Problem Set 6

Question 1

$$f_{X,Y}(x,y) = \begin{cases} 1/8 & x \in [0,1), y \in [0,1) \\ 1/8 & x \in [1,2), y \in [1,2) \\ 3/8 & x \in [0,1), y \in [1,2), \text{ or } x \in [1,2), y \in [0,1) \\ 0 & \text{otherwise.} \end{cases}$$

Need to find expectations and variances of individual and combinations of X and Y for covariance and correlation formulas.

$$f_X(x) = \int_0^2 f_{X,Y}(x,y)dy = \int_0^2 \frac{1}{8}dy = \frac{1}{4}$$

$$f_Y(y) = \int_0^2 f_{X,Y}(x,y)dx = \int_0^2 \frac{1}{8}dx = \frac{1}{4}$$

$$E(X) = \int_0^2 xf_X(x)dx = \int_0^2 \frac{1}{4}xdx = \frac{1}{8}x^2\Big|_0^2 = \frac{1}{2}$$

$$E(Y) = \int_0^2 yf_Y(y)dy = \int_0^2 \frac{1}{4}ydy = \frac{1}{8}x^2\Big|_0^2 = \frac{1}{2}$$

$$E(X^2) = \int_0^2 x^2f_X(x)dx = \int_0^2 \frac{1}{4}x^2dx = \frac{1}{12}x^3\Big|_0^2 = \frac{2}{3}$$

$$E(Y^2) = \int_0^2 x^2f_Y(y)dy = \int_0^2 \frac{1}{4}y^2dy = \frac{1}{12}y^3\Big|_0^2 = \frac{2}{3}$$

$$Var(X) = Var(X^2) - Var(X)^2 = \frac{2}{3} - \frac{1}{2}^2 = \frac{5}{12}$$

$$Var(Y) = Var(Y^2) - Var(Y)^2 = \frac{2}{3} - \frac{1}{2}^2 = \frac{5}{12}$$

$$E(XY) = \int_0^2\int_0^2 xy \cdot f_{X,Y}(x,y)dxdy = \int_0^2\int_0^2 \frac{1}{8}xydxdy + 2\int_0^1\int_1^2 \frac{3}{8}xydxdy = \frac{1}{2} + 2\cdot\frac{9}{32} = \frac{17}{16}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{17}{16} - \frac{1}{2}\cdot\frac{1}{2} = \frac{13}{16}$$

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{\frac{13}{16}}{\sqrt{\frac{5}{12}\cdot\frac{5}{12}}} = 1.95$$

Question 2

Stocks X and Y with normally distributed annual returns, with $\mu_1 = \mu_2 = 12$ and variance of $\sigma_1^2 = 9$, $\sigma_2^2 = 16$.

Part A

Assume X and Y are independent.

Since the sum of two normal distributions is also normally distributed, X and Y can be summed. Then, the probability can be found using a standard normal table.

$$\mu_{X+Y} = E(X+Y) = E(X) + E(Y) = 12 + 12 = 24$$

$$\sigma_{X+Y}^2 = Var(X+Y) = \sigma_1^2 + \sigma_2^2 = 9 + 16 = 25$$

$$P(X+Y > 25) = 1 - \Phi\Big(\frac{X-\mu}{\sigma}\Big) = 1 - \Phi\Big(\frac{25-24}{5}\Big) = 1 - .579 = .421$$

$$P(X+Y > 25) = .421$$

Part B

Assume X and Y are negatively correlated, where $\rho = -.6$.

Since the sum of two normal distributions is also normally distributed when negatively correlated, follow the same steps as before.

$$\mu_{X+Y} = E(X+Y) = E(X) + E(Y) = 12 + 12 = 24$$

$$\sigma_{X+Y}^2 = Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

$$\rho = \frac{Cov(X,Y)}{\sigma_1 \cdot \sigma_2} \quad \rightarrow \quad Cov(X,Y) = \rho \cdot \sigma_1 \cdot \sigma_2$$

$$\sigma_{X+Y}^2 = \sigma_1^2 + \sigma_2^2 + 2 \cdot \sigma_1 \cdot \sigma_2 = 9 + 16 + 2 \cdot -.6 \cdot 3 \cdot 4 = 25 - 14.4 = 10.6$$

$$P(X+Y > 25) = 1 - \Phi\Big(\frac{X-\mu}{\sigma}\Big) = 1 - \Phi\Big(\frac{25-24}{\sqrt{10.6}}\Big) = 1 - \Phi(.307) = 1 - .644 = .356$$

Part C

Assume X and Y are negatively correlated, where $\rho = -.6$.

