Recitation 2: Counting & Probability

1 Review

When attempting the exercises below, the following may be useful:

- If we sample without replacement and order does matter, then there are

$$\frac{n!}{(n-k)!} = n \times (n-1) \times \cdots \times (n-k+1)$$

  ways to sample k items without replacement from a collection of size n. This can be computed with the help of R's `factorial(n)` function.

- If we sample without replacement and order does not matter, then there are

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

  ways instead. This can be computed in R using `choose(n, k)`.

- If a population contains N_1 Type 1 members and N_2 Type 2 members. If we draw a sample of size n without replacement, then

$$P\left(\text{Sample contains } n_1 \text{ Type 1 Items}\right) = \frac{\binom{N_1}{n_1}\binom{N_2}{n-n_1}}{\binom{N_1+N_2}{n}}$$

  which is called the hypergeometric probability.

2 Exercises

1. A professor is hosting a dinner party. His current wine supply includes 4 bottles of zinfandel, 5 bottles of merlot, and 6 bottles of cabernet (he only drinks red wine), all from different wineries.

   (a) If he wants to serve 3 bottles of zinfandel and serving order is important, how many ways are there to do this?

$$4 \cdot 3 \cdot 2 = \frac{4!}{(4-3)!} = 24$$

   (b) If 3 bottles of wine are to be randomly selected, and order does not matter, how many ways are there to do this?

$$\binom{15}{3} = 455$$

(c) If 3 bottles are randomly selected, what is the probability that this results in a selection consisting of 1 bottle of each variety?

$$\frac{\binom{4}{1}\binom{5}{1}\binom{6}{1}}{455} \approx 0.264$$

(d) If 3 bottles are randomly selected, what is the probability that they are all of the same variety?

$$\frac{\binom{4}{3}+\binom{5}{3}+\binom{6}{3}}{\binom{15}{3}} = \frac{34}{455} \approx 0.075$$

(e) If 3 bottles are randomly selected, what is the probability that exactly 2 of the wines are zinfandels?

$$\frac{\binom{4}{2}\binom{11}{1}}{\binom{15}{3}} = \frac{4 \cdot 3 \cdot 3 \cdot 11}{15 \cdot 14 \cdot 13} = \frac{396}{2730} \approx 0.145.$$

(f) Suppose now that bottles are to be selected one by one until a cabernet is found. What is the probability that it is necessary to examine at least six bottles?
This is the probability of selecting 5 bottles (without replacement) in a row that are not cabernet:

$$p = \frac{9}{15} \cdot \frac{8}{14} \cdot \frac{7}{13} \cdot \frac{6}{12} \cdot \frac{5}{11} \approx 0.042.$$

2. A fair coin is flipped 10 times.

(a) Compute the probability that heads appears 3 times.
Since the coin is fair the probability is

$$\binom{10}{3} 0.5^{10}.$$

(b) Suppose that heads does appear 3 times. What is the probability that all three heads occurred within the first five flips?
From the previous part, the number of ways that 3 heads can occur in 10 flips is

$$\binom{10}{3}.$$

Now, the number of ways that 3 heads appear, all within the first five flips is

$$\binom{5}{3} \cdot 1,$$

where the 1 stands for the unique combination $TTTTT$ that must occur in the last 5 flips. Hence the desired probability is

$$\frac{\binom{5}{3}}{\binom{10}{3}} = \frac{5 \times 4 \times 3}{10 \times 9 \times 8} = \frac{1}{2} \cdot \frac{1}{3}\frac{1}{2} = \frac{1}{12}.$$

3. Suppose we have two fair four-sided dice: one green, one red.

   (a) Describe $\mathcal{S}$, the set of all possible outcomes. (There are 16 of them.)
   Denote by G_i, $i = 1, \dots, 4$, the outcome of rolling face i with the green die, so the possible outcomes for the green die are $\{G_1, G_2, G_3, G_4\}$. Similarly say the possible outcomes for the red die are $\{R_1, \dots, R_4\}$. Then

$$\begin{aligned} S = \{&(G_1, R_1), (G_1, R_2), (G_1 R_3), (G_1, R_4),\\ &(G_2, R_1), (G_2, R_2), (G_2, R_3), (G_2, R_4),\\ &(G_3, R_1), (G_3, R_2), (G_3, R_3), (G_3, R_4),\\ &(G_4, R_1), (G_4, R_2), (G_4, R_3), (G_4, R_4)\}. \end{aligned}$$

   (b) Suppose we define the following events:
   - A : The red die shows a 3.
   - B : The green die shows a 3.
   - C : Both dice show the same number.
   - D : The sum of the dice is 7.
   - E : The sum of the dice is even.

   Specify each event as a subset of the sample space.

   - $A = \{(G_1, R_3), (G_2, R_3), (G_3, R_3), (G_4, R_3)\}$.
   - $B = \{(G_3, R_1), (G_3, R_2), (G_3, R_3), (G_3, R_4)\}$.
   - $C = \{(G_1, R_1), (G_2, R_2), (G_3, R_3), (G_4, R_4)\}$.
   - $D = \{(G_3, R_4), (G_4, R_3)\}$.
   - $E = \{(G_1, R_1), (G_1, R_3), (G_2, R_2), (G_2, R_4), (G_3, R_1), (G_3, R_3), (G_4, R_2), (G_4, R_4)\}$.

   (c) Give the probability of each of the events from part (b).
   The probability of rolling any particular face, for either die, is $1/4$ and therefore the probability of each outcome is $1/16$. Hence,
   - $P(A) = 4/16 = 1/4$.
   - $P(B) = 4/16 = 1/4$.
   - $P(C) = 4/16 = 1/4$.
   - $P(D) = 2/16 = 1/8$.
   - $P(E) = 8/16 = 1/2$.