantum theory we can't define a trajectory due to the *uncertainty principle*. In order to solve this problem one of the formulations of the quantum mechanics assumes that the probability of finding the particle at point B if the particle starts from point A is given by the sum of probability over all paths that the particle can take (one of those paths is given by the red dashed line in Figure [1.3]). This formulation of the quantum mechanics, called *path-integral formulation*, assigns a non-zero probability to each possible path of the particle such that if the coordinate of point A is $x = q(t_0)$ and the coordinate of the point B is $y = q(t_1)$ with $t_0, t_1 \in \mathcal{R}$. Then the probability $U(x, y)$ that the particle travels from A towards B is given by

$$U(x,y) = \int \mathcal{D}q(t) e^{iS(q(t))}, \tag{1.17}$$

where $S(q(t))$ is the classical action associated to each path $q(t)$ of the particle and $\int \mathcal{D}q(t)$ means the integral over all paths. A problem here is to give a good definition of the path measure $\mathcal{D}q(t)$, but we will discuss this detail in the following chapters.

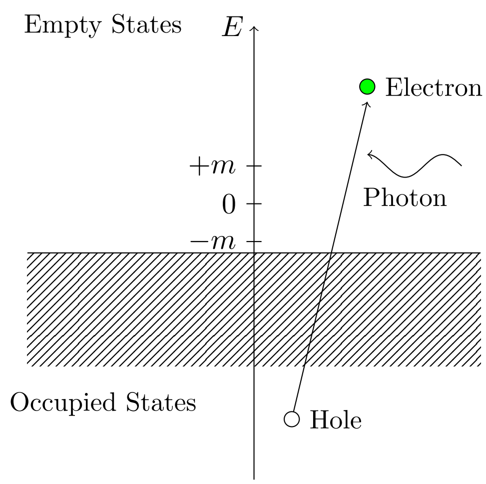


FIGURE 1.4. Antiparticle as an empty state (hole) of negative energy.

Let us come back to the ϕ^4-theory. In this case, we can think that the particle is described by the scalar field ϕ. Then the path integral is taken over all classical scalar fields, i.e $\int \mathcal{D}\phi$. However, in a quantum field theory, the state of the particle is given by[MD95, LDL77, VBBP82]

$$|\phi(x)\rangle = \hat{\phi}(x) |\Omega\rangle, \tag{1.18}$$

where $\hat{\phi}(x)$ is the quantized field and $|\Omega\rangle$ is the vacuum state of the theory. We remark that the vacuum state of a theory doesn't mean that all states of the theory are empty. For example in a theory given by *Paul Dirac* to describe the quantum electrodynamics, the vacuum state corresponds to a state where the states of positive energy are empty and the states of negative energy are occupied. In this theory, called *particle-hole theory*, an empty state of negative energy is interpreted as an anti-particle.

Now, the quantized field $\hat{\phi}$ is such that the probability that the particle travels from A towards B is[MD95]

$$\langle\Omega|\, \hat{\phi}(y)\hat{\phi}(x)\, |\Omega\rangle = \frac{\int \mathcal{D}\phi\ \phi(y)\phi(x)e^{iS(\phi)}}{\int \mathcal{D}\phi\ e^{iS(\phi)}}, \tag{1.19}$$

where $S(\phi) = \int_M \mathcal{L}_{\phi^4} d^4 z$. The factor in the denominator of the latter equation $Z := \int \mathcal{D}\phi\ e^{iS(\phi)}$, called sometimes *partition function*, is a normalization factor because each path ϕ has a weight $e^{iS(\phi)}$. The expression $\langle\Omega|\, \hat{\phi}(y)\hat{\phi}(x)\, |\Omega\rangle =: \langle\phi(y)\phi(x)\rangle$ is called *propagator* of the field or *correlation function of two-points* and more exactly corresponds to the probability that an initial state $|\phi(x)\rangle$ pass to a final state $|\phi(y)\rangle$. In general, a correlation function of n-points is given by

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \frac{1}{Z}\int \mathcal{D}\phi\ e^{iS(\phi)} \prod_{j=1}^{n} \phi(x_j) \tag{1.20}$$

1.3 Feynman Diagrams

Physically it's enough define the quantized field $\hat{\phi}$, through the correlation functions (1.19) and (1.20). The main problem with this formulation is the formal definition of the path integral $\int \mathcal{D}\phi$. However, we will not consider this problem and we will care about of how compute this kind of integrals. Nevertheless, it is possible consider the path-integral as heuristic probability measure over an infinite dimensional space[PAY03].

1.3.1 One point Space-Time

In order to understand how to compute the path integral we shall show a simple version of this integral. Let us suppose that the space-time M is a set with just one element, $M = \{p\}$. In this case, the action of the theory is given directly by the Lagrangian and the degrees of freedom in the theory are finite[Pol04]. In fact, if M is a one point space-time, the set of all scalar fields is $C^\infty(M) \simeq \mathbb{R}$, because every function $f : M \to \mathbb{R}$ can take only one value $f(p) \in \mathbb{R}$.

Due to the simple form of the scalar field, let us consider another set of fields, say $\mathbb{R}^d$ with d a positive integer. Thus, the Lagrangian $\mathcal{L}$ of a theory must be an operator from $\mathbb{R}^d$ to $\mathbb{R}$; a typical Lagrangian (and also the action) is given by

$$S(x) = \mathcal{L}(x) = -\frac{1}{2}\langle x, Ax\rangle + U(x), \tag{1.21}$$

where A is a square matrix of dimension d, $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{R}^d$ and $U : \mathbb{R}^d \to \mathbb{R}$. The first term in the above Lagrangian is similar to a dynamics term, $\frac{1}{2}(\partial_\mu\phi)^2 - m\phi^2$, in the ϕ^4-theory whereas the second term is a interacting potential similar to $V(\phi)$ in the ϕ^4-theory.

The advantage of considering the space-time as a single point set is that the path integral corresponds to the usual integration over $\mathbb{R}^d$ is a C^∞-function. Thus the partition function of the theory (1.21)is given by

$$Z = \int_{\mathbb{R}^d} dx e^{-\frac{1}{2}\langle x, Ax\rangle + U(x)}. \tag{1.22}$$

Let us consider a free theory[9], i.e $U(x) = 0$ for all $x \in \mathbb{R}^d$. Besides, suppose that A is an invertible symmetric matrix, and let us compute the partition function of the free theory,

$$Z_0 = \int_{\mathbb{R}^d} dx e^{-\frac{1}{2}\langle x, Ax\rangle}. \tag{1.23}$$

Since A is an invertible symmetric matrix, we can evaluate the product $\langle x, Ax\rangle$ through the eigenvalues of A, say $\lambda_1, \cdots, \lambda_d \in \mathbb{R}\backslash\{0\}$ such that $\langle x, Ax\rangle = \sum_{j=1}^d \lambda_j x_j^2$. Thus

$$Z_0 = \int_{\mathbb{R}^d} dx \prod_{j=1}^d e^{-\frac{1}{2}\lambda_j x_j^2}.$$

As $dx = \prod_{j=1}^d dx_j$, then

$$Z_0 = \prod_{j=1}^d \int_{-\infty}^{\infty} dx_j e^{-\frac{1}{2}\lambda_j x_j^2}.$$

Since for each $j = 1, \cdots, d$, $\int_{-\infty}^{\infty} dx_j e^{-\frac{1}{2}\lambda_j x_j^2} = \sqrt{\frac{2\pi}{\lambda_j}}$[Pol04], we obtain that

$$Z_0 = \prod_{j=1}^d \sqrt{\frac{2\pi}{\lambda_j}} = \left(\frac{\det A}{2\pi}\right)^{-\frac{1}{2}}. \tag{1.24}$$

Therefore, Z_0 is given by the determinant of the matrix A. The following proposition give us a formula to calculate the correlation function of n-points.

Proposition 1.3.1. *Let Z_b be the partition function of the following theory*

$$\mathcal{L} = -\frac{1}{2}\langle x, Ax\rangle + \langle b, x\rangle, \tag{1.25}$$

where $b \in \mathbb{R}^d$ and A is a dxd invertible matrix. Then,

$$\langle x_{j_1}, \cdots, x_{j_m}\rangle_{free} = \frac{1}{Z_0}\frac{\partial}{\partial b_{j_1}} \cdots \frac{\partial}{\partial b_{j_m}} Z_b\bigg|_{b=0}, \tag{1.26}$$

with $1 \leq j_k \leq d$ for each $k = 1, \cdots, m$.

[9] Or non-interacting theory

Proof. Note that for $1 \leq j_k \leq d$

$$\frac{\partial}{\partial b_{j_k}} \int_{\mathbb{R}^d} dx e^{-\frac{1}{2}\langle x, Ax\rangle + \langle b, x\rangle} = \int_{\mathbb{R}^d} dx e^{-\frac{1}{2}\langle x, Ax\rangle + \langle b, x\rangle} x_{j_k}.$$

Then,

$$\int_{\mathbb{R}^d} dx e^{-\frac{1}{2}\langle x, Ax\rangle} x_{j_k} = \frac{\partial}{\partial b_{j_k}} Z_b \bigg|_{b=0}. \tag{1.27}$$

Thus, the correlation function of the theory (1.25) is given by

$$\langle x_{j_1}, \cdots, x_{j_m}\rangle_b = \frac{1}{Z_0} \frac{\partial}{\partial b_{j_1}} \cdots \frac{\partial}{\partial b_{j_m}} Z_b. \tag{1.28}$$

In particular, for $b = 0$ we have that the correlation function of the free theory is,

$$\langle x_{j_1}, \cdots, x_{j_m}\rangle_{free} = \frac{1}{Z_0} \frac{\partial}{\partial b_{j_1}} \cdots \frac{\partial}{\partial b_{j_m}} Z_b \bigg|_{b=0}. \tag{1.29}$$

$\square$

Thus, in order to calculate any correlation function we need to compute the partition function Z_b. By definition, we have that

$$Z_b = \int_{\mathbb{R}^d} dx e^{-\frac{1}{2}\langle x, Ax\rangle + \langle b, x\rangle}.$$

with the change of variable $x \to x - A^{-1}b$, we obtain[Pol04]

$$Z_b = \int_{\mathbb{R}^d} dx e^{-\frac{1}{2}\langle x, Ax\rangle + \frac{1}{2}\langle b, A^{-1}b\rangle} = Z_0 e^{\frac{1}{2}\langle b, A^{-1}b\rangle}.$$

For example, a correlation function of two-points is given by

$$\langle x_{j_1}, x_{j_2}\rangle = \left(A^{-1}\right)_{j_1, j_2} \text{ with } 1 \leq j_1, j_2 \leq d. \tag{1.30}$$

In fact, note that

$$\begin{aligned}
\frac{1}{Z_0} \frac{\partial}{\partial b_{j_2}} e^{\frac{1}{2}\langle b, A^{-1}b\rangle} &= \frac{\partial}{\partial b_{j_2}} \exp\left\{\frac{1}{2}\sum_{j,k} b_j (A^{-1})_{j,k} b_k\right\} \\
&= \frac{1}{2}\left\{\sum_{j,k}\left(\frac{\partial b_j}{\partial b_{j_2}}(A^{-1})_{j,k} b_k + b_j (A^{-1})_{j,k}\frac{\partial b_k}{\partial b_{j_2}}\right)\right\} e^{\frac{1}{2}\langle b, A^{-1}b\rangle} \\
&= \frac{1}{2}\left\{\sum_k (A^{-1})_{j_2,k} b_k + (A^{-1})_{k,j_2} b_k\right\} e^{\frac{1}{2}\langle b, A^{-1}b\rangle}.
\end{aligned}$$

Since A is a symmetric matrix, we have

$$\frac{1}{Z_0}\frac{\partial}{\partial b_{j_2}}e^{\frac{1}{2}\langle b,A^{-1}b\rangle} = \left\{\sum_k (A^{-1})_{j_2,k}b_k\right\} e^{\frac{1}{2}\langle b,A^{-1}b\rangle}.$$

Note that the latter expression corresponds to the correlation function $\langle x_{j_2}\rangle_b$ and if $b=0$ we have that $\langle x_{j_2}\rangle_{free}=0$. Thus, the correlation function of one-point for the free theory is always zero. Now, the correlation function $\langle x_{j_1},x_{j_2}\rangle_b$ is given by

$$\begin{aligned}\langle x_{j_1},x_{j_2}\rangle_b &= \frac{1}{Z_0}\frac{\partial}{\partial b_{j_1}}\frac{\partial}{\partial b_{j_2}}e^{\frac{1}{2}\langle b,A^{-1}b\rangle}\\ &= \left\{\sum_k (A^{-1})_{j_2,k}\frac{\partial b_k}{\partial b_{j_1}}\right\} e^{\frac{1}{2}\langle b,A^{-1}b\rangle} + \left\{\sum_k (A^{-1})_{j_2,k}b_k\right\}\frac{\partial e^{\frac{1}{2}\langle b,A^{-1}b\rangle}}{\partial b_{j_1}}.\end{aligned}$$

Since $\frac{\partial e^{\frac{1}{2}\langle b,A^{-1}b\rangle}}{\partial b_{j_1}} = Z_0\langle x_{j_1}\rangle_b$, we obtain

$$\langle x_{j_1},x_{j_2}\rangle_b = (A^{-1})_{j_2,j_1} + Z_0\left\{\sum_k (A^{-1})_{j_2,k}b_k\right\}\langle x_{j_1}\rangle_b. \tag{1.31}$$

In particular, for $b=0$ we have that the correlation function of two-points of the free theory is given by

$$\langle x_{j_1},x_{j_2}\rangle_{free} = (A^{-1})_{j_1,j_2}. \tag{1.32}$$

Thus, in the free theory the correlation function of two-points depends only on the components of the inverse of the matrix A. The same process can be realized for computing other correlation functions. The correlation functions of k-points is given by the following theorem, *Wick theorem*[Pol04].

Theorem 1.3.1. *Let $1\leq j_1,\cdots,j_n\leq d$ with n a positive integer. Then, in the free theory we have that*

$$\langle x_{j_1},\cdots,x_{j_n}\rangle_{free} = \begin{cases}\sum (A^{-1})_{k_1,k_2}\cdots(A^{-1})_{k_{n-1},k_n} & , \text{ if } n \text{ is even,}\\ 0 & , \text{ if } n \text{ is odd,}\end{cases} \tag{1.33}$$

where the sum is taken over all partitions $\{(k_1,k_2),\cdots,(k_{n-1},k_n)\}$ in pairs of the set $\{j_1,\cdots,j_n\}$.

Thus, we have that in a free theory any correlation function is determined by the components of the inverse of A, whereas the partition function is given by the determinant of A. Those facts are also true in other theories; however, in general A is not a matrix but it is a differential or a pseudo-differential operator. Then, we it is necessary a particular definition of the inverse of that operator and its determinant[PAY03].

