# Recitation 5 - Addendum: Continuous Random Variables

## 1 Example: Uniform Distribution

A random variable $X$ having the *uniform distribution on* $[a, b]$ is defined by the following PDF:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

The *standard uniform distribution* is the uniform distribution on $[0, 1]$. Its CDF can be expressed as follows.

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

### 1.1 Relation to other continuous random variables

Any continuous random variable can be transformed into the standard uniform distribution. Consider a continuous random variable $X$ with cumulative distribution function $F_X(x)$. Then, the random variable $Y = F_X(x)$ has the standard uniform distribution. To see this, consider its CDF. For $0 \leq y \leq 1$,

$$\begin{aligned} F_Y(y) &= \mathbb{P}[Y \leq y] \\ &= \mathbb{P}[F_X(X) \leq y] \\ &= \mathbb{P}[X \leq F_X^{-1}(y)] \\ &= F_X(F_X^{-1}(y)) \\ &= y. \end{aligned}$$

By comparing with the expression above, see that the CDF of $Y$ coincides with the CDF of the standard uniform distribution.

This transformation is a useful tool in data analysis. (1) In certain situations, this gives simple and efficient methods for generating samples from a given distribution. (2) This transform can be used to test whether a dataset is well represented by a given distribution.

## 2 Working with Expectation and Variance

For random variable $X$ and constants $a, b$,

- *Linearity of Expectation*:
$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b,$$
- *Variance under linear transformations*:
$$\text{Var}(aX + b) = a^2\text{Var}(X).$$

# 3 Independence

Intuitively, two random variables $X$ and $Y$ are independent if they take values independently of each other. Therefore, we say $X$ and $Y$ are *independent* if for any two intervals $[a, b]$ and $[c, d]$, the event that $X \in [a, b]$ is independent of the event that $Y \in [c, d]$:

$$\mathbb{P}[a \leq X \leq b, c \leq Y \leq d] = \mathbb{P}[a \leq X \leq b]\mathbb{P}[c \leq Y \leq d].$$

**Remark:** We will discuss how to compute the probability on the LHS in a later discussion about joint probability distributions.

## 3.1 Correlation

If $X$ and $Y$ are independent random variables, then the expectation of their product is the product of their individual expectations.

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y].$$

In general, for two random variables $X$ and $Y$, we may be interested in how much this equality is violated. We define the *covariance* between $X$ and $Y$ to be the difference:

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$

We say that $X$ and $Y$ are *uncorrelated* if $\text{Cov}(X, Y) = 0$. In summary, if $X$ and $Y$ are independent, then they are uncorrelated. However, the following counterexample shows that the converse is not true.

## 3.2 Example: Uncorrelated but not Independent

Consider a discrete random variable $X$ with the following PMF:

$$f_X(x) = \begin{cases} \frac{1}{4} & \text{if } x = -1, +1 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$$

Define another random variable $Y = X^2$. The PMF of $Y$ is given by:

$$f_Y(y) = \begin{cases} \frac{1}{2} & \text{if } y = 1 \\ \frac{1}{2} & \text{if } y = 0 \end{cases}$$

Note that the product $XY$ has the same distribution as $X$. Then, $\mathbb{E}[X] = 0$, $\mathbb{E}[Y] = \frac{1}{2}$ and $\mathbb{E}[XY] = 0 = \mathbb{E}[X]\mathbb{E}[Y]$; $X$ and $Y$ are uncorrelated.

However, it is clear by construction that $Y$ is not independent of $X$. For example,

$$\mathbb{P}[X = 1, Y = 1] = \frac{1}{4} \neq \frac{1}{4}\frac{1}{2} = \mathbb{P}[X = 1]\mathbb{P}[Y = 1].$$

### 3.3 Sum of Independent Normal Random Variables

Let $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$ be two independent normal random variables. A remarkable fact is that the sum $X + Y$ of independent random variables is also normally distributed. We can compute its mean and variance as follows.

By linearity of expectation,

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = \mu_1 + \mu_2.$$

Since $X$ and $Y$ are independent, we have $\mathrm{Cov}(X, Y) = 0$. Then,

$$\begin{aligned}
\mathrm{Var}(X + Y) &= \mathbb{E}[((X + Y) - \mathbb{E}[X + Y])^2] \\
&= \mathbb{E}[((X - \mathbb{E}[X]) + (Y - \mathbb{E}[Y]))^2] \\
&= \mathbb{E}[(X - \mathbb{E}[X])^2 + 2(X - \mathbb{E}[X])(Y - \mathbb{E}[Y]) + (Y - \mathbb{E}[Y])^2] \\
&= \mathrm{Var}(X) + 2\mathrm{Cov}(X, Y) + \mathrm{Var}(Y) \\
&= \sigma_1^2 + \sigma_2^2.
\end{aligned}$$

We conclude that $X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. Similar arguments can generalize this result to any number of pairwise independent normal random variables $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ and constants $a_i$:

$$\sum_i X_i \sim \mathcal{N}\left(\sum_i a_i \mu_i, \sum_i a_i^2 \sigma_i^2\right).$$