ENGRD 2700: Basic Engineering Probability and Statistics
Fall 2019

Homework 2 Solutions

Due Friday, September 20 at 11:59 pm. Submit to Gradescope by clicking the name of the assignment. See `https://people.orie.cornell.edu/yudong.chen/engrd2700_2019fa.html#homework` for detailed submission instructions.

When completing this assignment (and all subsequent ones), keep in mind the following:

- You must complete the homework individually and independently.
- Provide evidence for each of your answers. If a calculation involves only very minor computation then explain the computation you performed and give the results. If a calculation involves more complicated steps on many many records then hand in the calculations and formulas for the first few records only.
- Write clearly and legibly. You are encouraged to *type* your work although you do not have to. We may deduct points if your answers are difficult to read or disorganized.
- For questions that you answer using R, attach any code that you write, along with the relevant plots. You may use other software, but the same condition applies.
- Submit your homework a single pdf file on Gradescope.

1. You need to give the final answer as an number, along with justification. This same rule applies to all questions in this homework that involves explicit numbers.

   (a) How many different ways are there to divide 12 people into 2 equal-sized groups? (Hint: the naming of the groups does not matter. For example, if we pick $\{1, 2, 3, 4, 5, 6\}$ to form a group "A" and $\{7, 8, 9, 10, 11, 12\}$ to form group "B", it is the same as picking $\{12, 11, 10, 9, 8, 7\}$ to be group "A" and $\{6, 5, 4, 3, 2, 1\}$ to be group "B".)

   The number of different ways to divide 12 people into 2 equal-sized groups is $\binom{12}{6}/2 = 462$.

   (b) How many different ways are there to divide 12 people into 3 equal-size groups? (Use the same hint in (a))

   The number of different ways to divide 12 people into 3 equal-sized groups is $\binom{12}{4}\binom{8}{4}\binom{4}{4}/3! = 495 \times 70 \times 1 \div 6 = 5,775$.

   (c) How many different ways are there to divide 12 people into 2 groups of any sizes? (A group can have zero people.)

   The number of different ways to divide 12 people into 2 groups of any sizes is

   $$\sum_{k=0}^{5} \binom{12}{k} + \binom{12}{6}/2 = 2048$$

   Alternative solution: We can consider whether to put each person into a group and then dividing by 2 to discount for the naming of the group. In this case, we have $2^{12}/2 = 2048$

2. Alice and Bob want to calculate how many distinct words can be made by permuting the letters in "ENGINEER". (Gibberish such as "EEENNGIR" is counted as a word.)

   (a) Alice computes $\binom{8}{3}\binom{5}{2} \cdot 3! = 3360$. Explain the reasoning for this calculation.

   Alice is taking the "start from scratch, build up a word" approach. There are 8 letters in total, 3 of which are "E"s, 2 of which are "N"s, and the remaining are other distinct characters "G", "I" and "R". Alice first decides where the 3 "E"s will belong in her new 8 letter word. Since there are 3 "E"s, she chooses 3 out of 8 positions to insert her Es. Because order does not matter, this is $\binom{8}{3}$. In the remaining 5 positions, she chooses 2 positions for the "N"s. Thus $\binom{5}{2}$. For the last three characters, order does matter because G, I and R are distinct characters. This gives us 3!.

(b) Bob computes $8!/3!/2! = 3360$. Explain the reasoning for this calculation.

Bob is taking the "find all permutations, then remove the same words" approach. Notice that if we treat all letters as distinct letters, there are 8! ways to permute them. Since there are 3 "E"s, we can shuffle the "E"s around and still obtain the same word. That is, if we label "E"s with E_1, E_2 and E_3, $E_1E_2E_3NNGIR$ is the same as $E_1E_3E_2NNGIR$, for example. Therefore, we need to divide 8! by the total number of ways in which we can permute the 3 "E"s. That gives us $8!/3!$. Similarly, since there are 2 "N"s, we need to divide this number again by 2! to obtain $8!/3!/2!$.

(c) This one is for you: How many distinct ways are there to arrange the letters in "ENGINEER" in a circular necklace?