1.3.2 Feynman Rules for ϕ^4-theory

The one-point space-time example gives us a general idea to compute the path-integral of a quantum field theory. In particular, for the ϕ^4-theory we have that the operator which corresponds to the matrix A is[10]

$$\Psi = \sum_{\mu=1}^{4} \partial_\mu^2 + m^2 = \frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + m^2. \tag{1.34}$$

Therefore, in order to compute the partition function and the correlation functions of the free theory, i.e taking $\lambda = 0$ in (1.16), we need to define the determinant and the inverse of the pseudo-differential operator Ψ. For the partition function, one expects that

$$Z_0 = \int \mathcal{D}\phi \; e^{iS_0(\phi)} = c\,(\det \Psi)^{-\frac{1}{2}} \,,$$

where $S_0(\phi) = \int_M -\frac{1}{2}\phi(x)\Psi\phi(x)d^4x$, $c \in \mathbb{C}$ is a constant. The $\det \Psi$ is called *functional determinant*[PAY03]. There are different ways to define this determinant, however we won't care about this definition since we want to define just the correlation functions. As we saw in the one-point space-time example, the correlation function depends only on the inverse of the operator Ψ. The common definition of a "inverse" operator Ψ^{-1} is given through the Green function associated to the operator Ψ. In this case, the Green function is given by[Has99]

$$G_\Psi(x-y) = \int_M \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{m^2 - k^2}. \tag{1.35}$$

Thus, the correlation function of two-points of the free theory, according to the Wick theorem, is given by

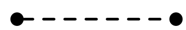
$$\langle \phi(x_1)\phi(x_2) \rangle_0 = \frac{1}{Z_0} \int \mathcal{D}\phi \; e^{iS_0} \phi(x_1)\phi(x_2) = G_\Psi(x_1 - x_2). \tag{1.36}$$

The other correlation functions are also given by the Wick theorem. For example, the 4-point correlation function is

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle_0 = \sum_{all\ pairs} G_\Psi(x_{j_1} - x_{j_2}) G_\Psi(x_{j_3} - x_{j_4})$$
$$= G_\Psi(x_1-x_2)G_\Psi(x_3-x_4) + G_\Psi(x_1-x_3)G_\Psi(x_2-x_4) + G_\Psi(x_1-x_4)G_\Psi(x_3-x_2). \tag{1.37}$$

[10] In a field theory, the classical solution of the principle of least action is given by the action S up to a shift of the action by a section f, i.e $\delta S = \delta S'$ being $S'(\phi) = S(\phi) + f$. In particular, the dynamic term $(\partial_\mu\phi)^2$ in the Lagrangian of the ϕ^4-theory can be written as $\sum_{\mu=1}^4 \partial_\mu(\phi\partial_\mu) - \phi\left(\sum_{\mu=1}^4 \partial_\mu^2\right)\phi$. Then, the Lagrangian $\mathcal{L} = -\frac{1}{2}\phi\left(\sum_{\mu=1}^4 \partial_\mu^2 + m^2\right)\phi + V(\phi)$ gives the same classical solution to the principle of least action.

Therefore, we can see that the expressions of the correlation function in the free theory are only an algebraic combinations of the correlation function of two-points. A brilliant form to express this algebraic combination was given by Richard Feynman[MD95] and consist in assigning to the two-points correlation function of each field in the theory, a diagrammatic representation called *Feynman diagram*. Each assignment is called a *Feynman rule*. In the case of the ϕ^4-theory, we have that

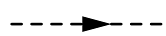
x  y

$$\rightarrow G_\Psi(x-y). \tag{1.38}$$

The above Feynman diagram can be interpreted as a particle, described by the scalar field which is represented with the dashed line, that is propagated from a point x to a point y in the space-time. It is common to give the Feynman rules in terms of the Fourier transformation, $\tilde{G}_\Psi(k)$, of the propagator G_Ψ. From Equation (1.35), we obtain that

$$\tilde{G}_\Psi(k) = \frac{1}{m^2 - k^2}. \tag{1.39}$$

Thus, the Feynman rule in the space of the k's, called *momentum space*[11], is written as follows


k

$$\rightarrow \frac{1}{m^2 - k^2} \tag{1.40}$$

The direction of the arrow in the Feynman diagram means that a **particle** is propagated with a total momentum k. Whereas, the inverse arrow means that an anti-particle is propagated with momentum $-k$.

1.3.3 Feynman Diagrams for an interacting theory

So far, we have shown how to compute the correlation functions for the free theory. In the interacting theory the problem of computing the correlation functions is not so different. We can compute the correlation functions of the interacting theory in terms of the correlation functions of the free theory. The method consists of considering the interacting potential $V(\phi)$ as a perturbation of the free theory, i.e we consider that the parameter λ in Equation (1.16) is small. In this approximation using a Taylor expansion, we have that

[11]More exactly *four-momentum space*

$$e^{iS(\phi)} = e^{iS_0} e^{-i\frac{\lambda}{4!}\int_M d^4z\phi^4(z)} \approx e^{iS_0}\left[1 - i\frac{\lambda}{4!}\int_M d^4z\phi^4(z) - \frac{\lambda^2}{(4!)^2}\left(\int_M d^4z\phi^4(z)\right)^2 + \cdots\right] \tag{1.41}$$

Let us consider just the first order of this approximation. The partition function of the interacting theory is given by

$$Z = \int \mathcal{D}\phi\ e^{iS(\phi)} = \int \mathcal{D}\phi e^{iS_0(\phi)} - i\frac{\lambda}{4!}\int_M d^4z \int \mathcal{D}\phi\ e^{iS_0(\phi)}\phi^4(z). \tag{1.42}$$

From the definition of the correlation function of 4-points in the free theory (Equations (1.37,1.36)), we have that

$$Z = Z_0\left(1 - i\frac{\lambda}{4!}\int_M d^4z\langle\phi(z)\phi(z)\phi(z)\phi(z)\rangle_0\right). \tag{1.43}$$

By the Wick theorem, we obtain.

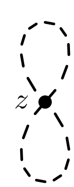
$$\frac{Z}{Z_0} = \left(1 - i\frac{\lambda}{4!}\int_M d^4z \sum_{allpairs} G_\Psi(z-z)G_\Psi(z-z)\right) = 1 - 3i\frac{\lambda}{4!}\int_M d^4z G_\Psi(0)G_\Psi(0). \tag{1.44}$$

The factor 3 on the right hand side of the equation becomes from the sum over all partitions in pairs of the four points in the correlation function (take $x_1 = x_2 = x_3 = x_4 = z$ in (1.37)). Moreover, we must note that the integral in the right hand side of the latter equation is divergent, since $\int_M d^4z G_\Psi(0)^2 = G_\Psi(0)^2 \int_M d^4z \to \infty$. However we will see below that this divergent quantity does not appear in the correlation functions. According to the Feynman rule, the propagator $G_\Psi(0) = G_\Psi(z-z)$ corresponds to the diagram

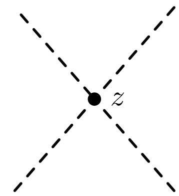
$$\to G_\Psi(0). \tag{1.45}$$

This kind of diagram is called *loop*, in particular we call *self-loop* a loop over one point. Therefore, the diagrammatic expression of the partition function for the interacting theory is

$$Z = Z_0 \cdot \left(1 + 3 \cdot \right), \tag{1.46}$$



where we have included the following Feynman rule to express the integration over the point z in each vertex.



$$\rightarrow -i\frac{\lambda}{4!}\int_M d^4z. \tag{1.47}$$

Similarly, the correlation function of the interacting theory is given by

$$\langle\phi(x)\phi(y)\rangle_\lambda = \frac{1}{Z}\left(\int \mathcal{D}(\phi)\ e^{iS_0}\phi(x)\phi(y) + i\frac{\lambda}{4!}\int_M d^4z \int \mathcal{D}(\phi)\ e^{iS_0}\phi(x)\phi(y)\phi^4(z)\right). \tag{1.48}$$

In terms of the correlation functions of the free theory, we have

$$\langle\phi(x)\phi(y)\rangle_\lambda = \frac{Z_0}{Z}\left[\langle\phi(x)\phi(y)\rangle_0 + \int_M d^4z\langle\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)\rangle_0\right]. \tag{1.49}$$

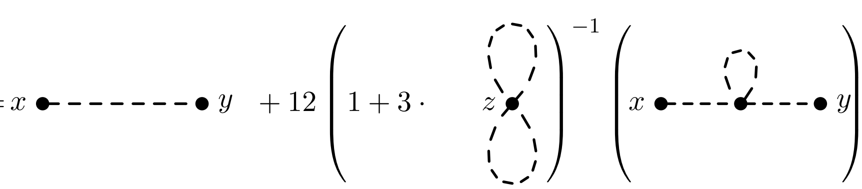
By the Wick theorem, we have

$$\begin{aligned}\langle\phi(x)\phi(y)\rangle_\lambda = \frac{Z_0}{Z}\Bigg[&G_\Psi(x-y) - 3i\frac{\lambda}{4!}G_\Psi(x-y)\int_M d^4zG_\Psi(z-z)G_\Psi(z-z)\\ &-12i\frac{\lambda}{4!}\int_M d^4zG_\Psi(x-z)G_\Psi(y-z)G_\Psi(z-z)\Bigg].\end{aligned}$$

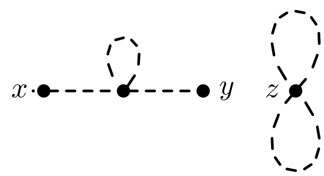
From Equation (1.44) we obtain

$$\begin{aligned}\langle\phi(x)\phi(y)\rangle_\lambda &= \frac{Z_0}{Z}\left[G_\Psi(x-y)\left(\frac{Z}{Z_0}\right) + -12i\frac{\lambda}{4!}\int_M d^4zG_\Psi(x-z)G_\Psi(y-z)G_\Psi(z-z)\right]\\ &= G_\Psi(x-y) - \frac{12Z_0}{Z}i\frac{\lambda}{4!}\int_M d^4zG_\Psi(x-z)G_\Psi(y-z)G_\Psi(z-z).\end{aligned}$$

Using the Feynman rules (1.38,1.47), we can express the correlation function of the interacting theory diagrammatically as

$\langle\phi(x)\phi(y)\rangle_\lambda =$  (1.50)

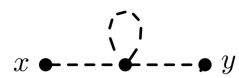
The second term in the latter expression is problematic since, as we said before, the value of the two self-loop over the point z is divergent. Nevertheless, if we consider the following order in the Taylor expansion of Equation (1.41), we obtain diagrams like

$12 \cdot 3$ 

. (1.51)

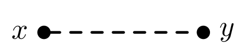
Therefore, the expression of the two-point correlation function is

$\langle\phi(x)\phi(y)\rangle_\lambda =$ 

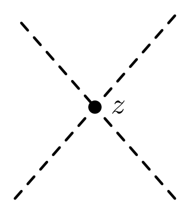
$+12$ 

(1.52)

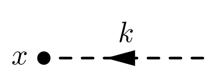
Diagrams like (1.51) are called *disconnected* diagrams and one can show that this kind of diagrams doesn't appear in the diagrammatic expression of any correlation function[MD95], i.e the diagram corresponding to a correlation function is a linear combination of connected diagrams. The kind of possible diagrams that can be formed in the ϕ^4-theory are restricted by the following Feynman rules

1) For each propagator, 

$\rightarrow G_\Psi(x-y)$,

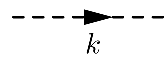
2) For each vertex, 

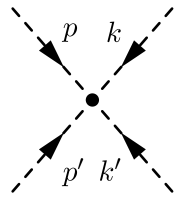
$\rightarrow -i\dfrac{\lambda}{4!}\displaystyle\int_M d^4z$,

3) For each external line, 

$\rightarrow e^{-k\cdot x}$.

The last Feynman rule is associated to scattering between particles. The external line is interpreted as a particle that comes with a total momentum k from a far point and interact with a particle in the point x. The corresponding Feynman rules in the momentum space are the following[MD95],

1) For each propagator,  $\rightarrow \frac{1}{m^2-k^2}$,

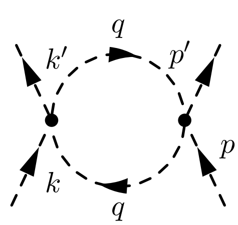
2) For each vertex,  $\rightarrow -i\frac{\lambda}{4!}$,

3) Impose momentum conservation in each vertex, i.e $p + k = p' + k$ in the rule 2,

4) For each external line,  $\rightarrow 1$,

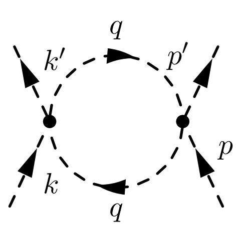
5) Integrate over each undetermined loop momentum, $\int \frac{d^4q}{(2\pi)^4}$.

The last rule makes sense in diagrams like,



(1.53)

where by momentum conservation the momentum q in the internal loop can take any value. According to the Feynman rules, we have that



$$\rightarrow \left(\frac{-i\lambda}{4!}\right)^2 \int \frac{d^4q}{(2\pi)^4}\frac{1}{m^2-q^2}\frac{1}{m^2-(q+t)^2}, \qquad (1.54)$$

being $t = k' - k = p - p'$ is the total transferred momentum. This diagram is one of the more important diagrams in the ϕ^4-theory, but it is divergent[MD95]. The problem with this integral appears when $q^2 \rightarrow \infty$, i.e the problem arises from high energy loops[12].

[12]Therefore, this kind of divergences are called *ultraviolet* divergences. In other field theories there are other kind of divergences, called *infrared divergences*, that appear as $q^2 \rightarrow 0$, i.e for low energy loops.

However, this divergence can be eliminated if the original theory is renormalized to a theory with just low energy fields. We regard this fact in Appendix B. In this perturbative analysis, the process of renormalization becomes effectively in a *regularization* of the integral (1.54). There are many ways to define this regularization; a canonical method to do this is called *dimensional regularization*. This method consists in evaluating the loop integral in a d-dimensional space-time and thereupon taking finite part of the limit as $d \to 4$.

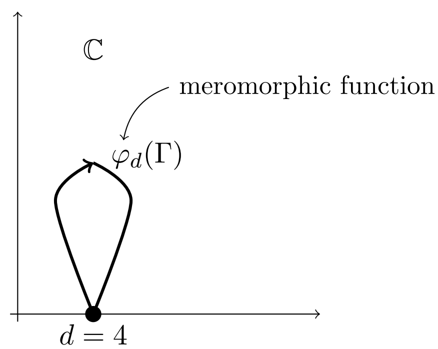


FIGURE 1.5. Scheme of the dimensional regularization

Let φ_d be the map that assigns the regularized value to each Feynman diagram given by the dimensional regularization, let H_{ϕ^4} be the set of all possible diagrams, according to the Feynman rules, of the ϕ^4-theory and $\mathcal{A}_d$ be the algebra of meromorphic functions. Then, $\varphi_d : H_{\phi^4} \to \mathcal{A}_d$. The objective of the following chapters is to describe formally how we can take the finite part of the possible divergent function $\varphi_d(\Gamma)$ for any Feynman diagram $\Gamma \in H_{\phi^4}$.

CHAPTER 2

Hopf Algebra and Feynman Graphs

In the latter chapter, we saw that the relevant observables in a quantum field theory are expressed as a linear combination of some kind of Feynman diagrams. In this chapter, we give an abstract definition of Feynman diagrams without leaving aside the physical intuition of the definition of the Feynman-path integral. Besides, we give the basic notions about the *Hopf algebra* structure which will be useful to develop the theory of Hopf Algebra Renormalization.

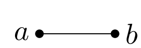
2.1 Graph Theory of Feynman Diagrams

The pictorial representation of the correlation functions and the partition function in Chapter 1 suggests to use graph theory to understand abstractly the Feynman diagram. According to the basic theory[Ros12] a **graph** G consists in a pair (V, E) where V is a nonempty set of **vertices** and a set E of **edges**. Each edge in a diagram is associated one or two vertices, called **endpoints** of the edge.

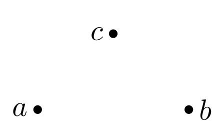
Each graph has a pictorial representation where each vertex is a represented as a single point while each edge is represented as a straight line that joins its endpoints.

Example 2.1.1.

1. Let $V = \{a, b\}$ and $E = \{\{a, b\}\}$. Then, the corresponding pictorial representation of the graph $G = (V, E)$ consist in two single points and one line.



2. Let $V = \{a, b, c\}$ and $E = \emptyset$. Since the set of edges is empty, then the graph $G = (\{a, b, c\}, \emptyset)$ consists in just three single points.



