Integral definido. Exercícios resolvidos.

a) Calcular os integrais definidos utilizando a fórmula de Barrow.

Exercício 1. $\int_0^1 \sqrt{1+x}\,dx$.

$$\int_0^1 \sqrt{1+x}\,dx = \int_0^1 \sqrt{1+x}\,d\,(1+x) = \int_0^1 (1+x)^{\frac{1}{2}}\,d\,(1+x) = \left.\left(\frac{(1+x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)\right|_0^1 = \left.\left(\frac{(1+x)^{\frac{3}{2}}}{\frac{3}{2}}\right)\right|_0^1 =$$

$$= \frac{2}{3}\cdot(1+x)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}\cdot\left[(1+1)^{\frac{3}{2}} - (1+0)^{\frac{3}{2}}\right] = \frac{2}{3}\cdot\left(2^{\frac{3}{2}} - 1\right) = \frac{2}{3}\cdot\left(2\sqrt{2}-1\right).$$

Exercício 2. $\int_{-1}^1 \frac{x\,dx}{\left(x^2+1\right)^2}$.

$$\int_{-1}^1 \frac{x\,dx}{\left(x^2+1\right)^2} = \int_{-1}^1 \frac{\frac{1}{2}d(x^2)}{\left(x^2+1\right)^2} = \frac{1}{2}\cdot\int_{-1}^1 \frac{d(x^2)}{\left(x^2+1\right)^2} = \frac{1}{2}\cdot\int_{-1}^1 \frac{d(x^2+1)}{\left(x^2+1\right)^2} = \frac{1}{2}\cdot\int_{-1}^1 \left(x^2+1\right)^{-2} d(x^2+1) =$$

$$= \frac{1}{2}\cdot\left.\left(\frac{\left(x^2+1\right)^{-2+1}}{-2+1}\right)\right|_{-1}^1 = -\frac{1}{2}\cdot\left(\left(x^2+1\right)^{-1}\right)\Big|_{-1}^1 = -\frac{1}{2}\cdot\left.\left(\frac{1}{x^2+1}\right)\right|_{-1}^1 = -\frac{1}{2}\cdot\left(\frac{1}{1^2+1} - \frac{1}{(-1)^2+1}\right) = 0$$

Exercício 3. $\int_1^e \frac{(1+lnx)\,dx}{x}$.

$$\int_1^e \frac{(1+lnx)\,dx}{x} = \int_1^e (1+lnx)\cdot\frac{dx}{x} = \int_1^e (1+lnx)\cdot d(lnx) = \int_1^e d(lnx) + \int_1^e lnx\,d(lnx) =$$

$$= (lnx)\Big|_1^e + \left.\left(\frac{ln^2 x}{2}\right)\right|_1^e = (ln\,e - ln1) + \left(\frac{ln^2 e}{2} - \frac{ln^2 1}{2}\right) = 1 + \frac{1}{2} = \frac{3}{2}.$$

Exercício 4. $\int_0^2 \frac{dx}{\sqrt{16-x^2}}$.

$$\int_0^2 \frac{dx}{\sqrt{16-x^2}} = \int_0^2 \frac{dx}{\sqrt{4^2-x^2}} = arcsen\left(\frac{x}{4}\right)\Bigg|_0^2 = arcsen\left(\frac{2}{4}\right) - arcsen\left(\frac{0}{4}\right) =$$

$$= arcsen\left(\frac{1}{2}\right) - arcsen(0) = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

Exercício 5. $\int_0^1 \frac{dx}{x^2+4x+5}$.

O polinómio do denominador não tem raizes reais:

$$x^2+4x+5=0 \quad \Rightarrow \quad x = \frac{-4\pm\sqrt{16-20}}{2} = \frac{-4\pm\sqrt{-4}}{2}.$$

Portanto é uma fracção elementar do terceiro tipo.

$$\int_0^1 \frac{dx}{x^2+4x+5} = \int_0^1 \frac{dx}{x^2+4x+4+1} = \int_0^1 \frac{dx}{(x+2)^2+1} = \int_0^1 \frac{d(x+2)}{1^2+(x+2)^2} =$$

$$= \left(arctg(x+2)\right)\Big|_0^1 = arctg(1+2) - arctg(0+2) = arctg(3) - arctg(2).$$

Exercício 6. $\int_0^{\frac{\pi}{2}} sen(2x)\,dx$.

$\underline{1^o \quad \text{método}:}$

$$\int_0^{\frac{\pi}{2}} sen(2x)\,dx = \int_0^{\frac{\pi}{2}} sen(2x)\cdot\frac{1}{2}\cdot d(2x) = \frac{1}{2}\cdot\int_0^{\frac{\pi}{2}} sen(2x)\cdot d(2x) = -\frac{1}{2}\cdot\left(cos(2x)\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= -\frac{1}{2}\cdot\left(cos\left(2\cdot\frac{\pi}{2}\right) - cos(2\cdot 0)\right) = -\frac{1}{2}\cdot\left(cos(\pi) - cos(0)\right) = -\frac{1}{2}\cdot(-1-1) = 1.$$

2^o método:

$$\int_0^{\frac{\pi}{2}} sen(2x)\,dx = \int_0^{\frac{\pi}{2}} 2\cdot senx\cdot\cos x\cdot dx = 2\cdot\int_0^{\frac{\pi}{2}} senx\cdot d(senx) = 2\cdot\left(\frac{sen^2x}{2}\right)\Bigg|_0^{\frac{\pi}{2}} =$$

$$= 2\cdot\left(\frac{sen^2\left(\frac{\pi}{2}\right)}{2} - \frac{sen^2 0}{2}\right) = 2\cdot\left(\frac{1^2}{2} - \frac{0^2}{2}\right) = 1.$$

Exercício 7. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx}{sen^3x}\,dx$.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{cosx}{sen^3x}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{d(senx)}{sen^3x} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (sen\,x)^{-3}\cdot d(sen\,x) = \left(\frac{(sen\,x)^{-3+1}}{-3+1}\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = -\frac{1}{2}\cdot\left(\frac{1}{sen^2x}\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}} =$$

$$= -\frac{1}{2}\cdot\left(\frac{1}{sen^2\left(\frac{\pi}{2}\right)} - \frac{1}{sen^2\left(\frac{\pi}{6}\right)}\right) = -\frac{1}{2}\cdot\left(\frac{1}{1^2} - \frac{1}{\left(\frac{1}{2}\right)^2}\right) = -\frac{1}{2}\cdot(1-4) = \frac{3}{2}.$$

Exercício 8. $\displaystyle\int_0^1 \left(x - 2e^x\right)\cdot dx$.

$$\int_0^1 \left(x-2e^x\right)\cdot dx = \int_0^1 x\cdot dx - \int_0^1 2e^x\cdot dx = \int_0^1 x\cdot dx - 2\cdot\int_0^1 e^x\cdot dx = \left(\frac{x^2}{2}\right)\Bigg|_0^1 - 2\cdot\left(e^x\right)\Big|_0^1 =$$

$$= \left(\frac{1^2}{2} - \frac{0^2}{2}\right) - 2\cdot\left(e^1 - e^0\right) = \frac{1}{2} - 2e + 2 = \frac{5}{2} - 2e.$$

Exercício 9. $\displaystyle\int_0^8 \left(\sqrt{2x} + \sqrt[3]{x}\right)\cdot dx$.

$$\int_0^8 \left(\sqrt{2x}+\sqrt[3]{x}\right)\cdot dx = \int_0^8 \sqrt{2x}\cdot dx + \int_0^8 \sqrt[3]{x}\cdot dx = \sqrt{2}\cdot\int_0^8 x^{\frac{1}{2}}\cdot dx + \int_0^8 x^{\frac{1}{3}}\cdot dx =$$

$$= \sqrt{2}\cdot\left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)\Bigg|_0^8 + \left(\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right)\Bigg|_0^8 = \sqrt{2}\cdot\frac{2}{3}\cdot\left(x^{\frac{3}{2}}\right)\Big|_0^8 + \frac{3}{4}\cdot\left(x^{\frac{4}{3}}\right)\Big|_0^8 =$$

$$= \frac{2\sqrt{2}}{3}\cdot\left(8^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) + \frac{3}{4}\cdot\left(8^{\frac{4}{3}} - 0^{\frac{4}{3}}\right) = \frac{2\sqrt{2}}{3}\cdot\sqrt{8^3} + \frac{3}{4}\cdot\sqrt[3]{8^4} = \frac{2\sqrt{2}}{3}\cdot 16\sqrt{2} + \frac{3}{4}\cdot 16 = \frac{100}{3}.$$

Exercício 10. $\int_1^2 \frac{2x-1}{x^3+x}\cdot dx$.

A função $\frac{2x-1}{x^3+x}$ é racional.

$$\frac{2x-1}{x^3+x} = \frac{2x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \quad \Rightarrow \quad 2x-1 = Ax^2 + A + Bx^2 + Cx = (A+B)x^2 + Cx + A$$

Obtemos o sistema:

$$\begin{cases} A+B=0, \\ C=2, \\ A=-1, \end{cases} \Leftrightarrow \begin{cases} A=-1, \\ B=1, \\ C=2. \end{cases}$$

Portanto

$$\int_1^2 \frac{2x-1}{x^3+x}\cdot dx = \int_1^2\left(\frac{-1}{x} + \frac{x+2}{x^2+1}\right)\cdot dx = -\int_1^2 \frac{1}{x}\cdot dx + \int_1^2 \frac{x+2}{x^2+1}\cdot dx =$$

$$= -\int_1^2 \frac{1}{x}\cdot dx + \int_1^2\left(\frac{x}{x^2+1} + \frac{2}{x^2+1}\right)\cdot dx = -\int_1^2 \frac{1}{x}\cdot dx + \int_1^2 \frac{x}{x^2+1}\cdot dx + \int_1^2 \frac{2}{x^2+1}\cdot dx =$$

$$= -\int_1^2 \frac{1}{x}\cdot dx + \frac{1}{2}\cdot\int_1^2 \frac{1}{x^2+1}\cdot d(x^2+1) + 2\cdot\int_1^2 \frac{1}{x^2+1}\cdot dx =$$

$$= -\left(ln|x|\right)\Big|_1^2 + \frac{1}{2}\cdot\left(ln|x^2+1|\right)\Big|_1^2 + 2\cdot\left(arctg\, x\right)\Big|_1^2 = -\left(ln2 - ln1\right) + \frac{1}{2}\cdot\left(ln5 - ln2\right) +$$

$$+ 2\cdot\left(arctg\,2 - arctg1\right) = \frac{1}{2}\cdot ln5 - \frac{3}{2}\cdot ln2 + 2\cdot\left(arctg\,2 - \frac{\pi}{4}\right) = \frac{1}{2}\cdot ln\left(\frac{5}{8}\right) + 2\cdot\left(arctg\,2 - \frac{\pi}{4}\right).$$

b) Calcular os integrais efectuando a substituição de variável.

Exercício 11. $\int_0^{\frac{\pi}{2}} senx \cdot cos^2 x \cdot dx$.

Fazemos a substituição $senx = t$. Então temos:

$senx = t \quad \Rightarrow \quad x = arcsen\, t \quad \Rightarrow \quad dx = \dfrac{dt}{\sqrt{1-t^2}}$ e $cos^2 x = 1 - sen^2 x = 1 - t^2$.

