Springer Proceedings in Mathematics & Statistics

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Lie Theory and Its Applications in Physics

Sofia, Bulgaria, June 2021

Springer

Springer Proceedings in Mathematics & Statistics

Volume 396

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Vladimir Dobrev
Editor

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Sofia, Bulgaria, June 2021

Springer

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and Nuclear Energy
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ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-981-19-4750-6 ISBN 978-981-19-4751-3 (eBook)
https://doi.org/10.1007/978-981-19-4751-3

Mathematics Subject Classification: 14A22, 14L17, 14R20, 16G30, 16T25, 17A70, 17B10, 17B25, 17B35, 17B37, 17B65, 17B80, 17C40, 20C33, 20C35, 20G41, 20G42, 22E46, 22E65, 22E70, 37K30, 58B34, 68T07, 68Q32, 70H06, 81P40, 81R10, 81R50, 81R60, 81T13, 81T30, 81T35, 81T40, 83C65, 83E50, 83F05, 91B80

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Preface

The Workshop series “Lie Theory and Its Applications in Physics” is designed to serve the community of theoretical physicists, mathematical physicists, and mathematicians working on mathematical models for physical systems based on geometrical methods and in the field of Lie theory.

The series reflects the trend toward a geometrization of the mathematical description of physical systems and objects. A geometric approach to a system yields in general some notion of symmetry which is very helpful in understanding its structure. Geometrization and symmetries are meant in their widest sense, i.e., representation theory, algebraic geometry, number theory infinite-dimensional Lie algebras and groups, superalgebras and supergroups, groups and quantum groups, noncommutative geometry, symmetries of linear and nonlinear PDE, special functions, and functional analysis. This is a big interdisciplinary and interrelated field.

The first three workshops were organized in Clausthal (1995, 1997, 1999), the 4th was part of the 2nd Symposium “Quantum Theory and Symmetries” in Cracow (2001), the 5th, 7th, 8th, 9th, 11th, and 13th were organized in Varna (2003, 2007, 2009, 2011, 2013, 2015, 2019), the 6th and the 12th were part of the 4th, resp., 10th, Symposium “Quantum Theory and Symmetries” in Varna (2005, 2017).

The 14-th Workshop of the series (LT-14) was organized by the Organizing Committee from the Institute of Nuclear Research and Nuclear Energy of the Bulgarian Academy of Sciences (BAS) in June 2021 (21–25). Due to the COVID-19 restrictions it was organized online, based in Sofia.

The overall number of participants was 97 and they came from 25 countries. The number of talks was 87.

The scientific level was very high as can be judged by the *plenary speakers*: Toshiyuki Kobayashi (Tokyo), Yang-Hui He (london & Tianjin), Ivan Todorov (Sofia), Patrizia Vitale (Napoli), Paolo Aschieri (Alessandria & Torino), Nikolay Bobev (Leuven), Tomasz Brzezinski (Swansea & Bialystok), Malte Henkel (Nancy & Lisboa), Volodymyr Mazorchuk (Uppsala), and George Zoupanos (Athens & CERN).

The topics covered the most modern trends in the field of the Workshop: Symmetries in String Theories, (Super-)Gravity Theories, Conformal Field Theory, Integrable Systems, Representation Theory, Quantum Computing and Deep Learning, Applications to Quantum Theory. Gauge Theories and Applications, Structures on Lie Groups and Lie Algebras.

The International Steering Committee was: C. Burdik (Prague) V. K. Dobrev (Sofia, Chair), H. D. Doebner (Clausthal), B. Dragovich (Belgrade), and G. S. Pogosyan (Yerevan & Guadalajara & Dubna).

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Sofia, Bulgaria Vladimir Dobrev
May 2022

Acknowledgments We express our gratitude to the Publisher, Springer Japan, represented by Mr. Masayuki Nakamura (Editor, Mathematics), for assistance in the publication. Last but not least, I thank the members of the Local Organizing Committee who, through their efforts, made the workshop run smoothly and efficiently.

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Plenary Talks

Multiplicity in Restricting Minimal Representations

Toshiyuki Kobayashi

Abstract We discuss the action of a subgroup on small nilpotent orbits, and prove a bounded multiplicity property for the restriction of minimal representations of real reductive Lie groups with respect to arbitrary reductive symmetric pairs.

Keywords Minimal representation · Branching law · Reductive group · Symmetric pair · Coisotropic action · Multiplicity · Coadjoint orbit

2020 MSC Primary 22E46 · Secondary 22E45 · 53D50 · 58J42 · 53C50

1 Statement of Main Results

This article is a continuation of our work [9, 13, 15, 17, 18, 21, 23] that concerns the restriction of irreducible representations Π of reductive Lie groups G to reductive subgroups G' with focus on the *bounded multiplicity property* of the restriction $\Pi|_{G'}$ (Definition 2). In this article we highlight the following specific setting:

- (G, G') is an arbitrary reductive symmetric pair;
- Π is of the smallest Gelfand–Kirillov dimension.

We refer to [14] for some motivation and perspectives in the general branching problems, see also Sect. 2 for some aspects regarding finite/bounded multiplicity properties of the restriction.

To be rigorous about "multiplicities" for infinite-dimensional representations, we need to fix the topology of the representation spaces. For this, let G be a real reductive Lie group, $\mathcal{M}(G)$ the category of smooth admissible representations of G of finite length with moderate growth, which are defined on Fréchet topological vector spaces [32, Chap. 11]. We denote by $\mathrm{Irr}(G)$ the set of irreducible objects in $\mathcal{M}(G)$.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_1

Suppose that G' is a reductive subgroup in G. For $\Pi \in \mathcal{M}(G)$, the *multiplicity* of $\pi \in \mathrm{Irr}(G')$ in the restriction $\Pi|_{G'}$ is defined by

$$[\Pi|_{G'} : \pi] := \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \in \mathbb{N} \cup \{\infty\},$$

where $\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi)$ denotes the space of *symmetry breaking operators*, i.e., continuous G'-homomorphisms between the Fréchet representations. For non-compact G', the multiplicity $[\Pi|_{G'} : \pi]$ may be infinite even when G' is a maximal subgroup of G, see Example 1 below.

By a *reductive symmetric pair* (G, G'), we mean that G is a real reductive Lie group and that G' is an open subgroup in the fixed point group G^σ of an involutive automorphism σ of G. The pairs $(SL(n, \mathbb{R}), SO(p, q))$ with $p + q = n$, $(O(p, q), O(p_1, q_1) \times O(p_2, q_2))$ with $p_1 + p_2 = p$, $q_1 + q_2 = q$, and the *group manifold case* $({}^\backprime G \times {}^\backprime G, \mathrm{diag}({}^\backprime G))$ are examples. For a reductive symmetric pair (G, G'), the subgroup G' is maximal amongst reductive subgroups of G.

One may ask for which pair (G, G') the finite multiplicity property

$$[\Pi|_{G'} : \pi] < \infty, \quad {}^\forall \Pi \in \mathrm{Irr}(G), {}^\forall \pi \in \mathrm{Irr}(G') \tag{1}$$

holds. Here are examples when (G, G') is a reductive symmetric pair:

Example 1 ([9, 19]) (1) For the symmetric pair $(SL(n, \mathbb{R}), SO(p, q))$ $(p + q = n)$, the finite multiplicity property (1) holds if and only if one of the following conditions holds: $p = 0$, $q = 0$, or $p = q = 1$.
(2) For the pair $(O(p, q), O(p_1, q_1) \times O(p_2, q_2))$ $(p_1 + p_2 = p, q_1 + q_2 = q)$, the finite multiplicity property (1) holds if and only if one of the following conditions holds: $p_1 + q_1 = 1$, $p_2 + q_2 = 1$, $p = 1$, or $q = 1$.
(3) For the group manifold case $({}^\backprime G \times {}^\backprime G, \mathrm{diag}({}^\backprime G))$ where ${}^\backprime G$ is a simple Lie group, the finite multiplicity property (1) holds if and only if ${}^\backprime G$ is compact or is locally isomorphic to $SO(n, 1)$.

See Fact 15 (2) for a geometric criterion of the pair (G, G') to have the finite multiplicity property (1). A complete classification of such symmetric pairs (G, G') was accomplished in Kobayashi–Matsuki [19].

On the other hand, if we confine ourselves only to "small" representations Π of G, there will be a more chance that the multiplicity $[\Pi|_{G'} : \pi]$ becomes finite, or even stronger, the restriction $\Pi|_{G'}$ has the bounded multiplicity property in the following sense:

Definition 2 Let $\Pi \in \mathcal{M}(G)$. We say the restriction $\Pi|_{G'}$ has the *bounded multiplicity property* if $m(\Pi|_{G'}) < \infty$, where we set

$$m(\Pi|_{G'}) := \sup_{\pi \in \mathrm{Irr}(G')} \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \in \mathbb{N} \cup \{\infty\}. \tag{2}$$

In the series of the papers, we have explored the bounded multiplicity property of the restriction $\Pi|_{G'}$ not only uniformly with respect to $\pi \in \mathrm{Irr}(G')$ for the subgroup

G' but also uniformly with respect to $\Pi \in \mathcal{M}(G)$, e.g., either Π runs over the whole set $\mathrm{Irr}(G)$ [9, 13, 23] or Π belongs to certain family of "relatively small" representations of the group G [15, 17, 18, 21]. See Sect. 2 for some general results, which tell that the smaller Π is, the more subgroups G' tends to satisfy the bounded multiplicity property $\Pi|_{G'}$. In this article, we highlight the extremal case where Π is the "smallest", and give the bounded multiplicity theorems for *all* symmetric pairs (G, G').

What are "small representations" amongst infinite-dimensional representations? For this, the Gelfand–Kirillov dimension serves as a coarse measure of the "size" of representations. We recall that for $\Pi \in \mathcal{M}(G)$ the Gelfand–Kirillov dimension $\mathrm{DIM}(\Pi)$ is defined as half the dimension of the associated variety of $\mathcal{I}$ where $\mathcal{I}$ is the annihilator of Π in the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$. The associated variety of $\mathcal{I}$ is a finite union of nilpotent coadjoint orbits in $\mathfrak{g}_{\mathbb{C}}^*$.

We recall for a complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$, there exists a unique non-zero minimal nilpotent $(\mathrm{Int}\, \mathfrak{g}_{\mathbb{C}})$-orbit in $\mathfrak{g}_{\mathbb{C}}^*$, which we denote by $\mathbb{O}_{\min,\mathbb{C}}$. The dimension of $\mathbb{O}_{\min,\mathbb{C}}$ is known as below, see [2] for example. We set $n(\mathfrak{g}_{\mathbb{C}})$ to be half the dimension of $\mathbb{O}_{\min,\mathbb{C}}$.

$\mathfrak{g}_{\mathbb{C}}$	A_n	B_n $(n \geq 2)$	C_n	D_n	$\mathfrak{g}_2^{\mathbb{C}}$	$\mathfrak{f}_4^{\mathbb{C}}$	$\mathfrak{e}_6^{\mathbb{C}}$	$\mathfrak{e}_7^{\mathbb{C}}$	$\mathfrak{e}_8^{\mathbb{C}}$
$n(\mathfrak{g}_{\mathbb{C}})$	n	$2n-2$	n	$2n-3$	3	8	11	17	29

For the rest of this section, let G be a non-compact connected simple Lie group without complex structure. This means that the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is still a simple Lie algebra. By the definition, the Gelfand–Kirillov dimension has the following property: $\mathrm{DIM}(\Pi) = 0 \iff \Pi$ is finite-dimensional, and

$$n(\mathfrak{g}_{\mathbb{C}}) \leq \mathrm{DIM}(\Pi) \leq \frac{1}{2}(\dim \mathfrak{g} - \mathrm{rank}\, \mathfrak{g}), \tag{3}$$

for any infinite-dimensional $\Pi \in \mathrm{Irr}(G)$. In this sense, if $\Pi \in \mathrm{Irr}(G)$ satisfies $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$, then such Π is thought of as the "smallest" amongst infinite-dimensional irreducible representations of G.

In this article, we prove the following bounded multiplicity theorem of the restriction:

Theorem 3 *If the Gelfand–Kirillov dimension of* $\Pi \in \mathrm{Irr}(G)$ *is* $n(\mathfrak{g}_{\mathbb{C}})$, *then* $m(\Pi|_{G'}) < \infty$ *for any symmetric pair* (G, G').

For $\Pi_1, \Pi_2 \in \mathrm{Irr}(G)$, we consider the tensor product representation $\Pi_1 \otimes \Pi_2$, and define the upper bound of the multiplicity in $\Pi_1 \otimes \Pi_2$ by

$$m(\Pi_1 \otimes \Pi_2) := \sup_{\Pi \in \mathrm{Irr}(G)} \dim_{\mathbb{C}} \mathrm{Hom}_G(\Pi_1 \otimes \Pi_2, \Pi) \in \mathbb{N} \cup \{\infty\}.$$

The tensor product representation of two representations is a special case of the restriction with respect to symmetric pairs. We also prove the bounded multiplicity property of the tensor product:

Theorem 4 *If the Gelfand–Kirillov dimensions of $\Pi_1, \Pi_2 \in \mathrm{Irr}(G)$ are $n(\mathfrak{g}_{\mathbb{C}})$, then one has $m(\Pi_1 \otimes \Pi_2) < \infty$.*

Remark 5 Since the upper bound of the multiplicity $m(\Pi|_{G'})$ is defined in the category of admissible representations of moderate growth, $m(\Pi|_{G'})$ also gives an upper bound in the category of unitary representations where the multiplicity in the direct integral of irreducible unitary representations is defined as a measurable function on the unitary dual of the subgroup G'.

These results apply to "minimal representations" of G, which we recall now. For a complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ other than $\mathfrak{sl}(n, \mathbb{C})$, Joseph [6] constructed a completely prime two-sided primitive ideal $\mathcal{J}$ in $U(\mathfrak{g}_{\mathbb{C}})$, whose associated variety is the closure of the minimal nilpotent orbit $\mathbb{O}_{\mathrm{min},\mathbb{C}}$. See also [3].

Definition 6 (*minimal representation, see* [4]) An irreducible admissible representation Π of G is called a *minimal representation* if the annihilator of the $U(\mathfrak{g}_{\mathbb{C}})$-module Π is the Joseph ideal $\mathcal{J}$ of $U(\mathfrak{g}_{\mathbb{C}})$.