There are two special kinds of graphs to consider in this work, the directed and undirected graphs. A **directed graph** G is a graph (V, E) where

$$E \subseteq \{\bar{e} = (e, e, \cdots, e, 0, \cdots) : e \in E^s,\ (\bar{e})_j = 0 \text{ for all } j \geq n \text{ for some positive integer } n\},$$

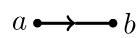
being $E^s := \{(a, b) \in V \times V : a \neq b\}$ the set of **single edges** of the directed graph. For each $\bar{e} \in E$ it is associated an ordered pair (a, b) (its single edge), a is said the start point of the edge and b is said its end point. An **undirected graph** is a graph where

$$E \subseteq \{\bar{e} = (e, e, \cdots, e, 0, \cdots) : e \in E^s,\ (\bar{e})_j = 0 \text{ for all } j \geq n \text{ for some positive integer } n\},$$

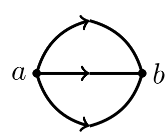
being $E^s = \{e \in X : X \subset V,\ |X| = 2\}$. Unlike to a directed graph, the edges in an undirected graph are not tuples of ordered pairs, i.e the single edge $\{a, b\}$ is the same that the edge $\{b, a\}$. We will use this kind of graphs to represents the Feynman diagrams in the *real space* while the directed graph will be used to describe the Feynman diagram in the *momentum space*.

Example 2.1.2.

1. Let $V = \{a, b\}$ and $E = \{(a, b)\}$. The edge (a, b) is drawing as an arrow that starts at a and ends at b.



2. In the definition of directed graphs an edge corresponds to an infinite tuple with finite support, say $n \geq 1$. When $n > 1$, we say that the edge $\overline{(a, b)}$ is an edge of *multiplicity* n. For example, the following diagram has a single edge of multiplicity 3.



The multiplicity of an edge in an undirected graph is defined in the same way.

2.1.1 Feynman Graphs

The pictorial representation of a Feynman diagram associated to a quantum field theory does not correspond to a graph in the above sense. In fact, a Feynman diagram has vertices and edges but there are diagrams with two kinds of edge (see Figure [2.1]). An edge is associated to the propagation of a field between two points which corresponds to the usual notion of an edge in a graph. Another edge, associated to a scattering process, corresponds to a new kind of edge, an *external edge*.

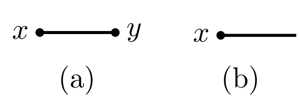


FIGURE 2.1. Two kind of edges in a Feynman diagram. (a) Edge associated to the propagation of a field from a point x to a point y in the space-time, (b) External edge associated to a scattering process.

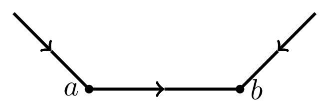
In order to consider this new kind of edges, we define an extended notion of a graph. A **Feynman graph** (resp. **directed Feynman graph**) is a tuple $\Gamma = (V, E_{int}, E_{ex})$ where (V, E_{int}) is an undirected graph (resp. directed graph) and the set E_{ext} of the **external edges** is such that

$$E_{ext} \subseteq \{\bar{e} = (e, e, \cdots, e, 0, \cdots) : e \in E^s_{ext},\ (\bar{e})_j = 0 \text{ for all } j \geq n \text{ for some positive integer } n\},$$

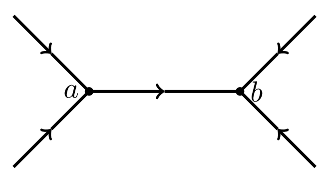
being $E^s_{ext} := \{e \in X : X \subset V,\ |X| = 1\}$ (resp. $E^s_{ext} := V$). Each $e \in E_{int^s}$(resp. $e \in E^s_{ext}$) is called a **single internal edge** (resp. a **single external edge**) of the graph.

Example 2.1.3.

1. The pictorial representation of an external edge in a Feynman graph (resp. directed Feynman graph) corresponds to a simple segment of a line (resp. arrow) that end at the corresponding vertices (resp. vertex). For $V = \{a, b\}$, $E_{int} = \{(a, b)\}$ and $E_{ext} = \{a, b\}$,



2. As in the case of the edges in a directed graph, there is a notion of multiplicity of an internal or external edge of a Feynman graph. For example, for an external edge this means that a given vertex can be joined with n-external edges, for some integer $n \geq 1$. When this happens, we say that the external edge is the *multiplicity* n. For example, the graph $(\{a, b\}, \{(a, b)\}, \{(a, a), (b, b)\})$ has two external edges of multiplicity 2.

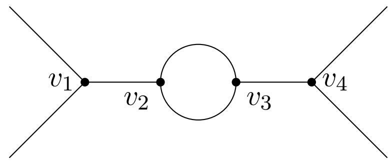


We say that a Feynman graph has **loops** if there is almost an internal edge of multiplicity $n > 1$ (the graph in part 2 of Example (2.1.2) presents two loops). Futhermore, the **loop number** L of a Feynman graph Γ is given by

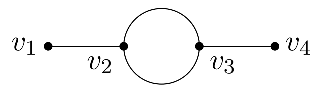
$$L(\Gamma) := I(\Gamma) - V(\Gamma) + 1, \tag{2.1}$$

where $V(\Gamma)$ is the number of vertices in Γ and $I(\Gamma)$ is the number of SINGLE INTERNAL EDGES in Γ. A Feynman graph without loops, i.e with $L = 0$, is called a **simple Feynman graph**. A Feynman graph $\gamma = (W, F_{int}, F_{ext})$ is called a **subgraph** of a Feynman graph $\Gamma = (V, E_{int}, E_{ext})$ (denoted by $\gamma \subseteq \Gamma$) if $W \subseteq V$, $F_{int} \subseteq E_{int}$ and $F_{ext} \subseteq E_{ext}$.

Example 2.1.4. Let Γ be the non-simple Feynman graph,

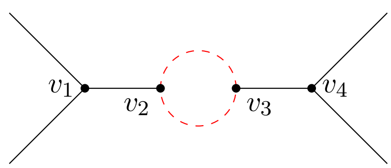


Let $W := V$, $F_{int} := E_{int}$ and $F_{ext} := \emptyset$, then $\gamma = (W, F_{int}, F_{ext})$ is a subgraph of Γ,



The notion of connectedness of a diagram is important as we mentioned in Chapter 1. Let m be a non-negative integer and Γ be a Feynman graph (resp. directed Feynman graph). A **path** of length m from a vertex u to a vertex v in Γ is a sequence of single edges $e_1, \cdots, e_m$ of Γ for which there exist a sequence $u = x_1, \cdots, x_n = v$ of vertices such that each edge e_j is associated to (x_{j-1}, x_j) (resp. it starts at u and it ends at v) for $j = 2, \cdots, n$. A **circuit** of length m is a path of length m that starts and ends at the same point.

Example 2.1.5. In Example 2.1.4, the internal loop is a circuit of length 2

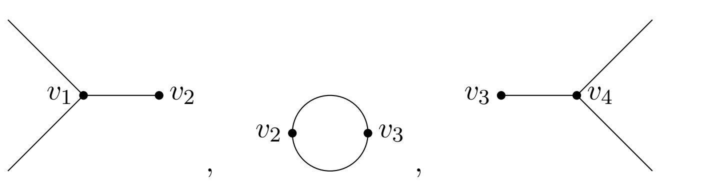


A Feynman graph is called **connected**, if there exists a path between every pair of distinct vertices of the Feynman graph.

2.1.2 Operation on Feynman Graph

There are certain operations that we can do over Feynamn graphs which will be important to define an algebraic structure over Feynman diagrams. For example, any Feynman graph can be seen as union of other graphs (see Example (2.1.6)). Given a collection $\{(V^{(j)}, E^{(j)}_{int}, E^{(j)}_{ext})\}_{j\in J}$ of Feynman graphs, the Feynman graph $\Gamma := \left(\bigcup_{j\in J} V^{(j)}, \bigcup_{j\in J} E^{(j)}_{int}, \bigcup_{j\in J} E^{(j)}_{ext}\right)$ is called the **union of Feynman graphs**.

Example 2.1.6. In Example (2.1.4), Γ is the union of the following Feynman graphs,

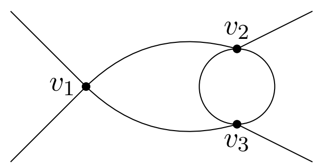


We can also obtain other graphs from a given Feynman graph. Let $\Gamma = (V, E_{int}, E_{ext})$ be a Feynman graph and v be a fixed vertex of Γ. The graph obtained by removing the vertex v and all edges (internal and external) associated to that vertex, denoted by $\Gamma - v$, is a subgraph of Γ.

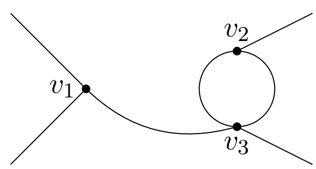
Example 2.1.7. In Example (2.1.6), the first subgraph is obtained by removing the vertices v_3 and v_4 of the Feynman graph Γ of Example (2.1.4).

We can also obtain other graphs by removing edges from a given Feynman graph. Let $\Gamma = (V, E_{int}, E_{ext})$ be a Feynman graph and let e be a single edge (internal or external) of Γ. The graph obtained by removing the single edge e, denoted by $\Gamma - e$, is a subgraph of Γ.

Example 2.1.8. Let Γ be the following non-simple Feynman graph,



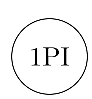
The subgraph,



is obtained by removing the single internal edge (v_1, v_2).

We mentioned in the introduction of this chapter that the observables in a quantum field theory are expressed as a linear combination of some kind of Feynman diagrams more exactly some kind of connected Feynamn diagrams. A **one-particle irreducible** (1PI) Feynman Graph is a connected Feynman graph $\Gamma = (V, E_{int}, E_{ext})$ which satisfy the following condition,

- The subgraph $\Gamma_{\int} := (V, E_{\int}, \emptyset)$ is such that any subgraph obtained by removing any internal single edge is also connected. The graph (V, E_{int}) associated to this kind of graphs is represented by



Note that by definition of 1PI Feynman graphs with only one vertex are 1PI, since those graphs have no internal edges.

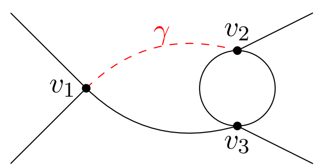
Example 2.1.9.

1. In Example (2.1.8), the Feynman graph Γ is 1PI.
2. In Example (2.1.4), the Feynman graph Γ is 1PI. Because, the any subgraph of internal edges obtained by removing any internal single edge is connected.

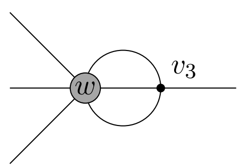
The following operation is important for us as we will see in Chapter 3. Let $\Gamma = (V, E_{int}, E_{ext})$ be a Feynman graph, e be a single internal edge of Γ with endpoints $u, v \in V$. The graph obtained by removing the edge e and merge the endpoints u, v into a *new* single vertex is called a **single edge contracted graph** of Γ. Given a subgraph γ of internal edges of Γ, the **contracted graph** Γ/γ is the Feynman graph obtained by contracting all edges of γ into a new single vertex.

Example 2.1.10.

1. In Example (2.1.8), let $\gamma = (\{v_1, v_2\}, \{\{v_1, v_2\}\}, \emptyset)$ (dashed edge),



The contracted graph Γ/γ corresponds to the Feynman graph,



2. Other contracted graphs Γ/γ of the Feynman graph Γ in Example (2.1.8) are given in the following table.

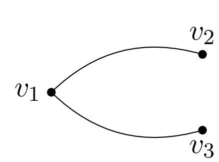
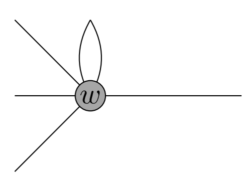
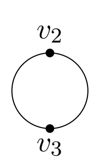
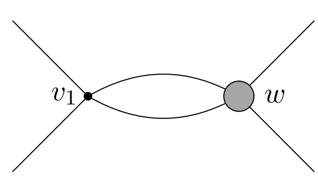
γ	Γ/γ
	
	

TABLE 2.1. Some contracted graphs of the Feynman graph Γ in Example (2.1.8).

The following proposition give us a simple way to calculate the contraction over subgraphs.

Proposition 2.1.1. Let Γ be a Feynman graph. Then, the map $\gamma \mapsto \tilde{\gamma} = \gamma/\delta$ from subgraphs of Γ containing δ onto subgraphs of Γ/δ, given by contracting δ, is a bijection. Moreover, we have the following *transitive contracting property*

$$\Gamma/\gamma = (\Gamma/\delta)/(\gamma/\delta).$$

Proof. Let Γ be a graph and δ be a subgraph of internal edges of Γ.

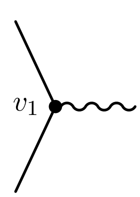
- (onto) Let $\tilde{\gamma}$ be a subgraph of Γ/δ, let v_δ be the vertex corresponding to the contracting of the vertices of δ and E_δ be the set of all edges that have endpoints in vertices of δ. Take $\gamma = \gamma' \cup \delta$ where γ' is the graph obtained by removing the vertex v_δ from $\tilde{\gamma}$ and adding the edges in E_δ. Note first that δ is a subgraph of γ and second that, from the definition of the contraction of a graph, $\gamma/\delta = \tilde{\gamma}$.
- (one-to-one) Let γ_1, γ_2 be subgraphs of Γ which contain δ and such that $\gamma_1/\delta = \gamma_2/\delta$. We shall show that $\gamma_1 = \gamma_2$, i.e. that γ_1 and γ_2 have the same edges and vertex. Like $\gamma_1/\delta = \gamma_2/\delta$ then all edges and vertices in γ_1 which are not in δ are edges and vertices of γ_2 and since both γ_1 and γ_2 contain δ then $\gamma_1 = \gamma_2$.

□

2.1.3 Theories and Feynman Diagrams

So far we have drawn the edges of a Feynman graph as a solid line. However, we saw in Chapter 1 that the *Feynman rules* in a quantum field theory give different types of edges, like the wavy line which represents the photon propagator (see example 2.1.11). Thus, a **Feynman diagram** is defined as a tuple $\Gamma := (V, E_{int} := \bigcup_{j=1}^m E_{int}^{(j)}, E_{ext} := \bigcup_{j=1}^m E_{ext}^{(j)})$ where $(V, E_{int}^{(j)}, E_{ext}^{j})$ is a Feynman graph for each $j = 1, \cdots, m$. The collection $\{E_{int}^{(j)}\}_{j=1}^m$ (resp. $\{E_{ext}^{(j)}\}_{j=1}^m$) is the collection of the **types** of internal edges (resp. external edges) in Γ.

Example 2.1.11. The Feynman diagram,



.

has two type of external edges (solid and wavy lines).

Let Γ be a Feynman diagram with m type of edges and e be a single edge of Γ, then $\bar{e} \in E^j_{int}$ for some $j = 1, \cdots m$. The number j, denoted by $\tau(e)$, is called the **type of the edge** e.

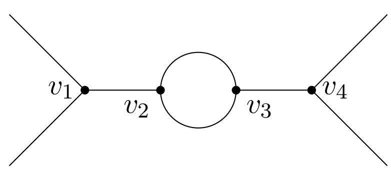
Note that a vertex of a given Feynman diagram can be the start or end point of several types of single edges. For any vertex v of a Feynman diagram (with m type of edges), the **star** of v, denoted by $st(v)$, is the set of all edges (internal and external) that start or end at v. The **valence** of the vertex is given by the cardinal of $st(v)$. The **type of a vertex** is a tuple $T(v) = (n_1, \cdots, n_m)$ of positive integers numbers where n_j is the number of edges of type j in $st(v)$ (with *self-loops* counted twice).

