THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: June 2019

Approximation Theory and Applications

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

COMPULSORY QUESTION

1. a) (i) Define the space of trigonometric polynomials of degree N. State the orthogonality property that the canonical basis satisfies. **[3]**

(ii) For $f \in C(\mathbb{T})$, (continuous, 2π-periodic), define the Fourier coefficients $\hat{f}_k$, the Fourier series, as well as the N-partial sum $\Pi_N f$. **[3]**

(iii) Use the orthogonality property from (i) to prove that **[3]**

$$\|f\|_{L^2(\mathbb{T})}^2 = \sum_{k \in \mathbb{Z}} |\hat{f}_k|^2.$$

(iv) Prove that $\Pi_N f$ is the best approximation to f in the $L^2(\mathbb{T})$-norm. **[3]**

b) Derive an error estimate for $\|\Pi_N f - f\|_{L^2(\mathbb{T})}$, where

(i) $|\hat{f}_k| \le C|k|^{-p}, p \ge 1$; **[2]**

(ii) $|\hat{f}_k| \le Ce^{-\alpha|k|}$. **[2]**

(iii) In each of the two cases (i, ii) state a regularity assumption on f that gives rise to this decay in the Fourier coefficients. **[2]**

(iv) For the following functions state without proof estimates on the rate of convergence for $\|\Pi_N f - f\|_{L^2(\mathbb{T})}$ based on (i, ii, iii): **[8]**

$$f_1(x) = |\sin(5x)|^5 \qquad f_2(x) = (1 + \cos^2(x))^{-1}$$
$$f_3(x) = \sin(\cos(x)) \qquad f_4(x) = x\sin^2(x).$$

All functions are defined for $x \in (-\pi, \pi]$ and extended 2π-periodically.

c) Let $\mathcal{P}_N$ denote the space of algebraic polynomials of degree up to N and let $E(\rho), \rho > 1$ denote a Bernstein ellipse. If f is analytic in $E(\rho)$ and $M = \|f\|_{L^\infty(E(\rho))}$ then

$$\inf_{p \in \mathcal{P}_N} \|f - p\|_\infty \le \frac{2M\rho^{-N}}{\rho - 1}. \tag{1}$$

(i) Define the set $E(\rho)$. **[2]**

(ii) Using (1), establish an explicit super-exponential convergence rate for $f(x) = e^x$, specifying all constants. *A sharp estimate is not required.* **[6]**

d) (i) Given interpolation nodes $x_0 < \cdots < x_N \in [-1, 1]$, define the associated nodal interpolation operator I_N for polynomials. *You need not show that it is well-defined.* **[2]**

(ii) Prove an interpolation error estimate of the form **[4]**

$$\|f - I_N f\|_{L^\infty(-1,1)} \le C_N \inf_{p \in \mathcal{P}_N} \|f - p\|_{L^\infty(-1,1)},$$

where C_N should be precisely defined but need not be calculated or estimated. Properties of I_N that you use should be stated, but need not be proven.

OPTIONAL QUESTIONS

2. a) (i) Let $f \in C(\mathbb{T})$ (continuous and 2π-periodic). Define what it means for f to have a modulus of continuity ω. **[1]**

(ii) State Jackson's first theorem for the approximation of f in the max-norm by trigonometric polynomials. **[2]**

b) Let $f \in C([-1,1])$ with modulus of continuity ω.

(i) Use a suitable coordinate transformation to derive the Chebyshev nodes and Chebyshev polynomial basis from equispaced nodes and the canonical trigonometric polynomial basis. *(You need not prove that the transformed functions are in fact algebraic polynomials.)* **[3]**

(ii) Derive an estimate for the approximation of f in the max-norm by algebraic polynomials from the result in (a)(ii). **[5]**

(iii) If $f \in C^1([-1,1])$, give a sharp bound for its modulus of continuity ω. **[1]**

c) (i) Let $E_N(f)$ denote the best approximation error of $f \in C([-1,1])$ by algebraic polynomials of degree N. Prove that, if $f \in C^1([-1,1])$, then there exists a generic constant C such that **[3]**

$$E_N(f) \le C N^{-1} E_{N-1}(f').$$

HINT: use (b)(ii) and (b)(iii) and the fact that $E_N(f) = E_N(f+q)$ *for all* $q \in \mathcal{P}_N$.

(ii) Let $f \in C^j([-1,1]), j \ge 1$ where $f^{(j)}$ has modulus of continuity ω. Prove a sharp max-norm approximation error estimate of the form **[5]**

$$\inf_{p \in \mathcal{P}_N} \big(\|f - p\|_\infty + \|f' - p'\|_\infty \big) \le C \epsilon(N),$$

where $\epsilon(N)$ is a rate that you should specify.