Determinamos os limites de integração para a variável t:

$$x_{inf} = 0 \quad \Rightarrow \quad t_{inf} = sen(0) = 0, \qquad x_{sup} = \frac{\pi}{2} \quad \Rightarrow \quad t_{sup} = sen\left(\frac{\pi}{2}\right) = 1$$

Portanto

$$\int_0^{\frac{\pi}{2}} senx \cdot cos^2 x \cdot dx = \int_0^1 t \cdot (1-t^2) \cdot \frac{1}{\sqrt{1-t^2}} \cdot dt = \int_0^1 t \cdot \sqrt{1-t^2} \cdot dt = \frac{1}{2} \cdot \int_0^1 \sqrt{1-t^2} \cdot d(t^2) =$$

$$= -\frac{1}{2} \cdot \int_0^1 \sqrt{1-t^2} \cdot d(1-t^2) = -\frac{1}{2} \cdot \int_0^1 \left(1-t^2\right)^{\frac{1}{2}} \cdot d(1-t^2) = -\frac{1}{2} \cdot \left. \left(\frac{\left(1-t^2\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) \right|_0^1 =$$

$$= -\frac{1}{2} \cdot \left. \left(\frac{\left(1-t^2\right)^{\frac{3}{2}}}{\frac{3}{2}} \right) \right|_0^1 = -\frac{1}{3} \cdot \left. \left(\left(1-t^2\right)^{\frac{3}{2}} \right) \right|_0^1 = -\frac{1}{3} \cdot \left[\left(1-1^2\right)^{\frac{3}{2}} - \left(1-0^2\right)^{\frac{3}{2}} \right] = \frac{1}{3}.$$

Exercício 12. $\int_0^1 \dfrac{1}{e^x + e^{-x}} \cdot dx$.

Fazemos a substituição $e^x = t$. Então temos:

$e^x = t \quad \Rightarrow \quad x = ln\, t \quad \Rightarrow \quad dx = \dfrac{dt}{t}$.

Determinamos os limites de integração para a variável t:

$$x_{inf} = 0 \quad \Rightarrow \quad t_{inf} = e^0 = 1, \qquad x_{sup} = 1 \quad \Rightarrow \quad t_{sup} = e^1 = e.$$

Portanto

$$\int_0^1 \frac{1}{e^x + e^{-x}} \cdot dx = \int_0^1 \frac{1}{e^x + \frac{1}{e^x}} \cdot dx = \int_1^e \frac{1}{t + \frac{1}{t}} \cdot \frac{1}{t} \cdot dt = \int_1^e \frac{1}{t^2 + 1} \cdot dt = \left(arctg\, t\right)\Big|_1^e =$$

$$= arctg\, e - arctg\, 1 = arctg\, e - \frac{\pi}{4}.$$

Exercício 13. $\int_0^4 x\cdot\sqrt{x^2+9}\cdot dx$.

Fazemos a substituição $\sqrt{x^2+9}=t$. Então temos:

$$\sqrt{x^2+9}=t \quad\Rightarrow\quad x^2+9=t^2 \quad\Rightarrow\quad x=\sqrt{t^2-9} \quad\Rightarrow\quad dx=\frac{t\cdot dt}{\sqrt{t^2-9}}.$$

Determinamos os limites de integração para a variável t:

$$x_{inf}=0 \quad\Rightarrow\quad t_{inf}=\sqrt{9}=3, \qquad x_{sup}=4 \quad\Rightarrow\quad t_{sup}=\sqrt{4^2+9}=5.$$

Portanto

$$\int_0^4 x\cdot\sqrt{x^2+9}\cdot dx=\int_3^5\sqrt{t^2-9}\cdot t\cdot\frac{t}{\sqrt{t^2-9}}\cdot dt=\int_3^5 t^2\cdot dt=\left(\frac{t^3}{3}\right)\Bigg|_3^5=\frac{5^3}{3}-\frac{3^3}{3}=\frac{98}{3}.$$

Exercício 14. $\int_0^2\frac{dx}{\sqrt{x+1}+\sqrt{(x+1)^3}}$.

Fazemos a substituição $\sqrt{x+1}=t$. Então temos:

$$\sqrt{x+1}=t \quad\Rightarrow\quad x+1=t^2 \quad\Rightarrow\quad x=t^2-1 \quad\Rightarrow\quad dx=2t\cdot dt.$$

Determinamos os limites de integração para a variável t:

$$x_{inf}=0 \quad\Rightarrow\quad t_{inf}=\sqrt{1}=1, \qquad x_{sup}=2 \quad\Rightarrow\quad t_{sup}=\sqrt{2+1}=\sqrt{3}.$$

Portanto

$$\int_0^2\frac{dx}{\sqrt{x+1}+\sqrt{(x+1)^3}}=\int_0^2\frac{dx}{\sqrt{x+1}+(\sqrt{x+1})^3}=\int_1^{\sqrt{3}}\frac{2t\cdot dt}{t+t^3}=2\cdot\int_1^{\sqrt{3}}\frac{t\cdot dt}{t(1+t^2)}=$$

$$=2\cdot\int_1^{\sqrt{3}}\frac{dt}{1+t^2}=2\cdot\left(arctg\,t\right)\Big|_1^{\sqrt{3}}=2\cdot\left(arctg\sqrt{3}-arctg\,1\right)=2\cdot\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\frac{\pi}{6}.$$

Exercício 15. $\int_0^{\frac{\pi}{2}}\frac{dx}{1+sen\,x+\cos x}$.

Fazemos a substituição $tg\left(\frac{x}{2}\right)=t$. Então temos:

$$senx=\frac{2t}{1+t^2}, \quad \cos x=\frac{1-t^2}{1+t^2}.$$

$$tg\left(\frac{x}{2}\right)=t \quad\Rightarrow\quad \frac{x}{2}=arctg\,t \quad\Rightarrow\quad x=2\cdot arctg\,t \quad\Rightarrow\quad dx=\frac{2\cdot dt}{1+t^2}.$$

Determinamos os limites de integração para a variável t:

$$x_{inf}=0 \quad\Rightarrow\quad t_{inf}=tg(0)=0, \qquad x_{sup}=\frac{\pi}{2} \quad\Rightarrow\quad t_{sup}=tg\left(\frac{\pi}{4}\right)=1.$$

Portanto

$$\int_0^{\frac{\pi}{2}}\frac{dx}{1+sen\,x+cos\,x}=\int_0^1\frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2\cdot dt}{1+t^2}=\int_0^1\frac{2\cdot dt}{1+t^2+2t+1-t^2}=\int_0^1\frac{dt}{1+t}=$$

$$=\int_0^1\frac{1}{1+t}\cdot d(1+t)=\left(ln\left|1+t\right|\right)\Big|_0^1=\left(ln\left|1+1\right|-ln\left|1+0\right|\right)=ln\,2.$$

Exercício 16. $\displaystyle\int_0^4\frac{dx}{1+\sqrt{x}}$.

Fazemos a substituição $\sqrt{x}=t$. Então temos:
$\sqrt{x}=t \quad\Rightarrow\quad x=t^2 \quad\Rightarrow\quad dx=2t\cdot dt$.

Determinamos os limites de integração para a variável t:

$$x_{inf}=0 \quad\Rightarrow\quad t_{inf}=\sqrt{0}=0, \qquad x_{sup}=4 \quad\Rightarrow\quad t_{sup}=\sqrt{4}=2.$$

Portanto

$$\int_0^4\frac{dx}{1+\sqrt{x}}=\int_0^2\frac{2t\cdot dt}{1+t}=2\cdot\int_0^2\frac{t\cdot dt}{1+t}=2\cdot\int_0^2\frac{1+t-1}{1+t}\cdot dt=2\cdot\int_0^2\left(\frac{1+t}{1+t}-\frac{1}{1+t}\right)\cdot dt=$$

$$=2\cdot\int_0^2 dt-2\cdot\int_0^2\frac{1}{1+t}\cdot d(1+t)=2\cdot(t)\Big|_0^2-2\cdot\left(ln\left|1+t\right|\right)\Big|_0^2=$$

$$=2\cdot(2-0)-2\cdot\left(ln\left|1+2\right|-ln\left|1+0\right|\right)=4-2\cdot ln\,3.$$

Exercício 17. $\displaystyle\int_{-1}^0\frac{dx}{1+\sqrt[3]{x+1}}$.

Fazemos a substituição $\sqrt[3]{x+1}=t$. Então temos:
$\sqrt[3]{x+1}=t \quad\Rightarrow\quad x+1=t^3 \quad\Rightarrow\quad x=t^3-1 \quad\Rightarrow\quad dx=3t^2\cdot dt$.

Determinamos os limites de integração para a variável t:

$$x_{inf} = -1 \Rightarrow t_{inf} = \sqrt[3]{-1+1} = 0, \quad x_{sup} = 0 \Rightarrow t_{sup} = \sqrt[3]{0+1} = 1.$$

Portanto

$$\int_{-1}^{0} \frac{dx}{1+\sqrt[3]{x+1}} = \int_{0}^{1} \frac{3t^2 dt}{1+t} = 3\cdot\int_{0}^{1} \frac{t^2 dt}{1+t} = 3\cdot\int_{0}^{1} \frac{1+t^2-1}{1+t}\cdot dt = 3\cdot\int_{0}^{1}\left(\frac{1}{1+t}+\frac{t^2-1}{1+t}\right)\cdot dt =$$

$$= 3\cdot\int_{0}^{1}\left(\frac{1}{1+t}+\frac{(t-1)(t+1)}{1+t}\right)\cdot dt = 3\cdot\int_{0}^{1}\left(\frac{1}{1+t}+t-1\right)\cdot dt =$$

$$= 3\cdot\int_{0}^{1}\frac{1}{1+t}\cdot dt + 3\cdot\int_{0}^{1} t\cdot dt - 3\cdot\int_{0}^{1} dt = 3\cdot\left(ln\left|1+t\right|\right)\Big|_{0}^{1} + 3\cdot\left(\frac{t^2}{2}\right)\Bigg|_{0}^{1} - 3\cdot(t)\Big|_{0}^{1} =$$

$$= 3\cdot\left(ln\left|1+1\right| - ln\left|1+0\right|\right) + 3\cdot\left(\frac{1^2}{2}-\frac{0^2}{2}\right) - 3\cdot(1-0) = 3\cdot ln2 - \frac{3}{2}.$$

Exercício 18. $\displaystyle\int_{0}^{3}\sqrt{\frac{x}{6-x}}\cdot dx$.

Fazemos a substituição $\sqrt{\dfrac{x}{6-x}} = t$. Então temos:

$$\sqrt{\frac{x}{6-x}} = t \Rightarrow \frac{x}{6-x} = t^2 \Rightarrow x = 6t^2 - x\cdot t^2 \Rightarrow x = \frac{6t^2}{1+t^2} \Rightarrow$$

$$dx = \left(\frac{6t^2}{1+t^2}\right)'\cdot dt = \frac{12t(1+t^2)-6t^2\cdot 2t}{\left(1+t^2\right)^2}\cdot dt = \frac{12t}{\left(1+t^2\right)^2}\cdot dt.$$

Determinamos os limites de integração para a variável t:

$$x_{inf} = 0 \Rightarrow t_{inf} = 0, \quad x_{sup} = 3 \Rightarrow t_{sup} = 1.$$

Portanto

$$\int_{0}^{3}\sqrt{\frac{x}{6-x}}\cdot dx = \int_{0}^{1} t\cdot\frac{12t}{\left(1+t^2\right)^2}\cdot dt = 12\cdot\int_{0}^{1}\frac{t^2}{\left(1+t^2\right)^2}\cdot dt = 6\cdot\int_{0}^{1}\frac{t\cdot 2t\cdot dt}{\left(1+t^2\right)^2} = 6\cdot\int_{0}^{1} t\cdot\frac{2t\cdot dt}{\left(1+t^2\right)^2} =$$

Na continuação integramos por partes:

$U = t, \quad dU = dt;$

$$dV = \frac{2t \cdot dt}{\left(1+t^2\right)^2}, \quad V = \int \frac{2t \cdot dt}{\left(1+t^2\right)^2} = \int \frac{d(t^2)}{\left(1+t^2\right)^2} = \int \frac{d(1+t^2)}{\left(1+t^2\right)^2} = \frac{\left(1+t^2\right)^{-2+1}}{-2+1} = -\frac{1}{1+t^2}.$$

Obtemos

$$= 6 \cdot \left[-\left(\frac{t}{1+t^2} \right)\Bigg|_0^1 + \int_0^1 \frac{dt}{1+t^2} \right] = 6 \cdot \left[-\left(\frac{1}{1+1^2} - \frac{0}{1+0^2} \right) + \left(arctg\, t \right)\Big|_0^1 \right] =$$

$$= 6 \cdot \left(-\frac{1}{2} + \frac{\pi}{4} \right) = \frac{3 \cdot (\pi - 2)}{2}.$$

Exercício 19. $\displaystyle\int_0^1 \frac{dx}{e^x + 1}$.