$$Var(wX + (1-w)Y) = Var(wX) + Var((1-w)Y) + 2Cov(wX, (1-w)Y)$$

$$Var(wX + (1-w)Y)w^2Var(X) + (1-w)^2Var(Y) + w(1-w)(2 \cdot \rho \cdot \sigma_1 \cdot \sigma_2)$$

$$Var(wX+(1-w)Y) = 9w^2+16(1-w)^2-.6(12)w(1-w) = 9w^2+16w^2-32w+16+14.4w^2-14.4w$$

$$Var(wX+(1-w)Y) = 39.4w^2 - 46.4w + 16$$

Variance is quadratic, so it can be minimized by looking at the minimum where $w = -\frac{b}{2a}$.

$$w = -\frac{-46.6}{2 \cdot 39.4} = .59$$

Thus, a w of .59 will minimize the variance of the total return.

Part D

Assume X and Y are perfectly negatively correlated where $\rho = -1$.

$$Var(wX+(1-w)Y) = Var(wX) + Var((1-w)Y) + 2Cov(wX, (1-w)Y)$$

$$Var(wX+(1-w)Y) = w^2Var(X) + (1-w)^2Var(Y) + w(1-w)(2 \cdot \rho \cdot \sigma_1 \cdot \sigma_2)$$

$$Var(wX+(1-w)Y) = 9w^2+16(1-w)^2-(24)w(1-w) = 9w^2+16w^2-32w+16+24w^2-24w$$

$$Var(wX+(1-w)Y) = 49w^2 - 56w + 16$$

$$w = -\frac{b}{2a} = -\frac{-56}{2 \cdot 49} = .57$$

Thus, a w of .57 will minimize the variance of the total return.

Part E

The variance associated with a w of .57 and a $\rho = -1$ is 0 compared to a higher variance for a w of .59 and $\rho = -.6$ based on graphing. The scenario with no variance is best since it ensures safer returns rather than variability and unexpectedness.