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3. a) For $f \in C^{N+1}([a,b])$, $N > 0$ prove that there exists a constant C_N and a polynomial $p_N \in \mathcal{P}_N$ such that **[4]**

$$\|f - p_N\|_{L^\infty(a,b)} \le C_N (b-a)^{N+1} \|f^{(N+1)}\|_{L^\infty(a,b)}, \qquad \text{and}$$
$$\|f' - p_N'\|_{L^\infty(a,b)} \le C_N (b-a)^{N} \|f^{(N+1)}\|_{L^\infty(a,b)}.$$

You may use, without proof that, for $g \in C^{(N+1)}([0,1])$, there exists $q_N \in \mathcal{P}_N$ such that

$$\|g - q_N\|_{L^\infty(0,1)} + \|g' - q_N'\|_{L^\infty(0,1)} \le C_N \|f^{(N+1)}\|_{L^\infty(0,1)}.$$

b) (i) Define the spaces $\mathcal{S}_N((y_m)_{m=0}^M)$ of splines of degree N and $\mathcal{S}_N^p((y_m)_{m=0}^M)$ of splines of degree N, regularity C^p and nodes y_m. **[2]**

(ii) For $y_m = hm$ where $h = 1/M$ prove that **[3]**

$$\inf_{s \in \mathcal{S}_N} \|f - s\|_{L^\infty(0,1)} \le C_N h^{N+1} \|f^{(N+1)}\|_{L^\infty(0,1)}. \tag{2}$$

(iii) Let $f_1(x) = \cos(x)$ and $f_2(x) = |\cos(2x)|^3$. For each of f_1, f_2 and for each $M \in \mathbb{N}$ and $N \in \mathbb{N}$ give a sharp upper bound for the approximation error (2). *You need not give a rigorous proof.* **[3]**

c) (i) Still consider $y_m = hm$, $h = 1/M$. Given $f \in C^1([0,1])$ prove that there exists a unique spline $s := I_h f \in \mathcal{S}_3^1$ satisfying **[4]**

$$s(y_m) = f(y_m) \quad \text{and} \quad s'(y_m) = f'(y_m) \qquad \text{for } m = 0, \dots, M.$$

HINT: For $x \in [y_{m-1}, y_m]$, write $s(x) = f_{m-1} + f_{m-1}'(x - y_{m-1}) + a_m (x - y_{m-1})^2 + b_m (x - y_{m-1})^3$, then show that you can uniquely determine a_m, b_m.

(ii) Prove that there exists a constant C_H such that, for $f \in C^3([0,1])$, **[4]**

$$\|f - I_h f\|_{L^\infty(0,1)} \le C_H h^4 \|f^{(4)}\|_{L^\infty(0,1)}.$$

You may use the following identity without proof:

$$\begin{pmatrix} h^2 & h^3 \\ 2h & 3h^2 \end{pmatrix}^{-1} = h^{-4} \begin{pmatrix} 3h^2 & -h^3 \\ -2h & h^2 \end{pmatrix}.$$

4. a) (i) Consider the uniform grid $x_n := n\pi/N$. Prove the identity **[2]**

$$\frac{1}{2N}\sum_{n=0}^{2N-1} e^{ikx_n} = \begin{cases} 1, & k \in 2N\mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

(ii) For $f \in C(\mathbb{T})$ define the trigonometric interpolant $I_N f \in \mathcal{T}_N'$ on the nodes x_n. Define also the modified space of trigonometric polynomials $\mathcal{T}_N'$ with only $2N$ degrees of freedom. Motivate this modification by referring to (3). **[4]**

(You may assume throughout the remainder of the question that $I_N f$ is well-defined.)

(iii) Let $F = (F_n)_{n=0}^{2N-1}$, where $F_n = f(x_n)$, and define the Discrete Fourier transform $\hat{F} \in \mathbb{R}^{2N}$. Briefly explain how this can be used to evaluate the trigonometric interpolant $I_N f$. **[4]**

b) Now consider a finer equi-spaced grid $y_m = m\pi/M$, $m = 0, \dots, 2M-1$ where $M > N$ is an integer multiple of N.