The two irreducible components of the Segal–Shale–Weil representation are classical examples of a minimal representation of the metaplectic group $Mp(n, \mathbb{R})$, the connected double cover of the real symplectic group $Sp(n, \mathbb{R})$, which play a prominent role in number theory. The solution space of the Yamabe Laplacian on $S^p \times S^q$ gives the minimal representation of the conformal transformation group $O(p+1, q+1)$ when $p + q$ (≥ 6) is even ([20]). In general, there are at most four minimal representations for each connected simple Lie group G if exist, and they were classified [4, 30].

By the definition of the Joseph ideal, one has $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$ if Π is a minimal representation. Thus Theorems 3 and 4 imply the following:

Theorem 7 *Let Π be a minimal representation of G. Then the restriction $\Pi|_{G'}$ has the bounded multiplicity property $m(\Pi|_{G'}) < \infty$ for any symmetric pair (G, G').*

Theorem 8 *Let Π_1, Π_2 be minimal representations of G. Then the tensor product representation has the bounded multiplicity property $m(\Pi_1 \otimes \Pi_2) < \infty$.*

Example 9 The tensor product representation of the two copies of the Segal–Shale–Weil representations of the metaplectic group $Mp(n, \mathbb{R})$ is unitarily equivalent to the phase space representation of $Sp(n, \mathbb{R})$ on $L^2(\mathbb{R}^{2n})$ via the Wigner transform, see [22, Sect. 2] for instance.

In general, it is rare that the restriction $\Pi|_{G'}$ of $\Pi \in \mathcal{M}(G)$ is *almost irreducible* in the sense that the G'-module $\Pi|_{G'}$ remains irreducible or a direct sum of finitely many irreducible representations of G'. In [12, Sect. 5], we discussed such rare phenomena and gave a list of the triples (G, G', Π) where the restriction $\Pi|_{G'}$ is almost irreducible, in particular, in the following settings: $\Pi \in \mathcal{M}(G)$ is a degenerate principal series representation or Zuckerman's derived functor module $A_{\mathfrak{q}}(\lambda)$, which is supposed to be a "geometric quantization" of a hyperbolic coadjoint orbit or an elliptic coadjoint orbit, respectively, in the orbit philosophy, see [12, Theorems 3.8 and 3.5]. As a corollary of Theorem 7, we also prove the following theorem where Π is "attached to" the minimal nilpotent coadjoint orbit $\mathbb{O}_{\min,\mathbb{C}}$.

Theorem 10 *Suppose that (G, G') is a symmetric pair such that the complexified Lie algebras $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ is in the list of Proposition 30 (vi). Then the restriction $\Pi|_{G'}$ is almost irreducible if Π is a minimal representation of G.*

Example 11 For the following symmetric pairs $(\mathfrak{g}, \mathfrak{g}')$, there exists a minimal representation Π of some Lie group G with Lie algebra $\mathfrak{g}$ (e.g., $G = Mp(n, \mathbb{R})$ for $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$), and Theorem 10 applies to (G, G', Π),

- $(\mathfrak{sp}(p+q, \mathbb{R}), \mathfrak{sp}(p, \mathbb{R}) \oplus \mathfrak{sp}(q, \mathbb{R}))$,
- $(\mathfrak{so}(p, q), \mathfrak{so}(p-1, q))$ or $(\mathfrak{so}(p, q), \mathfrak{so}(p, q-1))$ for "$p \geq q \geq 4$ and $p \equiv q$ mod 2", "$p \geq 5$ and $q = 2$", or "$p \geq 4$ and $q = 3$".
- $(\mathfrak{f}_{4(4)}, \mathfrak{so}(5, 4))$,
- $(\mathfrak{e}_{6(6)}, \mathfrak{f}_{4(4)})$, or $(\mathfrak{e}_{6(-14)}, \mathfrak{f}_{4(-20)})$.

We note that the upper bound $m(\Pi|_{G'})$ or $m(\Pi_1 \otimes \Pi_2)$ of the multiplicity can be larger than 1 in Theorems 3 and 4, see e.g. [21] for an explicit branching law of the restriction $\Pi|_{G'}$ when $(G, G') = (SL(n, \mathbb{R}), SO(p, q))$ with $p + q = n$. However, it is plausible that a multiplicity-free theorem holds in Theorems 7 and 8:

Conjecture 12 $m(\Pi|_{G'}) = 1$ in Theorem 7, and $m(\Pi_1 \otimes \Pi_2) = 1$ in Theorem 8.

Conjecture 12 holds when (G, G') is a Riemannian symmetric pair (G, K), see [4, Proposition 4.10].

Remark 13 (1) The Joseph ideal is not defined for $\mathfrak{sl}(n, \mathbb{C})$, hence there is no minimal representation in the sense of Definition 6 for $G = SL(n, \mathbb{R})$, for instance. However there exist continuously many $\Pi \in \mathrm{Irr}(G)$ (e.g., degenerate principal series representations induced from a mirabolic subgroup) for $G = SL(n, \mathbb{R})$ such that $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$, and Theorems 3 and 4 apply to these representations. The Plancherel-type theorem for the restriction $\Pi|_{G'}$ is proved in [21] for all symmetric pairs (G, G') when Π is a *unitarily* induced representation. See also Example 16 below.
(2) The inequality (3) depends only on the complexification $\mathfrak{g}_{\mathbb{C}}$, and is not necessarily optimal for specific real forms $\mathfrak{g}$. In fact, one has a better inequality $n(\mathfrak{g}) \leq \mathrm{DIM}(\Pi)$ where $n(\mathfrak{g})$ depends on the real form $\mathfrak{g}$, see Sect. 3.2. For most of real Lie algebras one has $n(\mathfrak{g}) = n(\mathfrak{g}_{\mathbb{C}})$, but there are a few simple Lie algebras $\mathfrak{g}$ satisfying $n(\mathfrak{g}) > n(\mathfrak{g}_{\mathbb{C}})$. For example, if $G = Sp(p, q)$, $n(\mathfrak{g}) = 2(p+q) - 1 > n(\mathfrak{g}_{\mathbb{C}}) = p + q$, hence there is no $\Pi \in \mathrm{Irr}(G)$ with $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$, however, there exists a countable family of $\Pi \in \mathrm{Irr}(G)$ with $\mathrm{DIM}(\Pi) = n(\mathfrak{g})$, to which another bounded multiplicity theorem (Theorem 34 in Sect. 3) applies.

(3) Concerning Theorem 3, the bounded property of the multiplicity in the tensor product representations $\Pi_1 \otimes \Pi_2$ still holds for some other "small representations" Π_1 and Π_2 whose Gelfand–Kirillov dimensions are greater than $n(\mathfrak{g}_{\mathbb{C}})$. See [17, Theorem 1.5 and Corollary 4.10] for example.

This paper is organized as follows. In Sect. 2 we give a brief review of some background of the problem, examples, and known theorems. Section 3 is devoted to the proof of Theorems 3, 4 and 10.

2 Background and Motivation

In this section, we explain some background, examples, and known theorems in relation to our main results.

If Π is an irreducible *unitary* representation of a group G, then one may consider the irreducible decomposition (*branching law*) of the restriction $\Pi|_{G'}$ to a subgroup G' by using the direct integral of Hilbert spaces. For non-unitary representations Π, such an irreducible decomposition does not make sense, but the computation of the multiplicity $[\Pi|_{G'} : \pi]$ for all $\pi \in \mathrm{Irr}(G')$ may be thought of as a variant of branching laws. Here we recall from Sect. 1 that for $\Pi \in \mathcal{M}(G)$ and $\pi \in \mathrm{Irr}(G')$ that the multiplicity $[\Pi|_{G'} : \pi]$ is the dimension of the space $\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi)$ of symmetry breaking operators.

By branching problems in representation theory, we mean the broad problem of understanding how irreducible (not necessarily, unitary) representations of a group behave when restricted to a subgroup. As viewed in [14], we may divide the branching problems into the following three stages:

Stage A. Abstract features of the restriction;
Stage B. Branching law;
Stage C. Construction of symmetry breaking operators.

The role of Stage A is to develop a theory on the restriction of representations as generally as possible. In turn, we may expect a detailed study of the restriction in Stages B (decomposition of representations) and C (decomposition of vectors) in the "promising" settings that are suggested by the general theory in Stage A.

The study of the upper estimate of the multiplicity in this article is considered as a question in Stage A of branching problems.

For a detailed analysis on the restriction $\Pi|_{G'}$ in Stages B and C, it is desirable to have the bounded multiplicity property $m(\Pi|_{G'}) < \infty$ (see Definition 2), or at least to have the finite multiplicity property

$$[\Pi|_{G'} : \pi] < \infty \quad \text{for } \pi \in \mathrm{Irr}(G'). \tag{4}$$

In the previous papers [10, 13, 16–18, 23] we proved some general theorems for bounded/finite multiplicities of the restriction $\Pi|_{G'}$, which we review briefly now.

2.1 Bounded Multiplicity Pairs (G, K') with K' Compact

Harish-Chandra's admissibility theorem tells the finiteness property (4) holds for any $\Pi \in \mathcal{M}(G)$ if G' is a maximal compact subgroup K of G. More generally, the finiteness property (4) for a compact subgroup plays a crucial role in the study of discretely decomposable restriction with respect to reductive subgroups [8, 10, 11, 16]. We review briefly the necessary and sufficient condition for (4) when G' is compact. In this subsection, we use the letter K' instead of G' to emphasize that G' is compact. Without loss of generality, we may and do assume that K' is contained in K.

Fact 14 ([10, 16]) *Suppose that K' is a compact subgroup of a real reductive group G. Let $\Pi \in \mathcal{M}(G)$. Then the following two conditions on the triple (G, K', Π) are equivalent:*
(i) *The finite multiplicity property* (4) *holds.*
(ii) $\mathrm{AS}_K(\Pi) \cap C_K(K') = \{0\}$.

Here $\mathrm{AS}_K(\Pi)$ is the asymptotic K-support of Π, and $C_K(K')$ is the momentum set for the natural action on the cotangent bundle $T^*(K/K')$. There are two proofs for the implication (ii) $\Rightarrow$ (i): by using the singularity spectrum (or the wave front set) [10] and by using symplectic geometry [16]. The proof for the implication (i) $\Rightarrow$ (ii) is given in [16]. See [24] for some classification theory.

2.2 Bounded/Finite Multiplicity Pairs (G, G')

We now consider the general case where G' is not necessarily compact. In [13] and [23, Theorems C and D] we proved the following geometric criteria that concern *all* $\Pi \in \mathrm{Irr}(G)$ and *all* $\pi \in \mathrm{Irr}(G')$:

Fact 15 *Let $G \supset G'$ be a pair of real reductive algebraic Lie groups.*
(1) **Bounded multiplicity** *for a pair* (G, G')*:*

$$\sup_{\Pi \in \mathrm{Irr}(G)} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty \tag{5}$$

if and only if $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\operatorname{diag} G'_{\mathbb{C}}$ *is spherical.*
(2) **Finite multiplicity** *for a pair* (G, G')*:*

$$[\Pi|_{G'} : \pi] < \infty, \quad {}^\forall \Pi \in \mathrm{Irr}(G), {}^\forall \pi \in \mathrm{Irr}(G')$$

if and only if $(G \times G')/\operatorname{diag} G'$ *is real spherical.*

Here we recall that a complex $G_{\mathbb{C}}$-manifold X is called *spherical* if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in X, and that a G-manifold Y is called *real spherical* if a minimal parabolic subgroup of G has an open orbit in Y.

A remarkable discovery in Fact 15 (1) was that the bounded multiplicity property (5) is determined only by the complexified Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}'_{\mathbb{C}}$. In particular, the classification of such pairs (G, G') is very simple, because it is reduced to a classical result when G is compact [27]: the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ is the direct sum of the following ones up to Abelian ideals:

$$(\mathfrak{sl}_n, \mathfrak{gl}_{n-1}), (\mathfrak{so}_n, \mathfrak{so}_{n-1}), \text{ or } (\mathfrak{so}_8, \mathfrak{spin}_7). \tag{6}$$

See [25, 26] e.g., for some recent developments in Stage C such as detailed analysis on symmetry breaking operators for some non-compact real forms of the pairs (6).

On the other hand, the finite multiplicity property in Fact 15 (2) depends on real forms G and G'. It is fulfilled for any Riemannian symmetric pair, which is Harish-Chandra's admissibility theorem. More generally for non-compact G', the finite-multiplicity property (4) often holds when the restriction $\Pi|_{G'}$ decomposes discretely, see [8, 10, 11] for the general theory of "G'-admissible restriction". However, for some reductive symmetric pairs such as $(G, G') = (SL(p+q, \mathbb{R}), SO(p, q))$, there exists $\Pi \in \mathrm{Irr}(G)$ for which the finite multiplicity property (4) of the restriction $\Pi|_{G'}$ fails, as we have seen in Example 1. Such Π is fairly "large".

2.3 Uniform Estimates for a Family of Small Representations

The classification in [19] tells that the class of the reductive symmetric pairs (G, G') satisfying the finite multiplicity property (1) is much broader than that of real forms (G, G') corresponding to those complex pairs in (5). However, there also exist pairs (G, G') beyond the list of [19] for which we can still expect fruitful branching laws of the restriction $\Pi|_{G'}$ in Stages B and C for some $\Pi \in \mathrm{Irr}(G)$. Such Π must be a "small representation". Here are some known examples:

Example 16 (1) (**Stage B**) See-saw dual pairs ([28]) yield explicit formulæ of the multiplicity for the restriction of small representations, with respect to some classical symmetric pairs (G, G').
(2) (**Stage C**) For $G = SL(n, \mathbb{R})$, any degenerate representation $\Pi = \mathrm{Ind}_P^G(\mathbb{C}_\lambda)$ induced from a "mirabolic subgroup" P of G has the smallest Gelfand–Kirillov dimension $n(\mathfrak{g}_{\mathbb{C}})$. For a unitary character $\mathbb{C}_\lambda$, the Plancherel-type formula of the restriction $\Pi|_{G'}$ is determined in [21] for *all* symmetric pairs (G, G'). The feature of the restriction $\Pi|_{G'}$ is summarized as follows: let $p + q = n$, and when n is even we write $n = 2m$.

