Problem Set 4

Question 1

$$f(x) = P(X = x) = \tfrac{1}{n} \qquad x = 0, 1, 2, ...(n-1)$$

Part A

$$F_X(x) = P_X(X \leq x) = \sum_{y \leq x} p_X(y)$$

$$F_X(x) = \sum_{y \leq x} \frac{1}{n} \text{ for } x = 0, 1, 2, ...(n-1)$$

$$\text{CDF, } F_X(x) = \frac{x}{n} \text{ for } x = 0, 1, 2, ...(n-1)$$

$$F_X(x) = \begin{cases} 1 & x \geq n \\ \frac{x}{n} & 0 \leq x \leq (n-1) \\ 0 & x < 0 \end{cases}$$

Part B

Question 2

Part A

Part B

Part C

Question 3

$$\text{PDF, } f(x) = \begin{cases} cx^{-6} & 1 \le x < \infty \\ 0 & x < 1 \end{cases}$$

Part A

$$P(a \le X \le b) = \int_a^b f(u)du$$

$$P(1 \le X \le \infty) = 1$$

$$P(1 \le X \le \infty) = \int_1^\infty cx^{-6}dx = \frac{c}{-5}x^{-5}\Big|_1^\infty = \frac{c}{-5}\infty^{-5} - \frac{c}{-5}1^{-5}$$

$$5 = c \cdot 1^{-5} \longrightarrow c = 5$$

Part B

$$\text{CDF, } F(x) = P(X \le x) = \int_1^x f(u)du$$

$$F(x) = \int_1^x 5x^{-6}dx = -x^{-5}\Big|_1^x = -x^{-5} + (1)^{-5}$$

$$F(x) = 1 - x^{-5}$$

$$\text{CDF, } F(x) = \begin{cases} 1 - x^{-5} & x \ge 1 \\ 0 & x < 1 \end{cases}$$

Part C

$$P(X \le \gamma) = .4$$

$$P(X \le \gamma) = .4 = 1 - \gamma^{-5}$$

$$\gamma^{-5} = .6 \longrightarrow \gamma = 1.12$$

Part D

$$E[X] = \int_{-\infty}^\infty x \cdot f(x)dx = \int_1^\infty x \cdot 5x^{-6}dx = -\frac{5}{4}x^{-4}\Big|_1^\infty = \frac{5}{4} = 1.25$$

$$E[X^2] = \int_{-\infty}^\infty x^2 \cdot f(x)dx = \int_1^\infty x^2 \cdot 5x^{-6}dx = -\frac{5}{3}x^{-3}\Big|_1^\infty = \frac{5}{3} = 1.67$$

$$Var[X] = E[X^2] - E[X]^2 = 1.67 - (1.25^2) = .104$$

Part E

$$E[X^5] = \int_{-\infty}^\infty x^5 \cdot f(x)dx = \int_1^\infty x^5 \cdot 5x^{-6}dx = 5ln(x)\Big|_1^\infty = \infty$$

Question 4

Since if $X < 5$, the time to departure T will be 5, and if $X > 5$ T will be X, so $T = \max 5, X$, with X uniformly distributed from 0 to 15.

$$E(T) = \int_0^5 5 \cdot \frac{1}{15} dx + \int_5^{15} x \cdot \frac{1}{15} dx$$

$$E(T) = \frac{x}{3}\Big|_0^5 + \frac{x^2}{30}\Big|_5^{15} = \frac{5}{3} - 0 + \frac{15^2}{30} - \frac{5^2}{30} = 8.33 \text{ min}$$

Question 5

Gain from one share of stock in company i over the next year is X, with $i = 1, ..., 10$.
X_i are independent normal random variables (Normal(100, 196)) with $\mu = 100$, $\sigma^2 = 196$.

Part A

$$P(X_1 \geq 90) = P\Big(\frac{X_1 - 100}{14} \geq \frac{90 - 100}{14}\Big)$$

$$N(0,1) \sim \frac{X_1 - 100}{14}$$

$$\Phi\Big(\frac{90 - 100}{14}\Big) = P\Big(\frac{X_1 - 100}{14} \leq \frac{90 - 100}{14}\Big)$$

$$P(X_1 \geq 90) = 1 - P\Big(\frac{X_1 - 100}{14} \leq \frac{90 - 100}{14}\Big) = 1 - \Phi\Big(\frac{90 - 100}{14}\Big)$$

$$P(X_1 \geq 90) = 1 - \Phi(-.71) = 1 - .761 = .239$$

Part B

For 10 companies, 100% of the expectation will be 1000, hence 900 for 90%.

$$P\Big(\sum_{i=1}^{10} X_i \geq 900\Big)$$

$$\sum X_i = N\Big(\sum \mu_i, \sum \sigma_1^2\Big)$$

$$\mu = 1000 \ , \ \sigma = \sqrt{1960}$$

$$\Phi\Big(\frac{X - \mu}{\sigma}\Big) = \Phi\Big(\frac{900 - 1000}{\sqrt{1960}}\Big) = \Phi(-2.26) =$$

Part C

independent $(P(X_1 - 2X_2 \geq 10))$ so $X_1 - 2X_2 \sim N(-100, 980)$

$$Y \sim N(-100, 980)$$

$$P(Y \geq 10)^{-}100$$

$$\text{CDF } ,P(Y \leq 1) = \Phi(\frac{X - \mu}{\sigma}) = \Phi(\frac{Y - (-100)}{\sqrt{980}})$$

$$\Phi = N(0,1)$$

$$P(Y \geq 10)$$

$$P(\frac{Y - (-100)}{\sqrt{980}} \geq)$$

Question 6

Part A

$\lambda = 3$

$$Y = -2X + 2$$

with X as $\exp(\lambda)$
CDF of Y, $F_Y(y)$

$$P(Y \leq y) = P(-2X + 2 \leq y) = P(X \geq \frac{-y}{2} + 1) = 1 - P(X \leq \frac{-y}{2} + 1)$$

CDF of $X = P(X \leq \frac{-y}{2} + 1)$

$$F_Y(y) = \begin{cases} 0 \\ < - - - don't actually need this CDF of X \\ 1 \end{cases}$$

Part B

Part C

Part D

FOR QUESTIONS 5 AND 6
$E(X_1 + X_2) = E(X_1) + E(X_2)$
$Var(X_1 + X_2) = Var(X_1) + Var(X_2)$
$Var(5X_1 - X_2) = 5^2 \cdot Var(X_1) + Var(X_2)$