Example 2.1.12. In Example (2.1.11), the vertex v_1 is of valence two (bivalent) and corresponds to a vertex of type $T(v_1) = (2, 1)$.

The type of vertex in a diagram is very important. For instance, we saw in Chapter 1 that the Feynman rules in a quantum field theory define a particular type of vertices which come from the interaction term of the Lagrangian of the physical theory. We can give an abstract definition a theory keeping this fact in mind. A **theory** of Feynman diagrams is a set $\mathcal{T} = \{T_1, \cdots, T_k\}$, being k a positive integer, of m-tuples of positive integer. We say that a set of Feynman diagrams, $\mathcal{F}$, is **described** by a theory $\mathcal{T}$ if for each $\Gamma \in \mathcal{F}$ and each vertex v of Γ, $T(v) \in \mathcal{T}$.

Example 2.1.13.

1. ϕ^4-THEORY: Let $\mathcal{F} := \{\Gamma : \Gamma \text{ is a Feynman graph}\}$ and let $\mathcal{T}_{\phi^4} := \{4\}$. This theory describes all Feynman graphs where each vertex v has only four edges (internal or external) in $st(v)$. Note that the Feynman graph



is not described by the theory $\mathcal{T}_{\phi^4}$, because the star of the vertex v_2 has three edges. Thus, the set $\mathcal{F}$ is not described by $\mathcal{T}_{\phi^4}$.

2. QED: Let $\mathcal{F} := \{\Gamma : \Gamma \text{ is a Feynman diagram with two types of edges}\}$ and let $\mathcal{T}_{QED} := \{(2, 1)\}$. This theory describes all Feynman diagrams with vertex of type $(2, 1)$ (see Example (2.1.11)).

2.2 Algebras and Co-algebras

In this section we give the basic definitions about some special abstract structures without care about the classification of them or another usual and important questions in the modern algebra. More details about of this structure can be consulted in [Kas95, Man06].

2.2.1 Algebras

Let $\mathbb{K}$ be a field of characteristic zero. A $\mathbb{K}$-**algebra** is a $\mathbb{K}$-vector space $\mathcal{A}$ together with a bilinear map, called *product*, $p : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$, which evaluation is denoted by $p(a\otimes b) = a\cdot b$ for $a, b \in \mathcal{A}$, such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "p\otimes I"] \arrow[d, "I\otimes p"'] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "p"] \\
\mathcal{A}\otimes\mathcal{A} \arrow[r, "p"'] & \mathcal{A}
\end{tikzcd}$$

where $I : \mathcal{A} \to \mathcal{A}$ is the identity map of $\mathcal{A}$. The commutativity of the latter diagram means that for each $a, b, c \in \mathcal{A}$ we have

$$p((p\otimes I)(a\otimes b\otimes c)) = p((I\otimes p)(a\otimes b\otimes c)),$$

i.e, $(a\cdot b)\cdot c = p(p(a\otimes b)\otimes c) = p(a\otimes p(b\otimes c)) = a\cdot(b\cdot c)$[1]. The algebra $\mathcal{A}$ is **unital** if there is a linear map, called *unit* of $\mathcal{A}$, $u : \mathbb{K} \to \mathcal{A}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathbb{K}\otimes\mathcal{A} \arrow[r, "u\otimes I"] \arrow[dr, "\kappa_2"'] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "p"] & \mathcal{A}\otimes\mathbb{K} \arrow[l, "I\otimes u"'] \arrow[dl, "\kappa_1"] \\
& \mathcal{A} &
\end{tikzcd}$$

where the map $\kappa_1 : \mathcal{A}\otimes\mathbb{K} \to \mathcal{A}$ (resp. $\kappa_2 : \mathbb{K}\otimes\mathcal{A} \to \mathcal{A}$) is defined by $\kappa_1(a\otimes\alpha) = a$ (resp. $\kappa_2(\alpha\otimes a) = a$) for each $a \in \mathcal{C}$ and $\alpha \in \mathbb{K}$. The fact that the latter diagram commutes implies that, in particular, for each $a \in \mathcal{A}$ and $1_{\mathbb{K}} \in \mathbb{K}$

$$p((u\otimes I)(1_{\mathbb{K}}\otimes a)) = \kappa_2(1_{\mathbb{K}}\otimes a) = a = \kappa_1(a\otimes 1_{\mathbb{K}}) = p((I\otimes u)(a\otimes 1_{\mathbb{K}})), \tag{2.2}$$

i.e, $u(1_{\mathbb{K}})\cdot a = p(u(1_{\mathbb{K}})\otimes a) = p(a\otimes u(1_{\mathbb{K}})) = a\cdot u(1_{\mathbb{K}}) = a$. This means that $1_{\mathcal{A}} := u(1_{\mathbb{K}})$ is the unit of the algebra in the normal sense.

Example 2.2.1.

1. Let $(G,\cdot)$ be a group and $\mathbb{K}G$ be the *vector space freely generated* by G, i.e. the set of formal $\mathbb{K}$-linear combinations of elements in G. The product on G is extended uniquely to a bilinear map $p : \mathbb{K}G \times \mathbb{K}G \to \mathbb{K}G$ by the distributive law

$$kg\cdot(k_1g_1 + k_2g_2) = k_1k(g\cdot g_1) + k_2k(g\cdot g_2),$$

[1] This corresponds to the associativity of the product in the usual definition of an algebra.

for $g, g_1, g_2 \in G$ and $k, k_1, k_2 \in \mathbb{K}$. The unit map is given by the neutral element e of G, $u(k) = ke$ for all $k \in \mathbb{K}$.

2. Let $\mathcal{F}$ be a set of Feynman Graphs, $V_{\mathcal{F}}$ be the vector space freely generated by $\mathcal{F}$. Then, the *free algebra generated* by $V_{\mathcal{F}}$, i.e. the set of formal products of elements in $V_{\mathcal{F}}$ which satisfy the distributive law is a $\mathbb{K}$-algebra with the unit map given by the empty graph $\Gamma_{\emptyset} = (V = \emptyset, E_{int} = \emptyset, E_{ext} = \emptyset)$, $u(k) = k\Gamma_{\emptyset}$ for all $k \in \mathbb{K}$.

The usual notion of a *commutative algebra* is given as follows. Let $\tau_{\mathcal{A}} = \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be the map defined by $\tau(a \otimes b) = b \otimes a$, called the **flip** map. A $\mathbb{K}$-algebra is called **commutative** if $p \circ \tau_{\mathcal{A}} = p$, that is $a \cdot b = p(a \otimes b) = p(b \otimes a) = b \cdot a$ for each $a, b \in \mathcal{A}$.

2.2.2 Co-algebras

A dual structure of an algebra can be defined as follows. A $\mathbb{K}$**-coalgebra** is a $\mathbb{K}$-vector space $\mathcal{C}$ together with a bilinear map, called *coproduct*, $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} & \mathcal{C} \otimes \mathcal{C} \arrow[l, "\Delta \otimes I"'] \\
\mathcal{C} \otimes \mathcal{C} \arrow[u, "I \otimes \Delta"] & \mathcal{C} \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

The commutativity of the latter diagram, i.e. $(I \otimes \Delta) \circ \Delta = (\Delta \otimes I) \circ \Delta$ is called **co-associativity**. The "associativity" in a coalgebra is a kind of associativity of the coproduct with respect to the tensor product. In order to express more clearly this fact it is useful write the coproduct as

$$\Delta(c) = \sum_{(c)} c_1 \otimes c_2. \tag{2.3}$$

This notation is called the **Sweedler's notation**. The sum in the Sweedler's notation is taking over all pairs $c_1, c_2 \in \mathcal{C}$ that represent $\Delta(c)$ in the tensor product $\mathcal{C} \otimes \mathcal{C}$[2]. In the Sweedler's notation, the coassociativity is written as

[2]By definition of the coproduct, for each $c \in \mathcal{C}$, $\Delta(c) \in \mathcal{C} \otimes \mathcal{C}$, then $\Delta(c) = \left(\sum_{j=1}^{n} c_1^{(j)}\right) \otimes \left(\sum_{k=1}^{m} c_2^{(k)}\right)$ where $c_1^j, c_2^k \in \mathcal{C}$ for each $j = 1, \cdots, n$ and $k = 1, \cdots, m$. The Sweedler's notation is a short notation for the coproduct.

$$\begin{aligned}
(\Delta \otimes I)(\Delta(c)) &= (I \otimes \Delta)(\Delta(c)) \\
\sum_{(c)} (\Delta \otimes I)(c_1 \otimes c_2) &= \sum_{(c)} (I \otimes \Delta)(c_1 \otimes c_2) \\
\sum_{(c)} \Delta(c_1) \otimes c_2 &= \sum_{(c)} c_1 \otimes \Delta(c_2).
\end{aligned}$$

A coalgebra $\mathcal{C}$ is called **co-unital** if there is a linear map, called *counit*, $\varepsilon : \mathcal{C} \to \mathbb{K}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{K} \otimes \mathcal{C} & \mathcal{C} \otimes \mathcal{C} \arrow[l, "\varepsilon \otimes I"'] \arrow[r, "I \otimes \varepsilon"] & \mathcal{C} \otimes \mathbb{K} \\
& \mathcal{C} \arrow[ul, "\mathrm{II}_2"] \arrow[u, "\Delta"] \arrow[ur, "\mathrm{II}_1"'] &
\end{tikzcd}$$

The map $\mathrm{II}_1 : \mathcal{C} \to \mathcal{C} \otimes \mathbb{K}$ (resp. $\mathrm{II}_2 : \mathcal{C} \to \mathbb{K} \otimes \mathcal{C}$) is defined by $\mathrm{II}_1(c) = c \otimes 1_{\mathbb{K}}$ (resp. $\mathrm{II}_2(c) = 1_{\mathbb{K}} \otimes c$) for each $c \in \mathcal{C}$.

Example 2.2.2. Let G be a group and $\mathbb{K}G$ be the group algebra over $\mathbb{K}$ given in Example (2.2.1). The algebra $\mathbb{K}G$ is also a coalgebra with the coproduct $\Delta : \mathbb{K}G \to \mathbb{K}G \otimes \mathbb{K}G$ and counit $\varepsilon : \mathbb{K}G \to \mathbb{K}$ given by

$$\begin{aligned}
\Delta\left(\sum_{i=1}^{n} \lambda_i g_i\right) &:= \sum_{i=1}^{n} (\lambda_i g_i) \otimes g_i, \\
\varepsilon\left(\sum_{i=1}^{n} \lambda_i g_i\right) &:= \sum_{i=1}^{n} \lambda_i,
\end{aligned}$$

where $\lambda_i \in \mathbb{K}$ and $g_i \in G$ for each $i = 1, \cdots, n$.

The commutativity in an algebra can be extended to a coalgebra in the same way as the associativity of an algebra. A $\mathbb{K}$-coalgebra is called **cocommutative** if $\tau_{\mathcal{C}} \circ \Delta = \Delta$. In Sweedler's notation, it is written as

$$\tau_{\mathcal{C}}(\Delta(c)) = \tau_{\mathcal{C}}\left(\sum_{(c)} c_1 \otimes c_2\right) = \sum_{(c)} c_2 \otimes c_1 = \sum_{(c)} c_1 \otimes c_2 = \Delta(c). \tag{2.4}$$

Thus the cocommutativity corresponds to some kind of commutativity with respect to the tensor product. We said at the beginning of this subsection that algebra and coalgebra structures are *duals* in some sense. The following proposition show that the coalgebra structure corresponds effectively to a dual structure of an algebra [Man06, Section 1.3].

Proposition 2.2.1. Let $\mathcal{C}$ be a unital $\mathbb{K}$-coalgebra. Then, the linear dual $\mathcal{C}^*$ is a unital $\mathbb{K}$-algebra being the product (resp. unit map) the transpose of the coproduct ${}^t\Delta : \mathcal{C}^* \otimes \mathcal{C}^* \to \mathcal{C}^*$ (resp. of the counity ${}^t\varepsilon : \mathbb{K} \to \mathcal{C}^*$) defined by

$$\begin{aligned} {}^t\Delta(f \otimes g) :& \mathcal{C} \longrightarrow \mathbb{K} \\ & c \longmapsto (f \otimes g)(\Delta(c)), \end{aligned} \tag{2.5}$$

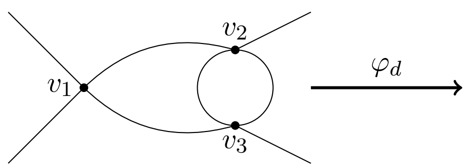
where $f, g \in \mathcal{C}^*$. Respectively,

$$\begin{aligned} {}^t\varepsilon(k) :& \mathcal{C} \longrightarrow \mathbb{K} \\ & c \longmapsto k \cdot \varepsilon(k), \end{aligned} \tag{2.6}$$

where $k \in \mathbb{K}$.

2.2.3 Convolution Product

At the end of Chapter 1, we mentioned that each Feynman diagram can be associated to a *regularized* value (see Figure [2.2]) given through the Feynman rules and the map φ_d, from the set of Feynman diagrams to the algebra $\mathcal{A}_d$ of meromorphic functions, defined by the method of dimensional regularization.



$$\varphi_d(\Gamma) = \sum_{n=-\infty}^{\infty} A_n (d - d_0)^n$$

FIGURE 2.2. Scheme of the action of the dimensional regularization.

We will see in Chapter 3 that a particular set of Feynman diagrams has a structure of Coalgebra. Thus, the map φ_d can be regarded as a map from a coalgebra to an algebra. Let $\mathcal{A}$ be a $\mathbb{K}$-algebra and $\mathcal{C}$ be a $\mathbb{K}$-coalgebra. A special product on $\mathcal{L}(\mathcal{C}, \mathcal{A}) := \{\varphi : \mathcal{C} \to \mathcal{A} : \varphi \text{ is a linear map}\}$ called **convolution product** is given by,

$$\varphi * \psi = p_{\mathcal{A}} \circ (\varphi \otimes \psi) \circ \Delta_{\mathcal{C}}. \tag{2.7}$$

In Sweedler's notation, the convolution product is written as

$$\begin{aligned}
\varphi * \psi(x) &= p_{\mathcal{A}} \circ (\varphi \otimes \psi) \circ \Delta_{\mathcal{C}}(x) \\
&= \sum_{(x)} p_{\mathcal{A}} \circ (\varphi \otimes \psi)(x_1 \otimes x_2) \\
&= \sum_{(x)} p_{\mathcal{A}}(\varphi(x_1) \otimes \psi(x_2)) \\
&= \sum_{(x)} \varphi(x_1) \cdot \psi(x_2) \qquad (2.8)
\end{aligned}$$

This product will be important to understand the map φ_d over Feynman diagrams. Before we discuss this fact, we need to explore the relation between the convolution product, the algebras and the coalgebras.