- $G' = S(GL(p, \mathbb{R}) \times GL(q, \mathbb{R}))$.
  $\cdots$ Only continuous spectrum appears with multiplicity one.
- $G' = SL(m, \mathbb{C}) \cdot \mathbb{T}$.
  $\cdots$ Only discrete spectrum appears with multiplicity one.

- $G' = SO(p, q)$.
  $\cdots$ Discrete spectrum appears with multiplicity one,
  and continuous spectrum appears with multiplicity two.
- $G' = Sp(m, \mathbb{R})$
  $\cdots$ Almost irreducible (See also Theorem 10).

The uniform bounded multiplicity property in all these cases (**Stage A**) is guaranteed by Theorem 3 in this article because $\mathrm{DIM}(\Pi)$ attains $n(\mathfrak{g}_{\mathbb{C}})$, and alternatively, by another general result [17, Theorem 4.2].
(3) (**Stage C**) For the symmetric pair $(G, G') = (O(p, q), O(p_1, q_1) \times O(p_2, q_2))$ with $p_1 + p_2 = p$ and $q_1 + q_2 = q$, by using the Yamabe operator in conformal geometry, discrete spectrum in the restriction $\Pi|_{G'}$ of the minimal representation Π was obtained geometrically in [20]. Moreover, for the same pair (G, G'), discrete spectrum in the restriction $\Pi|_{G'}$ was explicitly constructed and classified when Π belongs to cohomologically parabolic induced representation $A_{\mathfrak{q}}(\lambda)$ from a maximal θ-stable parabolic subalgebra $\mathfrak{q}$ in [15]. In contrast to Example 1 (2), the multiplicity is one for any p_1, q_1, p_2, and q_2.

In view of these nice cases, and also in search for further broader settings in which we could expect a detailed study of the restriction $\Pi|_{G'}$ in Stages B and C, we addressed the following:

Problem 17 ([14, Problem 6.2], [17, Problem 1.1]) Given a pair $G \supset G'$, find a subset Ω of $\mathcal{M}(G)$ such that $\sup_{\Pi \in \Omega} m(\Pi|_{G'}) < \infty$.

Since branching problems often arise for a family of representations Π, the formulation of Problem 17 is to work with the *triple* (G, G', Ω) rather than the pair (G, G') for the finer study of multiplicity estimates of the restriction $\Pi|_{G'}$. Fact 15 (1) deals with the case $\Omega = \mathrm{Irr}(G)$. In [15, 17], we have considered Problem 17 including the following cases:

(1) $\Omega = \mathrm{Irr}(G)_H$, the set of H-distinguished irreducible representations of G where (G, H) is a reductive symmetric pair;
(2) $\Omega = \Omega_P$, the set of induced representations from characters of a parabolic subgroup P of G;
(3) $\Omega = \Omega_{P,\mathfrak{q}}$, certain families of (vector-bundle valued) degenerate principal series representations.

For the readers' convenience, we give a flavor of the solutions to Problem 17 in the above cases by quoting the criteria from [17]. See [18] for a brief survey.

We write $G_{\mathbb{C}}$ for the complexified Lie group G, and G_U for the compact real form of $G_{\mathbb{C}}$. For a reductive symmetric pair (G, H), one can define a Borel subgroup $B_{G/H}$ which is a parabolic subgroup in $G_{\mathbb{C}}$, see [18, Definition 3.1]. Note that $B_{G/H}$ is not necessarily solvable. For $\Omega = \mathrm{Irr}(G)_H$ when (G, H) is a reductive symmetric pair, one has the following answer to Problem 17:

Fact 18 ([17, Theorem 1.4]) *Let $B_{G/H}$ be a Borel subgroup for G/H. Suppose G' is an algebraic reductive subgroup of G. Then the following three conditions on the triple (G, H, G') are equivalent:*

(i) $\sup_{\Pi \in \mathrm{Irr}(G)_H} m(\Pi|_{G'}) < \infty$.
(ii) *$G_{\mathbb{C}}/B_{G/H}$ is G'_U-strongly visible.*
(iii) *$G_{\mathbb{C}}/B_{G/H}$ is $G'_{\mathbb{C}}$-spherical.*

For $\Omega = \Omega_P$, one has the following answer to Problem 17:

Fact 19 ([17, Example 4.5], [31]) *Let $G \supset G'$ be a pair of real reductive algebraic Lie groups, and P a parabolic subgroup of G. Then one has the equivalence on the triple $(G, G'; P)$:*
(i) $\sup_{\Pi \in \Omega_P} m(\Pi|_{G'}) < \infty$.
(ii) *$G_{\mathbb{C}}/P_{\mathbb{C}}$ is strongly G'_U-visible.*
(iii) *$G_{\mathbb{C}}/P_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical.*

The following is a useful extension of Fact 19.

Fact 20 ([17, Theorem 4.2]) *Let $G \supset G'$ be a pair of real reductive algebraic Lie groups, P a parabolic subgroup of G, and Q a complex parabolic subgroup of $G_{\mathbb{C}}$ such that $\mathfrak{q} \subset \mathfrak{p}_{\mathbb{C}}$. One defines a subset $\Omega_{P,\mathfrak{q}}$ in $\mathcal{M}(G)$ that contains Ω_P (see [17] for details). Then the following three conditions on $(G, G'; P, Q)$ are equivalent:*

(i) $\sup_{\Pi \in \Omega_{P,\mathfrak{q}}} m(\Pi|_{G'}) < \infty$.
(ii) *$G_{\mathbb{C}}/Q$ is G'_U-strongly visible.*
(iii) *$G_{\mathbb{C}}/Q$ is $G'_{\mathbb{C}}$-spherical.*

These criteria lead us to classification results for the triples (G, G', Ω), see [15, 17, 18] and references therein.

The representations Π in $\Omega = \mathrm{Irr}(G)_H$ or Ω_P, $\Omega_{P,\mathfrak{q}}$ are fairly small, however, the classification results in [17] indicate that some symmetric pairs (G, G') still do not appear for such a family Ω. A clear distinction from these previous results is that Theorem 3 allows *all* symmetric pairs (G, G') for an affirmative answer to Problem 17 in the extremal case where $\Omega = \{\Pi\}$ with $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$.

Concerning the method of the proof, we utilized in [23] hyperfunction boundary maps for the "if" part (i.e., the sufficiency of the bounded multiplicity property) and a generalized Poisson transform [13] for the "only if" part in the proof of Fact 15. The proof in [17, 31] used a theory of holonomic $\mathcal{D}$-modules for the "if" part. Our proof in this article still uses a theory of $\mathcal{D}$-modules, and more precisely, the following:

Fact 21 ([7]) *Let $\mathcal{I}$ be the annihilator of $\Pi \in \mathcal{M}(G)$ in the enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$. Assume that the $G'_{\mathbb{C}}$-action on the associated variety of $\mathcal{I}$ is coisotropic (Definition 22). Then the restriction $\Pi|_{G'}$ has the bounded multiplicity property (Definition 2).*

We note that the assumption in Fact 21 depends only on the complexification of the pair $(\mathfrak{g}, \mathfrak{g}')$ of the Lie algebras. Thus the proof of Theorems 3 and 4 is reduced to a geometric question on *holomorphic* coisotropic actions on *complex* nilpotent coadjoint orbits, which will be proved in Theorem 23.

3 Coisotropic Action on Coadjoint Orbits

Let V be a vector space endowed with a symplectic form Ω. A subspace W is called *coisotropic* if $W^\perp \subset W$, where

$$W^\perp := \{v \in V : \Omega(v, \cdot) \text{ vanishes on } W\}.$$

The concept of coisotropic actions is defined infinitesimally as follows.

Definition 22 (*Huckleberry–Wurzbacher* [5]) Let H be a connected Lie group, and X a Hamiltonian H-manifold. The H-action is called *coisotropic* if there is an H-stable open dense subset U of X such that $T_x(H \cdot x)$ is a coisotropic subspace in the tangent space $T_x X$ for all $x \in U$.

Any coadjoint orbit of a Lie group G is a Hamiltonian G-manifold with the Kirillov–Kostant–Souriau symplectic form. The main result of this section is the following:

Theorem 23 *Let $\mathbb{O}_{\min,\mathbb{C}}$ be the minimal nilpotent coadjoint orbit of a connected complex simple Lie group $G_\mathbb{C}$.*

(1) *The diagonal action of $G_\mathbb{C}$ on $\mathbb{O}_{\min,\mathbb{C}} \times \mathbb{O}_{\min,\mathbb{C}}$ is coisotropic.*
(2) *For any symmetric pair $(G_\mathbb{C}, K_\mathbb{C})$, the $K_\mathbb{C}$-action on $\mathbb{O}_{\min,\mathbb{C}}$ is coisotropic.*

3.1 Generalities: Coisotropic Actions on Coadjoint Orbits

We begin with a general setting for a *real* Lie group. Suppose that $\mathbb{O}$ is a coadjoint orbit of a connected Lie group G through $\lambda \in \mathfrak{g}^*$. Denote by G_λ the stabilizer subgroup of λ in G, and by $Z_\mathfrak{g}(\lambda)$ its Lie algebra. Then the Kirillov–Kostant–Souriau symplectic form Ω on the coadjoint orbit $\mathbb{O} = \mathrm{Ad}^*(G) \simeq G/G_\lambda$ is given at the tangent space $T_\lambda \mathbb{O} \simeq \mathfrak{g}/Z_\mathfrak{g}(\lambda)$ by

$$\Omega : \mathfrak{g}/Z_\mathfrak{g}(\lambda) \times \mathfrak{g}/Z_\mathfrak{g}(\lambda) \to \mathbb{R}, \quad (X, Y) \mapsto \lambda([X, Y]). \tag{7}$$

Suppose that H is a connected subgroup with Lie algebra $\mathfrak{h}$. For $\lambda \in \mathfrak{g}^*$, we define a subspace of the Lie algebra $\mathfrak{g}$ by

$$Z_\mathfrak{g}(\mathfrak{h}; \lambda) := \{Y \in \mathfrak{g} : \lambda([X, Y]) = 0 \text{ for all } X \in \mathfrak{h}\}. \tag{8}$$

Clearly, $Z_{\mathfrak{g}}(\mathfrak{h}; \lambda)$ contains the Lie algebra $Z_{\mathfrak{g}}(\lambda) \equiv Z_{\mathfrak{g}}(\mathfrak{g}; \lambda)$ of G_λ.

We shall use the following:

Lemma 24 *The H-action on a coadjoint orbit $\mathbb{O}$ in $\mathfrak{g}^*$ is coisotropic if there exists a subset S (slice) in $\mathbb{O}$ with the following two properties:*

$$\begin{aligned} &\mathrm{Ad}^*(H)S \textit{ is open dense in } \mathbb{O}, \\ &Z_{\mathfrak{g}}(\mathfrak{h}; \lambda) \subset \mathfrak{h} + Z_{\mathfrak{g}}(\lambda) \textit{ for any } \lambda \in S. \end{aligned} \tag{9}$$

Proof It suffices to verify that $T_\lambda(\mathrm{Ad}^*(H)\lambda)$ is a coisotropic subspace in $T_\lambda\mathbb{O}$ for any $\lambda \in S$ because the condition (9) is H-invariant. Via the identification $T_\lambda\mathbb{O} \simeq \mathfrak{g}/Z_{\mathfrak{g}}(\lambda)$, one has $T_\lambda(\mathrm{Ad}^*(H)\lambda) \simeq (\mathfrak{h} + Z_{\mathfrak{g}}(\lambda))/Z_{\mathfrak{g}}(\lambda)$. By the formula (7) of the symplectic form Ω on $\mathbb{O}$, one has $T_\lambda(\mathrm{Ad}^*(H)\lambda)^\perp \simeq Z_{\mathfrak{g}}(\mathfrak{h}; \lambda)/Z_{\mathfrak{g}}(\lambda)$. Hence $T_\lambda(\mathrm{Ad}^*(H)\lambda)$ is a coisotropic subspace in $T_\lambda\mathbb{O}$ if and only if $Z_{\mathfrak{g}}(\mathfrak{h}; \lambda) \subset \mathfrak{h} + Z_{\mathfrak{g}}(\lambda)$, whence the lemma.

For semisimple $\mathfrak{g}$, the Killing form B induces the following G-isomorphism

$$\mathfrak{g}^* \simeq \mathfrak{g}, \quad \lambda \mapsto X_\lambda. \tag{10}$$

By definition, one has $\lambda([X, Y]) = B(X_\lambda, [X, Y]) = B([X_\lambda, X], Y)$, and thus

$$Z_{\mathfrak{g}}(\mathfrak{h}; \lambda) = [X_\lambda, \mathfrak{h}]^{\perp B},$$

where the right-hand side stands for the orthogonal complement subspace of $[X_\lambda, \mathfrak{h}] := \{[X_\lambda, X] : X \in \mathfrak{h}\}$ in $\mathfrak{g}$ with respect to the Killing form B. Hence we have the following.

Lemma 25 *For semisimple $\mathfrak{g}$, one may replace the condition* (9) *in Lemma 24 by*

$$(\mathfrak{h} + Z_{\mathfrak{g}}(\lambda))^{\perp B} \subset [X_\lambda, \mathfrak{h}] \quad \textit{for any } \lambda. \tag{11}$$

3.2 Real Minimal Nilpotent Orbits

Let G be a connected non-compact simple Lie group without complex structure. Denote by $\mathcal{N}$ the nilpotent cone in $\mathfrak{g}$, and $\mathcal{N}/G$ the set of nilpotent orbits, which may be identified with nilpotent coadjoint orbits in $\mathfrak{g}^*$ via (10). The finite set $\mathcal{N}/G$ is a poset with respect to the closure ordering, and there are at most two minimal elements in $(\mathcal{N} \setminus \{0\})/G$, which we refer to as *real minimal nilpotent (coadjoint) orbits*. See [1, 24, 29] for details. The relationship with the complex minimal nilpotent orbits $\mathbb{O}_{\min,\mathbb{C}}$ in the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is given as below. Let K be a maximal compact subgroup of G modulo center.