Fazemos a substituição $e^x = t$. Então temos:

$$e^x = t \quad \Rightarrow \quad x = ln\, t \quad \Rightarrow \quad dx = \frac{dt}{t}.$$

Determinamos os limites de integração para a variável t:

$$x_{inf} = 0 \quad \Rightarrow \quad t_{inf} = e^0 = 1, \qquad x_{sup} = 1 \quad \Rightarrow \quad t_{sup} = e^1 = e.$$

Portanto

$$\int_0^1 \frac{dx}{e^x + 1} = \int_1^e \frac{dt}{(t+1) \cdot t} = \int_1^e \frac{1+t-t}{(t+1) \cdot t} \cdot dt = \int_1^e \left(\frac{1+t}{(t+1) \cdot t} - \frac{t}{(t+1) \cdot t} \right) \cdot dt =$$

$$= \int_1^e \left(\frac{1}{t} - \frac{1}{(t+1)} \right) \cdot dt = \int_1^e \frac{1}{t} \cdot dt - \int_1^e \frac{1}{t+1} \cdot d(t+1) = \left(ln|\, t\, | \right)\Big|_1^e - \left(ln|\, t+1\, | \right)\Big|_1^e =$$

$$= \left(ln e - ln 1 \right) - \left(ln(e+1) - ln\, 2 \right) = ln e + ln\, 2 - ln(e+1) = ln\,(2e) - ln(e+1) = ln \frac{2e}{e+1}.$$

Exercício 20. $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{\sqrt{(1+x^2)^3}}$.

Fazemos a substituição $x = tg\, t$. Então temos:

$$dx = \frac{dt}{cos^2 t}, \quad 1 + x^2 = 1 + \left(tg\, t \right)^2 = 1 + \left(\frac{sen\, t}{\cos t} \right)^2 = 1 + \frac{sen^2\, t}{\cos^2 t} = \frac{\cos^2 t + sen^2\, t}{\cos^2 t} = \frac{1}{\cos^2 t}.$$

Porque $x = tg\, t \quad \Rightarrow \quad t = arctg\, x$, determinamos os limites de integração para a variável t:

$$x_{inf}=1 \;\Rightarrow\; t_{inf}=arctg\,(1)=\frac{\pi}{4}, \qquad x_{sup}=\sqrt{3} \;\Rightarrow\; t_{sup}=arctg\,(\sqrt{3})=\frac{\pi}{3}.$$

Portanto

$$\int_1^{\sqrt{3}}\frac{dx}{\sqrt{(1+x^2)^3}}=\int_1^{\sqrt{3}}\frac{dx}{\left(\sqrt{1+x^2}\right)^3}=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\frac{dt}{cos^2t}}{\left(\sqrt{\frac{1}{cos^2t}}\right)^3}=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\frac{dt}{cos^2t}}{\left(\frac{1}{cos\,t}\right)^3}=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\frac{dt}{cos^2t}}{\frac{1}{cos^3t}}=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{cos^3t\cdot dt}{cos^2t}=$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}cos\,t\cdot dt=\left(sent\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{3}}=sen\left(\frac{\pi}{3}\right)-sen\left(\frac{\pi}{4}\right)=\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}=\frac{\sqrt{3}-\sqrt{2}}{2}.$$

c) **Calcular os integrais aplicando o método de integração por partes.**

Exercício 21. $\int_0^1 xe^{-x}dx$.

Fazemos:

$$U=x \;\Rightarrow\; dU=dx, \quad dV=e^{-x}dx \;\Rightarrow\; V=\int e^{-x}dx=-e^{-x}.$$

Portanto

$$\int_0^1 xe^{-x}dx=\left(x\cdot\left(-e^{-x}\right)\right)\Big|_0^1-\int_0^1\left(-e^{-x}\right)dx=-\left(x\cdot\left(e^{-x}\right)\right)\Big|_0^1+\int_0^1 e^{-x}\,dx=-\left(x\cdot\left(e^{-x}\right)\right)\Big|_0^1+\left(-e^{-x}\right)\Big|_0^1=$$

$$=-\left(x\cdot\left(e^{-x}\right)\right)\Big|_0^1-\left(e^{-x}\right)\Big|_0^1=-\left(1\cdot e^{-1}-0\cdot e^0\right)-\left(e^{-1}-e^0\right)=-\frac{1}{e}-\frac{1}{e}+1=1-\frac{2}{e}.$$

Exercício 22. $\int_0^{\frac{1}{2}} arcsenx\,dx$.

Fazemos:

$$U=arcsen\,x \;\Rightarrow\; dU=\left(arcsen\,x\right)'dx=\frac{1}{\sqrt{1-x^2}}dx,$$

$$dV=dx \;\Rightarrow\; V=\int dx=x.$$

Portanto

$$\int_0^{\frac{1}{2}}arcsenx\,dx=\left(x\cdot arcsenx\right)\Big|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}\frac{x}{\sqrt{1-x^2}}dx=\left(x\cdot arcsenx\right)\Big|_0^{\frac{1}{2}}-\frac{1}{2}\cdot\int_0^{\frac{1}{2}}\frac{d(x^2)}{\sqrt{1-x^2}}=$$

$$=\left(x\cdot arcsenx\right)\Big|_0^{\frac{1}{2}}+\frac{1}{2}\cdot\int_0^{\frac{1}{2}}\frac{d(-x^2)}{\sqrt{1-x^2}}=\left(x\cdot arcsenx\right)\Big|_0^{\frac{1}{2}}+\frac{1}{2}\cdot\int_0^{\frac{1}{2}}\frac{d(1-x^2)}{\sqrt{1-x^2}}=$$

$$= \left(x \cdot arcsen x\right)\Big|_0^{\frac{1}{2}} + \frac{1}{2} \cdot \int_0^{\frac{1}{2}} \left(1 - x^2\right)^{-\frac{1}{2}} \cdot d(1 - x^2) = \left(x \cdot arcsen x\right)\Big|_0^{\frac{1}{2}} + \frac{1}{2} \cdot \left(\frac{\left(1 - x^2\right)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right)\Bigg|_0^{\frac{1}{2}} =$$

$$= \left(\frac{1}{2} \cdot arcsen\left(\frac{1}{2}\right) - 0 \cdot arcsen(0) \right) + \left(\left(1 - \left(\frac{1}{2}\right)^2\right)^{\frac{1}{2}} - \left(1 - 0^2\right)^{\frac{1}{2}} \right) = \frac{1}{2} \cdot \frac{\pi}{6} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

Exercício 23. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x \cdot dx}{sen^2 x}$.

Fazemos:

$U = x \;\Rightarrow\; dU = dx$, $\quad dV = \dfrac{dx}{sen^2 x} \;\Rightarrow\; V = \displaystyle\int \frac{dx}{sen^2 x} = -ctg\, x$.

Portanto

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x \cdot dx}{sen^2 x} = \left(-x \cdot ctg\, x\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (-ctg\, x) \cdot dx = \left(-x \cdot ctg\, x\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{cos\, x}{sen\, x} \cdot dx =$$

$$= -\left(x \cdot ctg\, x\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{d(sen\, x)}{sen\, x} = -\left(x \cdot ctg\, x\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \left(ln\left| sen x \right|\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} =$$

$$= -\left(\frac{\pi}{3} \cdot ctg\left(\frac{\pi}{3}\right) - \frac{\pi}{4} \cdot ctg\left(\frac{\pi}{4}\right) \right) + \left(ln\left| sen\left(\frac{\pi}{3}\right) \right| - ln\left| sen\left(\frac{\pi}{4}\right) \right| \right) = \frac{\pi(9 - 4\sqrt{3})}{36} + \frac{1}{2} \cdot ln\left(\frac{3}{2}\right).$$

Exercício 24. $\displaystyle\int_0^{\pi} x^3 \cdot sen x \cdot dx$.

Fazemos:

$U = x^3 \;\Rightarrow\; dU = 3x^2 dx$, $\quad dV = sen x\, dx \;\Rightarrow\; V = \displaystyle\int sen x\, dx = -cos\, x$.

Portanto

$$\int_0^{\pi} x^3 \cdot sen x \cdot dx = \left(-x^3 \cdot cos\, x\right)\Big|_0^{\pi} - \int_0^{\pi} 3x^2 \cdot (-cos\, x) \cdot dx = \left(-\pi^3 \cdot cos\, \pi + -0^3 \cdot cos\, 0\right) +$$

$$+ 3 \cdot \int_0^{\pi} x^2 \cdot cos\, x \cdot dx = \pi^3 + 3 \cdot \int_0^{\pi} x^2 \cdot cos\, x \cdot dx =$$

Integramos por partes:

$$U = x^2 \quad \Rightarrow \quad dU = 2xdx\,, \qquad dV = cos\,x\,dx \quad \Rightarrow \quad V = \int cos\,x\,dx = sen\,x\,.$$

Portanto na continuação temos:

$$= \pi^3 + 3 \cdot \left[\left(x^2 \cdot senx \right) \Big|_0^{\pi} - \int_0^{\pi} 2x \cdot sen\,x \cdot dx \right] =$$

$$= \pi^3 + 3 \cdot \left(\pi^2 \cdot sen(\pi) - 0^2 \cdot sen(0) \right) - 6 \cdot \int_0^{\pi} x \cdot sen\,x \cdot dx = \pi^3 - 6 \cdot \int_0^{\pi} x \cdot sen\,x \cdot dx =$$

Mais uma vez integramos por partes:

$$U = x \quad \Rightarrow \quad dU = dx\,, \qquad dV = sen\,x\,dx \quad \Rightarrow \quad V = \int sen\,x\,dx = -cos\,x\,.$$

Portanto obtemos:

$$= \pi^3 - 6 \cdot \left[\left(-x \cdot cos\,x \right) \Big|_0^{\pi} - \int_0^{\pi} (-cos\,x) \cdot dx \right] =$$

$$= \pi^3 + 6 \cdot \left(\pi \cdot cos\,(\pi) - 0 \cdot cos\,(0) \right) - 6 \cdot \int_0^{\pi} cos\,x \cdot dx =$$

$$= \pi^3 - 6\pi - 6 \cdot \left(sen\,x \right) \Big|_0^{\pi} = \pi^3 - 6\pi - 6 \cdot \left(sen\,(\pi) - sen\,(0) \right) = \pi^3 - 6\pi\,.$$

Exercício 25. $\int_0^{\frac{\pi}{2}} e^{2x} \cdot cos\,x \cdot dx\,.$

Fazemos:

$$U = e^{2x} \quad \Rightarrow \quad dU = 2 \cdot e^{2x} \cdot dx\,, \qquad dV = cos\,x\,dx \quad \Rightarrow \quad V = \int cos\,x\,dx = sen\,x\,.$$