2.3 Bialgebras

There is a special structure which makes the algebra and coalgebra structures be compatible. A $\mathbb{K}$-**bialgebra** is a $\mathbb{K}$-vector space $\mathcal{H}$ endowed with a structure of unital $\mathbb{K}$-algebra (p, u) and a structure of co-unital $\mathbb{K}$-coalgebra (Δ, ε) such that the following diagrams commute.

$$\begin{tikzcd}
\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H} \arrow[rr, "\tau_{23}"] & & \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H} \arrow[d, "p\otimes p"] \\
\mathcal{H}\otimes\mathcal{H} \arrow[u, "\Delta\otimes\Delta"] \arrow[r, "p"'] & \mathcal{H} \arrow[r, "\Delta"'] & \mathcal{H}\otimes\mathcal{H}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{H}\otimes\mathcal{H} \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "p"'] & \mathbb{K}\otimes\mathbb{K} \arrow[d, "\pi_1"] \\
\mathcal{H} \arrow[r, "\varepsilon"'] & \mathbb{K}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{H}\otimes\mathcal{H} & \mathbb{K}\otimes\mathbb{K} \arrow[l, "u\otimes u"'] \\
\mathcal{H} \arrow[u, "\Delta"] & \mathbb{K} \arrow[l, "u"] \arrow[u, "\amalg_1"']
\end{tikzcd}$$

τ_{23} is a flip map defined by $\tau_{23}(a \otimes b \otimes c \otimes d) \mapsto a \otimes c \otimes b \otimes d$. The first diagram gives a compatibility between the product and coproduct of the bialgebra. In the Sweedler's notation the commutativity of this diagram is expressed by

$$
\begin{gathered}
(p \otimes p) \circ \tau_{23} \circ (\Delta \otimes \Delta)(x \otimes y) = \Delta \circ p(x \otimes y) \\
\sum_{(x)(y)} (p \otimes p) \circ \tau_{23}(x_1 \otimes x_2 \otimes y_1 \otimes y_2) = \Delta(p(x \otimes y)) \\
\sum_{(x)(y)} (p \otimes p)(x_1 \otimes y_1 \otimes x_2 \otimes y_2) = \sum_{(xy)} (p(x \otimes y))_1 \otimes (p(x \otimes y))_2 \\
\sum_{(x)(y)} p(x_1 \otimes y_1) \otimes p(x_2 \otimes y_2) = \sum_{(xy)} (p(x \otimes y))_1 \otimes (p(x \otimes y))_2. \\
\sum_{(x)(y)} (x_1 \cdot y_1) \otimes (x_2 \cdot y_2) = \sum_{(xy)} (xy)_1 \otimes (xy)_2. \qquad (2.9)
\end{gathered}
$$

Similarly, the second diagram expresses a compatibility of the co-unit and unit with the product and the coproduct respectively,

$$
\varepsilon(x \cdot y) = \varepsilon(x)\varepsilon(y), \qquad (2.10)
$$

$$
\Delta(1_{\mathcal{H}}) = 1_{\mathcal{H}} \otimes 1_{\mathcal{H}}. \qquad (2.11)
$$

Example 2.3.1. The group algebra $\mathbb{K}G$ of a group G is a bialgebra (see Examples (2.2.1) and (2.2.2)).

2.3.1 Hopf Algebra

A particular kind of Bialgebra called *Hopf algebra* appears naturally in several branches of mathematics and mathematical physics. For example, if $\mathcal{H}$ is a bialgebra and V is a vector space with an $\mathcal{H}$-left action then the duality

$$
\langle v, w \rangle = v(w) \text{ for } v, w \in V, \qquad (2.12)
$$

induces an $\mathcal{H}$-right action over the linear dual V^* defined for each $g \in \mathcal{H}$ by,

$$
(v \cdot g)(w) = \langle v \cdot g, w \rangle := \langle v, g \cdot w \rangle \text{ for } v, w \in V. \qquad (2.13)
$$

A natural question is if we can define the $\mathcal{H}$-left action over V^* in the same way that the $\mathcal{H}$-right action (2.13). The answer is positive when there is a linear map $S : \mathcal{H} \to \mathcal{H}$ which is compatible with the algebra structure, i.e. $S(1_{\mathcal{H}}) = 1_{\mathcal{H}}$ and $S(g \cdot h) = S(g) \cdot S(h)$ for all $g, h \in \mathcal{H}$. If the map S exists we have an $\mathcal{H}$-left action over V^* given by,

$$
(g \cdot v)(w) = \langle v, S(g)w \rangle . \qquad (2.14)
$$

If we want a complete compatibility with the duality (2.12), then we should have for each $v \in V^*$, $w \in V$, $g \in \mathcal{H}$ and $\Delta(g) = \sum_{(g)} g_1 \otimes g_2$ that

$$\sum_{(g)} (g_1 \cdot v)(g_2 \cdot w) = \sum_{(g)} (g_2 \cdot v)(g_1 \cdot w) = \varepsilon(g)v(w).$$

This condition is satisfied if and only if for each $g \in \mathcal{H}$ we have

$$\sum_{(g)} S(g_1) \cdot g_2 = \sum_{(g)} g_1 \cdot S(g_2) = \varepsilon(g). \tag{2.15}$$

In a common application V can be a Hilbert space and $\mathcal{H}$ can be a group of symmetries in a quantum system. Thus, Condition (2.15) is necessary to define a good action of $\mathcal{H}$ over V. The particular structure of this kind of bialgebra will be very important in the following chapter. A $\mathbb{K}$-bialgebra $\mathcal{H}$ is called a **Hopf algebra** if there is a linear map $S : \mathcal{H} \to \mathcal{H}$, called **antipode**, such that the following diagram commutes

$$\begin{tikzcd}
& \mathcal{H}\otimes\mathcal{H} \arrow[rr, "S\otimes I"] & & \mathcal{H}\otimes\mathcal{H} \arrow[dr, "p"] & \\
\mathcal{H} \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \mathbb{K} \arrow[rr, "u"] & & \mathcal{H} \\
& \mathcal{H}\otimes\mathcal{H} \arrow[rr, "I\otimes S"'] & & \mathcal{H}\otimes\mathcal{H} \arrow[ur, "p"'] &
\end{tikzcd}$$

In the Sweedler's notation this condition is written as

$$\sum_{(x)} S(x_1)x_2 = \sum_{(x)} x_1 S(x_2) = u \circ \varepsilon(x). \tag{2.16}$$

Note that the compatibility of the antipode can be written in terms of the convolution product on $\mathcal{L}(\mathcal{H}, \mathcal{H})$ as follows,

$$S * I(x) = I * S(x) = u \circ \varepsilon(x), \tag{2.17}$$

i.e. the antipode is the inverse of the map $u \circ \varepsilon$ with respect to the convolution product on $\mathcal{L}(\mathcal{H}, \mathcal{H})$.

Example 2.3.2. The bialgebra $\mathcal{H} = \mathbb{K}G$ is a Hopf algebra being the antipode given by,

$$S(g) = g^{-1}, \ g \in G. \tag{2.18}$$

In fact, note that for each $g \in G$, $u \circ \varepsilon(g) = u(1_{\mathbb{K}}) = 1_G$ and $p \circ (S \otimes I)(\Delta(g)) = p \circ (S \otimes I)(g \otimes g) = p(S(g) \otimes g) = S(g)g = g^{-1}g = 1_G$.

2.4 Filtered and Graded Bialgebras

A usual known problem is if a given bialgebra is a Hopf algebra. However, there is a kind of bialgebras which have a natural structure of Hopf algebra, the *connected filtered bialgebras*. A **filtered bialgebra** is a $\mathbb{K}$-vector space $\mathcal{H}$ together with a filtration:

$$\mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^n \subset \cdots \bigcup_{n\geq 0} \mathcal{H}^n = \mathcal{H}. \tag{2.19}$$

endowed with a structure of algebra (p, u) and a structure of coalgebra (Δ, ε) such that

$$\begin{gathered}\mathcal{H}^p \cdot \mathcal{H}^q := \{a \in \mathcal{H} : a = a_1 \cdot a_2 \text{ with } a_1 \in \mathcal{H}^p, a_2 \in \mathcal{H}^q\} \subseteq \mathcal{H}^{p+q},\\ \Delta(\mathcal{H}^n) \subseteq \sum_{p+q=n} \mathcal{H}^p \otimes \mathcal{H}^q.\end{gathered}$$

A filtered bialgebra $\mathcal{H}$ is called **connected** if $\mathcal{H}^0$ is one-dimensional, i.e $\mathcal{H}^0 = \mathbb{K}$. The coproduct in a connected filtered bialgebra takes a particular form as the following Proposition shows

Proposition 2.4.1. Let $\mathcal{H}$ be a connected filtered bialgebra. Then, for any $x \in \mathcal{H}^n$, $n \geq 1$

$$\Delta(x) = x \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes x + \tilde{\Delta}x, \qquad \tilde{\Delta}x \in \sum_{p,q\neq 0, p+q=n} \mathcal{H}^p \otimes \mathcal{H}^q. \tag{2.20}$$

Proof. First note that the connectedness of the bialgebra implies that $\dim(\mathcal{H}^n) \geq 1$ for every $n \geq 1$. Then,

$$Ker(\varepsilon) = \mathcal{H} \backslash \mathcal{H}^0. \tag{2.21}$$

Since the bialgebra is filtered, for each $x \in \mathcal{H}^n$ we have

$$\Delta(x) = k_1 \otimes x_1 + x_2 \otimes k_2 + \tilde{\Delta}x = 1_H \otimes (k_1 x_1) + (k_2 x_2) \otimes 1_H + \tilde{\Delta}x, \tag{2.22}$$

where $k_1, k_2 \in \mathbb{K} = \mathcal{H}^0$, $x_1, x_2 \in \mathcal{H}^n$ and $\tilde{\Delta}x \in \sum_{p,q\neq 0, p+q=n} \mathcal{H}^p \otimes \mathcal{H}^q$. Now, from Equation (2.21) and the counit property we obtain

$$\begin{gathered}1_{\mathbb{K}} \otimes x = \varepsilon \otimes I(\Delta(x)) = 1_{\mathbb{K}} \otimes (k_1 x_1)\\ x \otimes 1_{\mathbb{K}} = I \otimes \varepsilon(\Delta(x)) = (k_2 x_2) \otimes 1_{\mathbb{K}}.\end{gathered}$$

Thus, $x = k_1x_1 = k_2x_2$ and the coproduct of x is given by,

$$\Delta(x) = x \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes x + \tilde{\Delta}x, \qquad \tilde{\Delta}x \in \sum_{p,q\neq 0, p+q=n} \mathcal{H}^p \otimes \mathcal{H}^q.$$

□

The map $x \mapsto \tilde{\Delta}x$ in the latter proposition has two special properties.[Man06, Proposition 2.1.1].

Lemma 2.4.1. *The map $\tilde{\Delta}$ is coassociative on $Ker\varepsilon$ and $\tilde{\Delta}_k = (I^{\otimes k-1} \otimes \tilde{\Delta})(I^{\otimes k-2} \otimes \tilde{\Delta}) \cdots \tilde{\Delta}$ sends $\mathcal{H}^n$ into $\left(\mathcal{H}^{n-k}\right)^{\otimes k+1}$.*

The following theorem gives us a relation between the convolution product and the connected filtered bialgebras.

Theorem 2.4.1. *Let $\mathcal{H}$ be a connected filtered bialgebra and $\mathcal{A}$ be an algebra. Then, the set $G := \{\varphi \in \mathcal{L}(\mathcal{H}, \mathcal{A}), \varphi(1_{\mathcal{H}}) = 1_{\mathcal{A}}\}$ endowed with the convolution product is a group.*

Proof. Note that the unit to the convolution product is $u \circ \varepsilon$. In fact, for any $\varphi \in \mathcal{L}(\mathcal{H}, \mathcal{A})$ we have

$$\begin{aligned}\varphi * (u_{\mathcal{A}} \circ \varepsilon)(x) &= \sum_{(x)} \varphi(x_1) \cdot u_{\mathcal{A}}(\varepsilon(x_2)) = \sum_{(x)} \varphi(x_1) \cdot 1_{\mathcal{A}}\varepsilon(x_2) \\ &= \sum_{(x)} \varphi(\varepsilon(x_2)x_1) = \varphi\left(\sum_{(x)} \varepsilon(x_2)x_1\right). \end{aligned} \tag{2.23}$$

By the counit property,

$$p_{\mathcal{H}} \circ (I \otimes \varepsilon) \circ \Delta_{\mathcal{H}}(x) = \sum_{(x)} \varepsilon(x_2)x_1 = x = p_{\mathcal{H}} \circ \text{II}_1(x). \tag{2.24}$$

Thus, $\varphi * (u_{\mathcal{A}} \circ \varepsilon)(x) = \varphi(x)$ for every $x \in \mathcal{H}$. Similarly, $(u_{\mathcal{A}} \circ \varepsilon) * \varphi = \varphi$. To the second part, let $e = u_{\mathcal{A}} \circ \varepsilon$ and let us consider the formal series

$$\varphi^{*-1}(x) = (e - (e - \varphi))^{*-1}(x) = \sum_{k\geq 0} (e - \varphi)^{*k}(x), \tag{2.25}$$

where the notation φ^{*-1} and φ^{*k} means the inverse and the k-th integer power with respect to the convolution product respectively. As $e(1_{\mathcal{H}}) = 1_{\mathcal{A}} = \varphi(1_{\mathcal{H}})$ then $(e - \varphi)^{*k}(1_{\mathcal{H}}) = 0$ for every $k \geq 0$, and for $x \in Ker\varepsilon$, we have

$$(e - \varphi)^{*k}(x) = p_{\mathcal{A},k-1}(\varphi \otimes \cdots \otimes \varphi)\tilde{\Delta}_{k-1}(x), \tag{2.26}$$

where $p_{\mathcal{A},1}(a_1 \otimes a_2) := a_1 \cdot_{\mathcal{A}} a_2$ and for $k \geq 1$ $p_{\mathcal{A},k+1}(a_1 \otimes \cdots \otimes a_{k+2}) := p_{\mathcal{A},k}(p_{\mathcal{A},k}(a_1 \otimes \cdots \otimes a_{k+1}) \otimes a_{k+2})$ for $a_1, \cdots, a_{k+2} \in \mathcal{A}$ and $k \geq 1$. By Lemma (2.4.1), if $x \in \mathcal{H}^n$ the latter expression vanishes when $k \geq n+1$. Thus, the formal series (2.25) is finite for any x. □

The latter theorem gives also a sufficient condition for a bialgebra to be a Hopf algebra. A **filtered Hopf algebra**, $\mathcal{H} = \bigcup_n \mathcal{H}^n$ is a filtered bialgebra endowed with an antipode $S : \mathcal{H} \to \mathcal{H}$ satisfying the usual axioms of a Hopf algebra and such that

$$S(\mathcal{H}^n) \subset \mathcal{H}^n.$$

Corollary 2.4.1. *Any connected filtered bialgebra $\mathcal{H}$ is a filtered Hopf algebra. The antipode is given by,*

$$S(x) = \sum_{k\geq 0} (u\varepsilon - I)^k(x). \tag{2.27}$$

Proof. As we mentioned before, the antipode is the inverse of the map $u \circ \varepsilon$ with respect of the convolution product on $\mathcal{L}(\mathcal{H}, \mathcal{H})$. Thus, it is enough to set $\mathcal{A} = \mathcal{H}$ in the Theorem (2.4.1). □

2.4.1 Graded Bialgebras

There is another particular kind of bialgebras which have a natural structure of Hopf algebra. A **graded bialbegra** $\mathcal{H}$ is a graded $\mathbb{K}$-vector space

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n, \tag{2.28}$$

endowed with a structure of algebra (p, u) and a structure of coalgebra (Δ, ε) such that

$$\mathcal{H}_p \cdot \mathcal{H}_q := \{a \in \mathcal{H} : a = p(a_1 \otimes a_2) \text{ with } a_1 \in \mathcal{H}_p, a_2 \in \mathcal{H}_q\} \subseteq \mathcal{H}_{p+q},$$
$$\Delta(\mathcal{H}_n) \subseteq \bigoplus_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q$$

A graded bialgebra $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ is said to be **connected** if $\mathcal{H}_0$ is one-dimensional.

Example 2.4.1. Any graded bialgebra is a filtered bialgebra by the canonical filtration

$$\mathcal{H}^n := \bigoplus_{p=1}^{n} \mathcal{H}_p. \tag{2.29}$$

Thus, by Corollary (2.4.1), we have that any connected graded bialgebra is also a connected filtered Hopf algebra.