(i) Formulate the least squares problem to fit a trigonometric polynomial $t \in \mathcal{T}_N'$ to data $(y_m, f(y_m))$ with weights $w_m = 1$, in the form $\min \|Ac - Y\|_2^2$ where c are the coefficients of t. **[3]**

(ii) Derive the normal equations and explicitly compute the $2N \times 2N$ system matrix that needs to be inverted to solve them. **[3]**

(iii) Deduce an $O(M \log M)$-scaling algorithm to solve the least squares problem from (i). **[4]**

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5. a) Consider a multi-variate polynomial in a tensor product Chebyshev basis,

$$p(\mathbf{x}) = \sum_{\mathbf{k}\in\{0,\dots,N\}^d} c_{\mathbf{k}} T_{\mathbf{k}}(\mathbf{x}), \qquad \text{where } \mathbf{x} \in \mathbb{R}^d.$$

(i) Define what we mean by $T_{\mathbf{k}}(\mathbf{x})$, where $\mathbf{k} \in \mathbb{N}^d, \mathbf{x} \in \mathbb{R}^d$. **[1]**

(ii) Estimate the computational cost of evaluating $p(\mathbf{x})$ at a single point $\mathbf{x} \in \mathbb{R}^d$. **[3]**

(iii) Suppose that for all $i = 1, \dots, d$, $x_i \mapsto f(\mathbf{x})$ is analytic in $E(\rho)$ (with $x_j, j \neq i$ remaining fixed) where $E(\rho)$ is the usual Bernstein ellipse. Prove the following tensor product polynomial approximation in the max-norm, defining what we mean by $\tilde{\Pi}_N^{(i)}$, **[5]**

$$\|f - \tilde{\Pi}_N^{(d)} \cdots \tilde{\Pi}_N^{(1)} f\|_\infty \leq C(\log N)^d \rho^{-N}.$$

You may use (1) *without proof, as well as the bound* $\|\tilde{\Pi}_N g\|_\infty \leq C \log N \|g\|_\infty$ *for* $g \in C([-1, 1])$. *Compare with (Q1c.)*

(iv) Convert the result from (iii) into an estimate in terms of the cost of storing and evaluating the multivariate polynomial, and thus explain the term "curse of dimensionality". **[3]**

b) Consider a function $f \in C([-1, 1]^d) \to \mathbb{R}$ with its multivariate Chebyshev expansion coefficients satisfying

$$\sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^{d} (1 + k_i)^2 |f_{\mathbf{k}}| =: M_f < \infty.$$

(i) Show that there is a non-tensor truncation of this expansion that leads to the error estimate **[3]**

$$\|f - p_N\|_\infty \leq M_f (1 + N)^{-2}.$$

(ii) By estimating the number of retained coefficients, convert the error estimate in (i) to an estimate in terms of the cost of evaluating the approximant p_N.

You may use, without proof that, if $c_0 n \log n \leq m \leq c_1 n \log n$ *then for* m, n *sufficiently large,* $n \geq c_2 m / \log m$ *where* $c_j > 0$. **[5]**

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Comments for external examiner: (see also solutions)
The exam has a mixture of testing knowledge, proofs, minor variations of proofs (to test understanding), a few concrete examples. This is a new module, so there is no risk of overlap with previous exams. Except on Q1, at most 5 points on each Q require deeper modification of the ideas learned in the lectures.

- Q. 1 is meant to be easy and check the most basic knowledge and understanding of the material. It is almost exclusively bookwork (BW), only (biv) and (cii) are (straightforward) variations of examples seen in the lectures. Every student who has studied seriously should be able to obtain about 80–90% of this question.
- Q.2: (a) is BW and tests one of the fundamental results covered in the lectures; (b) is BW and tests a key idea emphasized several times; (ci) is an exercise; (cii) is variation of an exercise.
- Q.3: (a) is 1/2 BW and 1/2 a trivial extension of the same idea; (bi) and (bii) are BW testing the basic results on splines; (biii) is an immediate application with similar examples seen in the lectures and exercises; (ci) is an exercise; (cii) is new and requires a clear head - the main point is to connect this with the concept of the Lebesgue constant, then it is easy (but there are also less direct ways to prove this).
- Q.4: This exercise mixes knowledge about trigonometric interpolation and least squares methods. (a) is standard BW; (bi) is still bookwork; (bii) is a direct application of (ai) and a minimal variation of what was seen in the lectures; (biii) requires them to realise that a certain operation can be performed via the FFT. It requires them to make a connection but is technically not dificicult.
- Q.5: (a) is almost verbatim bookwork; (b) is new: (bi) is an application of the greedy truncation idea discussed in the lectures; (bii) requires a bit of legwork, but similar calculations have been performed in the lectures many times.