Lemma 26 *In the setting above, exactly one of the following cases occurs.*

(1) $(\mathfrak{g}, \mathfrak{k})$ *is not of Hermitian type, and* $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g} = \emptyset$.
(2) $(\mathfrak{g}, \mathfrak{k})$ *is not of Hermitian type, and* $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g}$ *is a single orbit of* G.
(3) $(\mathfrak{g}, \mathfrak{k})$ *is of Hermitian type, and* $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g}$ *consists of two orbits of* G.

As the G-orbit decomposition of $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g}$, we write $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g} = \{\mathbb{O}_{\min,\mathbb{R}}\}$ in Case (2), $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g} = \{\mathbb{O}^+_{\min,\mathbb{R}}, \mathbb{O}^-_{\min,\mathbb{R}}\}$ in Case (3). Then they exhaust all real minimal nilpotent orbits in Cases (2) and (3). Real minimal nilpotent orbits are unique in Case (1), to be denoted by $\mathbb{O}_{\min,\mathbb{R}}$. We set

$$n(\mathfrak{g}) := \begin{cases} \frac{1}{2} \dim \mathbb{O}_{\min,\mathbb{R}} & \text{in Cases (1) and (2),} \\ \frac{1}{2} \dim \mathbb{O}^+_{\min,\mathbb{R}} = \frac{1}{2} \dim \mathbb{O}^-_{\min,\mathbb{R}} & \text{in Case (3).} \end{cases} \tag{12}$$

Then $n(\mathfrak{g}) = n(\mathfrak{g}_\mathbb{C})$ in Cases (2) and (3), and $n(\mathfrak{g}) > n(\mathfrak{g}_\mathbb{C})$ in Case (1). The formula of $n(\mathfrak{g})$ in Case (1) is given in [29] as follows:

$\mathfrak{g}$	$\mathfrak{su}^*(2n)$	$\mathfrak{so}(n-1,1)$	$\mathfrak{sp}(m,n)$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{e}_{6(-26)}$
$n(\mathfrak{g})$	$4n-4$	$n-2$	$2(m+n)-1$	11	16

For any $\Pi \in \mathrm{Irr}(G)$, the Gelfand–Kirillov dimension $\mathrm{DIM}(\Pi)$ satisfies $n(\mathfrak{g}) \le \mathrm{DIM}(\Pi)$, which is equivalent to $n(\mathfrak{g}_\mathbb{C}) \le \mathrm{DIM}(\Pi)$ in Cases (2) and (3). We shall give a brief review of several conditions that are equivalent to $n(\mathfrak{g}) > n(\mathfrak{g}_\mathbb{C})$ in Proposition 30.

We prove the following.

Theorem 27 *Let* $\mathbb{O}$ *be a real minimal nilpotent coadjoint orbit in* $\mathfrak{g}^*$. *Then the* K-*action on* $\mathbb{O}$ *is coisotropic.*

For the proof, we recall some basic facts on real minimal nilpotent orbits.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition, and θ the corresponding Cartan involution. We take a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and fix a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ of the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. We denote by μ the highest element in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, and $A_\mu \in \mathfrak{a}$ the coroot of μ. It is known (e.g., [29]) that any minimal nilpotent coadjoint orbit $\mathbb{O}$ is of the form $\mathbb{O} = \mathrm{Ad}(G)X$ via the identification $\mathfrak{g}^* \simeq \mathfrak{g}$ for some nonzero element $X \in \mathfrak{g}(\mathfrak{a}; \mu) := \{X \in \mathfrak{g} : [H, X] = \mu(H)X \text{ for all } H \in \mathfrak{a}\}$. Let G_X be the stabilizer subgroup of X in G. Then one has the decomposition:

Lemma 28 $G = K \exp(\mathbb{R}A_\mu) G_X$.

Proof We set $\mathfrak{a}^{\perp\mu} := \{H \in \mathfrak{a} : \mu(H) = 0\}$, $\mathfrak{n} = \bigoplus_{\nu \in \Sigma^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}(\mathfrak{a}; \nu)$, and $\mathfrak{m} := Z_\mathfrak{k}(\mathfrak{a})$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. We note that $\mathfrak{a} = \mathbb{R}A_\mu \oplus \mathfrak{a}^{\perp\mu}$ is the orthogonal direct sum decomposition with respect to the Killing form.

Since μ is the highest element in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, the Lie algebra $Z_\mathfrak{g}(X)$ of G_X contains $\mathfrak{a}^{\perp\mu} \oplus \mathfrak{n}$. In particular, G_X contains the subgroup $\exp(\mathfrak{a}^{\perp\mu})N$. Since $A = \exp(\mathbb{R}A_\mu)\exp(\mathfrak{a}^{\perp\mu})$, the Iwasawa decomposition $G = KAN$ implies $G = K \exp(\mathbb{R}A_\mu) G_X$.

Proof (*Proof of Theorem* 27) Retain the above notation and convention. In particular, we write as $\mathbb{O} = \mathrm{Ad}^*(G)X$. By Lemmas 25 and 28, it suffices to verify

$$(\mathfrak{k} + Z_{\mathfrak{g}}(X'))^{\perp} \subset [X', \mathfrak{k}] \quad \text{for any } X' \in \mathrm{Ad}(\exp \mathbb{R}A_{\mu})X. \tag{13}$$

Since $X \in \mathfrak{g}(\mathfrak{a}; \mu)$, any $X' \in \mathrm{Ad}(\exp \mathbb{R}A_{\beta})X$ is of the form $X' = cX$ for some $c > 0$. Thus it is enough to show (13) when $X' = X$. Since $Z_{\mathfrak{g}}(X) \supset \mathfrak{a}^{\perp\mu} \oplus \mathfrak{n}$, one has $\mathfrak{k} + Z_{\mathfrak{g}}(X) \supset \theta\mathfrak{n} \oplus \mathfrak{a}^{\perp\mu} \oplus \mathfrak{m} \oplus \mathfrak{n}$, hence $(\mathfrak{k} + Z_{\mathfrak{g}}(X))^{\perp\mu} \subset \mathbb{R}A_{\mu}$. In view that $(A_{\mu}, X, c'\theta X)$ forms an $\mathfrak{sl}_2(\mathbb{R})$-triple for some $c' \in \mathbb{R}$, one has $A_{\mu} \in [X, \mathfrak{k}]$. Thus (13) is verified for $X' = X$. Hence the K-action on $\mathbb{O}$ is coisotropic by Lemma 24.

3.3 Complex Minimal Nilpotent Orbit

In this section we give a proof of Theorem 23.

Suppose that $G_{\mathbb{C}}$ is a connected complex simple Lie group. We take a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$, choose a positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, and set $\mathfrak{n}_{\mathbb{C}}^+ := \bigoplus_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \mathfrak{g}_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}}; \alpha)$, $\mathfrak{n}_{\mathbb{C}}^- := \bigoplus_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \mathfrak{g}_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}}; -\alpha)$. Let β be the highest root in $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, and $H_{\beta} \in \mathfrak{h}_{\mathbb{C}}$ the coroot of β. Then one has the direct sum decomposition $\mathfrak{h}_{\mathbb{C}} = \mathbb{C}H_{\beta} \oplus \mathfrak{h}_{\mathbb{C}}^{\perp\beta}$ where $\mathfrak{h}_{\mathbb{C}}^{\perp\beta} := \{H \in \mathfrak{h}_{\mathbb{C}} : \beta(H) = 0\}$. The minimal nilpotent coadjoint orbit $\mathbb{O}_{\min,\mathbb{C}}$ is of the form $\mathbb{O}_{\min,\mathbb{C}} = \mathrm{Ad}(G_{\mathbb{C}})X \simeq G_{\mathbb{C}}/(G_{\mathbb{C}})_X$ for any non-zero $X \in \mathfrak{g}(\mathfrak{h}_{\mathbb{C}}; \beta)$ via the identification $\mathfrak{g}_{\mathbb{C}}^* \simeq \mathfrak{g}_{\mathbb{C}}$. One can also write as $\mathbb{O}_{\min,\mathbb{C}} = \mathrm{Ad}(G_{\mathbb{C}})Y \simeq G_{\mathbb{C}}/(G_{\mathbb{C}})_Y$ for any non-zero $Y \in \mathfrak{g}(\mathfrak{h}_{\mathbb{C}}; -\beta)$.

By an elementary representation theory of $\mathfrak{sl}_2$, one sees (e.g., [2]) that the Lie algebras $Z_{\mathfrak{g}_{\mathbb{C}}}(X)$ and $Z_{\mathfrak{g}_{\mathbb{C}}}(Y)$ of the isotropy subgroups $(G_{\mathbb{C}})_X$ and $(G_{\mathbb{C}})_Y$ are given respectively by

$$\begin{aligned} Z_{\mathfrak{g}_{\mathbb{C}}}(X) &= \bigoplus_{\substack{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \\ \alpha \perp \beta}} \mathfrak{g}_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}}; -\alpha) \oplus \mathfrak{h}_{\mathbb{C}}^{\perp\beta} \oplus \mathfrak{n}_{\mathbb{C}}^+, \\ Z_{\mathfrak{g}_{\mathbb{C}}}(Y) &= \mathfrak{n}_{\mathbb{C}}^- \oplus \mathfrak{h}_{\mathbb{C}}^{\perp\beta} \oplus \bigoplus_{\substack{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \\ \alpha \perp \beta}} \mathfrak{g}_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}}; \alpha). \end{aligned} \tag{14}$$

Proof (*Proof of Theorem* 23 *(1)*)

We set $S := \exp \mathbb{C}(H_{\beta}, -H_{\beta}) \cdot (X, Y)$ in $\mathbb{O}_{\min,\mathbb{C}} \times \mathbb{O}_{\min,\mathbb{C}}$. We claim that $\mathrm{diag}(G_{\mathbb{C}})S$ is open dense in $\mathbb{O}_{\min,\mathbb{C}} \times \mathbb{O}_{\min,\mathbb{C}}$. To see this, we observe that $(G_{\mathbb{C}})_X \exp(\mathbb{C}H_{\beta})(G_{\mathbb{C}})_Y$ contains the open Bruhat cell $N_{\mathbb{C}}^+ H_{\mathbb{C}} N_{\mathbb{C}}^- = N_{\mathbb{C}}^+ \exp(\mathfrak{h}_{\mathbb{C}}^{\perp\beta}) \exp(\mathbb{C}H_{\beta}) N_{\mathbb{C}}^-$ in $G_{\mathbb{C}}$ as is seen from (14), and thus

$$\mathrm{diag}(G_{\mathbb{C}}) \exp \mathbb{C}(H_{\beta}, 0)((G_{\mathbb{C}})_X \times (G_{\mathbb{C}})_Y)$$

is open dense in the direct product group $G_{\mathbb{C}} \times G_{\mathbb{C}}$ via the identification $\mathrm{diag}(G_{\mathbb{C}})\backslash(G_{\mathbb{C}} \times G_{\mathbb{C}}) \simeq G_{\mathbb{C}}$, $(x, y) \mapsto x^{-1}y$.

By Lemma 25, Theorem 23 (1) will follow if we show

$$(\mathrm{diag}(\mathfrak{g}_{\mathbb{C}}) + Z_{\mathfrak{g}_{\mathbb{C}}\oplus\mathfrak{g}_{\mathbb{C}}}(\mathrm{Ad}(a)X, \mathrm{Ad}(a)^{-1}Y))^{\perp B} \subset [(\mathrm{Ad}(a)X, \mathrm{Ad}(a)^{-1}Y), \mathrm{diag}(\mathfrak{g}_{\mathbb{C}})] \tag{15}$$

for any $a \in \exp(\mathbb{C}H_\beta)$. Since $\mathrm{Ad}(a)X = cX$ and $\mathrm{Ad}(a)^{-1}Y = c^{-1}Y$ for some $c \in \mathbb{C}^\times$, and since X and Y are arbitrary non-zero elements in $\mathfrak{g}_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}}; \beta)$ and $\mathfrak{g}_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}}; -\beta)$, respectively, it suffices to verify (15) for $a = e$. By (14), one has

$$(\mathrm{diag}(\mathfrak{g}_{\mathbb{C}}) + (Z_{\mathfrak{g}_{\mathbb{C}}}(X) \oplus Z_{\mathfrak{g}_{\mathbb{C}}}(Y)))^{\perp B} = \mathbb{C}(H_\beta, -H_\beta).$$

Since $[X, Y] = c'H_\beta$ for some $c' \in \mathbb{C}^\times$, one has $[(X, Y), (X + Y, X + Y)] = c'(H_\beta, -H_\beta)$, showing $(H_\beta, -H_\beta) \in [(X, Y), \mathrm{diag}(\mathfrak{g}_{\mathbb{C}})]$. Thus Theorem 23 (1) is proved.

Next, we consider the setting in Theorem 23 (2). Let $(G_{\mathbb{C}}, K_{\mathbb{C}})$ be a symmetric pair defined by a holomorphic involutive automorphism θ of $G_{\mathbb{C}}$. Then there is a real form $\mathfrak{g}_{\mathbb{R}}$ of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$ such that $\theta|_{\mathfrak{g}_{\mathbb{R}}}$ defines the Cartan decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}$ of the real simple Lie algebra $\mathfrak{g}_{\mathbb{R}}$ with $\mathfrak{k}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ being the Lie algebra $\mathfrak{k}_{\mathbb{C}}$ of $K_{\mathbb{C}}$. We denote by $G_{\mathbb{R}}$ the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{R}}$.

We take a maximal Abelian subspace $\mathfrak{a}_{\mathbb{R}}$ in $\mathfrak{p}_{\mathbb{R}}$, and apply the results of Sect. 3.2 by replacing the notation $\mathfrak{g}, \mathfrak{k}, \mathfrak{p}, \mathfrak{a}, \cdots$ with $\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}, \mathfrak{a}_{\mathbb{R}}$, etc.

Let $\mathcal{N}_{\mathbb{C}}$ be the nilpotent cone in $\mathfrak{g}_{\mathbb{C}}$, and $\mathcal{N}_{\mathbb{R},\mathbb{C}} := \{X \in \mathcal{N}_{\mathbb{C}} : \mathrm{Ad}(G_{\mathbb{C}})X \cap \mathfrak{g}_{\mathbb{R}} \neq \emptyset\}$. Then there exists a unique $G_{\mathbb{C}}$-orbit, to be denoted by $\mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}}$, which is minimal in $(\mathcal{N}_{\mathbb{R},\mathbb{C}} \setminus \{0\})/G_{\mathbb{C}}$ with respect to the closure relation, and $\mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}} = \mathrm{Ad}(G_{\mathbb{C}})X$ for any non-zero $X \in \mathfrak{g}_{\mathbb{R}}(\mathfrak{a}_{\mathbb{R}}; \beta)$ ([29]).