CHAPTER 3

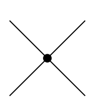
Hopf Algebra Renormalization

The ultraviolet divergences in a quantum field theory come from several diagrams with loops when the undetermined loop momenta increases. Those divergences can be eliminated by a *renormalization process* where the original quantum field theory is renormalized to a theory with low energy fields called *effective theory*[MD95, Cos11]. In particular, we saw in Chapter 1 that the map φ_d from a set of Feynman diagrams into the algebra of meromorphic functions defined by the *dimensional regularization* corresponds to a specific view of that renormalization problem. In this chapter, we give explicitly the bialgebra structure of the abstract Feynman diagrams defined in the latter chapter. Besides, we show a relation between the map φ_d and the convolution product called *Birkhoff decomposition* which let us give an own interpretation of the problem of renormalization of a quantum field theory with respect to the algebraic structure of the Feynman diagrams.

3.1 Hopf Algebra of a Feynman Graph

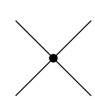
In order to understand the relation between the renormalization problem of a quantum field theory and the abstract definition of the Feynman graph we need to take a particular theory over Feynman diagrams. Fixed this theory, we are regarded the set of Feynman diagrams which are described by this theory as the diagrams associated to a specific quantum field theory.

$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$ $\qquad$ $\mathcal{F} = \{\Gamma : \Gamma \text{ is a Feynman graph}\}$, $\mathcal{T}_{\phi^4} = \{4\}$



$\longrightarrow -i\frac{\lambda}{4!}\int_M d^4z.$

(a)

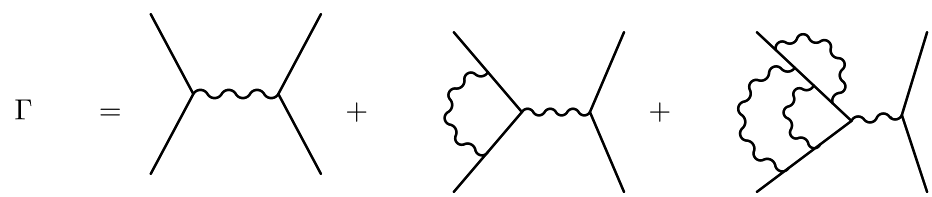


(b)

Figure 3.1. Theory associated to a Feynman Diagram. (a) Usual Feynman rule for vertex in the ϕ^4-theory. (b)Abstract Feynman Graph described by the theory $\mathcal{T}_{\phi^4}$.

Let $\mathcal{T}$ be a theory of Feynman diagrams, $V_{\mathcal{T}}$ be the vector space generated by all **1PI Feynman diagrams**[1] with vertex types in $\mathcal{T}$ and $\mathcal{B}_{\mathcal{T}}$ be the **free commutative $\mathbb{C}$-algebra** generated by $\mathcal{V}_{\mathcal{T}}$. As before (see Example (2.2.1)), we identify the unit 1 with the empty Feynman graph. The algebra structure of $\mathcal{B}_{\mathcal{T}}$ is given by definition in a such way that any element of $\mathcal{B}_{\mathcal{T}}$ is a $\mathbb{C}$-linear combination of disconnected locally 1PI Feynman diagrams, i.e Feynman diagrams whose connected components are 1PI Feynman diagrams.

Example 3.1.1. Let $\mathcal{T} = \mathcal{T}_{QED}$ be the theory associated to the quantum electrodynamics. Then the following diagram belongs to $\mathcal{B}_{\mathcal{T}_{QED}}$,



We should note that the algebra $\mathcal{B}_{\mathcal{T}}$ is graded. In fact, one of the possible gradings over connected 1PI diagram Γ is given by the **loop number**[2],

$$L(\Gamma) = I(\Gamma) - V(\Gamma) + 1,$$

where $I(\Gamma)$ is the number of internal edges and $V(\Gamma)$ is the number of vertices of the diagram Γ. In Example (3.1.1), the diagram Γ is the sum of three diagrams with loop number $0, 1$ and 3 respectively. The loop number is extended to non connected diagrams as follows,

$$L(\Gamma_1 \cdot \Gamma_2) = L(\Gamma_1) + L(\Gamma_2) \text{ for any } \Gamma_1, \Gamma_2 \in V_{\mathcal{T}}.$$

The structure of coalgebra is given as follows. The counit is given by $\varepsilon(1) := 1$ and $\varepsilon(\Gamma) := 0$ for any non-empty Feynman diagram Γ and the coproduct of connected 1PI digrams is given by,

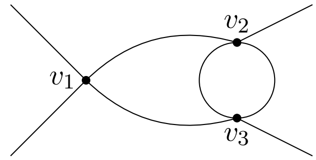
$$\Delta(\Gamma) := \begin{cases} \sum_{\gamma \subseteq \Gamma, \Gamma/\gamma \in V_{\mathcal{T}}} \gamma \otimes \Gamma/\gamma, & \text{if } L(\Gamma) \geq 1, \\ \Delta(\Gamma) = \Gamma \otimes \Gamma, & \text{if } L(\Gamma) = 0. \end{cases} \tag{3.1}$$

The coproduct for non-connected diagrams is extended by multiplicativity, i.e. such the coproduct and the product be compatible.

[1] A 1PI Feynman diagram is defined similarly to the 1PI Feynman graph.

[2] Note that the empty diagram has a non zero loop number

Example 3.1.2. Let $\mathcal{T} = \mathcal{T}_{\phi^4}$ be the theory associated to the ϕ^4-theory and Γ be the following diagram,



The subdiagrams γ such that $\Gamma/\gamma \in V_{\mathcal{T}}$ are given in the following table. Note that the other contracted subdiagrams have vertices of valence different to 4 (see Example (2.1.10) where one contracted subgraph has valence 6.)

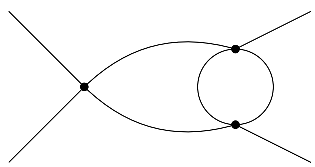
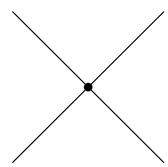
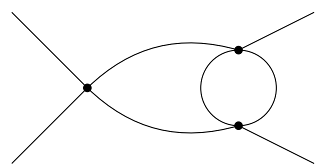
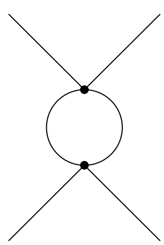
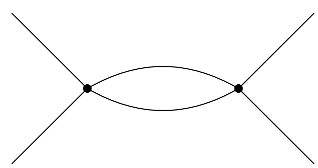
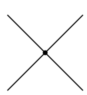
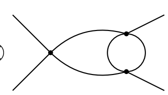
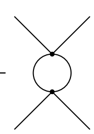
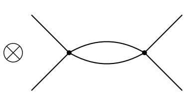
γ	Γ/γ
	
1	
	

TABLE 3.1. Contracted diagrams with vertex type in $\mathcal{T}_{\phi^4}$.

Thus, the coproduct of the diagram Γ is

$\Delta(\Gamma) = \Gamma \otimes$  $+ 1 \otimes$  $+$  $\otimes$  (3.2)

Note that the definition of the coproduct and counit on $\mathcal{B}_{\mathcal{T}}$ contains itself the compatibility with the algebra structure.

Theorem 3.1.1. *The algebra $\mathcal{B}_{\mathcal{T}}$ is a graded bialgebra algebra.*

Proof. It is enough show the coassociativity of the coproduct and the compatibility of the coproduct with the grading. The coassociativity on diagrams with vanishing loop number is intermediate from the definition,

$$(\Delta\otimes I)(\Delta(\Gamma)) = (\Delta\otimes I)(\Gamma\otimes\Gamma) = \Delta(\Gamma)\otimes\Gamma = \Gamma\otimes\Gamma\otimes\Gamma = \Gamma\otimes\Delta(\Gamma) = (I\otimes\Delta)(\Gamma\otimes\Gamma) = (I\otimes\Delta)(\Delta(\Gamma)).$$

Let Γ be a diagram in $V_{\mathcal{T}}$ with $L(\Gamma) \geq 1$. By definition we have,

$$\begin{aligned}(\Delta \otimes I)(\Delta(\Gamma)) &= (\Delta \otimes I)\left(\sum_{\gamma\subset\Gamma,\Gamma/\gamma\in V_{\mathcal{T}}} \gamma \otimes \Gamma/\gamma\right)\\ = \sum_{\gamma\subset\Gamma,\Gamma/\gamma\in V_{\mathcal{T}}} \Delta(\gamma) \otimes \Gamma/\gamma &= \sum_{\gamma\subset\Gamma,\Gamma/\gamma\in V_{\mathcal{T}}} \left(\sum_{\delta\subset\gamma,\gamma/\delta\in V_{\mathcal{T}}} \delta \otimes \gamma/\delta \otimes \Gamma/\gamma\right).\end{aligned} \tag{3.3}$$

On the other hand, we have

$$\begin{aligned}(I \otimes \Delta)(\Delta(\Gamma)) &= (I \otimes \Delta)\left(\sum_{\delta\subset\Gamma,\Gamma/\delta\in V_{\mathcal{T}}} \delta \otimes \Gamma/\delta\right)\\ = \sum_{\delta\subset\Gamma,\Gamma/\delta\in V_{\mathcal{T}}} \delta \otimes \Delta(\Gamma/\delta) &= \sum_{\delta\subset\Gamma,\Gamma/\delta\in V_{\mathcal{T}}} \left(\sum_{\gamma\subset\Gamma/\delta,(\Gamma/\delta)/\gamma\in V_{\mathcal{T}}} \delta \otimes \gamma \otimes (\Gamma/\delta)/\gamma\right).\end{aligned} \tag{3.4}$$

In Chapter 2, we saw that for a fixed subgraph δ of a graph Γ there is a bijection, $\gamma \mapsto \tilde{\gamma} = \gamma/\delta$, between *subgraphs* γ of Γ containing δ and the subgraphs of Γ/δ such that

$$\Gamma/\gamma = (\Gamma/\delta)/\tilde{\gamma}.$$

This fact can be extended to 1PI diagrams in a such way that for each subdiagram $\delta \subset \gamma \subset \Gamma$ there is unique subdiagram $\tilde{\gamma}$ of Γ/δ such that

$$\delta \otimes \gamma/\delta \otimes \Gamma/\gamma = \delta \otimes \tilde{\gamma} \otimes (\Gamma/\delta)/\tilde{\gamma}. \tag{3.5}$$

Comparing with Equations (3.3) and (3.4) we obtain,

$$(\Delta \otimes I) \circ \Delta(\Gamma) = (I \otimes \Delta) \circ \Delta(\Gamma) \text{ for any } \Gamma \in V_{\mathcal{T}} \text{ with } L(\Gamma) \geq 1. \tag{3.6}$$

The compatibility of the coproduct with the loop number is also given straightforward by the definition. In fact, given a Feynman diagram Γ with loop number $L(\Gamma) \geq 1$, if γ is a subdiagram with loop number $L(\gamma)$ such that $\Gamma/\gamma \in V_{\mathcal{T}}$, then the contraction Γ/γ has a loop number $L(\Gamma) - L(\gamma)$.

Since the contraction Γ/γ is the diagram obtained by removing the internal edges of γ and merge its vertices into a new vertex, there are no loops of γ in the loops in Γ/γ, i.e $L(\Gamma/\gamma) = L(\Gamma) - L(\gamma)$. Therefore, $L(\gamma \otimes \Gamma/\gamma) = L(\Gamma)$. Thus, if $\mathcal{B}_{\mathcal{T},n}$ is the set of all Feynman diagrams Γ of $\mathcal{B}_{\mathcal{T}}$ with $L(\Gamma) = n$ we have

$$\Delta(\mathcal{B}_{\mathcal{T},n}) \subseteq \bigoplus_{p+q=n} \mathcal{B}_{\mathcal{T},p} \otimes \mathcal{B}_{\mathcal{T},q}. \tag{3.7}$$

□

In order to obtain a structure of Hopf algebra over the graded bialgebra $\mathcal{B}_{\mathcal{T}}$ it is enough to build up a connected graded bialgebra from $\mathcal{B}_{\mathcal{T}}$. Then, let $\mathcal{Z}$ be the set of 1PI diagrams with zero loop number and K be the ideal generated by the set $\{\gamma - 1 : \gamma \in \mathcal{Z}\}$. Thus, the quotient bialgebra $\mathcal{H}_{\mathcal{T}} := \mathcal{B}_{\mathcal{T}}/K$ is a connected graded bialgebra and by Corollary (2.4.1) a Hopf algebra of Feynman diagrams described by the theory $\mathcal{T}$. The coproduct in the Hopf algebra $\mathcal{H}_{\mathcal{T}}$ is given by,

$$\Delta(\Gamma) = 1 \otimes \Gamma + \Gamma \otimes 1 + \sum_{\gamma \subset \Gamma, \Gamma/\gamma \in V_{\mathcal{T}}} \gamma \otimes \Gamma/\gamma. \tag{3.8}$$

Note that this definition is the canonical form of the coproduct in a connected filtered bialgebra given in Proposition (2.4.1). Thus, this definition of the coproduct on $\mathcal{H}_{\mathcal{T}}$ is the more natural one.

3.2 External Structure of Feynman Diagrams

So far we have given only the algebraic structure over undirected Feynman diagrams. However, the bialgebra structure over directed diagrams can be extended directly. It is important to consider this kind of diagrams because they correspond to the Feynman diagrams in the *momentum space*. Nevertheless, this structure does not describe one of the more important Feynman rule of quantum field theories, the *momentum conservation*. In order to obtain a bialgebra structure compatible with this Feynman rule we need to define what the momentum conservation in this abstract context means. Let W be a vector space of finite dimension, called **the momentum space**. A **specified Feynman diagram** is a couple (Γ, σ) where Γ is connected directed diagram in $V_{\mathcal{T}}$ with k external edges and σ belongs to some subset M_k, called **momentum distribution**, of W^k such that

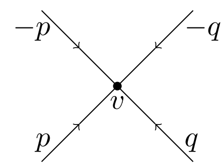
$$M_k = \{(p_1, \cdots, p_k) \in W^k : \sum_{j=1}^{k} p_j = 0\}. \tag{3.9}$$

The set M_k correspond to the momentum distribution of a given diagram in the physical momentum space. According to the usual rule of momentum conservation we need to define the momentum distribution in each vertex of the theory, i.e. we need to impose momentum conservation in each vertex. In order to consider the momentum distribution associated to a vertex we add an extra index $\sigma \in W^k$ being k the number of edges in $st(v)$ of a vertex v.

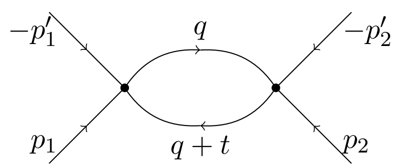
Therefore, a theory $\mathcal{T}$ over specified diagrams must be extended in order to include the several kind of vertices of same type with different momentum distributions in such way that a **theory over specified diagrams** is the set $\mathcal{T}$ of pairs (T,σ) called **kinds** of vertices where T is a type of vertex and σ is a specific momentum distribution.

Example 3.2.1. ϕ^4-THEORY FOR SPECIFIED DIAGRAMS

1. Let $\mathcal{T}_{\phi^4} := \{(4,\sigma) : \sigma \in M_4 \subset \mathbb{R}^4\}$. Then, the following diagram is described by the ϕ^4-theory.



2. Let Γ be the following diagram,



   is described by the ϕ^4-theory if $p_1+p_2=p_1'+p_2'$, where $t=p_1-p_1'=p_2'-p_2$. Note that there is not a restriction over q.