$\left[\binom{8}{3}\binom{5}{2}\cdot 3!\right]/8 = 420$

3. There are 36 Master of Engineering students to assign to projects this year. Suppose project teams must consist of exactly 4 people.

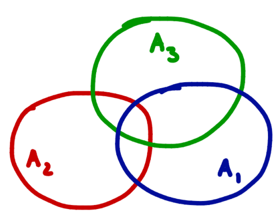
(a) How many different ways are there to construct a single project team?
The number of different ways to construct a single project team is $\binom{36}{4} = 58,905$.

(b) Suppose that there are 16 men and 20 women students. How many different ways are there to construct a single project team consisting of 2 men and 2 women?
The number of different ways to form a team with 2 men and 2 women is $\binom{16}{2}\binom{20}{2} = 22,800$.

4. A person can like any number of sports. We consider three sports in particular. Let A_1 denote the event that a person likes football. Let A_2 denote the event that a person likes basketball. Let A_3 denote the event that a person has likes swimming. Suppose we have determined the following probabilities:

$$P(A_1) = 0.24 \qquad P(A_2) = 0.18 \qquad P(A_3) = 0.1$$
$$P(A_1 \cup A_2) = 0.3 \qquad P(A_1 \cup A_3) = 0.28 \qquad P(A_2 \cup A_3) = 0.24$$
$$P(A_1 \cap A_2 \cap A_3) = 0.02$$

(a) Draw a Venn diagram representing the 3 events.



(b) What is the probability that a person does not like football?
$P(A_1^c) = 1 - P(A_1) = 0.76$

(c) What is the probability that a person likes both football and basketball?
$P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2) = 0.24 + 0.18 - 0.3 = 0.12$

(d) What is the probability that a person likes football and basketball, but not swimming?
$P(A_1 \cap A_2 \cap A_3^c) = P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3) = 0.12 - 0.02 = 0.1$

(e) What is the probability that a person likes at most two of the three sports?
$1 - P(A_1 \cap A_2 \cap A_3) = 1 - 0.02 = 0.98$

5. An office building contains three unreliable elevators, A, B, and C, which are down 35%, 30%, and 10% of the time, respectively.

(a) Let A, B, and C denote the events in which elevators A, B, and C are down, respectively. Describe the following by taking unions, intersections, and/or complements of A, B, and C.

- All three elevators are down. $A \cap B \cap C$
- Elevator A is down, but not B or C. $A \cap B^c \cap C^c = A \cap (B \cup C)^c$
- None of the elevators is down. $A^c \cap B^c \cap C^c = (A \cup B \cup C)^c$ by DeMorgan's Law
- At least one elevator is down. $A \cup B \cup C$
- Exactly one elevator is down. $(A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap A^c \cap B^c)$

(b) What are the maximum and minimum possible values for $P(A \cup B \cup C)$?

$P(A \cup B \cup C)$ attains its maximum if A, B, and C are disjoint. In this case, $P(A \cup B \cup C) = P(A) + P(B) + P(C) = 0.35 + 0.3 + 0.1 = 0.75$. On the other hand, $P(A \cup B \cup C)$ attains its minimum if A contains B, which in turn contains C. In this case, $P(A \cup B \cup C) = P(A) = 0.35$.

(c) You learn that $P(A \cap B) = 0.2$. What is the probability that either Elevator A or B is down?

By inclusion-exclusion, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.35 + 0.3 - 0.2 = 0.45$

(d) Using the information from part (c), compute $P(A|B)$, the probability that Elevator A is down, given that Elevator B is down.

By the definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.3} = 0.67$$

6. (Based on a real application). A certain oil refinery keeps intermediate products in 8 tanks. There are 15 pumps of varying capacity that can be assigned to pump the intermediate products from the 8 tanks into a final-product tank.

Question: How many ways can you assign the 15 pumps to the 8 tanks so that each tank gets at least one pump? Not all pumps need to be assigned. (Each tank may contain a different intermediate product, so assigning e.g., pump 1 to tank 1 and pump 2 to tank 2 is different from assigning pump 2 to tank 1 and pump 1 to tank 2.)

To answer this question, we can proceed as follows: Let $d(k, n)$ be the number of ways to assign k pumps to n tanks in such a way that each tank gets at least one pump, but not all pumps need to be assigned.