We extend $\mathfrak{a}_{\mathbb{R}}$ to a maximally split Cartan subalgebra $\mathfrak{h}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$ where $\mathfrak{t}_{\mathbb{R}} := \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{k}_{\mathbb{R}}$, write $\mathfrak{h}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}}$ for the complexification, and take a positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ which is compatible with $\Sigma^+(\mathfrak{g}_{\mathbb{R}}, \mathfrak{a}_{\mathbb{R}})$.

The proof of Theorem 27 shows its complexified version as follows.

Theorem 29 *The action of $K_{\mathbb{C}}$ on $\mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}}$ is coisotropic.*

This confirms Theorem 23 (2) when $\mathbb{O}_{\min,\mathbb{C}} = \mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}}$, or equivalently, in Cases (2) and (3) of Lemma 26.

Let us verify Theorem 23 (2) in the case $\mathbb{O}_{\min,\mathbb{C}} \neq \mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}}$.

We need the following:

Proposition 30 ([24, Corollary 5.9], [29, Proposition 4.1]) *Let $\mathfrak{g}_{\mathbb{R}}$ be a real form of a complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and $\mathfrak{k}_{\mathbb{C}}$ the complexified Lie algebra of $\mathfrak{k}_{\mathbb{R}}$, the Lie algebra $\mathfrak{k}_{\mathbb{R}}$ of a maximal compact subgroup $K_{\mathbb{R}}$ of the analytic subgroup $G_{\mathbb{R}}$ in* Int $\mathfrak{g}_{\mathbb{C}}$. *Then the following six conditions on $\mathfrak{g}_{\mathbb{R}}$ are equivalent:*

(i) $\mathbb{O}_{\min} \cap \mathfrak{g}_{\mathbb{R}} = \emptyset$.
(ii) $\mathbb{O}_{\min,\mathbb{C}} \neq \mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}}$.

(iii) $\theta\beta \neq -\beta$.
(iv) $n(\mathfrak{g}) > n(\mathfrak{g}_{\mathbb{C}})$.
(v) $\mathfrak{g}_{\mathbb{R}}$ *is compact or is isomorphic to* $\mathfrak{su}^*(2n)$, $\mathfrak{so}(n-1,1)$ $(n \geq 5)$, $\mathfrak{sp}(m,n)$, $\mathfrak{f}_{4(-20)}$, *or* $\mathfrak{e}_{6(-26)}$.
(vi) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}$ *or the pair* $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ *is isomorphic to* $(\mathfrak{sl}(2n,\mathbb{C}), \mathfrak{sp}(n,\mathbb{C}))$, $(\mathfrak{so}(n,\mathbb{C}), \mathfrak{so}(n-1,\mathbb{C}))$ $(n \geq 5)$, $(\mathfrak{sp}(m+n,\mathbb{C}), \mathfrak{sp}(m,\mathbb{C}) \oplus \mathfrak{sp}(n,\mathbb{C}))$, $(\mathfrak{f}_4^{\mathbb{C}}, \mathfrak{so}(9,\mathbb{C}))$, *or* $(\mathfrak{e}_6^{\mathbb{C}}, \mathfrak{f}_4^{\mathbb{C}})$.

Remark 31 The equivalence (i) $\iff$ (v) was stated in [1, Proposition 4.1] without proof, and Okuda [29] supplied a complete proof.

Lemma 32 *Suppose X is a highest root vector, namely,* $0 \neq X \in \mathfrak{g}_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}}; \beta)$. *If* $\theta\beta \neq -\beta$, *then* $H_\beta \in \mathfrak{k}_{\mathbb{C}} + Z_{\mathfrak{g}_{\mathbb{C}}}(X)$.

Proof Since $\theta\beta \neq -\beta$, one has $\beta|_{\mathfrak{t}_{\mathbb{C}}} \not\equiv 0$, namely, $\mathfrak{t}_{\mathbb{C}} \not\subset \mathfrak{h}_{\mathbb{C}}^{\perp\beta}$. Since $\mathfrak{h}_{\mathbb{C}}^{\perp\beta}$ is of codimension one in $\mathfrak{h}_{\mathbb{C}}$, we get $\mathfrak{t}_{\mathbb{C}} + \mathfrak{h}_{\mathbb{C}}^{\perp\beta} = \mathfrak{h}_{\mathbb{C}}$. Thus $H_\beta \in \mathfrak{h}_{\mathbb{C}} \subset \mathfrak{k}_{\mathbb{C}} + Z_{\mathfrak{g}_{\mathbb{C}}}(X)$.

Proposition 33 *If one of (and therefore any of) the equivalent conditions in Proposition 30 holds, then $K_{\mathbb{C}}$ has a Zariski open orbit in $\mathbb{O}_{\min,\mathbb{C}}$. In particular, the $K_{\mathbb{C}}$-action on $\mathbb{O}_{\min,\mathbb{C}}$ is coisotropic.*

Proof Since $\mathbb{O}_{\min,\mathbb{C}} = \mathrm{Ad}(G_{\mathbb{C}})X$ for a non-zero $X \in \mathfrak{g}_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}}; \beta)$, the proposition is clear.

Proof (*Proof of Theorem* 23 *(2)*) The Case (1) in Lemma 26 is proved in Proposition 33, and the Cases (2) and (3) are proved in Theorem 29.

3.4 *Proof of Theorems in Sect. 1*

As we saw at the end of Sect. 2, Theorems 3 and 4 are derived from the geometric result, namely, from Theorem 23, and thus the proof of these theorems has been completed.

In the same manner, one can deduce readily from Theorem 29 the following bounded multiplicity property which is not covered by Theorem 3 for the five cases in Proposition 30 where $n(\mathfrak{g}) > n(\mathfrak{g}_{\mathbb{C}})$.

Theorem 34 *Suppose that the Gelfand–Kirillov dimension of $\Pi \in \mathrm{Irr}(G)$ is $n(\mathfrak{g})$. If (G, G') is a symmetric pair such that $\mathfrak{g}'_{\mathbb{C}}$ is conjugate to $\mathfrak{k}_{\mathbb{C}}$ by $\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}$, then $m(\Pi|_{G'}) < \infty$.*

Proof We write $G'_{\mathbb{C}}$ and $K_{\mathbb{C}}$ for the analytic subgroups of $G_{\mathbb{C}} = \mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}$ with Lie algebras $\mathfrak{g}'_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}$, respectively. Then the $K_{\mathbb{C}}$-action on $\mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}}$ is coisotropic by Theorem 29, and so is the $G'_{\mathbb{C}}$-action on $\mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}}$ because $G'_{\mathbb{C}}$ and $K_{\mathbb{C}}$ are conjugate by an element of $G_{\mathbb{C}}$. Hence the theorem follows from Fact 21.

Finally, we give a proof of Theorem 10.

Proof *(Proof of Theorem 10)* Let $\mathcal{J}$ be the Joseph ideal. Let $(U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J})^{\mathfrak{g}'_{\mathbb{C}}}$ be the algebra of $\mathfrak{g}'_{\mathbb{C}}$-invariant elements in $U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J}$ via the adjoint action. Then one has

$$(U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J})^{\mathfrak{g}'_{\mathbb{C}}} = \mathbb{C}$$

if one of (therefore, all of) the equivalent conditions in Proposition 30 is satisfied, see [30, Lemma 3.4]. In particular, the center $Z(\mathfrak{g}'_{\mathbb{C}})$ of the enveloping algebra $U(\mathfrak{g}'_{\mathbb{C}})$ of the subalgebra $\mathfrak{g}'_{\mathbb{C}}$ acts as scalars on the minimal representation Π because the action factors through the following composition of homomorphisms:

$$Z(\mathfrak{g}'_{\mathbb{C}}) \to U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J} \to \mathrm{End}_{\mathbb{C}}(\Pi).$$

Since any minimal representation is unitarizable by the classification [30], and since there are at most finitely many elements in $\mathrm{Irr}(G')$ having a fixed $Z(\mathfrak{g}'_{\mathbb{C}})$-infinitesimal character, the restriction $\Pi|_{G'}$ splits into a direct sum of at most finitely many irreducible representations of G', with multiplicity being finite by Theorem 7. Thus the proof of Theorem 10 is completed.

Acknowledgements The author warmly thanks Professor Vladimir Dobrev for his hospitality during the 14th International Workshop: Lie Theory and its Applications in Physics, held online in Bulgaria, 20–26 June 2021. This work was partially supported by Grant-in-Aid for Scientific Research (A) (18H03669), JSPS.

References

1. R. Brylinski, Geometric quantization of real minimal nilpotent orbits. Diff. Geom. Appl. **9**, 5–58 (1998)
2. D.H. Collingwood, W.M. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras* Van Nostrand Reinhold Mathematics Series. (Van Nostrand Reinhold Co., New York, 1993)
3. W.T. Gan, G. Savin, Uniqueness of Joseph ideal. Math. Res. Lett. **11**, 589–597 (2004)
4. W.T. Gan, G. Savin, On minimal representations definitions and properties. Represent. Theory **9**, 46–93 (2005)
5. A.T. Huckleberry, T. Wurzbacher, Multiplicity-free complex manifolds. Math. Ann. **286**, 261–280 (1990)
6. A. Joseph, The minimal orbit in a simple Lie algebra and its associated maximal ideal. Ann. Sci. Ecole Norm. Sup. **9**, 1–29 (1976)
7. M. Kitagawa, Uniformly bounded multiplicities, polynomial identities and coisotropic actions, arXiv:2109.05555v2
8. T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications. Invent. Math. **117**, 181–205 (1994). https://doi.org/10.1007/BF01232239
9. T. Kobayashi, Introduction to harmonic analysis on real spherical homogeneous spaces, in Proceedings of the 3rd Summer School on Number Theory "Homogeneous Spaces and Automorphic Forms" in Nagano, ed. by F. Sato, pp. 22–41 (1995)

10. T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups II—micro-local analysis and asymptotic K-support. Ann. Math. **147**, 709–729 (1998). https://doi.org/10.2307/120963
11. T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups III—restriction of Harish-Chandra modules and associated varieties. Invent. Math. **131**, 229–256 (1998). https://doi.org/10.1007/s002220050203
12. T. Kobayashi, Branching problems of Zuckerman derived functor modules, In: Representation Theory and Mathematical Physics, in Honor of G. Zuckerman, *Contemporary Mathematics*, vol. 557, ed. by J. Adams, B. Lian, S. Sahi (American Mathematical Society, Providence, RI, 2011), pp. 23–40. http://www.ams.org/books/conm/557/11024
13. T. Kobayashi, Shintani functions, real spherical manifolds, and symmetry breaking operators. Dev. Math. **37**, 127–159 (2014)
14. T. Kobayashi, A program for branching problems in the representation theory of real reductive groups, in *Representations of Reductive Groups—In Honor of the 60th Birthday of* David A. Vogan, Jr., *Progress in Mathematics*, vol. 312 (Birkhäuser, 2015), pp. 277–322
15. T. Kobayashi, Branching laws of unitary representations associated to minimal elliptic orbits for indefinite orthogonal group $O(p, q)$. Adv. Math., **388**, Paper No. 107862, 38pp (2021)
16. T. Kobayashi, Admissible restrictions of irreducible representations of reductive Lie groups: Symplectic geometry and discrete decomposability. Pure Appl. Math. Q. **17**, 1321–1343 (2021). Special volume in memory of Bertram Kostant
17. T. Kobayashi, Bounded multiplicity theorems for induction and restriction. J. Lie Theory **32**, 197–238 (2022)
18. T. Kobayashi, Multiplicity in restricting small representations, Proc. Jpn. Acad. **98**, Ser. A, 19–24 (2022)
19. T. Kobayashi, T. Matsuki, Classification of finite-multiplicity symmetric pairs. Transform. Groups **19**, 457–493 (2014). https://doi.org/10.1007/s00031-014-9265-x. Special issue in honor of Dynkin for his 90th birthday
20. T. Kobayashi, B. Ørsted, Analysis on the minimal representation of $O(p, q)$. II. Branching laws. Adv. Math. **180**, 513–550 (2003)
21. T. Kobayashi, B. Ørsted, M. Pevzner, Geometric analysis on small unitary representations of $GL(N, \mathbb{R})$. J. Funct. Anal. **260**, 1682–1720 (2011)
22. T. Kobayashi, B. Ørsted, M. Pevzner, A. Unterberger, Composition formulas in the Weyl calculus. J. Funct. Anal. **257**, 948–991 (2009)
23. T. Kobayashi, T. Oshima, Finite multiplicity theorems for induction and restriction. Adv. Math. **248**, 921–944 (2013). https://doi.org/10.1016/j.jfa.2010.12.008
24. T. Kobayashi, Y. Oshima, Classification of symmetric pairs with discretely decomposable restrictions of $(\mathfrak{g}, K)$-modules. J. Reine Angew. Math. **703**, 201–223 (2015)
25. T. Kobayashi, B. Speh, Symmetry Breaking for Representations of Rank One Orthogonal Groups. Mem. Am. Math. Soc. **238**(1126), 118 pages (2015). https://doi.org/10.1090/memo/1126
26. T. Kobayashi, B. Speh, Symmetry Breaking for Representations of Rank One Orthogonal Groups, Part II. *Lecture Notes Mathematics*, vol. 2234 (Springer, 2018), xv+342 pages. https://arxiv.org/abs/1801.00158
27. M. Krämer, Multiplicity free subgroups of compact connected Lie groups. Arch. Math. (Basel) **27**, 28–36 (1976)
28. S. S. Kudla, Seesaw dual reductive pairs. Automorphic forms of several variables, Progr. Math., **46**, 244–268 (1984)
29. T. Okuda, Smallest complex nilpotent orbits with real points. J. Lie Theory **25**, 507–533 (2015)
30. H. Tamori, Classification of minimal representations of real simple Lie groups. Math. Z. **292**, 387–402 (2019)
31. T. Tauchi, A generalization of the Kobayashi-Oshima uniformly bounded multiplicity theorem. Internat. J. Math. **32**, 2150107 (2021)
32. N.R. Wallach, Real Reductive Groups. I, II, *Pure and Applied Mathematics*, vol. 132 (Academic Press, Inc., Boston, MA, 1988), 132-II, ibid, 1992

From the String Landscape to the Mathematical Landscape: A Machine-Learning Outlook

Yang-Hui He

Abstract We review the recent programme of using machine-learning to explore the landscape of mathematical problems. With this paradigm as a model for human intuition—complementary to and in contrast with the more formalistic approach of automated theorem proving—we highlight some experiments on how AI helps with conjecture formulation, pattern recognition and computation.