In order to build up a Hopf algebra of specified Feynman diagrams which are described by a theory $\mathcal{T}$ we proceed in the same way that the latter section. Let $\tilde{V}_{\mathcal{T}}$ be the vector space generated by all 1PI Feynman diagrams with vertex kinds in $\mathcal{T}$, $\tilde{V}_{\mathcal{T},k}$ be the subspace of $\tilde{V}_{\mathcal{T}}$ of diagrams with k external edges and $\tilde{W}_{\mathcal{T}}$ be the space of specified diagrams given by

$$\tilde{W}_{\mathcal{T}} := \bigoplus_{k\geq 0} \tilde{V}_{\mathcal{T},k} \otimes M_k. \tag{3.10}$$

Let $\tilde{\mathcal{B}}_{\mathcal{T}}$ be the free commutative $\mathbb{C}$-algebra generated by $\tilde{W}_{\mathcal{T}}$. The grading is defined with the loop number as before, as the coalgebra structure. The Hopf algebra of specified diagrams is obtained similarly to in the quotient $\tilde{\mathcal{H}}_{\mathcal{T}} := \tilde{\mathcal{B}}/\tilde{K}$ where $\tilde{K} = \{(\gamma,\sigma)-1 : (\gamma,\sigma) \in \tilde{\mathcal{Z}}\}$ being $\tilde{\mathcal{Z}}$ the set of all specified diagrams with vanishing loop number. The coproduct for connected specified diagrams is given by

$$\Delta(\Gamma,\sigma) = (\Gamma,\sigma)\otimes 1 + 1\otimes(\Gamma,\sigma) + \sum_{(\gamma,\sigma')\subset(\Gamma,\sigma)} \left(\sum_{\Gamma/\gamma\in V_{\mathcal{T}}} (\gamma,\sigma')\otimes(\Gamma/\gamma,\sigma') \right). \tag{3.11}$$

3.3 Birkhoff Decomposition

Now we are ready to give an abstract definition of the map φ_d given by the dimensional regularization. As we mentioned in Chapter 1 the principal idea behind the dimensional regularization is to compute the divergent integrals associated to diagrams with loops in a quantum field theory as a function of the dimension d of the space-time and thereupon take a limit as $d \to d_0$ where d_0 is the dimension of the real physical space-time[3] to obtain a regularized value of the integral.

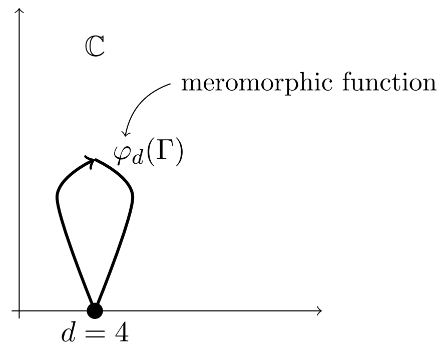


FIGURE 3.2. Scheme of the dimensional regularization

Let $d_0 \in \mathbb{R}$ be a fixed value, $\mathcal{A} := \mathbb{C}[[d - d_0]]$ be the algebra of meromorphic functions, $\mathcal{T}$ be a theory over Feynman diagrams[4] and $RG_{\mathcal{A}} := \{\varphi : \mathcal{H}_{\mathcal{T}} \to \mathcal{A}\varphi(1_{\mathcal{H}_{\mathcal{T}}}) = 1_{\mathcal{A}}\}$. Note that the map φ_d is a element of $RG_{\mathcal{A}}$. Besides, note that the algebra $\mathbb{C}[[d - d_0]]$ has a splitting in two subalgebras,

$$\mathbb{C}[[d - d_0, (d - d_0)^{-1}]] = (d - d_0)^{-1}\mathbb{C}[(d - d_0)^{-1}] \oplus \mathbb{C}[[d - d_0]],$$

where $\mathcal{A}_- = (d - d_0)^{-1}\mathbb{C}[(d - d_0)^{-1}]$ is called **the field of Laurent series** and $\mathcal{A}_+ = \mathbb{C}[[d - d_0]]$ is the usual algebra of formal series. Note that if $\varphi_d(\Gamma) \in \mathcal{A}_-$ for some Feynman diagram Γ then the limit $\lim_{d \to d_0} \varphi_d(\Gamma)$ is divergent, i.e the integral associated to the Feynman diagram is not regularized. In order to obtain a finite value from a given diagram Γ we need to take the value of the integral associated to that Feynman diagram as $\lim_{d \to d_0} \varphi_{d,+}(\Gamma)$ for some $\varphi_{d,+}(\Gamma) \in \mathcal{A}_+$ associated to the map φ_d. The following theorem tell us how to define the function $\varphi_{d,+}(\Gamma)$ in the general case.

Theorem 3.3.1. *(*BIRKHOFF DECOMPOSITION - HOPF ALGEBRA RENORMALIZATION*) Let $\mathcal{H}$ be a connected filtered Hopf algebra, $\mathcal{A}$ be an algebra that admits a splitting $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ and $G = \{\varphi \in \mathcal{L}(\mathcal{H}, \mathcal{A}) : \varphi(1_{\mathcal{H}}) = 1_{\mathcal{A}}\}$. Then, any $\varphi \in G$ admits a unique* ***Birkhoff decomposition****[Man06]*

$$\varphi = \varphi_-^{*-1} * \varphi_+, \tag{3.12}$$

where φ_- sends $1_{\mathcal{H}}$ to $1_{\mathcal{A}}$ and $Ker\varepsilon$ into $\mathcal{A}_-$, and where φ_+ sends $\mathcal{H}$ into $\mathcal{A}_+$. The maps φ_- and φ_+ are given on $Ker\varepsilon$ by

[3]In the usual quantum field theories $d_0 = 4$, but there are another theories like the string theory where $d_0 > 10$.

[4]The following discussion is extended straightforward to specified Feynman diagrams

$$\varphi_-(x) = -\pi\left(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\right), \tag{3.13}$$

$$\varphi_+(x) = (I-\pi)\left(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\right), \tag{3.14}$$

where $\pi : \mathcal{A} \to \mathcal{A}_-$ is the projection parallel to $\mathcal{A}_+$ and the sum in each equation corresponds to a short notation to the function $p_{\mathcal{A}} \circ (\varphi_- \otimes \varphi) \circ \tilde{\Delta}_{\mathcal{H}} : \mathcal{H} \to \mathcal{A}$ being $\tilde{\Delta}_{\mathcal{H}}$ the associative coproduct on $Ker\varepsilon$ defined in the lemma (2.4.1).

Proof. Fist note that from the definition of the projection $\pi : \mathcal{A} \to \mathcal{A}_-$ we have that φ_- sends $Ker\varepsilon$ into $\mathcal{A}_-$, and that φ_+ sends $Ker\varepsilon$ into $\mathcal{A}_+$. Since $\mathcal{H}$ is a connected filtered Hopf algebra then it is enough to show that $\varphi_+ = \varphi_- * \varphi$ (see Theorem 2.4.1). In fact,

$$\begin{aligned}
(\varphi_- * \varphi)(x) &= p_{\mathcal{A}} \circ (\varphi_- \otimes \varphi) \circ \Delta(x) \\
&= p_{\mathcal{A}} \circ (\varphi_- \otimes \varphi)(x \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes x + \tilde{\Delta}(x)) \\
&= p_{\mathcal{A}}(\varphi_-(1_{\mathcal{H}}) \otimes \varphi(x) + \varphi_-(x) \otimes \varphi(1_{\mathcal{H}}) + \sum_{(x)} \varphi_-(x') \otimes \varphi(x'')) \\
&= \varphi_-(1_{\mathcal{H}})\varphi(x) + \varphi_-(x)\varphi(1_{\mathcal{H}}) + \sum_{(x)} \varphi_-(x')\varphi(x'').
\end{aligned}$$

Since $\varphi(1_{\mathcal{H}}) = 1_{\mathcal{H}}$ and $\varphi_-(1_{\mathcal{H}}) = -\pi(\varphi(1_{\mathcal{H}})) = 1_{\mathcal{A}}$ we have,

$$\begin{aligned}
(\varphi_- * \varphi)(x) &= \varphi_-(x) + \varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'') \\
&= -\pi\left(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\right) + \varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'') \\
&= (I-\pi)\left(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\right) = \varphi_+(x)
\end{aligned}$$

□

In particular if $\mathcal{H} = \mathcal{H}_{\mathcal{T}}$ and $\mathcal{A} = \mathbb{C}[[d-d_0]]$, we have the corresponding Birkhoff decomposition to the map φ_d[CK00].

Corollary 3.3.1. *(*Connes-Kreimer*) The map $\varphi_d : \mathcal{H}_{\mathcal{T}} \to \mathbb{C}[[d-d_0]]$ admits the following Birkhoff decomposition.*

$$\varphi_d = \varphi_{d,-}^{*-1} * \varphi_{d,+},$$

where $\varphi_{d,-}$ (resp. $\varphi_{d,+}$) sends $Ker\varepsilon = \{\Gamma \in \mathcal{H}_{\mathcal{T}} : L(\Gamma) \geq 1\}$ into the Laurent series algebra $(d-d_0)^{-1}\mathbb{C}[(d-d_0)^{-1}]$ (resp. into the formal series algebra $\mathbb{C}[[d-d_0]]$). Moreover, for any Feynman diagram $\Gamma \in \mathcal{H}_{\mathcal{T}}$ the maps $\varphi_{d,-}$ and $\varphi_{d,+}$ are given by the following recursive formulas

$$\varphi_{d,-}(\Gamma) = -\pi\left(\varphi(\Gamma) + \sum_{\gamma\subset\Gamma,\Gamma/\gamma\in V} \varphi_-(\gamma)\otimes\varphi(\Gamma/\gamma)\right), \tag{3.15}$$

$$\varphi_{d,+}(\Gamma) = (I-\pi)\left(\varphi(\Gamma) + \sum_{\gamma\subset\Gamma,\Gamma/\gamma\in V} \varphi_-(\gamma)\otimes\varphi(\Gamma/\gamma)\right). \tag{3.16}$$

(3.17)

Proof. It is enough to note from Equation (3.8) that the associative coproduct $\tilde{\Delta}$ on $Ker\varepsilon$ is given by

$$\tilde{\Delta}(\Gamma) = \sum_{\gamma\subset\Gamma,\Gamma/\gamma\in V} \gamma\otimes\Gamma/\gamma. \tag{3.18}$$

□

This corollary allows us to give an abstract interpretation about the renormalization problem of a quantum field theory. We have that for a given Feynman diagram Γ its corresponding finite value is $\varphi_{d,+}(\Gamma)$. However, the value $\varphi_{d,+}$ must be computed recursively from the subdiagrams of Γ, i.e. if Γ has a loop number Γ $L(\Gamma)$ then its corresponding regularized value depends on all subdiagrams in the theory with a loop number less than $L(\Gamma)$. Thus, it is enough to see the behavior of the diagrams with smallest loop number to understand the problem of regularization in an abstract theory over Feynman diagrams. This fact is common in any quantum field theory where the notion of renormalization is defined as follows (see Appendix B):

1. A quantum field theory is called **super-renormalizable** if only a finite number of Feynman diagrams associated to the quantum field theory diverge, i.e if only a few diagrams need a regularization process.

2. A quantum field theory is called **renormalizable** if only a finite number of Feynman diagrams associated to the quantum field theory diverge. However, the divergence is presented in any order, i.e. any Feynman diagram needs a regularization process but the regularized value depends only on few subdiagrams (with small loop number).

3. A quantum field theory is called **non-renormalizable** if any diagrams diverge at a sufficiently high order, i.e. it is impossible to compute a regularized value of diagrams with a loop number sufficiently high.

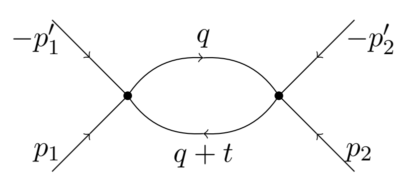
As we mentioned before, a Feynman diagram Γ does not have a regularized value if $\varphi_d(\Gamma) \in \mathcal{A}_-$, i.e. in terms of the Birkhoff decomposition, if $\varphi_{d,+}(\Gamma) = u \circ \varepsilon(\Gamma) = 1_*(\Gamma)$ being 1_* the unit with respect to the convolution product. Reciprocally, the diagram Γ does not need a regularization process if $\varphi_d(\Gamma) \in \mathcal{A}_+$, i.e. $\varphi_{d,-}(\Gamma) = S(\Gamma)^5$ being S the antipode. Thus, the latter definitions can be interpreted in terms of the Birkhoff decomposition as follows

1. The theory is called **super-renormalizable** if $\varphi_d(\gamma) \in \mathcal{A}_+$ for some diagram γ with sufficiently big loop number.
2. The theory is called **renormalizable** if $\varphi_d(\Gamma) \notin \mathcal{A}_-$ for any Feynman diagram Γ.
3. The theory is called **non-renormalizable** if $\varphi_d(\gamma) \in \mathcal{A}_-$ for some diagram γ with sufficiently big loop number.

This definition of super-renormalizable (renormalizable, non-renormalizable) theories are equivalent to the first one. In fact, if in a theory there is a diagram γ with enough big loop number such that $\varphi_d \in \mathcal{A}_+$ then for any diagram Γ that contains γ we have that $\varphi_d \in \mathcal{A}_+$ since the regularized value $\varphi_{d,+}(\Gamma)$ depends only on γ, i.e. only the diagrams Γ with $L(\Gamma) < L(\gamma)$ need a regularization process. Similarly, if in a theory $\varphi_d(\Gamma) \notin \mathcal{A}_-$ for any diagram then each diagram Γ needs a regularization process but it depends only on the subdiagrams γ with small loop number. Finally, if there is a diagram γ with enough big loop number such that $\varphi_d(\gamma) \in \mathcal{A}_-$ then $\varphi_d(\Gamma) \in \mathcal{A}_-$ for any diagram Γ that contains γ since the regularized value $\varphi_{d,+}(\Gamma)$ depends only on γ.

The fact that a theory is renormalizable or non-renormalizable depends on the fixed dimension d_0. For example, the quantum electrodynamics as well as the ϕ^4-theory are renormalizable for $d = 4$.

Example 3.3.1. Let us consider the following Feynman diagrams of the ϕ^4-theory,



This diagram corresponds to the unique diagram with one loop in the ϕ^4-theory. According to the Feynman rules defined in Chapter 1 we have that this diagram corresponds to the expression,

$$\left(\frac{-i\lambda}{4!}\right)^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{m^2 - q^2} \frac{1}{m^2 - (q+t)^2}.$$

[5] Remember that the antipode is the inverse of the unit map with respect to the convolution product.

Applying the dimensional regularization to this integral[MD95] we obtain,

$$\varphi_d\left(\tilde{\gamma}\right) = \frac{i\lambda^2}{2} \frac{\overbrace{\Gamma(2-d/2)}^{\text{Gamma Function}}}{(4\pi)^{d/2}} \int_0^1 \frac{dx}{(m^2 - x(1-x)t^2)^{2-d/2}}$$

Note that this expression is finite if $d < 4$ $(2 - d/2 > 0)$[6]. Thus, for dimensions less than 4 we have that this diagram does not need a regularization process. Therefore, by the latter discussion, we have that the ϕ^4-theory is super-renormalizable. Now, if $d \geq 4$, $\varphi_d(\tilde{\gamma})$ diverges (for integer dimension). However, if $\epsilon = d - 4$ from a Taylor expansion around $\epsilon = 0$ we obtain[MD95]

$$\varphi_d(\tilde{\gamma}) = \frac{i\lambda^2}{32\pi^2} \int_0^1 \left[\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log(m^2 - x(1-x)t^2)\right] dx,$$

where γ is the *Euler-Mascheroni* number. Since $\tilde{\gamma}$ has no proper simple subdiagrams from Corollary (3.3.1) we have that,

$$\varphi_{d,-}(\gamma) = -\pi(\varphi_d(\gamma)) = \frac{i\lambda^4}{16\pi^2\epsilon}.$$

Thus, the finite value of this diagram is given by,

$$\varphi_{d,+}(\gamma) = (I - \pi)(\varphi_d(\gamma)) = \frac{-i\lambda^2}{32\pi^2} \int_0^1 \left[\gamma - \log(4\pi) + \log(m^2 - x(1-x)t^2)\right] dx.$$

This corresponds to the regularized value of the diagram as $d \to 4$. Any other diagram in the ϕ^4-theory contains the particular diagram considered in this example and it is possible to compute its corresponding regularized value, i.e. the ϕ^4-theory is *renormalizable* as $d = 4$. One can also see that the theory is *non-renormalizable* for $d > 4$ (see Appendix B).