Portanto

$$\int_0^{\frac{\pi}{2}} e^{2x} \cdot cos\,x \cdot dx = \left(e^{2x} \cdot sen\,x \right) \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2 \cdot e^{2x} \cdot sen\,x \cdot dx =$$

$$= \left(e^{2 \cdot \frac{\pi}{2}} \cdot sen\left(\frac{\pi}{2} \right) - e^{2 \cdot 0} \cdot sen\,(0) \right) - 2 \cdot \int_0^{\frac{\pi}{2}} e^{2x} \cdot sen\,x \cdot dx = e^{\pi} - 2 \cdot \int_0^{\frac{\pi}{2}} e^{2x} \cdot sen\,x \cdot dx =$$

Mais uma vez integramos por partes:

$$U = e^{2x} \quad \Rightarrow \quad dU = 2 \cdot e^{2x} \cdot dx\,, \qquad dV = sen\,x\,dx \quad \Rightarrow \quad V = \int sen\,x\,dx = -cos\,x\,.$$

Na continuação temos:

$$= e^{\pi} - 2 \cdot \left[\left(-e^{2x} \cdot \cos x \right)\Big|_{0}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} 2 \cdot e^{2x} \cdot (-\cos x) \cdot dx \right] =$$

$$= e^{\pi} + 2 \cdot \left[\left(e^{2x} \cdot \cos x \right)\Big|_{0}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} 2 \cdot e^{2x} \cdot \cos x \cdot dx \right] = e^{\pi} - 2 - 4 \cdot \int_{0}^{\frac{\pi}{2}} e^{2x} \cdot \cos x \cdot dx \,.$$

Portanto obtemos:

$$\int_{0}^{\frac{\pi}{2}} e^{2x} \cdot \cos x \cdot dx = e^{\pi} - 2 - 4 \cdot \int_{0}^{\frac{\pi}{2}} e^{2x} \cdot \cos x \cdot dx \,.$$

Resolvemos a equação obtida em relação ao integral:

$$5 \cdot \int_{0}^{\frac{\pi}{2}} e^{2x} \cdot \cos x \cdot dx = e^{\pi} - 2 \quad \text{e} \quad \int_{0}^{\frac{\pi}{2}} e^{2x} \cdot \cos x \cdot dx = \frac{e^{\pi} - 2}{5} \,.$$

Exercício 26. $\int_{1}^{e} sen(\ln x)\, dx$.

Fazemos:

$$U = sen(\ln x) \quad \Rightarrow \quad dU = \left(sen(\ln x) \right)' \cdot dx = \cos(\ln x) \cdot \frac{1}{x} \cdot dx \,,$$

$$dV = dx \quad \Rightarrow \quad V = \int dx = x \,.$$

Portanto

$$\int_{1}^{e} sen(\ln x)\, dx = \left(x \cdot sen(\ln x) \right)\Big|_{1}^{e} - \int_{1}^{e} x \cdot \cos(\ln x) \cdot \frac{1}{x} \cdot dx =$$

$$= \left(e \cdot sen(\ln(e)) - 1 \cdot sen(\ln(1)) \right) - \int_{1}^{e} \cos(\ln x) \cdot dx = e \cdot sen(1) - \int_{1}^{e} \cos(\ln x) \cdot dx =$$

Mais uma vez integramos por partes:

$$U = \cos\,(\ln x) \quad \Rightarrow \quad dU = \left(\cos\,(\ln x) \right)' \cdot dx = -sen\,(\ln x) \cdot \frac{1}{x} \cdot dx \,,$$

$$dV = dx \quad \Rightarrow \quad V = \int dx = x \,.$$

Na continuação temos:

$$= e \cdot sen(1) - \left[\left. \left(x \cdot cos\,(lnx)\right) \right|_1^e - \int_1^e (-sen\,(lnx)) \cdot dx \right] =$$

$$= e \cdot sen(1) - \left(e \cdot cos\,(ln(e)) - 1 \cdot cos\,(ln(1))\right) - \int_1^e sen\,(lnx) \cdot dx =$$

$$= e \cdot sen(1) - e \cdot cos\,(1) + 1 - \int_1^e sen\,(lnx) \cdot dx\,.$$

Portanto obtemos:

$$\int_1^e sen\,(lnx) \cdot dx = e \cdot sen(1) - e \cdot cos\,(1) + 1 - \int_1^e sen\,(lnx) \cdot dx\,.$$

Resolvemos a equação obtida em relação ao integral:

$$2 \cdot \int_1^e sen\,(lnx) \cdot dx = e \cdot sen(1) - e \cdot cos\,(1) + 1 \quad \text{e} \quad \int_1^e sen\,(lnx) \cdot dx = \frac{e \cdot sen(1) - e \cdot cos\,(1) + 1}{2}\,.$$

Exercício 27. $\displaystyle\int_0^1 \frac{arcsen\,x}{\sqrt{1+x}} \cdot dx\,.$

Fazemos:

$$U = arcsen\,x \quad \Rightarrow \quad dU = \left(arcsen\,x\right)' \cdot dx = \frac{1}{\sqrt{1-x^2}} \cdot dx\,,$$

$$dV = \frac{1}{\sqrt{1+x}} \cdot dx \quad \Rightarrow \quad V = \int \frac{1}{\sqrt{1+x}} \cdot dx = \int \left(1+x\right)^{-\frac{1}{2}} \cdot d\,(1+x) = \frac{\left(1+x\right)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2\sqrt{1+x}\,.$$

Portanto

$$\int_0^1 \frac{arcsen\,x}{\sqrt{1+x}} \cdot dx = \left. \left(2\sqrt{1+x} \cdot arcsen\,x\right) \right|_0^1 - \int_0^1 2\sqrt{1+x} \cdot \frac{1}{\sqrt{1-x^2}} \cdot dx =$$

$$= \left(2\sqrt{1+1} \cdot arcsen\,(1) - 2\sqrt{1+0} \cdot arcsen\,(0)\right) - \int_0^1 2\sqrt{1+x} \cdot \frac{1}{\sqrt{1-x} \cdot \sqrt{1+x}} \cdot dx =$$

$$= 2\sqrt{2} \cdot \frac{\pi}{2} - 2 \cdot \int_0^1 \frac{1}{\sqrt{1-x}} \cdot dx = \sqrt{2}\pi + 2 \cdot \int_0^1 \frac{1}{\sqrt{1-x}} \cdot d\,(1-x) = \sqrt{2}\pi + 2 \cdot \left. \left(\frac{(1-x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right) \right|_0^1 =$$

$$= \sqrt{2}\pi + 4 \cdot \left. \left((1-x)^{\frac{1}{2}} \right) \right|_0^1 = \sqrt{2}\pi + 4 \cdot \left((1-1)^{\frac{1}{2}} - (1-0)^{\frac{1}{2}} \right) = \sqrt{2}\pi - 4\,.$$

Exercício 28. $\int_0^1 arctg\sqrt{x}\cdot dx$.

Fazemos:

$$U = arctg\sqrt{x} \quad\Rightarrow\quad dU = \left(arctg\sqrt{x}\right)'\cdot dx = \frac{1}{1+\left(\sqrt{x}\right)^2}\cdot\left(\sqrt{x}\right)'\cdot dx = \frac{dx}{2\sqrt{x}\cdot(1+x)},$$

$$dV = dx \quad\Rightarrow\quad V = x.$$

Portanto

$$\int_0^1 arctg\sqrt{x}\cdot dx = \left(x\cdot arctg\sqrt{x}\right)\Big|_0^1 - \int_0^1 x\cdot\frac{1}{2\sqrt{x}\cdot(1+x)}\cdot dx =$$

$$= \left(1\cdot arctg\sqrt{1} - 0\cdot arctg\sqrt{0}\right) - \frac{1}{2}\cdot\int_0^1\frac{\sqrt{x}}{1+x}\cdot dx = \frac{\pi}{4} - \frac{1}{2}\cdot\int_0^1\frac{\sqrt{x}}{1+x}\cdot dx =$$

Fazemos a substituição

$$\sqrt{x} = t \quad\Rightarrow\quad x = t^2 \quad\Rightarrow\quad dx = 2t\,dt;$$

Determinamos os limites de integração para a variável t:

$$x_{inf} = 0 \quad\Rightarrow\quad t_{inf} = \sqrt{0} = 0, \qquad x_{sup} = 1 \quad\Rightarrow\quad t_{sup} = \sqrt{1} = 1.$$

Na continuação temos:

$$= \frac{\pi}{4} - \frac{1}{2}\cdot\int_0^1\frac{t}{1+t^2}\cdot 2t\cdot dt = \frac{\pi}{4} - \int_0^1\frac{t^2}{1+t^2}\cdot dt = \frac{\pi}{4} - \int_0^1\frac{1+t^2-1}{1+t^2}\cdot dt =$$

$$= \frac{\pi}{4} - \int_0^1\left(\frac{1+t^2}{1+t^2} - \frac{1}{1+t^2}\right)\cdot dt = \frac{\pi}{4} - \int_0^1\left(1 - \frac{1}{1+t^2}\right)\cdot dt = \frac{\pi}{4} - \int_0^1 dt + \int_0^1\frac{1}{1+t^2}\cdot dt =$$

$$= \frac{\pi}{4} - (t)\Big|_0^1 + (arctg\,t)\Big|_0^1 = \frac{\pi}{4} - (1-0) + \left(arctg\,(1) - arctg\,(0)\right) = \frac{\pi}{2} - 1.$$

Exercício 29. $\int_0^1 x\cdot ln(1+x^2)\cdot dx$.

Fazemos:

$$U = ln(1+x^2) \quad\Rightarrow\quad dU = \left(ln(1+x^2)\right)'\cdot dx = \frac{1}{1+x^2}\cdot\left(1+x^2\right)'\cdot dx = \frac{2x\cdot dx}{1+x^2},$$

$$dV = x\,dx \quad\Rightarrow\quad V = \frac{x^2}{2}.$$

Portanto

$$\int_0^1 x\cdot ln(1+x^2)\cdot dx = \left(\frac{x^2}{2}\cdot ln(1+x^2)\right)\Bigg|_0^1 - \int_0^1 \frac{x^3}{1+x^2}\cdot dx =$$

$$= \left(\frac{1^2}{2}\cdot ln(1+1^2) - \frac{0^2}{2}\cdot ln(1+0^2)\right) - \int_0^1 \frac{x+x^3-x}{1+x^2}\cdot dx =$$

$$= \frac{1}{2}\cdot ln(2) - \int_0^1\left(x - \frac{x}{1+x^2}\right)\cdot dx = \frac{1}{2}\cdot ln(2) - \int_0^1 x\cdot dx + \int_0^1 \frac{x}{1+x^2}\cdot dx =$$

$$= \frac{1}{2}\cdot ln(2) - \left(\frac{x^2}{2}\right)\Bigg|_0^1 + \frac{1}{2}\cdot\int_0^1 \frac{d(1+x^2)}{1+x^2} = \frac{1}{2}\cdot ln(2) - \frac{1}{2} + \frac{1}{2}\cdot\left(ln(1+x^2)\right)\Big|_0^1 =$$

$$= \frac{1}{2}\cdot ln(2) - \frac{1}{2} + \frac{1}{2}\cdot\left(ln(1+1^2) - ln(1+0^2)\right) = ln(2) - \frac{1}{2}.$$

Exercício 30. $\int_0^1 ln(1+x)\cdot dx$.

Fazemos:

$$U = ln(1+x) \quad\Rightarrow\quad dU = \left(ln(1+x)\right)'\cdot dx = \frac{1}{1+x}\cdot dx\,,$$

$$dV = dx \quad\Rightarrow\quad V = x\,.$$

Portanto

$$\int_0^1 ln(1+x)\cdot dx = \left(x\cdot ln(1+x)\right)\Big|_0^1 - \int_0^1 \frac{x}{1+x}\cdot dx = \left(1\cdot ln(1+1) - 0\cdot ln(1+0)\right) - \int_0^1 \frac{1+x-1}{1+x^2}\cdot dx =$$

$$= ln(2) - \int_0^1\left(1 - \frac{1}{1+x}\right)\cdot dx = ln(2) - \int_0^1 dx + \int_0^1 \frac{1}{1+x}\cdot d(1+x) =$$

$$= ln(2) - \int_0^1 dx + \int_0^1 \frac{1}{1+x}\cdot d(1+x) = ln(2) - (x)\Big|_0^1 + \left(ln(1+x)\right)\Big|_0^1 =$$

$$= ln(2) - (1-0) + \left(ln(1+1) - ln(1+0)\right) = 2\cdot ln(2) - 1.$$

d) **Calcular os integrais.**

Exercício 31. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x+sen\,x}{1+cos\,x}\cdot dx$.