Keywords Machine-learning · AI · Mathematical structures · String theory

1 The String Landscape

Perhaps the greatest theoretical challenge to string theory as a theory of everything is the vast proliferation of possible vacuum solutions, each of which is a possible 4-dimensional "universe" that descends from the 10 spacetime dimensions of the superstring. This is the so-called "vacuum degeneracy problem", or the "string landscape problem". The reason for this multitude is the vast number of possible geometries for the missing 6 dimensions. Whether we consider compactification, where the a Calabi–Yau manifold constitutes the missing dimensions, or configurations of branes whose world-volumes complement these dimensions, we are inevitably confronted with the heart of the problem: geometrical structures, often due to an underlying combinatorial problem, tend to grow exponentially with dimension.

We can see this from estimates of possible vacua, which engender such astronomical numbers as 10^{500} to 10^{10^5} [1–3]. These estimates come from tallying "typical"

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_2

number of topologies of "typical" manifolds, as governed by the number of holes (or more strictly, algebraic cycles) of various dimensions within the manifolds. Such topological quantities are immanently combinatorial in nature.

Lacking a fundamental "selection principle" [4]—which would find *our universe* among the myriad—the traditional approaches have been statistical valuations [5], or brute-force searching for the Standard Model [6–11] as a needle in the haystack. Whilst these approaches have met some success, the overwhelming complexity (especially in the computational sense [12]) of, and the want of a canonical measure [19] on, the string landscape, beckon for a prismatically different method of attack.

As the Zeitgeist of Artificial Intelligence (AI) breathes over all disciplines of science [13] in recent times, and as we firmly enter the era of Big Data and Machine-Learning (ML), it is only natural that such a perspective be undertaken to explore the string landscape. This was indeed done in 2017 when ML was introduced into string theory [14–18]. In particular, the proposal of [14] was to see whether ML could be used to study the databases in algebraic geometry, which have been compiled over the last few decades for the sake of studying string theory in physics and concepts such as mirror symmetry in mathematics. To some details of this programme let us now turn.

1.1 Calabi–Yau Manifolds: From Geometry to Physics

The classification of (compact, smooth, boundary-less) surfaces Σ goes back to at least Euler, who realized that a single integer, called *genus*, completely characterizes the topological type of Σ. Roughly, the genus g counts the number of "holes": a sphere S^2 has genus 0, a torus $T^2 = S^1 \times S^1$ has genus 1, etc. The *Theorema Egregium* of Gauss then relates topology to metric geometry:

$$2 - 2g = \frac{1}{2\pi} \int_{\Sigma} R \ . \tag{1}$$

In the above, the combination $\chi = 2 - 2g$ is the Euler number and R is the (Gaussian) curvature. We therefore see a natural trichotomy of surfaces, as summarized in Fig. 1: negative, zero and positive curvature, with the boundary case of $R = 0$, or Ricci-flatness, being the torus T^2.

With Riemann enters complex geometry: Σ is not merely a real dimension 2 manifold, but a complex dimension 1 manifold. The trichotomy in this context manifests as Riemann Uniformization. Complexification allows us to employ the powers of algebraic geometry over $\mathbb{C}$ and Σ can thus be realized as a complex algebraic curve. For instance, it can be the vanishing locus of a complex polynomial in the three projective variables $[x : y : z]$ of $\mathbb{CP}^2$. The torus, in particular, can be realized as the famous cubic elliptic curve. In modern parlance, the Gaussian integral is thought of as the intersection theory between homology (the class $[\Sigma]$) with cohomology (the first Chern class $c_1(\Sigma)$). Likewise, χ, by the index theorem, is the alternating sum

$g(\Sigma) = 0$	$g(\Sigma) = 1$	$g(\Sigma) > 1$
$\chi(\Sigma) = 2$	$\chi(\Sigma) = 0$	$\chi(\Sigma) < 0$
Spherical	Ricci-Flat	Hyperbolic
$+$ curvature	0 curvature	$-$ curvature

Fig. 1 The trichotomy of (smooth, compact, boundary-less) surfaces, organized according to topological type and related curvature

$\chi(\Sigma) = 2 - 2g(\Sigma) =$	$= [c_1(\Sigma)] \cdot [\Sigma] =$	$= \frac{1}{2\pi} \int_\Sigma R =$	$= \sum_{i=0}^{2} (-1)^i h^i(\Sigma)$
Topology	Algebraic Geometry	Differential Geometry	Index Theorem (co-)Homology
Invariants	Characteristic classes	Curvature	Betti Numbers

Fig. 2 The index theorem relating differential/algebraic geometry/toplogy for surfaces as complex algebraic curves

of dimensions of appropriate (co-)homology groups. We summarize this beautiful story, spanning the two centuries from Euler to Chern, Atiyah, Singer et al., in Fig. 2.

Generalizing Figs. 1 and 2 to complex dimension higher than 1 is, understandably, difficult. However, at least for a class of complex manifolds, called Kähler, whose (Hermitian) metric $g_{\bar{\mu}\nu}$ comes from a single scalar potential K as $g_{\bar{\mu}\nu} = \partial_{\bar{\mu}}\partial_\nu K$, the story does extend nicely: the Chern class governs the curvature. This is roughly the content and significance of the Calabi conjecture [21], which Yau proved some 20 years later in his Fields-Medal-winning work [22].

It is serendipitous that when string theorists worked out the conditions for compactification in the incipience of string phenomenology [23], one of the solutions (and today still standard) for the extra 6-dimensions is a complex, Kähler, Ricci-flat 3-fold. Furthermore, Strominger, one of the authors, was Yau's visitor at the IAS. And thus the world of high-energy theoretical physics intermingled with the world of complex algebraic geometry. In fact, the physicists named such manifolds "Calabi–Yau" (CY), and the rest, was history. The torus T^2, is thus a premium example of a Calabi–Yau 1-fold, of complex dimension 1. The reader interested in further details of the Calabi–Yau landscape as a confluence between physics, mathematics and modern data science, is referred to the pedagogical book [24].

Over the decades since the mid-1980s, a host of activity ensued in creating large data-bases of CY manifolds for the intention of sifting through to find the Standard Model. Perhaps it was unexpected that the number[1] of CY 3-folds reached billions by the turn of the century (and still growing!) [25]. Furthermore, the sophisticated machinery of modern geometry, much of which was inherited from the Bourbaki School, was used to compute the various quantities (q.v. the classic [26] and for

[1] By contrast, a CY 1-fold can only be T^2, a CY 2-fold can only be T^4 and K3. We therefore see the aforementioned exponential growth of possibilities as we increase in dimension. Nevertheless, it is a standing conjecture of Yau that the number of possible topological types of CY in every dimension is *finite*.

physicists, [27]), particularly the topological ones such as Euler, Betti and Hodge numbers, which have precise interpretation as Standard Model particles.

1.2 Machine-Learning Algebraic Geometry

The *point d'appui* of [14] was that these large sets of CY manifolds constituted labelled datasets ripe for machine-learning. In fact, the situation is even more general and *any* mathematical computation can be thought of this way. We shall not delve into the details of CY topological invariants or string phenomenology, but the idea can be construed as follows. The purpose of algebraic geometry is to realize a manifold as the vanishing loci of a system of multi-variate polynomials where the variables are the coordinates of some appropriate ambient space such as projective space. We can thus represent a manifold as a list (tensor) of coefficients.[2] Traditional methods such as exact sequences and Gröbner bases (q.v. [20] for ML on selecting S-pairs) then computes desired geometrical quantities such as Hilbert series or Betti numbers. In the special case of extracting topological quantities, the coefficients are irrelevant (topology does not depend on shape) and we have even simpler representations. For instance, one could record just the degrees of the various defining polynomials.

But a tensor can naturally be interpreted as a pixelated image (up to some normalization and padding if necessary), and thus the general statement of [14, 15] is that

Observation 1 *Computation in algebraic geometry is an image—recognition problem.*

To make this observation concrete, let us give an example. Suppose we are given a CY 3-fold,[3] defined by the intersection of 8 polynomials in a product $(\mathbb{CP}^1)^6 \times (\mathbb{CP}^2)^2$ of projective spaces given by the configuration below.[4] The topological quantity, a so-called Hodge number $h^{2,1}$ was computed (see [27]) to be 22 using long exact sequence in cohomology induced by an Euler sequence (quite a difficult and expensive computation!). However, we could associate 0 to, say, purple, green to 1 and red to 2. After padding with 0 (to normalize over the full CY dataset of which this

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[2] These coefficients determine the "shape" of the manifold. In Mathematica, there is a convenient command for this, viz., CoefficientList[ ].

[3] Strictly, this is a family of manifolds since we are not specifying the coefficient which dictate complex structure (shape).

[4] This is an example of a complete intersection CY in product of project spaces (CICY), which was possibly the first database in algebraic geometry [28]. To read it, each column is a defining polynomial. For example, the first column corresponds to a polynomial which is multi-linear in the first and second $\mathbb{CP}^1$ factors and also linear in the first $\mathbb{CP}^2$ factor.

is one case), and the computation of $h^{2,1}$ becomes an image-processing problem no different than hand-writing recognition:

$$h^{2,1}\left(\begin{pmatrix} 1&1&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&2&0\\ 0&1&1&0&0&0&0&1\\ 1&0&0&0&0&1&1&0\\ 0&0&0&1&1&0&0&1 \end{pmatrix}\right) = 22 \qquad \text{becomes}$$

$$\longrightarrow 22\ . \qquad (2)$$

A surprising result of [14] is that such labeled data, consisting of typically around $10^5 \sim 10^6$ points, when fed into a standard ML algorithm, such as a fairly shallow feed-forward neural network (otherwise known as an MLP) with sigmoid activation functions, or a support vector machine (SVP), achieves over 90% accuracy in a standard 80-20 cross-validation[5] in a matter of seconds on an ordinary laptop. Since then, more sophisticated neural networks (NNs) have achieved over 99.9% accuracy [29–32] (q.v. recognition of elliptic fibration within the data using ML [33]). How could a relatively simple ML algorithm *guess* at a cohomology computation, without any knowledge of the underlying mathematics? At some level, this is the Universal Approximation Theorem of NNs at work [34], which states that at sufficiently large depth/width, a NN can approximate almost any map, much like the way a Taylor series can approximate any analytic function. Yet, the relative simplicity of the architecture of the NN is highly suggestive of a method which bypasses the sophistication and computational complexity of the standard algorithms of algebraic geometry. To this point let us now turn.

2 The Landscape of Mathematics

The great *utility* of our paradigm to the string landscape, and indeed to problems in theoretical physics, is obvious. Even when not reaching 100% accuracy, a rapid and highly accurate NN estimate could reduce practical computations, say, of searching the exact Standard Model within string, many orders of magnitude faster. Utility aside, the unexpected success of machine learning of algebraic geometry beckons a deeper question: can one machine learn mathematics [24, 35]? By this we mean several levels: can ML/AI (1) extract patterns from mathematical data, supervised and unsupervised, patterns which have not been noted by the human eye? (2) help formulate new conjectures and find easier formulae (q.v. recent collaboration on how AI can help with mathematical intuition [36])? (3) help with new pathways in a proof? (4) help understand the structure of mathematics across the disciplines?

It is expedient to digress momentarily on some speculations upon the nature of mathematics whilst we are planning to explore Her landscape. The turn of the

[5] In machine-learning, this means we take the full labelled data, train on 80% randomly selected, and validate—meaning we check what the output is as predicted by the NN versus the actual value—on the *unseen* 20%.

20th century witnessed a tension between two Schools of thought: (i) the logicism-formalism of Hilbert and (ii) the intuitionism-constructivism of Poincaré. The first, rested in the tradition of Leibniz, Frege, Peano, Russell–Whitehead, Wittgenstein et al., and attempted to logically build all of mathematics, without contradiction, symbol by symbol. The second, propelled by Brouwer, Heyting, Poincaré et al., sought for a more "human" and experiential element to mathematics.

The advent of computers in mathematics has dawned a new era. More importantly, they are becoming more than a mere aid to computation. There is a growing number of major results—championed by e.g., the 4-colour theorem, or the classification of finite simple groups—which *could not* have been possible without computer work. The reason is simple: the rate of growth of mathematical knowledge and the requisite length of many a proof have perhaps already exceeded the capacity of the human mind. The full details of the proof of Fermat is hundreds of pages of highly technical mathematics understandable by a small community, that of the classification of simple groups, thousands. It is entirely conceivable that the proof of the Riemann Hypothesis will take longer than several human lifetimes to construct or digest, even if we take into account the cumulative nature of research.

Consequently, Buzzard, Davenport et al. [37, 38] have been emphasising how essential the Automated Theorem Proving programme (ATP) is to the future of mathematics. Software such as Lean is currently constructing all statements and proofs of mathematics, symbol by symbol, line by line. Their optimistic estimate is that within 10 years, all of undergraduate level mathematics will be built from scratch automatically. More strikingly, some at Google Deepmind suspect that as computers defeated humans at chess in the 1990s and Go in the 2010s, they will beat us at producing new mathematics by 2030.

The ATP programme can be thought of as being along the formalistic skein of Hilbert, and, to borrow terminology from physics, one could call this "bottom-up mathematics" [35]. Our foregoing discussion of using ML which attempts to extract patterns from data or extrapolate methods from heuristics, on the other hand, is much more along the intuitionistic line of Poincaré. Again, to borrow from physics, one could call this "top-down mathematics" [35]. These two threads should indeed be pursued in parallel and here, we shall summarize some recent experiments in the latter.

