Problem Set 5

Question 1

Joint density function given by $f_{X,Y}(x,y) = \begin{cases} k(2y+xy) & \text{if } 0 \leq x \leq 1 \text{ and } x \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

Part A

$$\int\int f_{X,Y}(x,y)dxdy = 1 \rightarrow 1 = \int_0^1 \int_x^1 k(2y+xy)dydx + \int\int 0dxdy$$

$$1 = \int_0^1 \int_x^1 [2ky + kxy]dydx = \int_0^1 [ky^2 + \frac{1}{2}kxy^2]\Big|_x^1 dx$$

$$1 = \int_0^1 [k - kx^2 + \frac{1}{2}kx - \frac{1}{2}kx^3]dx = [kx - \frac{1}{3}kx^3 + \frac{1}{4}kx^2 - \frac{1}{8}kx^4]\Big|_0^1$$

$$1 = k[1-0] - \frac{1}{3}[k-0] + \frac{1}{4}[k-0] - \frac{1}{8}[k-0] = 1 = k - \frac{k}{3} + \frac{k}{4} - \frac{k}{8}$$

$$k\frac{19}{24} = 1 \rightarrow k = \frac{24}{19}$$

Part B

$$\text{Marginal PDF of } X \;\; f_X(x) = \int f_{X,Y}(x,y)dy$$

$$f_X(x) = \int_0^1 k(2y+xy)dy = k(y + \frac{1}{2}xy^2)\Big|_0^1$$

$$f_X(x) = k\Big[(1 + \frac{1}{2}x) - (0 + \frac{1}{2}0)\Big] = \frac{24}{19}[1 + \frac{1}{2}x] = \frac{24}{19} + \frac{12}{19}x$$

$$\text{Marginal PDF of } X \;\; f_X(x) = \begin{cases} \frac{24}{19} + \frac{12}{19}x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Part C

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int f_{X,Y}(x,y)dx = \int_0^y k(2y+xy)dx = k(2yx + \frac{1}{2}x^2y)\Big|_0^y = \frac{24}{19}(2y^2 + \frac{1}{2}y^3)$$

$$f_{X|Y}(x|y) = \frac{\frac{24}{19}(2y+xy)}{\frac{24}{19}(2y^2 + \frac{1}{2}y^3)} = \frac{2+x}{2y + \frac{1}{2}y^2)}$$

$$f_{X|Y}(x|\frac{1}{4}) = \frac{2+x}{\frac{1}{2} + \frac{1}{32}} = \frac{2+x}{\frac{17}{32}} = \frac{64+32x}{17}$$

$$f_{X|Y} = \begin{cases} \frac{64+32x}{17} & 0 \leq x \leq \frac{1}{4} \\ 0 & \text{otherwise} \end{cases}$$

Question 2

Lifetime X and Y of two batteries given by joint PDF $f_{X,Y}(x,y) = \begin{cases} 8e^{-4x}e^{-2y} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Part A

$$P(X,Y \geq 2) = \int_2^\infty \int_2^\infty 8e^{-4x}e^{-2y}dxdy$$

$$P(X,Y \geq 2) = \int_2^\infty -2e^{-4x}e^{-2y}\Big|_2^\infty dy = \int_2^\infty -2e^{-\infty}e^{-2y} + 2e^{-8}e^{-2y}dy$$

$$P(X,Y \geq 2) = \int_2^\infty 2e^{-8}e^{-2y}dy = -e^{-8}e^{-2y}\Big|_2^\infty = -e^{-8}e^{-\infty} + e^{-8}e^{-4} = e^{-12} = 6.14 \cdot 10^{-6}$$

Part B

$$f_X(x) = \int_0^\infty 8e^{-4x}e^{-2y}dy = -4e^{-4x}e^{-2y}\Big|_0^\infty = 4e^{-4x-4}\text{for } x \geq 0$$

$$f_Y(y) = \int_0^\infty 8e^{-4x}e^{-2y}dx = -2e^{-4x}e^{-2y}\Big|_0^\infty = 2e^{-2y-8}\text{for } x \geq 0$$