[6] The divergence in $\varphi_d(\tilde{\gamma})$ comes from the poles of the *Gamma function*, i.e when $2 - d/2$ is a negative integer.

APPENDIX A

Renormalization and Quantum Electrodynamics

The ultraviolet divergences that appear in diagrams like (1.54) seem to be a problem of the perturbative approach of the interacting theory. However, those divergences are not just a problem that we need to eliminate, but they are a better way to understand the phenomena described by the quantum field theory. In fact, we mentioned above that the regularization of the ultraviolet divergences is really a renormalization process. We saw Example 5 of the CLASSICAL RENORMALIZATION, that there is a renormalization of the parameters of the theory due to the renormalization process.

Quantum Electrodynamics

In order to understand why the regularization of ultraviolet divergences returns in physical information of the phenomena described by the quantum field theory, we regard a more common quantum field theory called *Quantum Electrodynamics* which describe the quantum interaction between the electromagnetic field and a charged particle.

Definition A.0.1. QUANTUM ELECTRODYNAMICS Let M be the Minkowski space-time, ψ a Dirac four-spinor[1] and $(A_\mu)_{\mu=1}^4$ a contravariant four-vector[2]. The theory of the quantum electrodynamics (QED) is defined by the following Lagrangian,

$$\mathcal{L}_{QED} = \bar{\psi}\left(i\sum_{\mu=1}^{4} i\gamma^\mu\partial_\mu - m\right)\psi - e_0\bar{\psi}\sum_{\mu=1}^{4}\gamma^\mu\psi A_\mu - \frac{1}{4}\sum_{\mu,\nu=1}^{4} F_{\mu\nu}F^{\mu\nu}, \tag{A.1}$$

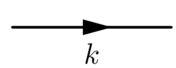
where m is the mass of the charged particle, e_0 is its charge, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\{\gamma^\mu\}_{\mu=1}^4$ are the Dirac matrices,

[1] For instance, a four-spinor (four-vector) is an irreducible representation of some Clifford algebra associated to the Lorentz group.

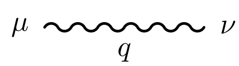
[2] A contravariant four-vector (rep. covariant four-vector) is represented by a scalar function $\phi \in C^\infty(M)$ and a vector field $\mathbf{A} \in C^\infty(M, \mathbb{R}^3)$ as follows, $(A_\mu)_{\mu=1}^4 = (\phi, -\mathbf{A})$ (resp. $(A^\mu)_{\mu=1}^4 = (\phi, \mathbf{A})$)

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

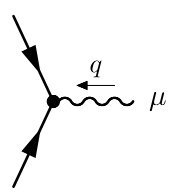
The Feynman rules associated to the quantization of the fields ψ and A_μ for the internal lines are the following,

1) For each propagator of the field ψ, 

$= \frac{m+\not k}{m^2-k^2}$,

2) For each propagator of the field A_μ, 

$= \frac{-g^{\mu\nu}}{q^2}$,

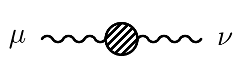
3) For each vertex, 

$= -ie_0\gamma^\mu$,

4) Impose momentum conservation in each vertex,

5) Integrate over each undetermined loop momentum.

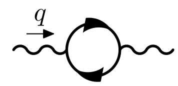
Rules 1 and 2 correspond to the electron and photon propagator rule respectively, whereas Rule 3 is characteristic of a scattering between two electrons or the annihilation of a pair electron-anti-electron. Now, one of the more interesting diagrams is the correction of the photon propagator in the interacting theory

$\Pi^{\mu\nu} =$ 

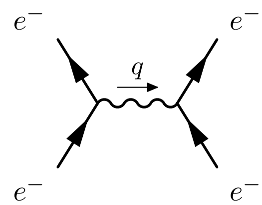
, (A.2)

where $\mu, \nu = 1, 2, 3, 4$ and the dashed circle means that the diagram corresponds to the sum over all Feynman diagrams with loops in the photon propagator. This sum does not include diagrams with self-loops[3]. For example, the first diagram with loops in the expansion of the latter diagram is

$$\Pi_2^{\mu\nu} = \quad \mu \quad \nu \quad . \tag{A.3}$$



Those diagrams are part of the scattering process between electrons. The respective diagram to this process is given by two vertex electron-photon-electron (see rule 3) and one photon propagator,



$$. \tag{A.4}$$

This diagram corresponds to the simple process of scattering between an electron of charge e_0 mediated by a photon. If Γ is the value associated to this diagram one can show that Γ is proportional to $\frac{e_0^2}{q^2}$[MD95]. Now, if the photon propagator in the latter diagram include loops as Diagram (A.3), from the process of regularization of this diagram we obtain

$$\Gamma \longrightarrow \Gamma \propto \frac{\zeta e_0^2}{q^2}, \tag{A.5}$$

where ζ is a quantity that comes from the finite part of Diagram (A.2). If we compare the latter expression with the vale of Diagram (A.4), we have that the value of the electric charge e_0 can be effectively identified with $\sqrt{\zeta}e_0$.

Therefore, in a complete interacting theory we have that the charge of the electron in the scattering process is $e = \sqrt{\zeta}e_0$ and not the original parameter e_0.

[3] Any diagram with self-loops is divergent. However, Lehemann, Symanzik and Zimmermann showed in 1995 that the Feynman diagrams that appear in any expansion of a physical quantity, for instance correlation functions, are only the fully connected without self-loops diagrams. This statement is called the *LSZ formula*[LSZ55].

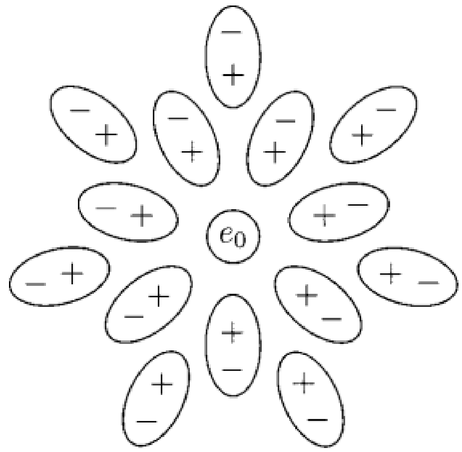


Figure A.1. Scheme of the *shielding* of the electric charge due to the electron/anti-electron loops(vacuum fluctuations). Image taken and modified from [MD95].

This fact is called *Renormalization of the electric charge* and it's due to the presence of electron/anti-electron loops in the photon propagator (see figure [A.1]).

APPENDIX B

Superficial Degree of Divergence

We gave a definition of the ϕ^4-theory in a space-time of dimension $d = 4$. We also said that the ultraviolet divergences in diagrams like (1.54) can be regularized by using dimensional regularization with respect to this fixed dimension. A natural question here is if we can perform the same procedure for other values of d. The answer, in general, is not affirmative and from this fact one can define when a quantum field theory is renormalizable.

Renormalizable Theories

Let us regard the dimension of the space-time arbitrary, say d. The ultraviolet divergences in the ϕ^4-theory come from integrals like

$$\int \frac{d^d k_1 \cdots d^d k_L}{(m^2 - k_1^2) \cdots (m^2 - k_n^2)}, \tag{B.1}$$

where $n, L \in \mathbb{N}$, each integral corresponds to one loop of the corresponding diagram and each fraction $1/(m^2 - k_j^2)$, with $j = 1, \cdots n$, corresponds to one scalar propagator in the diagram. In a Euclidean form of this integral[1] this is an integral of a rational function. Note that for each integral, $\int d^d k_l$ with $l = 1, \cdots, L$, the numerator is a polynomial of degree d, whereas for each factor $1/(m^2 - k_j^2)$ the denominator is a polynomial of degree 2.

Thus, if a given diagram in the ϕ^4-theory has L loops and n propagators (internal lines) then its corresponding analytic expression is the integration of a rational fraction $p(k_1, \cdots, k_L)/q(k_1, \cdots, k_n)$ where $degree(p) = dL$ and $degree(q) = 2n$.

Definition B.0.2. The *superficial degree of divergence*, D, of a Feynman diagram in the ϕ^4-theory is given by

$$D = dL - 2n, \tag{B.2}$$

[1] Taking $k_{j,E}^0 = -ik_j^0$.

where d is the dimension of the space-time, L the number of loops[2] in the diagram and n the number of internal lines of the diagram.

The superficial degree of divergence gives us a clue to know if a diagram has or not ultraviolet divergences. For example, if $D < 0$ the integral (B.1) corresponds to integration of a rational function as $1/q(k_1, \cdots, k_m)$ with $degree(q) \geq 2$[3]. Thus, one expects that the undefined integral corresponds to a rational function $1/r(k_1, \cdots, k_m)$ with $degree(r) \geq 1$. If we evaluate the undefined integral, the problem when the momentum goes to ∞ seem to disappear. In the other case, $D \geq 0$ the divergence seems to arise. Thus, diagrams with a non-negative superficial of divergence can present ultraviolet divergences. According to the definition of D, diagrams with a high number of loops can have a positive value of D. However, this is more clear if we express the superficial degree of divergence in another way.

Proposition B.0.1. Let us regard a Feynman diagram of the ϕ^4-theory with V vertices, n internal lines (scalar propagator), L loops (that are not self-loop) and N external lines. Then, the superficial degree of divergence is given by

$$D = d + (d-4)\,V - \left(\frac{d-2}{2}\right) N. \tag{B.3}$$

Proof. First note that the number of loops that are not self-loops in a diagram is $L = 1 + n - V$. Second, according to the Feynman rules of the ϕ^4-theory, there are four lines meeting at each vertex. Then $4V = N + 2n$. Thus,

$$\begin{aligned} D = dL - 2n &= d\,(1 + n - V) - 4V + N \\ &= d\left(1 + \frac{4V - N}{2} - V\right) - 4V + N \\ &= d\left(1 + V - \frac{N}{2}\right) - 4V + N \\ &= d + dV - 4V - \left(\frac{d}{2} - 1\right) N \\ &= d + (d-4)V - \left(\frac{d-2}{2}\right) N. \end{aligned}$$

□

Above we saw that each diagram corresponds to some order in the expansion of the partition function of the interacting theory. The higher orders correspond to diagrams with a high number of internal lines or high number of loops or equivalently a high number of vertices. The problem with diagrams with a big number of loops ensures that the possible divergences can be eliminated. It is usual think that a high number of loops in a diagram means a high number of divergences in the corresponding integral, however this is not necessarily true. For example, if $d < 4$ from (B.3) we have that $D < 0$ for a high number of vertices. Thus, the divergences of the ϕ^4-theory disappear in high order

[2] No self-loops.

[3] At least there is two internal lines in a loop.

diagrams, i.e it is not necessary a renormalization process for a theory in dimension less than four.

Now, if $d > 4$ the superficial degree of divergence is always positive for high order diagrams, i.e the divergences in the theory increases as well as the order of the expansion. Thus, the complete partition function and the correlation functions are divergent, i.e. the divergence of the theory can not be eliminated. At last if $d = 4$, the superficial degree of divergence does not depend on the number of vertices of the diagram. Then, for a fixed number of external lines, the divergences of the diagram are the same in any order of the expansion. This three cases give us the main idea of the definition of a renormalizable theory

Definition B.0.3.

1. A quantum field theory is called *super-renormalizable* if only a finite Feynman diagrams superficially diverge, i.e $D > 0$ for a finite number of Feynman diagrams.
2. A quantum field theory is called *renormalizable* if only a finite Feynman diagrams superficially diverge. However, the divergence is presented at any order.
3. A quantum field theory is called *non-renormalizable* if any diagrams diverge at a sufficiently high order.

Under this definition, we have that the ϕ^4-theory is super-renormalizable for $d < 4$, renormalizable for $d = 4$ and non-renormalizable for $d > 4$. Then, from the superficial degree of divergence of a diagram, we can calculate for what value of the dimension of the space-time in where the corresponding theory is renormalizable. For example for the quantum electrodynamics the superficial degree of divergence of a diagram with N_γ photon external lines, N_e electron external lines and V vertices is[MD95]

$$D = d + \left(\frac{d-4}{2}\right) V - \left(\frac{d-2}{2}\right) N_\gamma - \left(\frac{d-2}{2}\right) N_e. \tag{B.4}$$

Therefore, as with the ϕ^4-theory, the quantum electrodynamics is renormalizable for a space-time of dimension $d = 4$, super-renormalizable for $d < 4$ and non-renormalizable for $d > 4$.

Conclusions

We give an abstract definition of Feynman diagrams as a kind of graph with different types of edges and vertices. The types of edges and vertices are fixed by the Feynman rules associated to a quantum field theory in such way we leave a side the problems in the definition of the path-integral formulation of a quantum theory and the quantization problem. Moreover, a particular set $\mathcal{H}_{\mathcal{T}}$ of Feynman diagrams described by a theory, regarded as a set $\mathcal{T}$ of types of vertices, is endowed with a natural connected graded bialgebra structure.

In Chapter 1, we saw that the method of dimiensional regularization define a map φ_d from $\mathcal{H}_{\mathcal{T}}$ to the algebra of meromorphic function $\mathcal{A} := \mathbb{C}[[d-d_0, (d-d_0)^{-1}]]$ where d_0 a fixed value of the space-time dimension. In Chapter 3, we regarded the regularization problem studying the properties of the set $RG_{\mathcal{A}}$ of linear functions $\varphi : \mathcal{H}_{\mathcal{T}} \to \mathbb{C}[[d-d_0, (d-d_0)^{-1}]]$. The bialgebra structure of $\mathcal{H}_{\mathcal{T}}$ and the splitting $\mathbb{C}[[d-d_0, (d-d_0)^{-1}]] = (d-d_0)^{-1}\mathbb{C}[(d-d_0)^{-1}] \oplus \mathbb{C}[[d-d_0]]$ endows with a group structure to the set $RG_{\mathcal{A}}$ with respect to the convolution product and gives to the map φ_d a decomposition

$$\varphi_d = \varphi_{d,-}^{*-1} * \varphi_{d,+},$$

where $\varphi_{d,+}$ sends the non-simple Feynman diagrams into the algebra $\mathbb{C}[[d-d_0]]$. If d_0 is such that $\varphi_d(\Gamma) \notin (d-d_0)^{-1}\mathbb{C}[(d-d_0)^{-1}]$ for any Feynman diagram Γ. The function $\varphi_{d,+}(\Gamma)$ corresponds to a regularized value of the Feynman diagram. This means that we can find a regular vale of any Feynman diagram with loops as $d \to d_0$, i.e the quantum field theory associated to the theory $\mathcal{T}$ is renormalizable.

This characterization of the regularization in a quantum field theory through the Hopf algebra suppose that the map φ_d is already given from the Feynman rules, i.e suppose that the quantization of the theory. Due to this fact, this work does not correspond to an alternative method to quantized a field theory, but give a method to introduce formally the notion of renormalization in a quantum field theory. Another problem with this point of view is that we had fixed a regularization method. However, there are more methods like the *Paul-Villars regularization* or the *Zeta function regularization*. This problem can be solved to study the relation between Wilson's renormalization group and the Hopf algebra renormalization[CK01].

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