2.1 Methodology

For concreteness, let us focus on calculations of the form of (2), which should be ubiquitous in mathematics. We shall let $\mathcal{D} := \{T_i \to p_i\}$ be a set of input tensors T_i with output property p_i, typically obtained from some exhaustive and intensive computation. We then split $\mathcal{D} := \mathcal{T} \sqcup \mathcal{V}$ into training set $\mathcal{T}$ and validation set $\mathcal{V}$ where $\mathcal{T}$ is a random sample of, say, 80%. Such data, representing "experience and intuition" of the practitioner, could then be passed to standard machine-learning algorithms such as neural classifiers/regressors, MLPs, SVMs, decisions trees, etc.

Importantly, these algorithms have *no prior knowledge* of the mathematics.[6] Once validation reaches high precision (especially 100%), one could start formulating conjectures. On the other hand, if one could not reach any good results exhausting a multitude of algorithms, it would indicate an inherent difficulty in the problem whence the data came.

2.2 Across Disciplines

With this method of attack it is natural to scan through the available data of mathematics, as a reconnaissance onto the topography of Her territory. We saw in the above that algebraic geometry over $\mathbb{C}$ responds well to ML and speculate that the reason for this is that all computations inherent thereto reduce to finding (co-)kernels of matrices. Over the past 5 years, there have been various excursions into a variety of disciplines and we shall highlight some representative cases, and refer the reader to the citations as well as the summary in [35].

Algebra: In [46], the question was posed as to whether one could "see" a finite group being simple or not, by direct inspection of its Cayley multiplication table. Surprisingly, an SVM could do so to more than 0.98 precision, instigating the curious conjecture that simple and non-simple groups could be separable when plotting their flattened Cayley tables. For continuous groups, the tensor decomposition into irreps for simple Lie algebras of type ABCDG_2 is computationally exponential as one goes up in weight. Yet, numerical quantities such as the number of terms in the decomposition can be quickly machine-learnt by an MLP with only a few layers to 0.96 precision [47]. In [48], MLPs, decision trees and graph NNs could distinguish table/non-table ideals to 100% accuracy, whereby suggesting the existence of a yet-unknown formula.

Graphs and Combinatorics: Various properties of finite graphs, such as cyclicity, genus, existence of Euler or Hamilton cycles, etc., were explored by "looking" at the adjacency matrix with MLPs and SVMs [49]. The algorithms determining some of these quantities are quite involved indeed. For instance, Hamilton cycle detection is that of the traveling salesman problem, which is NP-hard. Typically, for these problems, one could reach 80–90% accuracies, which could be related to the fact that detecting matrix permutations—and hence graph isomorphism—is currently a challenge to ML. However, when more structures are put in, such as quiver representations [50], or tropical geometry [51], accuracies in the high 90s can once more be attained. Explorations in lattice polytopes [52, 53] and knot invariants [54, 55] also yield good results.

[6] Of course, building activation functions which *know* some of the underlying theory is effective and computationally helpful, as was done in, e.g., [39–45], but the true surprises lie in blind tests. This was performed in the initial experiments of [14] and the ones we shall shortly report, could lead to conjectures unfathomed by human thought.

Analytic Number Theory: As one might imagine, uncovering patterns in arithmetic functions, such as prime characteristic, or the likes of Mobius μ and Liouville λ, would be very hard. And it turns out to be so not only for the human eye, but also for any standard ML algorithm [14, 24, 35]. Likewise, one would imagine finding new patterns in the Riemann zeta function [56, 57] to be a formidable challenge.

Arithmetic Geometry: Yet, with a mixture of initial astonishment and a posteriori reassurance, problems in arithmetic geometry are very much amenable to ML. Properties such as the arithmetic of L-functions [58, 59], degree of Galois extensions for dessins d'enfants [62], or even the quantities pertaining to the strong Birch–Swinnerton–Dyer conjecture [60, 61] (interestingly, the most difficult Tate-Shafaverich group is the least responsive) can all be learnt to high accuracies. Indeed, as exemplified by countless historical cases, translating Diophantine problems to geometry, especially that of (hyper-)elliptic curves, renders them much more tractable. In this sense, our ML methodology and results on the data are consistent with this notion that arithmetic geometry is closer to geometry than to arithmetic.

With these experiments, we conclude with the remark and speculation that there is a "hierarchy" of mathematical problems, perhaps in tune with our expectations:

Observation 2 *Across the disciplines of mathematics,*

$$\big[\textit{numerical analysis}\big] < \big[\textit{algebraic geometry over } \mathbb{C} \sim \textit{arithmetic geometry}\big] < \big[\textit{algebra/representation theory}\big] < [\textit{combinatorics}] < \big[\textit{analytic number theory}\big]$$

where $a < b$ means patterns from problem from a are more easily extractable than those from b, or indeed that problems in a are more easily solvable.

Above all, we encourage the readers to take their favourite problems and data and see how well ML performs on them.

Acknowledgements *Ad Musam Alphabeticam meam tenebris luscis siderium.* The author is grateful to STFC UK for grant ST/J00037X/2, as well as the kind hospitality—virtual and in person (!) during 2021—of Cambridge (Winter School on ML), Cairo (BSM 22), Sofia (Lie Theory in Physics XIV), Toulouse (Geometry, Topology and AI), Lisboa (BH, BPS, and QI), Trento (ML for HEP), Singapore (M-theory and Beyond), Johannesburg (String Data 22), Bangalore (KAWS 22), Hang-Zhou, Pisa, Minnesota (IMA), Tokyo (NaapingClass), Leicester, and Galway, on the talks at which this review is based.

References

1. S. Kachru, R. Kallosh, A.D. Linde, S.P. Trivedi, De Sitter vacua in string theory. Phys. Rev. D **68**, 046005 (2003). arXiv:hep-th/0301240 [hep-th]
2. W. Taylor, Y.N. Wang, The F-theory geometry with most flux vacua. JHEP **12**, 164 (2015). arXiv:1511.03209 [hep-th]

3. J. Halverson, C. Long, B. Sung, Algorithmic universality in F-theory compactifications. Phys. Rev. D **96**(12), 126006 (2017). arXiv:1706.02299 [hep-th]
4. P. Candelas, X. de la Ossa, Y.H. He, B. Szendroi, Triadophilia: a special corner in the landscape. Adv. Theor. Math. Phys. **12**(2), 429–473 (2008). arXiv:0706.3134 [hep-th]
5. M.R. Douglas, The Statistics of string/M theory vacua. JHEP **05**, 046 (2003). arXiv:hep-th/0303194 [hep-th]
6. V. Braun, Y.H. He, B.A. Ovrut, T. Pantev, The Exact MSSM spectrum from string theory. JHEP **05**, 043 (2006). arXiv:hep-th/0512177 [hep-th]
7. V. Bouchard, R. Donagi, An SU(5) heterotic standard model. Phys. Lett. B **633**, 783–791 (2006). arXiv:hep-th/0512149 [hep-th]
8. F. Gmeiner, R. Blumenhagen, G. Honecker, D. Lust, T. Weigand, One in a billion: MSSM-like D-brane statistics. JHEP **01**, 004 (2006). arXiv:hep-th/0510170 [hep-th]
9. L.B. Anderson, J. Gray, A. Lukas, E. Palti, Two Hundred Heterotic Standard Models on Smooth Calabi-Yau Threefolds. Phys. Rev. D **84**, 106005 (2011). arXiv:1106.4804 [hep-th]
10. M. Cvetic, J. Halverson, L. Lin, C. Long, Constraints on standard model constructions in F-theory. Phys. Rev. D **102**(2), 026012 (2020). arXiv:2004.00630 [hep-th]
11. A. Constantin, Y.H. He, A. Lukas, Counting string theory standard models. Phys. Lett. B **792**, 258–262 (2019). arXiv:1810.00444 [hep-th]
12. J. Halverson, F. Ruehle, Computational complexity of vacua and near-vacua in field and string theory. Phys. Rev. D **99**(4), 046015 (2019). arXiv:1809.08279 [hep-th]
13. Y. LeCun, Y. Bengio, G. Hinton, Deep learning. Nature 521(7553), 436–444 (2015)
14. Y.H. He, Deep-Learning the Landscape. Feature article, M. Hutson, Science (2019). https://doi.org/10.1126/science.aay9534. arXiv:1706.02714 [hep-th]. q.v
15. Y.H. He, Machine-learning the string landscape. Phys. Lett. B **774**, 564–568 (2017)
16. D. Krefl, R.K. Seong, Machine learning of Calabi-Yau volumes. Phys. Rev. D **96**(6), 066014 (2017). arXiv:1706.03346 [hep-th]
17. J. Carifio, J. Halverson, D. Krioukov, B.D. Nelson, Machine learning in the string landscape. JHEP **09**, 157 (2017). arXiv:1707.00655 [hep-th]
18. F. Ruehle, Evolving neural networks with genetic algorithms to study the String Landscape. JHEP **08**, 038 (2017). arXiv:1706.07024 [hep-th]
19. Y.H. He, S. Lal, M.Z. Zaz, The World in a Grain of Sand: Condensing the String Vacuum Degeneracy. arXiv:2111.04761 [hep-th]
20. D. Peifer, M. Stillman, D. Halpern-Leistner, Learning sElection Strategies in Buchberger's Algorithm. arXiv:2005.01917
21. E. Calabi, The space of Kähler metrics, in *Proceedings of International Congress of Mathematicians, Amsterdam*, vol. 2, pp. 206—207 (1954); On Kähler manifolds with vanishing canonical class, in Fox, Spencer, Tucker, *Algebraic Geometry and Topology. A Symposium in Honor of S. Lefschetz*, Princeton Mathematical Series, vol. 12, PUP, pp. 78–89 (1957)
22. S.-T. Yau, Calabi's conjecture and some new results in algebraic geometry. Proc. Ntl Acad. USA **74**(5), 1798–1799 (1977); On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I. Comm. Pure Appl. Math. **31**(3), 339–411 (1978)
23. P. Candelas, G.T. Horowitz, A. Strominger, E. Witten, Vacuum configurations for superstrings. Nucl. Phys. B **258**, 46 (1985)
24. Y.H. He, *The Calabi–Yau Landscape: From Geometry, to Physics, to Machine Learning*. Springer LNM 2293 (2021). https://doi.org/10.1007/978-3-030-77562-9. arXiv:1812.02893 [hep-th]
25. M. Kreuzer, H. Skarke, On the classification of reflexive polyhedra. Commun. Math. Phys. **185**, 495 (1997) arXiv:hep-th/9512204; Reflexive polyhedra, weights and toric Calabi-Yau fibrations. Rev. Math. Phys. **14**, 343 (2002). arXiv:math/0001106 [math-ag]
26. R. Hartshorne, *Algebraic Geometry* (GTM, 1997). ISBN 13: 9780387902449
27. T. Hubsch, *Calabi-Yau Manifolds: A Bestiary for Physicists* (WS, 1994). ISBN 9810206623
28. P. Candelas, A.M. Dale, C.A. Lutken, R. Schimmrigk, Complete intersection Calabi-Yau manifolds. Nucl. Phys. B **298**, 493 (1988)

29. K. Bull, Y.H. He, V. Jejjala, C. Mishra, Machine learning CICY threefolds. Phys. Lett. B **785**, 65–72 (2018). arXiv:1806.03121 [hep-th]
30. S. Krippendorf, M. Syvaeri, Detecting symmetries with neural networks. arXiv:2003.13679 [physics.comp-ph]
31. H. Erbin, R. Finotello, ML for complete intersection Calabi-Yau manifolds: a methodological study. PRD **103**(12), 126014 (2021). arXiv:2007.15706 [hep-th]
32. R. Altman, J. Carifio, X. Gao, B. Nelson, Orientifold Calabi-Yau Threefolds with Divisor Involutions and String Landscape. arXiv:2111.03078 [hep-th]
33. Y.H. He, S.J. Lee, Distinguishing elliptic fibrations with AI. Phys. Lett. B **798**, 134889 (2019). arXiv:1904.08530 [hep-th]
34. K. Hornik, M. Tinchcombe, H. White, Multilayer feedforward networks are universal approximators, in *Neural Networks*, vol. 2. Pergamon Press, pp. 359–366
35. Y.H. He, Machine-Learning Mathematical Structures. arXiv:2101.06317 [cs.LG]
36. A. Davies, P. Velickovic, L. Buesing et al., Advancing mathematics by guiding human intuition with AI. Nature **600**, 70–74 (2021). https://doi.org/10.1038/s41586-021-04086-x
37. K. Buzzard, The Future of Mathematics. https://wwwf.imperial.ac.uk/~buzzard/one_off_lectures/msr.pdf, https://www.youtube.com/watch?v=Dp-mQ3HxgDE; The Xena Project https://wwwf.imperial.ac.uk/~buzzard/xena/
38. J. Davenport, B. Poonen, J. Maynard, H. Helfgott, P.H. Tiep, L. Cruz-Filipe, Machine-Assisted Proofs, ICM Panel 2018
39. R. Deen, Y.H. He, S.J. Lee, A. Lukas, ML String SMs. arXiv:2003.13339 [hep-th]
40. K. Heal, A. Kulkarni, E. Sertoz, Deep Learning Gauss-Manin Connections. arXiv:2007.13786 [cs.LG]
41. V. Jejjala, D.K. Mayorga Pena, C. Mishra, Neural Network Approximations for Calabi-Yau Metrics. arXiv:2012.15821 [hep-th]
42. M.R. Douglas, S. Lakshminarasimhan, Y. Qi, Numerical Calabi-Yau Metrics from Holomorphic Networks. arXiv:2012.04797 [hep-th]
43. L.B. Anderson, M. Gerdes, J. Gray, S. Krippendorf, N. Raghuram, F. Ruehle, Moduli-dependent Calabi-Yau and SU(3)-structure metrics from Machine Learning. JHEP **05**, 013 (2021). arXiv:2012.04656 [hep-th]
44. M. Larfors, A. Lukas, F. Ruehle, R. Schneider, Learning Size and Shape of Calabi-Yau Spaces. arXiv:2111.01436 [hep-th]
45. X. Gao, H. Zou, ML to the Orientifold CY with String Vacua. arXiv:2112.04950 [hep-th]
46. Y.H. He, M. Kim, Learning algebraic structures: preliminary investigations. arXiv:1905.02263 [cs.LG]
47. H.Y. Chen, Y.H. He, S. Lal, S. Majumder, Machine learning Lie structures & applications to physics. Phys. Lett. B **817**, 136297 (2021). arXiv:2011.00871 [hep-th]
48. L. Amoros, O. Gasanova, L. Jakobsson, A Machine Learning Approach to Commutative Algebra: Distinguishing Table vs Non-table Ideals. arXiv:2109.11417 [math.AC]
49. Y.H. He, S.T. Yau, Graph Laplacians, Riemannian Manifolds and their Machine-Learning. arXiv:2006.16619 [math.CO]
50. J. Bao, S. Franco, Y.H. He, E. Hirst, G. Musiker,Y. Xiao, Quiver Mutations, Seiberg Duality and ML. PRD **102**(8), 086013 (2020). arXiv:2006.10783 [hep-th]
51. J. Bao, Y.H. He, E. Hirst, Neurons on Amoebae. arXiv:2106.03695 [math.AG]
52. J. Bao, Y.H. He, E. Hirst, J. Hofscheier, A. Kasprzyk, S. Majumder, Polytopes and ML. arXiv:2109.09602 [math.CO]
53. P. Berglund, B. Campbell, V. Jejjala, Machine Learning Kreuzer-Skarke Calabi-Yau Threefolds. arXiv:2112.09117 [hep-th]
54. S. Gukov, J. Halverson, F. Ruehle, P. Sułkowski, Learning to Unknot. Mach. Learn. Sci. Tech. **2**(2), 025035 (2021). arXiv:2010.16263 [math.GT]
55. J. Craven, M. Hughes, V. Jejjala, A. Kar, Learning knot invariants across dimensions. arXiv:2112.00016 [hep-th]
56. O. Shanker, NN prediction of Riemann zeta zeros. Adv. Model. & Optim. **14**, 3 (2012)