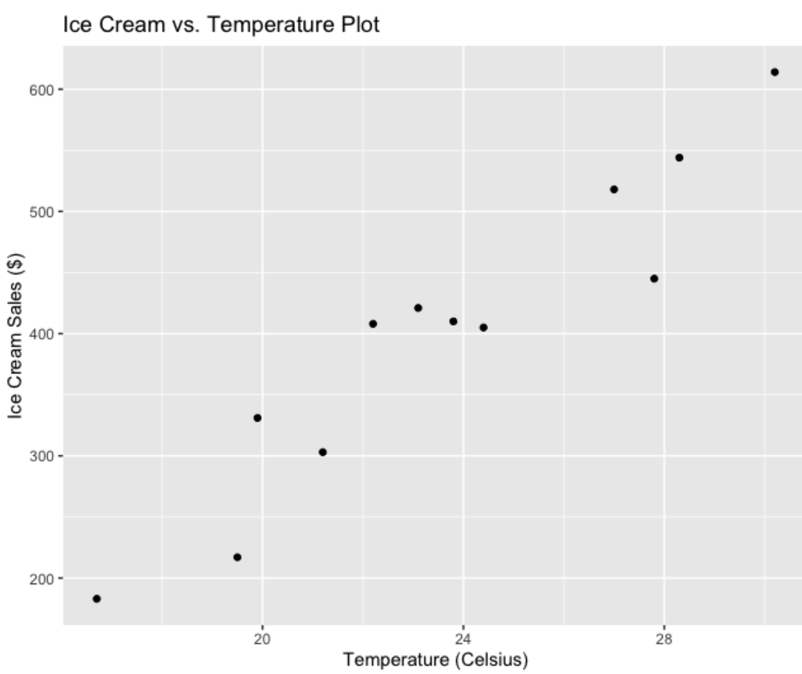
Question 3

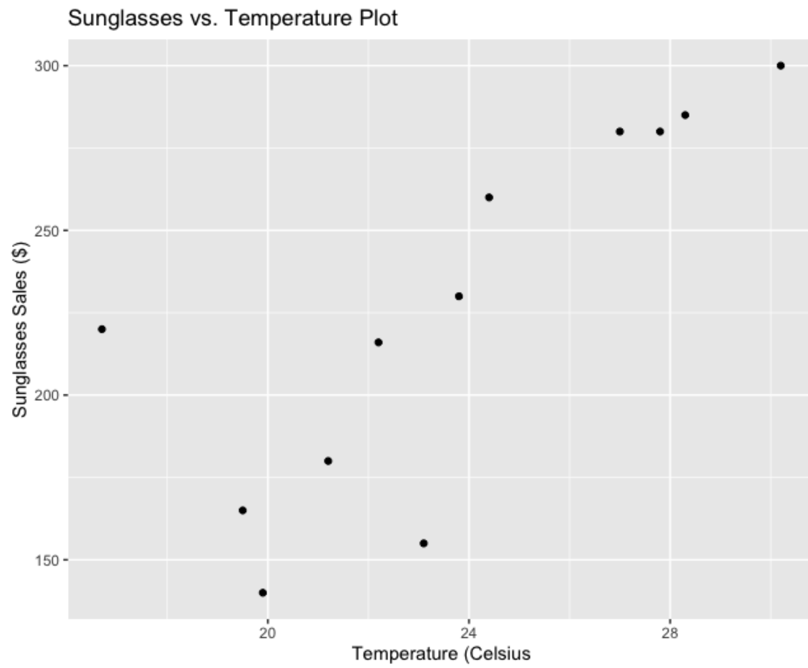
```
library(tidyverse)
data <- read_csv("DataForSunglasses.csv")
temp <- select(data, ’Temperature(Celsius)’)
sunglasses <- select(data, ’Sunglass Sales($)’)
icecream <- select(data, ’Ice Cream Sales($)’)
data <- tibble(temp, sunglasses, icecream)
```

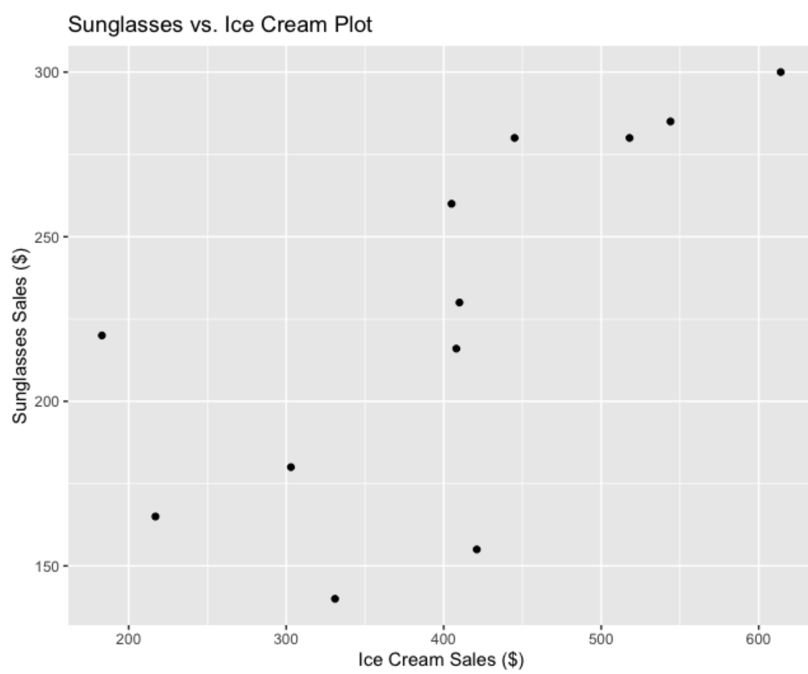
Part A

Correlation between ice cream sales and temperature is .950, between sunglass sales and temperature is .790, and between sunglass sales and ice cream sales is .699.

```
ggplot(data = data) + geom_point(aes(x = data$‘Temperature(Celsius)‘, y = data$‘Ice Cream Sales($)‘)) + xlab("Temperature (Celsius)") + ylab("Ice Cream Sales ($)") + ggtitle("Ice Cream vs. Temperature Plot")

ggplot(data = data) + geom_point(aes(x = data$‘Ice Cream Sales($)‘, y = data$‘Sunglass Sales($)‘)) + xlab("Ice Cream Sales ($)") + ylab("Sunglasses Sales ($)") + ggtitle("Sunglasses vs. Ice Cream Plot")

ggplot(data = data) + geom_point(aes(x = data$‘Temperature(Celsius)‘, y = data$‘Sunglass Sales($)‘)) + xlab("Temperature (Celsius") + ylab("Sunglasses Sales ($)") + ggtitle("Sunglasses vs. Temperature Plot")

cor(data$‘Temperature(Celsius)‘, data$‘Ice Cream Sales($)‘)
cor(data$‘Ice Cream Sales($)‘, data$‘Sunglass Sales($)‘)
cor(data$‘Temperature(Celsius)‘, data$‘Sunglass Sales($)‘)
```







Part B

No, positive correlation does not mean that one event causes another, just that a certain often happens when another event also happens too, such as buying sunglasses and ice cream.