Fazemos a substituição $tg\left(\frac{x}{2}\right)=t$. Então temos:

$$senx=\frac{2t}{1+t^2}, \quad cos\,x=\frac{1-t^2}{1+t^2} .$$

$$tg\left(\frac{x}{2}\right)=t \quad \Rightarrow \quad \frac{x}{2}=arctg\,t \quad \Rightarrow \quad x=2\cdot arctg\,t \quad \Rightarrow \quad dx=\frac{2\cdot dt}{1+t^2} .$$

Determinamos os limites de integração para a variável t:

$$x_{inf}=\frac{\pi}{6} \quad \Rightarrow \quad t_{inf}=tg\left(\frac{\pi}{12}\right), \qquad x_{sup}=\frac{\pi}{2} \quad \Rightarrow \quad t_{sup}=tg\left(\frac{\pi}{4}\right)=1.$$

Portanto

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x+sen\,x}{1+cos\,x}\cdot dx = \int_{tg\left(\frac{\pi}{12}\right)}^{1} \frac{2\cdot arctg\,t+\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{2\cdot dt}{1+t^2} = \int_{tg\left(\frac{\pi}{12}\right)}^{1} \frac{2\cdot arctg\,t+\frac{2t}{1+t^2}}{\frac{1+t^2+1-t^2}{1+t^2}}\cdot\frac{2\cdot dt}{1+t^2} =$$

$$= \int_{tg\left(\frac{\pi}{12}\right)}^{1} \frac{2\cdot arctg\,t+\frac{2t}{1+t^2}}{\frac{2}{1+t^2}}\cdot\frac{2\cdot dt}{1+t^2} = \int_{tg\left(\frac{\pi}{12}\right)}^{1} \left(2\cdot arctg\,t+\frac{2t}{1+t^2}\right)\cdot dt =$$

$$= 2\cdot \int_{tg\left(\frac{\pi}{12}\right)}^{1} arctg\,t\cdot dt + 2\cdot \int_{tg\left(\frac{\pi}{12}\right)}^{1} \frac{t}{1+t^2}\cdot dt =$$

Integramos por partes o primeiro integral:

$$U=arctg\,t \quad \Rightarrow \quad dU=\frac{1}{1+t^2}\cdot dt; \qquad dV=dt \quad \Rightarrow \quad V=t .$$

Na continuação temos:

$$= 2 \cdot \left[\left(t \cdot arctg(t) \right) \Big|_{tg\left(\frac{\pi}{12}\right)}^{1} - \int_{tg\left(\frac{\pi}{12}\right)}^{1} \frac{t}{1+t^2} \cdot dt \right] + 2 \cdot \int_{tg\left(\frac{\pi}{12}\right)}^{1} \frac{t}{1+t^2} \cdot dt =$$

$$= 2 \cdot \left(1 \cdot arctg(1) - tg\left(\frac{\pi}{12}\right) \cdot arctg\left(tg\left(\frac{\pi}{12}\right)\right)\right) - 2 \cdot \int_{tg\left(\frac{\pi}{12}\right)}^{1} \frac{t}{1+t^2} \cdot dt + 2 \cdot \int_{tg\left(\frac{\pi}{12}\right)}^{1} \frac{t}{1+t^2} \cdot dt =$$

$$= 2 \cdot \left(\frac{\pi}{4} - \frac{\pi}{12} \cdot tg\left(\frac{\pi}{12}\right)\right) = \frac{\pi}{2} - \frac{\pi}{6} \cdot tg\left(\frac{\pi}{12}\right).$$

Exercício 32. $\int_0^{\frac{\pi}{2}} sen^3 x \cdot \sqrt{cos\, x} \cdot dx$.

1^o método:

$$\int_0^{\frac{\pi}{2}} sen^3 x \cdot \sqrt{cos\, x} \cdot dx = \int_0^{\frac{\pi}{2}} sen^2 x \cdot \sqrt{cos\, x} \cdot sen\, x \cdot dx = \int_0^{\frac{\pi}{2}} \left(1 - cos^2 x\right) \cdot \sqrt{cos\, x} \cdot d(-cos\, x) =$$

$$= -\int_0^{\frac{\pi}{2}} \left(1 - cos^2 x\right) \cdot \sqrt{cos\, x} \cdot d(cos\, x) = -\int_0^{\frac{\pi}{2}} (cosx)^{\frac{1}{2}} \cdot d(cos\, x) + \int_0^{\frac{\pi}{2}} (cosx)^{\frac{5}{2}} \cdot d(cos\, x) =$$

$$= -\left(\frac{(cosx)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) \Bigg|_0^{\frac{\pi}{2}} + \left(\frac{(cosx)^{\frac{5}{2}+1}}{\frac{5}{2}+1} \right) \Bigg|_0^{\frac{\pi}{2}} = -\frac{2}{3} \cdot \left((cosx)^{\frac{3}{2}} \right) \Big|_0^{\frac{\pi}{2}} + \frac{2}{7} \cdot \left((cosx)^{\frac{7}{2}} \right) \Big|_0^{\frac{\pi}{2}} =$$

$$= -\frac{2}{3} \cdot \left(\left(cos\left(\frac{\pi}{2}\right)\right)^{\frac{3}{2}} - (cos(0))^{\frac{3}{2}} \right) + \frac{2}{7} \cdot \left(\left(cos\left(\frac{\pi}{2}\right)\right)^{\frac{7}{2}} - (cos(0))^{\frac{7}{2}} \right) = \frac{2}{3} - \frac{2}{7} = \frac{8}{21}.$$

2^o método:

Porque a função $sen\, x$ tem exponente impar fazemos a seguinte substituição de variável:

$$cos\, x = t, \qquad sen^2 x = 1 - cos^2 x = 1 - t^2, \qquad senx\, dx = -d(cos\, x).$$

Determinamos os limites de integração para a variável t:

$$x_{inf}=0 \;\Rightarrow\; t_{inf}=cos(0)=1, \qquad x_{sup}=\frac{\pi}{2} \;\Rightarrow\; t_{sup}=cos\left(\frac{\pi}{2}\right)=0$$

Portanto

$$\int_0^{\frac{\pi}{2}} sen^3 x\cdot\sqrt{cos\,x}\cdot dx=\int_0^{\frac{\pi}{2}} sen^2 x\cdot\sqrt{cos\,x}\cdot sen\,x\cdot dx=\int_0^{\frac{\pi}{2}}\left(1-cos^2 x\right)\cdot\sqrt{cos\,x}\cdot(-d(cos\,x))=$$

$$=-\int_1^0\left(1-t^2\right)\cdot\sqrt{t}\cdot dt=\int_0^1\left(1-t^2\right)\cdot\sqrt{t}\cdot dt=\int_0^1\left(t^{\frac{1}{2}}-t^{\frac{5}{2}}\right)\cdot dt=$$

$$=\int_0^1 t^{\frac{1}{2}}\cdot dt-\int_0^1 t^{\frac{5}{2}}\cdot dt=\left.\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)\right|_0^1-\left.\left(\frac{t^{\frac{5}{2}+1}}{\frac{5}{2}+1}\right)\right|_0^1=\frac{2}{3}\cdot\left.\left(t^{\frac{3}{2}}\right)\right|_0^1-\frac{2}{7}\cdot\left.\left(t^{\frac{7}{2}}\right)\right|_0^1=\frac{2}{3}-\frac{2}{7}=\frac{8}{21}.$$

Exercício 33. $\int_1^2 (x^3-2x)\cdot ln\,x\cdot dx$.

Integramos por partes. Fazemos:

$$U=ln\,x \;\Rightarrow\; dU=\left(ln\,x\right)'\cdot dx=\frac{1}{x}\cdot dx,$$

$$dV=(x^3-2x)\cdot dx \;\Rightarrow\; V=\int(x^3-2x)\cdot dx \;\Rightarrow\; V=\frac{x^4}{4}-x^2.$$

Portanto

$$\int_1^2 (x^3-2x)\cdot ln\,x\cdot dx=\left.\left(\left(\frac{x^4}{4}-x^2\right)\cdot ln\,x\right)\right|_1^2-\int_1^2\left(\frac{x^4}{4}-x^2\right)\cdot\frac{1}{x}\cdot dx=$$

$$=\left(\left(\frac{2^4}{4}-2^2\right)\cdot ln\,2-\left(\frac{1^4}{4}-1^2\right)\cdot ln\,1\right)-\int_1^2\left(\frac{x^3}{4}-x\right)\cdot dx=-\int_1^2\left(\frac{x^3}{4}-x\right)\cdot dx=$$

$$=-\int_1^2\frac{x^3}{4}\cdot dx+\int_1^2 x\cdot dx=-\left.\left(\frac{x^4}{16}\right)\right|_1^2+\left.\left(\frac{x^2}{2}\right)\right|_1^2=-\left(\frac{2^4}{16}-\frac{1^4}{16}\right)+\left(\frac{2^2}{2}-\frac{1^2}{2}\right)=-\frac{15}{16}+\frac{3}{2}=\frac{9}{16}.$$

Exercício 34. $\int_0^2 \frac{4\sqrt{2-x}-\sqrt{2+x}}{\left(\sqrt{2+x}+4\sqrt{2-x}\right)\cdot(x+2)^2}\cdot dx$.

Transformamos a função integranda de modo a obter expressões de forma $\sqrt{\frac{2-x}{2+x}}$.

$$\int_0^2 \frac{4\sqrt{2-x}-\sqrt{2+x}}{\left(\sqrt{2+x}+4\sqrt{2-x}\right)\cdot(x+2)^2}\cdot dx=\int_0^2 \frac{4\cdot\sqrt{\frac{2-x}{2+x}}-1}{\left(1+4\cdot\sqrt{\frac{2-x}{2+x}}\right)\cdot(x+2)^2}\cdot dx=(*)$$

Fazemos a substituição $\frac{2-x}{2+x}=t^2$.