(a) Explain why $d(k, n) = 0$ for $n > k \geq 1$.
For $n > k \geq 1$, it means that there are more tanks than pumps and therefore the condition of having each tank with at least one pump cannot be satisfied.

(b) Explain why $d(k, k) = k!$ for all $k \geq 1$.
For $n = k$, each tank will get exactly one pump. The first tank has k different pumps to use, and the second tank can have any of the $k - 1$ remaining pumps. This continues until the k^{th} tank for which there is only one pump left, and therefore we have

$$d(k, k) = k \times (k - 1) \times (k - 1) \cdots 1 = k!\,.$$

(c) Explain why $d(k, 1) = 2^k - 1$ for all $k \geq 1$.
(First solution) This is a single tank case. For each pump, there are two states, namely being assigned or not being assigned to the tank, giving rise to 2^k possible assignments. Yet, we have to exclude the case where all pumps are not assigned, and hence the total number of ways is $2^k - 1$.
(Alternative solution) We can approach the problem in the following way. The tank can have either 1 pump or 2 pumps or $\cdots$ or k pumps. In general, the tank can have j pump(s), with $j = 1, 2, \cdots, k$. The number of ways to choose 1 pump to assign to the tank is $\binom{k}{1}$, and the number of ways to choose 2 pumps to assign to the tank is $\binom{k}{2}$. In general, the number of ways to choose j pumps to assign to

the tank is $\binom{k}{j}$, with $j = 1, 2, \cdots, k$. The total number of ways to choose the pump(s) is therefore

$$\begin{aligned} d(k,1) &= \binom{k}{1} + \binom{k}{2} \cdots + \binom{k}{k} \\ &= \left(\binom{k}{0} + \binom{k}{1} \cdots + \binom{k}{k} \right) - \binom{k}{0} \\ &= (1+1)^k - 1 \qquad \text{(by binomial formula)} \\ &= 2^k - 1\,. \end{aligned}$$

(d) Explain why, for $k > 1$ and $n \geq 1$,

$$d(k,n) = \sum_{j=1}^{k-(n-1)} \binom{k}{j} d(k-j, n-1).$$

Hint: Think which pumps can be assigned to the first tank.
Consider the first tank, which must receive at least one pump. It can't receive more than $k-(n-1)$ pumps, because otherwise the other $n-1$ pumps couldn't receive at least one each. So the possibilities for the number of pumps to assign to the first tank range from 1 through $k-(n-1)$. There are $\binom{k}{j}$ ways to assign j pumps to the first tank. For each of these assignments of j pumps, there are $k-j$ pumps left to assign to the remaining $n-1$ tanks. We multiply $\binom{k}{j}$ by $d(k-j, n-1)$ because every combination is possible.

(e) Answer the **question**. Feel free to use some software, e.g., R.(Hint: you can write recursive function d(k,n) to solve this problem. The base cases would be the results we get from (a), (b), and (c) and the recursive step is the expression in (d). For recursive step you could use "for loop". Examples for recursion: `https://data-flair.training/blogs/r-recursive-function/`. Examples for "for loop": `https://www.r-bloggers.com/how-to-write-the-first-for-loop-in-r/`)
We need to compute $d(15, 8)$. We can write a recursive function using the formula in (d), together with the conditions in (a),(b),(c). Plugging in $k = 15$ and $n = 8$ in the code will yield the desired answer. The answer is 3.3094×10^{13}. See the R file for further reference. Note that the function may be slow for large values of k and n, and one can unpack the recursion and the for loop to lessen the computation time. Example code is in 2700hw2.R

(f) We want to determine the assignment of pumps to tanks that minimizes the time required to pump the intermediate products. Assuming we have a computer that can analyze one million assignments each second, how long in years would it take the computer to analyze all possible assignments?
The required time is $3.3094 \times 10^{13}/1,000,000 = 3.3094 \times 10^{7}$ seconds $= 1.0$ year.

P.S. It turns out that using integer programming (a technique learned in ORIE 3300), we can solve this problem almost instantaneously using off-the-shelf computer packages. Shouldn't you be planning to major in ORIE, or at least take ORIE 3300?