Part C

$$E[XY] = \int_0^\infty \int_0^\infty xyf(x,y)dxdy = \int_0^\infty \int_0^\infty 8xye^{-4x}e^{-2y}dxdy$$

$$E[XY] = \int_0^\infty 8xe^{-4x}dx \int_0^\infty ye^{-2y}dy = -\frac{1}{2}(4x+1)e^{-4x}\Big|_0^\infty \cdot -\frac{1}{4}(2y+1)e^{-2y}\Big|_0^\infty$$

$$E[XY] = -\frac{1}{2}\Big[(4(\infty)+1)e^{-\infty} - (0+1)e^0\Big] \cdot -\frac{1}{4}\Big[(2(\infty)+1)e^{-\infty} - (0+1)e^0\Big]$$

$$E[XY] = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

Part D

$$P(X \leq \frac{Y}{2}) = \int_0^\infty \int_0^{2x} 8e^{-4x}e^{-2y}dydx$$

$$P(X \leq \frac{Y}{2}) = \int_0^\infty -4e^{-4x}e^{-2y}\Big|_0^{2x} dx = \int_0^\infty -4e^{-4x}e^{-4x} + 4e^{-4x}dx$$

$$P(X \leq \frac{Y}{2}) = \frac{1}{2}e^{-8x} - e^{-4x}\Big|_0^\infty = \frac{1}{2}e^{-\infty} - e^{-\infty} - \frac{1}{2}e^0 + e^0 = \frac{1}{2}$$

Part E

X and Y are independent based on the fact that their joint probability is the same as the product of their probabilities.

$$f_Y(y) = -2e^{-4x}e^{-2y}$$

$$f_X(x) = -4e^{-4x}e^{-2y}$$

$$f_{X,Y}(x, y) = 8e^{-4x}e^{-2y}$$

$$f_Y(y)f_X(x) = -4e^{-4x}e^{-2y} \cdot -2e^{-4x}e^{-2y} = 8e^{-4x}e^{-2y}$$

$$\text{Therefore, } f_{X,Y}(x, y) = f_Y(y)f_X(x)$$

Question 3

R is the number of red velvet cupcakes and S the number of salted caramel cupcakes a customer buys, distributed according to the joint PMF:

		s	
$f_{R,S(r,s)}$	0	1	2
0	0	0.25	0.15
1	0.20	0.10	0.10
2	0.10	0.05	0.05

Part A

$$f_S(1) = .25 + .1 + .05 = .4$$

$$f_R(1) = .2 + .1 + .1 = .4$$

$$f_{R|S}(1|1) = \frac{f_{R|S}(1,1)}{f_S(1)} = \frac{.1}{.4} = .25$$

$f_{R|S}(1|1)$ is less than $f_R(1)$ since $.25 < .4$, which is somewhat expected since if someone already chose one cupcake they likely do not want a second to eat.

Part B

$$f_S(2) = .15 + .1 + .05 = .3$$

$$f_{R|S}(0|2) = \frac{f_{R|S}(0,2)}{f_S(2)} = \frac{.1}{.3} = .333$$

$$f_{R|S}(1|2) = \frac{f_{R|S}(1,2)}{f_S(2)} = \frac{.05}{.3} = .166$$

$$f_{R|S}(2|2) = \frac{f_{R|S}(2,2)}{f_S(2)} = \frac{.05}{.3} = .166$$

$$f_{R|S}(r|2) = .333 + .166 + .166 = \frac{2}{3}$$

Part C

Conditional expectation is given by $E[X|Y=y] = \sum_x x f_{X|Y}(x|y)$

$$E[R|S=2] = \sum_r r f_{R|S}(r|2) = (0 \cdot .333) + (1 \cdot .166) + (2 \cdot .166) = \frac{1}{2}$$

Question 4

$$\text{Joint PDF given by } f_{X|Y}(x,y) = \begin{cases} 2ye^{-y(2+x)} & x,y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_0^\infty 2ye^{-y(2+x)}dx = -2\int_0^\infty e^u dx \text{ with } u = -y(2+x)$$

$$f_Y(y) = -2e^u\Big|_0^\infty = -2e^{-y(2+x)}\Big|_0^\infty$$

$$f_Y(y) = -2e^{-y(2+\infty)} + 2e^{-y(2+0)} = 2e^{-2y}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2ye^{-y(2+x)}}{2e^{-2y}}$$