Então

$$\sqrt{\frac{2-x}{2+x}}=t^2 \quad\Rightarrow 2-x=t^2\cdot(2+x) \quad\Rightarrow 2-x=2\cdot t^2+t^2\cdot x \quad\Rightarrow 2-2\cdot t^2=x\cdot(t^2+1) \quad\Rightarrow$$

$$x=\frac{2-2\cdot t^2}{t^2+1}=\frac{4-2-2\cdot t^2}{t^2+1}=\frac{4-(2+2\cdot t^2)}{t^2+1}=\frac{4}{t^2+1}-\frac{(2+2\cdot t^2)}{t^2+1}=\frac{4}{t^2+1}-2.$$

Portanto

$$dx=d\left(\frac{4}{t^2+1}-2\right)=d\left(\frac{4}{t^2+1}\right)=-\frac{8t}{\left(t^2+1\right)^2}\cdot dt.$$

Determinamos os limites de integração para a variável t:

$$x_{inf}=0 \quad\Rightarrow\quad t_{inf}=\sqrt{\frac{2-0}{2+0}}=1, \qquad x_{sup}=2 \quad\Rightarrow\quad t_{sup}=\sqrt{\frac{2-2}{2+2}}=0.$$

Na continuação temos:

$$(*)=\int_1^0 \frac{4\cdot t-1}{(1+4\cdot t)\cdot\left(\frac{4}{t^2+1}-2+2\right)^2}\cdot\left(-\frac{8t}{\left(t^2+1\right)^2}\cdot dt\right)=\int_1^0 \frac{(4\cdot t-1)\cdot 8t}{16\cdot(1+4\cdot t)}\cdot dt=$$

$$=\int_1^0 \frac{(4\cdot t-1)\cdot 8t}{16\cdot(1+4\cdot t)}\cdot dt=-\frac{1}{2}\cdot\int_0^1 \frac{4\cdot t^2-t}{4\cdot t+1}\cdot dt=$$

A função integranda é racional irregular. Dividimos o polinómio do numerador pelo polinómio do denominador e obtemos:

$$=-\frac{1}{2}\cdot\int_0^1\left(t-\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{4\cdot t+1}\right)\cdot dt=-\frac{1}{2}\cdot\int_0^1 t\cdot dt-\frac{1}{2}\cdot\int_0^1\left(-\frac{1}{2}\right)\cdot dt+\frac{1}{2}\cdot\int_0^1\left(\frac{1}{2}\cdot\frac{1}{4\cdot t+1}\right)\cdot dt=$$

$$= -\frac{1}{2}\cdot\int_0^1 t\cdot dt + \frac{1}{4}\cdot\int_0^1 dt + \frac{1}{4}\cdot\int_0^1 \frac{1}{4\cdot t+1}\cdot dt = -\frac{1}{2}\cdot\int_0^1 t\cdot dt + \frac{1}{4}\cdot\int_0^1 dt + \frac{1}{16}\cdot\int_0^1 \frac{1}{4\cdot t+1}\cdot d(4t+1) =$$

$$= -\frac{1}{2}\cdot\left(\frac{t^2}{2}\right)\Bigg|_0^1 + \frac{1}{4}\cdot\left(t\right)\Big|_0^1 + \frac{1}{16}\cdot\left(ln\left|4\cdot t+1\right|\right)\Big|_0^1 =$$

$$= -\frac{1}{2}\cdot\left(\frac{1^2}{2}-\frac{0^2}{2}\right)+\frac{1}{4}\cdot(1-0)+\frac{1}{16}\cdot ln\left|4\cdot 1+1\right| - ln\left|4\cdot 0+1\right| = -\frac{1}{4}+\frac{1}{4}+\frac{ln5}{16} = \frac{ln5}{16}.$$

Exercício 35. Calcular a área da região plana

$$A = \left\{(x,y)\in R^2 : \quad y \le 2-x^2 \wedge y \ge x \right\}.$$

A região é limitada pelo gráfico da parábola $y = 2-x^2$ orientada em baixo e pela recta $y = x$. O esboço da região é apresentado na figura 1.

Os limites de variação (integração) da variável x é o segmento que é a projecção da região sobre o eixo Ox. Portanto para determinar os limites de integração determinamos as abcissas dos pontos de intersecção da parábola $y = 2-x^2$ com a recta $y = x$:

$$y = x \wedge y = 2-x^2 \quad \Rightarrow \quad x = 2-x^2 \quad \Rightarrow \quad x^2+x-2=0 \quad \Rightarrow$$

$$x = \frac{-1\pm\sqrt{9}}{2} = \frac{-1\pm 3}{2} \quad \Rightarrow \quad x=-2 \vee x=1.$$

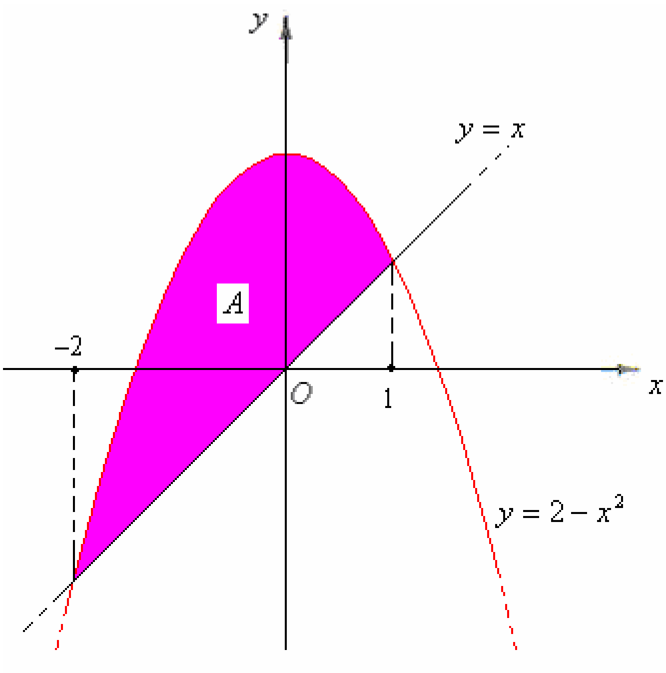

figura 1

Porque a linha que delimita a região na parte de baixo é dada analiticamente só por uma função e a linha que delimita a região na parte de cima também é dada analiticamente só por uma função temos:

$$S_A = \int_{-2}^{1} \left(2 - x^2 - x\right) dx = \int_{-2}^{1} 2\, dx - \int_{-2}^{1} x^2\, dx - \int_{-2}^{1} x\, dx = (2x)\Big|_{-2}^{1} - \left(\frac{x^3}{3}\right)\Bigg|_{-2}^{1} - \left(\frac{x^2}{2}\right)\Bigg|_{-2}^{1} =$$

$$= (2\cdot 1 - 2\cdot(-2)) - \left(\frac{1^3}{3} - \frac{(-2)^3}{3}\right) - \left(\frac{1^2}{2} - \frac{(-2)^2}{2}\right) = 6 - \frac{9}{3} + \frac{3}{2} = \frac{9}{2}.$$

Exercício 36. Calcular a área da região plana

$$A = \left\{(x, y) \in R^2 : \quad y^2 \geq 2x + 4 \wedge y \geq x - 2 \right\}.$$

A região é limitada pelo gráfico da parábola $y^2 = 2x + 4$ orientada no sentido positivo do eixo Ox e pela recta $y = x - 2$. O esboço da região é apresentado na figura 2.

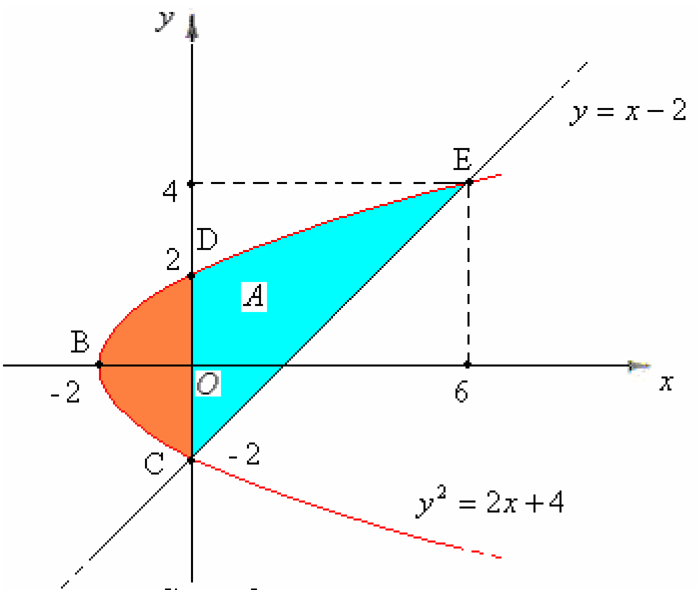

figura 2

Porque

$y^2 = 2x + 4 \quad \Leftrightarrow \quad x = \frac{1}{2} y^2 - 2$ concluímos que o vértice B da parábola tem as coordenadas $\left(-2, 0\right)$ e a parábola intersecta o eixo Oy nos pontos $C = \left(0, -2\right)$ e $D = \left(0, 2\right)$.

Determinamos as coordenadas dos pontos de intersecção da parábola $y^2 = 2x + 4$ com a recta $y = x - 2$:

$$y^2 = 2x + 4 \wedge y = x - 2 \quad \Rightarrow \quad (x-2)^2 = 2x + 4 \quad \Rightarrow \quad x^2 - 4x + 4 = 2x + 4 \quad \Rightarrow$$

$x^2 - 6x = 0 \quad \Rightarrow \quad x = 0 \vee x = 6$.
Calculemos a área da região.

1^o método

A projecção da região sobre o eixo Ox é o segmento $[-2, 6]$.

Porque a linha que delimita a região na parte de baixo é dada analiticamente por duas funções a área da região representa a soma das áreas das regiões (BCD) e (CDE).

A região (BCD) é limitada na parte de baixo pelo ramo $y = -\sqrt{2x+4}$, na parte de cima pelo ramo $y = \sqrt{2x+4}$ da parábola e a sua projecção sobre o eixo Ox é o segmento $[-2, 0]$. Portanto

$$S_{(BCD)} = \int_{-2}^{0}\left(\sqrt{2x+4} - \left(-\sqrt{2x+4}\right)\right)dx = \int_{-2}^{0} 2\cdot\sqrt{2x+4}\,dx = \int_{-2}^{0}\sqrt{2x+4}\,d(2x+4) =$$

$$= \int_{-2}^{0}(2x+4)^{\frac{1}{2}}\,d(2x+4) = \left.\left(\frac{(2x+4)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)\right|_{-2}^{0} = \frac{2}{3}\cdot\left.\left((2x+4)^{\frac{3}{2}}\right)\right|_{-2}^{0} = \frac{2}{3}\cdot 4^{\frac{3}{2}} = \frac{2}{3}\cdot 4\sqrt{4} = \frac{16}{3}.$$

A região (CDE) é limitada na parte de baixo pela recta $y = x - 2$, na parte de cima pelo ramo $y = \sqrt{2x+4}$ da parábola e a sua projecção sobre o eixo Ox é o segmento $[0, 6]$. Portanto

$$S_{(CDE)} = \int_{0}^{6}\left(\sqrt{2x+4} - (x-2)\right)dx = \int_{0}^{6}\left(\sqrt{2x+4} - x + 2\right)dx = \int_{0}^{6}\sqrt{2x+4}\,dx - \int_{0}^{6}x\,dx + 2\int_{0}^{6}dx =$$

$$= \frac{1}{2}\cdot\int_{0}^{6}(2x+4)^{\frac{1}{2}}\,d(2x+4) - \int_{0}^{6}x\,dx + 2\int_{0}^{6}dx = \frac{1}{2}\cdot\left.\left(\frac{(2x+4)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)\right|_{0}^{6} - \left.\left(\frac{x^2}{2}\right)\right|_{0}^{6} + 2\cdot(x)\Big|_{0}^{6} =$$

$$= \frac{1}{3}\cdot\left.\left((2x+4)^{\frac{3}{2}}\right)\right|_{0}^{6} - \left.\left(\frac{x^2}{2}\right)\right|_{0}^{6} + 2\cdot(x)\Big|_{0}^{6} = \frac{1}{3}\cdot\left((2\cdot 6+4)^{\frac{3}{2}} - (2\cdot 0+4)^{\frac{3}{2}}\right) - \left(\frac{6^2}{2} - \frac{0^2}{2}\right) + 2\cdot(6-0) =$$

$$= \frac{1}{3}\cdot(64-8) - 18 + 12 = \frac{56}{3} - 6 = \frac{38}{3}.$$

Portanto

$$S_A = S_{(BCD)} + S_{(CDE)} = \frac{16}{3} + \frac{38}{3} = \frac{54}{3} = 18.$$