Question 4

Part A

The Central Limit Theorem gives that $L \sim N(n\mu, n\sigma^2)$ so that $P(L \leq X) = \Phi\left(\frac{X-n\mu}{\sqrt{n}\sigma}\right)$

Flipping a coin is a Bernouli random variable, so $X \sim Bernouli(.5)$.

$$E(X) = .5 \quad \text{and} \quad Var(X) = p(1-p) = .5(1-.5) = .25$$

$$P(100X \leq 46) = \Phi\left(\frac{X - n\mu}{\sqrt{n}\sigma}\right) = \Phi\left(\frac{46-50}{\sqrt{10}\cdot .5}\right) = \Phi(-.8) = .212$$

Part B

$$X \sim Binomial(100, .5)$$

$$P(X \leq 46) = \sum_{i=1}^{46} \binom{100}{i} \cdot .5^i \cdot .5^{100-i} = \sum_{i=1}^{46} \binom{100}{i} \cdot .5^{100}$$

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pbinom(46, 100, .5)
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$$P(X \leq 46) = .242$$

Part C

$$P(100X \leq 46.5) = \Phi\left(\frac{46.5-50}{\sqrt{10}\cdot .5}\right) = \Phi(-.7) = -.242$$

Question 5

Part A

Profit for a house without fire would be \$120 with $P = .99$ while for a house with fire is \$120 minus \$10000 payout, so \$-9880 with $P = .01$.

$$E(X) = .99 \cdot 120 + .01 \cdot -9880 = 20$$

$$E(X_{10}) = 10 \cdot 20 = \$200$$

Part B

Since there are only 10 houses currently sold insurance, the revenue is \$1200 while the payout for a house with a fire is \$10000 which is significantly more, making any fire a cause of bankruptcy. Since the probability of fire is independent for each house the probability of bankruptcy is:

$$P(fire) = 1 - .99^{10} = .0956$$

Part C

The Central Limit Theorem gives that $L \sim N(n\mu, n\sigma^2)$ so that $P(L \leq X) = \Phi\left(\frac{X-n\mu}{\sqrt{n}\sigma}\right)$

For 1 million homes, the revenue of the company will be \$120000000, so the sum of payouts must be greater than the revenue for bankruptcy, so X will be the payouts.

$$E(X) = .01 \cdot 10000 + .99 \cdot 0 = 100$$

$$Var(X) = E(X^2) - E(X)^2 = .01 \cdot 10000^2 - 100^2 = 990000$$

$$P(X \geq 120000000) = 1 - \Phi\left(\frac{X - n\mu}{\sqrt{n}\sigma}\right) = 1 - \Phi\left(\frac{120000000 - 1000000 \cdot 100}{\sqrt{1000000} \cdot \sqrt{990000}}\right) = 1 - \Phi(20.1) = 0$$

Since z score of 20 is very far to the right of the normal distribution.

Question 6

Light bulb with lifetime exponentially distributed with rate parameter $\lambda = 5$, where L is a random variable for the sum of the lifetimes of 50 independent light bulbs.

Part A

For one bulb, $L_1 \sim Exp(5)$ so $E(L_1) = 5$ and $Var(L_1) = \frac{1}{25}$.

Since all of the bulbs are independent they sum together:

$$E(L) = \sum^{50} \frac{1}{5} = 50 \cdot \frac{1}{5} = 10$$

$$Var(L) = \sum^{50} \frac{1}{25} = 50 \cdot \frac{1}{25} = 2$$

Part B

The Central Limit Theorem gives that $L \sim N(n\mu, n\sigma^2)$ so that $P(L \leq X) = \Phi\Big(\frac{X-n\mu}{\sqrt{n}\sigma}\Big)$

$$P(8 \leq L \leq 12) = P(L \leq 12) - P(L \leq 8) = \Phi\Big(\frac{12-10}{\sqrt{2}}\Big) - \Phi\Big(\frac{8-10}{\sqrt{2}}\Big)$$

$$P(8 \leq L \leq 12) = \Phi(1.41) - \Phi(-1.41) = .9207 - .0793 = .8414$$

Part C

Interval (a, b) centered on $E(L)$ where $P(a \leq L \leq b) = .95$.

The normal is centered at $E(L)$ so if the contained integral between a and b must be .95 then .05 is left over for the tails of the normal. The probability that L is greater or less than either a or b will be .05 split, so .025.

$$P(E(L) - c \leq .025) = \Phi\Big(\frac{X - n\mu}{\sqrt{n}\sigma}\Big) = \Phi\Big(\frac{10 - c - 10}{\sqrt{2}}\Big) = \Phi\Big(-\frac{c}{\sqrt{2}}\Big)$$

Based on the standard normal table, a z score of -1.96 will give .025.

$$-\frac{c}{\sqrt{2}} = -1.96 \quad \rightarrow \quad c = 1.96 \cdot \sqrt{2} = 2.77$$

The interval will then be $(10 - 2.77, 10 + 2.77)$, so that $(a, b) = (7.23, 12.77)$