$$f_{X|Y}(x|y) = \frac{ye^{-2y}e^{-yx}}{e^{-2y}} = ye^{-yx}$$

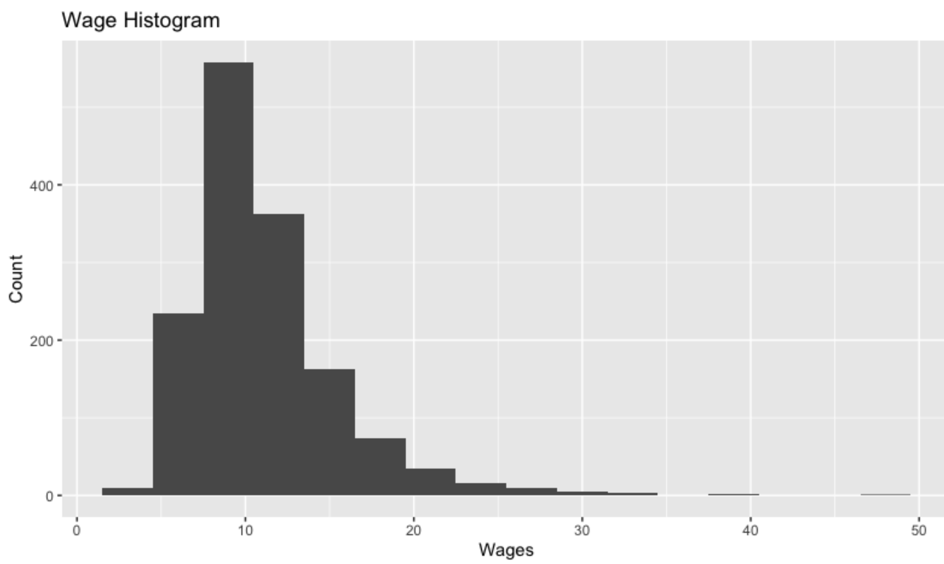
$$f_{X|Y}(x|y) = \begin{cases} ye^{-yx} & x,y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Question 5

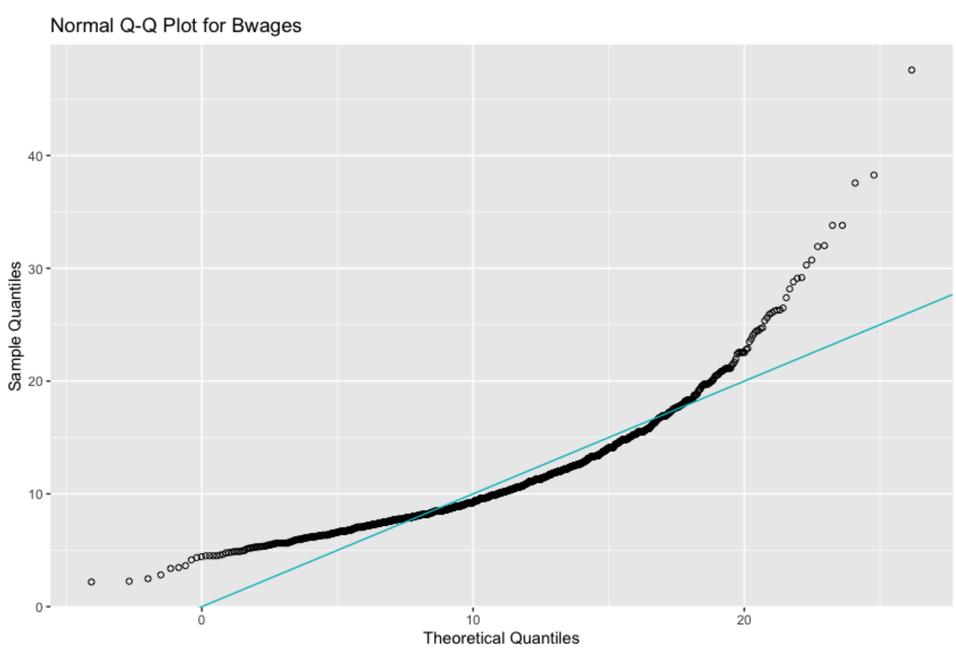
```r
Bwages <- read_csv("Bwages.csv")
```

Part A

```r
ggplot(data = Bwages) + geom_histogram(mapping = aes(x = wage), binwidth = 3) +
xlab("Wages") + ylab("Count") + ggtitle("Wage Histogram")
```



