Problem Set 3

Question 1

Lifetime of a light bulb T is given by $P(T > t) = e^{-t/3}$ for all $t \geq 0$. The bulb has lasted x years, so the conditional probability that it will last at most $x+2$ years is given by the conditional probability equation.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = 1 - e^{-(x+2)/3} - 1 + e^{-x/3}$$

$$P(B) = e^{-x/3}$$

$$P(A|B) = \frac{e^{-x/3} - e^{-(x+2)/3}}{e^{-x/3}} = \frac{e^{-x/3} - e^{-x/3}e^{-2/3}}{e^{-x/3}} = 1 - e^{-2/3} = .487$$

Thus the probability that given the bulb has lasted x years, it will last at most $x + 2$ years is .487, which is independent of x.

Question 2

Part A

The probability that both children are female, assuming the first is female, is given by the conditional probability equation $P(A|B)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.25}{.5} = .5$$

There is a probability of .5 that given the first child is female that the second will also be female.

Part B

Here the conditional probability equation is used again, but the order of children no longer matters.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.25}{.75} = .333$$

There is a probability of .333 that given as least one child is female that the other will also be female.

Part C

$$P(\text{ff} \mid \geq 1 \text{ Katie}) = \frac{P(\text{ff} \cap \geq 1 \text{ Katie})}{P(\geq 1 \text{ Katie})} = \frac{.25 \cdot (2p(1-p) + p^2)}{.5p + .25 \cdot (2p(1-p) + p^2)} = \frac{.5p - .5p^2 + .25p^2}{.5p + .5p - .5p^2 + .25p^2}$$

$$= \frac{.5p - .25p^2}{p - .25p^2} = \frac{.5 - .25p}{1 - .25p}$$

Question 3

Part A

A = bag is identified as dangerous
B = bag contains explosives
C = false positive, safe bag identified as containing explosives

$$P(A \cap B) = P(B|A) \cdot P(A) = .9 \cdot (10/4000000) = 2.25 \times 10^{-6}$$

$$P(B) = P(B|A) \cdot P(A) + P(C) = .9 \cdot (10/4000000) + (1 - .99) = .01$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2.25 \times 10^{-6}}{.01} = .000225$$

Part B

D = bag without explosives correctly identified

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2.25 \times 10^{-6}}{2.25 \times 10^{-6} + (1 - P(D))} \geq .5$$

$$2.25 \times 10^{-6} \geq .5\big[2.25 \times 10^{-6} + (1 - P(D))\big]$$

$$2.25 \times 10^{-6} \geq .5 \cdot 2.25 \times 10^{-6} + .5 - .5 \cdot P(D)$$

$$1 - 2.25 \times 10^{-6} \leq P(D)$$

$$P(D) \geq .99999775$$

Part C

No, it is not possible to make $P(A|B)$ be at least .5 by increasing the chance of correclty identifying bags containing explosives. It would require increasing $P(A)$ so significantly that it would make it above 1, thus breaking the standard laws of probabilities ranging from 0 to 1.

Question 4

Part A

The set $\mathcal{X}$ of possible values for X is given by $S = \{1, 2, 4, 8, 16, 32, ...\}$ (or by 2^x where x can take all integers starting at 0).

Part B

Since each coin flip has a .5 chance of heads and a .5 chance of tails, the probability of n number of successive tails will be given by $.5^{n+1}$ since the first probability is .5, thus the PMF of X is given by:

$$p_X(2^n) = .5^{n+1}$$

Part C

$$P(X > 40) = 1 - \sum_{i}^{40} P_X(2^n) = 1 - .5^1 - .5^2 - .5^3 - .5^4 - .5^5 = .03125$$

Part D

$$E(X) = \sum_{x \in \mathcal{X}} x \cdot P_X(X = x) = \sum_{x \in \mathcal{X}} 2^n \cdot .5^{n+1} = \sum_{x \in \mathcal{X}} .5 = \infty$$

Part E

Possible values for Y are 2^n for values of n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
The PMF of Y will just be $P_X(X = x) = .5^{n+1}$ for n 1 through 9, and $2^10 for n = 10$.

Question 5

Part A

$$\text{f(x)} = \begin{cases} P(X=-2) & = .2 \\ P(X=-1) & = .1 \\ P(X=0) & = .4 \\ P(X=1) & = .1 \\ P(X=2) & = .2 \end{cases}$$

Part B

$$E[X] = \sum_{x \in \mathcal{X}} x \cdot P_X(X=x) = (-2 \cdot .2) + (-1 \cdot .1) + (0 \cdot .4) + (1 \cdot .1) + (2 \cdot .2) = -.4 - .1 + 0 + .1 + .4 = 0$$

$$E[X^2] = \sum_{x \in \mathcal{X}} x^2 \cdot P_X(X=x) = (4 \cdot .2) + (1 \cdot .1) + (0 \cdot .4) + (1 \cdot .1) + (4 \cdot .2) = .4 + .1 + 0 + .1 + .4 = 1$$

$$Var(X) = E(X^2) - E(X)^2 = 1 - 0 = 1$$

Part C

$$E[sin(X)] = \sum_{x \in (X)} sin(x) \cdot P_X(X=x) = (sin(-2) \cdot .2) + (sin(-1) \cdot .1) + (sin(0) \cdot .4) + (sin(1) \cdot .1) + (sin(2) \cdot .2) = 0$$

Question 6

Part A

Making or missing a freethrow is classified as a Bernoulli Trial since there are only two outcomes, modelled by a binomial distribution. Since X is a binomial distribution, it has parameters: $n = 1000$ and $\rho = .07$.

$$E(X) = n\rho = 1000 \cdot .07 = 70$$

$$Var(X) = n\rho(1-\rho) = 1000 \cdot .07(1-.07) = 65.1$$

Part B

$$P_X(X = 61) = \binom{n}{x}\rho^x(1-\rho)^{n-x} = \binom{1000}{61}.07^{61}(1-.07)^{1000-61} = 9.046\times10^{-101}\cdot3.042\times10^{98} = .0275$$

Part C

$$P_X(X = 61) = \frac{e^{-n\rho}(n\rho)^x}{x!} = \frac{e^{-1000\cdot.07}(1000\cdot.07)^61}{61!} = .0279$$

Part D

$$P(\geq 60 \text{ misses}|55 \text{ misses}) = \frac{P(\geq 60 \text{ misses} \cap 55 \text{ misses})}{P(55 \text{ misses})} =$$

Part E