2^o método

Observamos que em relação ao eixo Oy a região é limitada à esquerda pela parábola $x = \frac{1}{2}y^2 - 2$, à direita pela recta $x = y + 2$ e a sua projecção sobre o eixo Oy é o segmento $[-2, 4]$. Portanto

$$S_A = \int_{-2}^{4}\left(y+2-\frac{1}{2}y^2+2\right)dy = \int_{-2}^{4}\left(4+y-\frac{1}{2}y^2\right)dy = \int_{-2}^{4}(4+y)\,d(4+y)-\frac{1}{2}\cdot\int_{-2}^{4}y^2\,dy =$$

$$=\left(\frac{(4+y)^2}{2}\right)\Bigg|_{-2}^{4}-\frac{1}{2}\cdot\left(\frac{y^3}{3}\right)\Bigg|_{-2}^{4}=\left(\frac{8^2}{2}-\frac{2^2}{2}\right)-\frac{1}{2}\cdot\left(\frac{4^3}{3}-\frac{(-2)^3}{3}\right)=30-12=18\,.$$

Exercício 37. Calcular a área da região plana

$$A=\left\{(x,y)\in R^2:\quad x^2+(y-1)^2\ge 1 \ \wedge\ x^2+(y-2)^2\le 4 \ \wedge\ y\ge x^2\ \right\}.$$

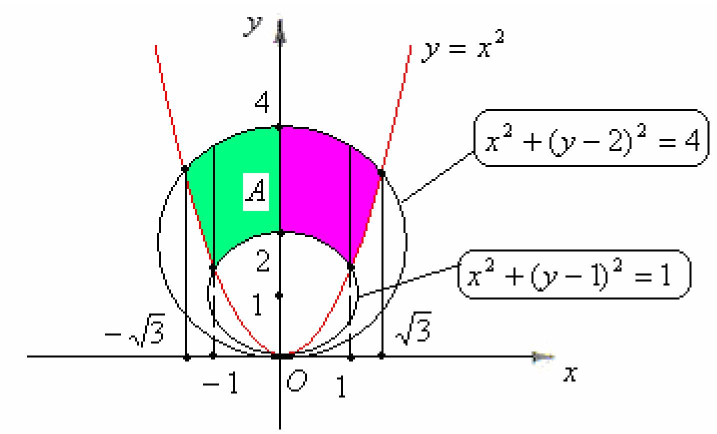

figura 3

A região é situada fora da circunferência $x^2+(y-1)^2=1$, dentro da circunferência $x^2+(y-2)^2=4$ e entre os ramos da parábola $y=x^2$. O esboço da região é dado na figura 3 e porque as funções que delimitam a região são pares concluímos que a região é simétrica em relação ao eixo Oy. Portanto para determinar a área S_A da região calculemos a área S da metade da região situada á direita do eixo Oy e multiplicamos o resultado obtido por dois.

A metade da região situada á direita do eixo Oy é limitada na parte de baixo pela circunferência $x^2+(y-1)^2=1$ (mais precisamente pela semicircunferência $y=1+\sqrt{1-x^2}$) e pela parábola $y=x^2$. Na parte de cima é limitada pela semicircunferência $y=2+\sqrt{4-x^2}$.

Determinamos os pontos de intersecção da parábola com as circunferências:

$$\begin{cases} y=x^2 \\ x^2+(y-1)^2=1 \end{cases} \Leftrightarrow \begin{cases} y=x^2 \\ y+y^2-2y+1=1 \end{cases} \Leftrightarrow \begin{cases} y=x^2 \\ y^2-y=0 \end{cases} \Leftrightarrow \begin{cases} y=x^2 \\ y=0 \vee y=1 \end{cases} \Leftrightarrow$$

$$\begin{cases} y=x^2 \\ y=0 \end{cases} \vee \begin{cases} y=x^2 \\ y=1 \end{cases} \Leftrightarrow (x,y)=(0,0)\vee(x,y)=(-1,1)\vee(x,y)=(1,1)\,.$$

Para metade da região situada á direita do eixo Oy temos os pontos $(x,y)=(0,0)$ e $(x,y)=(1,1)$ de intersecção da parábola com a circunferência $x^2+(y-1)^2=1$.

$$\begin{cases} y=x^2 \\ x^2+(y-2)^2=4 \end{cases} \Leftrightarrow \begin{cases} y=x^2 \\ y+y^2-4y+4=4 \end{cases} \Leftrightarrow \begin{cases} y=x^2 \\ y^2-3y=0 \end{cases} \Leftrightarrow \begin{cases} y=x^2 \\ y=0 \vee y=3 \end{cases} \Leftrightarrow$$

$$\begin{cases} y=x^2 \\ y=0 \end{cases} \vee \begin{cases} y=x^2 \\ y=3 \end{cases} \Leftrightarrow (x,y)=(0,0) \vee (x,y)=(-\sqrt{3},3) \vee (x,y)=(\sqrt{3},3).$$

Para metade da região situada á direita do eixo Oy temos os pontos $(x,y)=(0,0)$ e $(x,y)=(\sqrt{3},3)$ de intersecção da parábola com a circunferência $x^2+(y-2)^2=4$.

Portanto a projecção da parte direita da região sobre o eixo Ox é o segmento $\left[0,\sqrt{3}\right]$. Calculemos a área dela S:

$$S=\int_0^1\left(2+\sqrt{4-x^2}-1-\sqrt{1-x^2}\right)dx+\int_1^{\sqrt{3}}\left(2+\sqrt{4-x^2}-x^2\right)dx=$$

$$=\int_0^1\left(1+\sqrt{4-x^2}-\sqrt{1-x^2}\right)dx+\int_1^{\sqrt{3}}\left(2+\sqrt{4-x^2}-x^2\right)dx=$$

$$=\int_0^1 d\,x+\int_0^1\sqrt{4-x^2}\,dx-\int_0^1\sqrt{1-x^2}\,dx+2\int_1^{\sqrt{3}}d\,x+\int_1^{\sqrt{3}}\sqrt{4-x^2}\,dx-\int_1^{\sqrt{3}}x^2\,dx=$$

$$=\int_0^1 d\,x+\int_0^{\sqrt{3}}\sqrt{4-x^2}\,dx-\int_0^1\sqrt{1-x^2}\,dx+2\int_1^{\sqrt{3}}d\,x-\int_1^{\sqrt{3}}x^2\,dx=$$

$$=(x)\Big|_0^1+\int_0^{\sqrt{3}}\sqrt{4-x^2}\,dx-\int_0^1\sqrt{1-x^2}\,dx+2\cdot(x)\Big|_1^{\sqrt{3}}-\left(\frac{x^3}{3}\right)\Bigg|_1^{\sqrt{3}}=$$

$$=1+\int_0^{\sqrt{3}}\sqrt{4-x^2}\,dx-\int_0^1\sqrt{1-x^2}\,dx+2\sqrt{3}-2-\sqrt{3}+\frac{1}{3}=$$

$$=\sqrt{3}-\frac{2}{3}+\int_0^{\sqrt{3}}\sqrt{4-x^2}\,dx-\int_0^1\sqrt{1-x^2}\,dx=\sqrt{3}-\frac{2}{3}+2\cdot\int_0^{\sqrt{3}}\sqrt{1-\left(\frac{x}{2}\right)^2}\,dx-\int_0^1\sqrt{1-x^2}\,dx=(*)$$

No primeiro integral fazemos a substituição $x = 2\,sen\,t$.

Então

$d\,x = 2\cdot cos\,t\cdot d\,t$ e $\sqrt{1-\left(\frac{x}{2}\right)^2} = \sqrt{1-sen^2 t} = \sqrt{cos^2 t} = cos\,t$

Determinamos os limites de integração para a variável t:

$$x_{inf} = 0 \quad \Rightarrow \quad t_{inf} = arcsen\left(\frac{x_{inf}}{2}\right) = arcsen(0) = 0,$$

$$x_{sup} = \sqrt{3} \quad \Rightarrow \quad t_{sup} = arcsen\left(\frac{x_{sup}}{2}\right) = arcsen\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}.$$

No segundo integral fazemos a substituição $x = sen\,u$.

Então

$d\,x = cos\,u\cdot d\,u$ e $\sqrt{1-x^2} = \sqrt{1-sen^2 u} = \sqrt{cos^2 u} = cos\,u$

Determinamos os limites de integração para a variável u:

$$x_{inf} = 0 \quad \Rightarrow \quad u_{inf} = arcsen(x_{inf}) = arcsen(0) = 0,$$

$$x_{sup} = 1 \quad \Rightarrow \quad u_{sup} = arcsen(x_{sup}) = arcsen(1) = \frac{\pi}{2}.$$

Na continuação temos:

$$(*) = \sqrt{3} - \frac{2}{3} + 2\cdot\int_0^{\frac{\pi}{3}} 2\cdot cos^2 t\cdot dt - \int_0^{\frac{\pi}{2}} cos^2 u\,du =$$

Como $cos^2\alpha = \frac{1+cos(2\alpha)}{2} = \frac{1}{2} + \frac{cos(2\alpha)}{2}$ temos:

$$= \sqrt{3} - \frac{2}{3} + 2\cdot\int_0^{\frac{\pi}{3}} (1+cos(2t))\cdot dt - \int_0^{\frac{\pi}{2}}\left(\frac{1}{2} + \frac{cos(2u)}{2}\right)du =$$

$$= \sqrt{3} - \frac{2}{3} + 2\cdot\int_0^{\frac{\pi}{3}} d\,t + \int_0^{\frac{\pi}{3}} cos(2t)\cdot d(2t) - \frac{1}{2}\cdot\int_0^{\frac{\pi}{2}} du - \frac{1}{4}\cdot\int_0^{\frac{\pi}{2}} cos(2u)\cdot d(2u) =$$

$$= \sqrt{3} - \frac{2}{3} + 2\cdot\left(t\right)\Big|_0^{\frac{\pi}{3}} + \left(sen(2t)\right)\Big|_0^{\frac{\pi}{3}} - \frac{1}{2}\cdot\left(u\right)\Big|_0^{\frac{\pi}{2}} - \frac{1}{4}\cdot\left(sen(2u)\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= \sqrt{3} - \frac{2}{3} + 2\cdot\left(\frac{\pi}{3} - 0\right) + \left(sen\left(\frac{2\pi}{3}\right) - sen(0)\right) - \frac{1}{2}\cdot\left(\frac{\pi}{2} - 0\right) - \frac{1}{4}\cdot\left(sen(\pi) - sen(0)\right) =$$

$$= \sqrt{3} - \frac{2}{3} + 2\cdot\frac{\pi}{3} + \frac{\sqrt{3}}{2} - \frac{\pi}{4} = \frac{9\sqrt{3}-4}{6} + \frac{5\pi}{12}.$$

Portanto

$$S_A = 2S = 2\cdot\left(\frac{9\sqrt{3}-4}{6} + \frac{5\pi}{12}\right) = \frac{9\sqrt{3}-4}{3} + \frac{5\pi}{6}.$$

Exercício 38. Calcular o comprimento da linha dada pela função $f(x) = ln\,x$, com $x \in \left[2\sqrt{2}\,,\, 2\sqrt{6}\,\right]$.

Para calcular o comprimento da linha utilizamos a fórmula $l = \int\limits_a^b \sqrt{1+\left[f'(x)\right]^2}\cdot dx$.

Temos: $a = 2\sqrt{2}$, $b = 2\sqrt{6}$, $f'(x) = \frac{1}{x}$.

Portanto

$$l = \int\limits_{2\sqrt{2}}^{2\sqrt{6}} \sqrt{1+\left(\frac{1}{x}\right)^2}\cdot dx = \int\limits_{2\sqrt{2}}^{2\sqrt{6}} \sqrt{\frac{x^2+1}{x^2}}\cdot dx = \int\limits_{2\sqrt{2}}^{2\sqrt{6}} \frac{\sqrt{x^2+1}}{x}\cdot dx = (*)$$

Integramos por substituição.