Part B



```r
n = length(Bwages$wage)
qi = (1:n - 0.5)/n
mu = mean(Bwages$wage)
sigma = sd(Bwages$wage)

x = qnorm(qi, mu, sigma)
y = sort(Bwages$wage)
wages <- tibble(x, y)

ggplot(data = wages) + geom_point(mapping = aes(x = x, y = y), shape = 1) + geom_abline(intercept = 0, slope = 1) + xlab("Theoretical Quantiles") + ylab("Sample Quantiles") + ggtitle("Normal Q-Q Plot for Bwages")
```

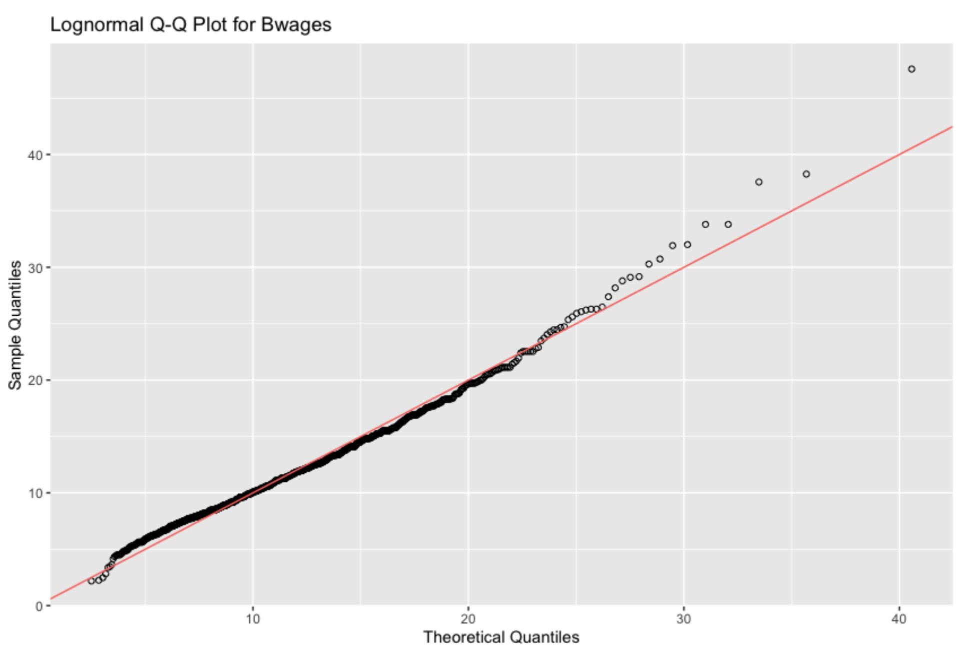
Part C

The normal distribution looks to be a poor fit of the wage data provided. The long tails on both the left and right indicate that the normal distribution is especially poor at modeling edge points. This is likely due to the combined effect of wages never going negative even though the normal distribution does (seen on the axes) and on the right side especially high wages that do not follow the normal distribution well.

Part D

Fit a lognormal distribution to Bwages, using $X \sim \text{Lognormal}(\mu, \sigma^2)$ with PDF of:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-(ln(x)-\mu)^2/(2\sigma^2)} \text{when } x > 0$$




```r
meanlog <- 2.31
sdlog <- .41

n = length(Bwages$wage)
qi = (1:n - 0.5)/n
```

```

x = qlnorm(qi, meanlog, sdlog)
y = sort(Bwages$wage)
wages <- tibble(x, y)

ggplot(data = wages) + geom_point(mapping = aes(x = x, y = y), shape = 1) + geom_abline(intercept = 0, slope = 1, color = "#F8766D") + xlab("Theoretical Quantiles") + ylab("Sample Quantiles") + ggtitle("Lognormal Q-Q Plot for Bwages")
```

Part E

The lognormal distribution is a much better fit than the normal distribution for Bwages. For the bulk of the data, the points now sit very close to the $y = x$ line and the tail on the left hand side where the normal distribution was previously negative is now gone, indicating an improved fit. The right hand side could be improved slightly since the wages are still larger than the lognormal distribution, though these are only outliers and thus could likely be ignored without significant consequence.