Fazemos $\sqrt{x^2+1} = t \;\Rightarrow\; x^2+1 = t^2 \;\Rightarrow\; x^2 = t^2-1 \;\Rightarrow\; x = \sqrt{t^2-1}$.

$$d\,x = \left(\sqrt{t^2-1}\right)'\cdot dt = \frac{t}{\sqrt{t^2-1}}\cdot dt.$$

Determinamos os limites de integração para a variável t:

$$x_{inf} = 2\sqrt{2} \;\Rightarrow\; t_{inf} = \sqrt{\left(x_{inf}\right)^2+1} = \sqrt{\left(2\sqrt{2}\right)^2+1} = \sqrt{9} = 3,$$

$$x_{sup} = 2\sqrt{6} \;\Rightarrow\; t_{sup} = \sqrt{\left(x_{sup}\right)^2+1} = \sqrt{\left(2\sqrt{6}\right)^2+1} = \sqrt{25} = 5.$$

Na continuação temos:

$$(*) = \int\limits_3^5 \frac{t}{\sqrt{t^2-1}}\cdot\frac{t}{\sqrt{t^2-1}}\cdot dt = \int\limits_3^5 \frac{t^2}{t^2-1}\cdot dt = \int\limits_3^5 \frac{t^2-1+1}{t^2-1}\cdot dt = \int\limits_3^5 \left(\frac{t^2-1}{t^2-1} + \frac{1}{t^2-1}\right)\cdot dt =$$

$$= \int\limits_3^5 \left(1+\frac{1}{t^2-1}\right)\cdot dt = \int\limits_3^5 dt + \int\limits_3^5 \frac{1}{t^2-1}\cdot dt =$$

Representamos a fracção racional como soma de fracções elementares:

$$\frac{1}{t^2-1}=\frac{A}{t-1}+\frac{B}{t+1} \quad \Rightarrow \quad 1=(A+B)\cdot t+(A-B) \quad \Rightarrow \quad \begin{cases} A+B=0 \\ A-B=1 \end{cases} \quad \Rightarrow \quad \begin{cases} A=\frac{1}{2}, \\ B=-\frac{1}{2}. \end{cases}$$

Então:

$$=\int_3^5 dt+\int_3^5\left(\frac{\frac{1}{2}}{t-1}-\frac{\frac{1}{2}}{t-1}\right)\cdot dt=\int_3^5 dt+\frac{1}{2}\cdot\int_3^5\frac{1}{t-1}\cdot dt-\frac{1}{2}\cdot\int_3^5\frac{1}{t+1}\cdot dt=$$

$$=\left(t\right)\Big|_3^5+\frac{1}{2}\cdot\left(ln(t-1)\right)\Big|_3^5-\frac{1}{2}\cdot\left(ln(t+1)\right)\Big|_3^5=$$

$$=(5-3)+\frac{1}{2}\cdot\left[ln(5-1)-ln(3-1)\right]-\frac{1}{2}\cdot\left[ln(5+1)-ln(3+1)\right]=$$

$$=2+\frac{1}{2}\cdot\left[ln(4)-ln(2)-ln(6)+ln(4)\right]=2+\frac{1}{2}\cdot\left[ln(16)-ln(12)\right]=2+\frac{1}{2}\cdot ln\left(\frac{16}{12}\right)=2+ln\frac{2}{\sqrt{3}}.$$

Exercício 39. Calcular o comprimento da linha dada pela função $f(x)=ln(cos x)$, com $x\in\left[0,\frac{\pi}{6}\right]$.

Para calcular o comprimento da linha utilizamos a fórmula $l=\int_a^b\sqrt{1+\left[f'(x)\right]^2}\cdot dx$.

Temos: $a=0$, $b=\frac{\pi}{6}$, $f'(x)=\left(ln(cos x)\right)'=-\frac{sen x}{cos x}$.

Portanto

$$l=\int_0^{\frac{\pi}{6}}\sqrt{1+\left(-\frac{sen x}{cos x}\right)^2}\cdot dx=\int_0^{\frac{\pi}{6}}\sqrt{1+\frac{sen^2 x}{cos^2 x}}\cdot dx=\int_0^{\frac{\pi}{6}}\sqrt{\frac{cos^2 x+sen^2 x}{cos^2 x}}\cdot dx=\int_0^{\frac{\pi}{6}}\frac{1}{cos x}\cdot dx=(*)$$

Integramos por substituição.

Fazemos $tg\left(\frac{x}{2}\right)=t \quad \Rightarrow \quad x=2\cdot arctg(t) \quad \Rightarrow \quad dx=\frac{2}{1+t^2}\cdot dt$. $cos x=\frac{1-t^2}{1+t^2}$.

Determinamos os limites de integração para a variável t:

$x_{inf} = \quad \Rightarrow \quad t_{inf} = tg\left(\frac{x_{inf}}{2}\right) = tg\,(0) = 0\,,$

$x_{sup} = \frac{\pi}{6} \quad \Rightarrow \quad t_{sup} = tg\left(\frac{x_{sup}}{2}\right) = tg\left(\frac{\pi}{12}\right).$

Na continuação temos:

$$(*) = \int_0^{tg\left(\frac{\pi}{12}\right)} \frac{1}{\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} \cdot dt = \int_0^{tg\left(\frac{\pi}{12}\right)} \frac{2}{1-t^2} \cdot dt =$$

Representamos a fracção racional como soma de fracções elementares:

$$\frac{2}{1-t^2} = \frac{A}{1-t} + \frac{B}{1+t} \quad \Rightarrow \quad 2 = (A-B)\cdot t + (A+B) \quad \Rightarrow \quad \begin{cases} A-B=0 \\ A+B=2 \end{cases} \quad \Rightarrow \quad \begin{cases} A=1, \\ B=1. \end{cases}$$

Então:

$$= \int_0^{tg\left(\frac{\pi}{12}\right)} \left(\frac{1}{1-t} + \frac{1}{1+t}\right) \cdot dt = \int_0^{tg\left(\frac{\pi}{12}\right)} \frac{1}{1-t} \cdot dt + \int_0^{tg\left(\frac{\pi}{12}\right)} \frac{1}{1+t} \cdot dt =$$

$$= -\int_0^{tg\left(\frac{\pi}{12}\right)} \frac{1}{1-t} \cdot d(1-t) + \int_0^{tg\left(\frac{\pi}{12}\right)} \frac{1}{1+t} \cdot d(1+)t = -\left(ln|1-t|\right)\Bigg|_0^{tg\left(\frac{\pi}{12}\right)} + \left(ln|1+t|\right)\Bigg|_0^{tg\left(\frac{\pi}{12}\right)} =$$

$$= -\left(ln\left|1-tg\left(\frac{\pi}{12}\right)\right| - ln|1-0|\right) + \left(ln\left|1+tg\left(\frac{\pi}{12}\right)\right| - ln|1+0|\right) =$$

$$= ln\left|1+tg\left(\frac{\pi}{12}\right)\right| - ln\left|1-tg\left(\frac{\pi}{12}\right)\right| = ln\left(\frac{1+tg\left(\frac{\pi}{12}\right)}{1-tg\left(\frac{\pi}{12}\right)}\right) = ln\left(\frac{cos\left(\frac{\pi}{12}\right)+sen\left(\frac{\pi}{12}\right)}{cos\left(\frac{\pi}{12}\right)-sen\left(\frac{\pi}{12}\right)}\right) =$$

$$= ln\left(\frac{\left(cos\left(\frac{\pi}{12}\right)+sen\left(\frac{\pi}{12}\right)\right)\cdot\left(cos\left(\frac{\pi}{12}\right)-sen\left(\frac{\pi}{12}\right)\right)}{\left[cos\left(\frac{\pi}{12}\right)-sen\left(\frac{\pi}{12}\right)\right]^2}\right) =$$

$$= ln\left(\frac{cos^2\left(\frac{\pi}{12}\right)-sen^2\left(\frac{\pi}{12}\right)}{cos^2\left(\frac{\pi}{12}\right)-2\cdot cos\left(\frac{\pi}{12}\right)\cdot sen\left(\frac{\pi}{12}\right)+sen^2\left(\frac{\pi}{12}\right)}\right) = ln\left(\frac{cos\left(\frac{\pi}{6}\right)}{1-sen\left(\frac{\pi}{6}\right)}\right) = ln\left(\frac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}}\right) = ln\sqrt{3}.$$

Exercício 40. Calcular o comprimento da linha dada pela função $f(x)=\sqrt{x^3}$, com $x\in\left[\,0\,,\frac{4}{3}\,\right]$.

Para calcular o comprimento da linha utilizamos a fórmula $l=\int_a^b\sqrt{1+[f'(x)]^2}\cdot dx$.

Temos: $a=0$, $b=\frac{4}{3}$, $f'(x)=\left(\sqrt{x^3}\right)'=\left(x^{\frac{3}{2}}\right)'=\frac{3}{2}\cdot x^{\frac{1}{2}}$.

Portanto

$$l=\int_0^{\frac{4}{3}}\sqrt{1+\left(\frac{3}{2}\cdot x^{\frac{1}{2}}\right)^2}\cdot dx=\int_0^{\frac{4}{3}}\sqrt{1+\frac{9}{4}x}\cdot dx=\frac{4}{9}\cdot\int_0^{\frac{4}{3}}\left(1+\frac{9}{4}x\right)^{\frac{1}{2}}\cdot d\left(1+\frac{9}{4}x\right)=$$

$$=\frac{4}{9}\cdot\left.\left(\frac{\left(1+\frac{9}{4}x\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)\right|_0^{\frac{4}{3}}=\frac{4}{9}\cdot\frac{2}{3}\cdot\left.\left(\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}\right)\right|_0^{\frac{4}{3}}=\frac{8}{27}\cdot\left(\left(1+\frac{9}{4}\cdot\frac{4}{3}\right)^{\frac{3}{2}}-\left(1+\frac{9}{4}\cdot 0\right)^{\frac{3}{2}}\right)=\frac{56}{27}.$$

Exercício 41. Calcular o comprimento da linha dada pela função $f(x)=\sqrt{1-x^2}+arcsenx$, com $x\in\left[\,0\,,\frac{1}{2}\,\right]$.

Para calcular o comprimento da linha utilizamos a fórmula $l=\int_a^b\sqrt{1+[f'(x)]^2}\cdot dx$.

Temos: $a=0$, $b=\frac{1}{2}$,

$$f'(x)=\left(\sqrt{1-x^2}+arcsenx\right)'=\left(\sqrt{1-x^2}\right)'+\left(arcsenx\right)'=\frac{-x}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x^2}}=$$

$$=\frac{-x}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x^2}}=\frac{1-x}{\sqrt{1-x^2}}=\frac{1-x}{\sqrt{(1-x)\cdot(1+x)}}=\frac{1-x}{\sqrt{1-x}\cdot\sqrt{1+x}}=\frac{\sqrt{1-x}}{\sqrt{1+x}}=\sqrt{\frac{1-x}{1+x}}.$$

Portanto

$$l=\int_0^{\frac{1}{2}}\sqrt{1+\left(\sqrt{\frac{1-x}{1+x}}\right)^2}\cdot dx=\int_0^{\frac{1}{2}}\sqrt{1+\frac{1-x}{1+x}}\cdot dx=\int_0^{\frac{1}{2}}\sqrt{\frac{1+x+1-x}{1+x}}\cdot dx=\int_0^{\frac{1}{2}}\sqrt{\frac{2}{1+x}}\cdot dx=$$

$$=\sqrt{2}\cdot\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1+x}}\cdot dx=\sqrt{2}\cdot\int_0^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}\cdot d(1+x)=\sqrt{2}\cdot\left.\left(\frac{(1+x)^{\frac{1}{2}}}{\frac{1}{2}}\right)\right|_0^{\frac{1}{2}}=2\sqrt{2}\cdot\left(\sqrt{\frac{3}{2}}-1\right)=2(\sqrt{3}-\sqrt{2}).$$