57. J. Kampe, A. Vysogorets, Predicting Zeros of the Riemann Zeta Function Using Machine Learning: A Comparative Analysis. http://www.sci.sdsu.edu/math-reu/2018-2.pdf
58. Y.H. He, K.H. Lee, T. Oliver, ML the Sato–Tate Conjecture. arXiv:2010.01213 [math.NT]. J. Symb. Comput. **111**, 61
59. Y.H. He, K.H. Lee, T. Oliver, ML Number Fields. arXiv:2011.08958 [math.NT]
60. L. Alessandretti, A. Baronchelli, Y.H. He, ML meets Number Theory: The Data Science of BSD. arXiv:1911.02008 [math.NT]. To appear, in Stemma Journal
61. Y.H. He, K.H. Lee, T. Oliver, ML Arithmetic Curves. arXiv:2012.04084 [math.NT]
62. Y.H. He, E. Hirst, T. Peterken, ML dessins d'enfants: explorations via modular and Seiberg-Witten curves. J. Phys. A **54**(7), 075401 (2021). arXiv:2004.05218 [hep-th]

Octonionic Clifford Algebra for the Internal Space of the Standard Model

Ivan Todorov

Abstract We explore the $\mathbb{Z}_2$ graded product $C\ell_{10} = C\ell_4 \hat{\otimes} C\ell_6$ as a finite internal space algebra of the Standard Model of particle physics. The gamma matrices generating $C\ell_{10}$ are expressed in terms of left multiplication by the imaginary octonion units and the Pauli matrices. The subgroup of $Spin(10)$ that fixes an imaginary unit (and thus allows to write $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ expressing the quark-lepton splitting) is the Pati-Salam group $G_{\rm PS} = Spin(4) \times Spin(6)/\mathbb{Z}_2 \subset Spin(10)$. If we identify the preserved imaginary unit with the $C\ell_6$ pseudoscalar $\omega_6 = \gamma_1 \cdots \gamma_6$, $\omega_6^2 = -1$, then $\mathcal{P} = \frac{1}{2}(1 - i\omega_6)$ will be the projector on the extended particle subspace, including the right-handed (sterile) neutrino. We express the generators of $C\ell_4$ and $C\ell_6$ in terms of fermionic oscillators $a_\alpha, a_\alpha^*, \alpha = 1, 2$ and $b_j, b_j^*, j = 1, 2, 3$ describing flavour and colour, respectively. The internal space observables belong to the Jordan subalgebra of hermitian elements of the complexified Clifford algebra $\mathbb{C} \otimes C\ell_{10}$ which commute with the weak hypercharge $\frac{1}{2}Y = \frac{1}{3}\sum_{j=1}^{3} b_j^* b_j - \frac{1}{2}\sum_{\alpha=1}^{2} a_\alpha^* a_\alpha$. We only distinguish particles from antiparticles if they have different eigenvalues of Y. Thus the sterile neutrino and antineutrino (both with $Y = 0$) are allowed to mix into Majorana neutrinos. Restricting $C\ell_{10}$ to the particle subspace, which consists of leptons with $Y < 0$ and quarks, allows a natural definition of the Higgs field Φ, the scalar of Quillen's superconnection, as an element of $C\ell_4^1$, the odd part of the first factor in $C\ell_{10}$. As an application we express the ratio $\frac{m_H}{m_W}$ of the Higgs and the W-boson masses in terms of the cosine of the *theoretical* Weinberg angle.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_3

1 Introduction

The elaboration of the Standard Model (SM) of particle physics was completed in the early 1970s. To quote John Baez [3] 50 "years trying to go beyond the Standard Model hasn't yet led to any clear success". The present paper belongs to an equally long albeit less fashionable effort to clarify the algebraic (or geometric) roots of the SM, more specifically, to find a natural framework featuring its internal space properties. After discussing some old ideas motivating our approach among others, we review some recent developments, clarifying on the way the role of different projection operators, expressed in terms of Clifford algebra pseudoscalars and their interrelations.

Most ideas on the natural framework of the SM originate in the 1970s, the first decade of its existence. (Two exceptions: the Jordan algebras were introduced and classified in the 1930s [41, 42]; the noncommutative geometry approach originated in the late 1980s, [15, 16, 25] and is still vigorously developed by Connes and collaborators [11–13, 49].)

First, early in 1973, the ultimate division algebra, the octonions[1] were introduced by Gürsey[2] and his student Günaydin [37, 38] for the description of quarks and their $SU(3)$ colour symmetry. The idea was taken up and extended to incorporate all four division algebras by G. Dixon (see [17, 18] and earlier work cited there) and is further developed by Furey [29–35]. Dubois-Violette (D-V) arrives at the octonions via the quark-lepton symmetry and the unimodularity of the colour group [23]. Thus, the octonions appear with an additional complex structure,

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3 , \tag{1}$$

preserved by the subgroup $SU(3)$ of the automorphism group G_2 of $\mathbb{O}$.

1.1 Octonions as a Composition Algebra. the Cayley-Dickson Construction

One can in fact provide a basis free definition of the octonions starting with the splitting (1). To this end one uses the skew symmetric vector product and the standard inner product on $\mathbb{C}^3$ to define a noncommutative and non-associative distributive product xy on $\mathbb{O}$ and a real valued nondegenerate symmetric bilinear form $\langle x, y\rangle = \langle y, x\rangle$ such that the quadratic norm $N(x) = \langle x, x\rangle$ is multiplicative:

$$N(xy) = N(x)N(y) \quad \text{for} \quad N(x) = \langle x, x\rangle \tag{2}$$

[1] For a pleasant to read review of octonions, their history and applications—see [1].

[2] See Witten's eloquent characterization of his personality and work in the Wikipedia entry on Feza Gürsey (1921–1991).

(cf. [23, 61]). Furthermore, defining the real part of $x \in \mathbb{O}$ by $\mathrm{Re}\, x = \langle x, 1\rangle$ and the octonionic conjugation $x \to x^* = 2\langle x, 1\rangle - x$, we shall have

$$xx^* = N(x)\mathbb{1} \Leftrightarrow x^2 - 2\langle x, 1\rangle x + N(x)\mathbb{1} = 0\,. \tag{3}$$

A unital algebra with a non-degenerate quadratic norm obeying (2) is called a *composition algebra*.

Another basis free definition of the octonions $\mathbb{O}$ and of their split version $\widetilde{\mathbb{O}}$ can be given in terms of quaternions by the Cayley-Dickson construction. We represent the quaternion as scalars plus vectors

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3,\ x = u + U,\ y = v + V,\ u, v \in \mathbb{R},\ U, V \in \mathbb{R}^3,$$

$$xy = uv - \langle U, V\rangle + uV + Uv + U \times V \tag{4}$$

with the vector product $U \times V \in \mathbb{R}^3$ satisfying

$$U \times V = -V \times U,\ (U \times V) \times W = \langle U, W\rangle V - \langle V, W\rangle U\,. \tag{5}$$

The product (4) is clearly noncommutative but one verifies that it is associative. The Cayley-Dickson construction defines the octonions $\mathbb{O}$ and the split octonions $\widetilde{\mathbb{O}}$ in terms of a pair of quaternions and a new "imaginary unit" ℓ as:

$$x = u + U + \ell(v + V),\ \ell(v + V) = (v - V)\ell\,,$$

$$\ell^2 = \begin{cases} -1 \Rightarrow & x \in \mathbb{O} \\ \ \ 1 \Rightarrow & x \in \widetilde{\mathbb{O}}\,. \end{cases} \tag{6}$$

1.2 Jordan Algebras; Grand Unified Theories; Clifford Algebras

D-V suggests that classical observables (real valued functions) are replaced by an algebra of functions on space-time with values in a *finite dimensional euclidean Jordan algebra*.[3] As a particularly attractive choice, which incorporates the idea of quark-lepton symmetry, D-V proposes [23] the exceptional Jordan algebra of 3×3 hermitian matrices with octonionic entries,

$$J_3^8 = \mathcal{H}_3(\mathbb{O})\,. \tag{7}$$

This approach is further pursued in [26, 27, 59, 61, 62].

[3] These algebras are defined and classified in [42]; for concise reviews see Sect. 3.2 in [23] and Sect. 2 of [59].

A second development, *Grand Unified theory* (GUT), anticipated during the same 1973 by Pati and Salam [50], became for a time mainstream.[4] Fundamental chiral fermions fit the complex spinor representation of $Spin(10)$, introduced as a GUT group by Fritzsch and Minkowski and by Georgi. A preferred symmetry breaking yields the maximal rank semisimple Pati-Salam subgroup,

$$G_{\mathrm{PS}} = \frac{Spin(4) \times Spin(6)}{\mathbb{Z}_2} \subset Spin(10),$$

$$Spin(4) = SU(2)_L \times SU(2)_R, \; Spin(6) = SU(4) . \tag{8}$$

We note that G_{PS} is the only GUT group which does not predict a gauge triggered proton decay. It is also encountered in the noncommutative geometry approach to the SM [8, 12]. In general, GUTs provide a nice home for the fundamental fermions, as displayed by the two 16-dimensional complex conjugate "Weyl spinors" of $Spin(10)$. Their other representations, however, like the 45-dimensional adjoint representation of $Spin(10)$ are much too big, involve unobserved beasts like leptoquarks which cause difficulties.

A central role in our approach will be given to the Clifford algebra[5] $C\ell_{10}$, viewed as a $\mathbb{Z}_2$-graded tensor product [31, 32, 60]:

$$C\ell_{10} = C\ell_4 \widehat{\otimes} C\ell_6 . \tag{9}$$

The complexified Clifford algebra has a single faithful irreducible representation (IR) of dimension $2^5 = 32$ which fits precisely the fundamental (anti)fermions of one generation. Clifford algebras were also applied to the SM in the 1970s—see [10] and references therein. There are two new points in our approach.

(1) We use the presence of the octonions with a preferred complex structure in $C\ell_{8+\nu}$, $\nu = 0, 1, 2$ to derive the gauge group of the SM (for $C\ell_9$),

$$G_{\mathrm{SM}} = S(U(2) \times U(3)) \tag{10}$$

and its left right symmetric extension (for $C\ell_{10}$) [7] (see also the talks of Baez [3], Krasnov [44] and L. Boyle at the Perimeter Institute Workshop, as well as [34, 35, 60]). One relies, in particular, on the nonassociativity of the octonions (as emphasized in [43]) which implies noncommutativity of left and right multiplication L_x, R_y $(x, y \in \mathbb{O})$.

(2) We make essential use of the $\mathbb{Z}_2$ grading of the Clifford algebra. The Higgs field, which intertwines left and right chiral fermions, belongs to the odd part of the factor $C\ell_4$ in (9) [27, 60]. This fits perfectly the *super-connection* approach to the SM, pioneered by Ne'eman [48] and Fairlie [28] well before the notion was coined (and named) by mathematicians [46, 51].

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[4] For an enlightening review of the algebra of GUTs and some 40 references see [4].

[5] Aptly called *geometric algebra* by its inventor—see [19].

Octonions by themselves are not fitted to describe observables. Their Jordan subalgebra of hermitian elements consists just of the real numbers. They do enter however the Jordan *spin factors* J_2^ν of degree $\nu \geq 7$ whose associative envelopes are $C\ell_{\nu+1}$ (as well as the exceptional Jordan algebra (7)):

$$J_2^\nu \subset C\ell'_{\nu+1}(\nu = 7, 8, 9, \ldots),\ \dim(J_2^\nu) = \nu + 2,\ J_2^8 \subset J_3^8 . \tag{11}$$

As already noted, for $\nu = 8, 9$ their Clifford envelopes may describe the internal space observables of one generation of fundamental fermions. It will be recalled in Sect. 3 that the gauge group of the SM (10) is recovered by considering the restriction of J_3^8 to J_2^8. More precisely, G_{SM} appears as the intersection of two subgroups of the automorphism group F_4 of J_3^8: the centralizer F_4^ω of $\omega \in SU(3)_c \subset F_4$, $\omega^2 + \omega + 1 = 0$ and $Spin(9)$, the stabilizer of J_2^8, the subalgebra of 3×3 matrices in J_3^8 with zero first row and first column:

$$G_{\mathrm{SM}} = F_4^\omega \cap Spin(9) \subset F_4 , \tag{12}$$

$$F_4^\omega = \frac{SU(3)_c \times SU(3)}{\mathbb{Z}_3},\ \omega(z + Z) = z + exp\left(\frac{2\pi i}{3}\right) Z,\ z \in \mathbb{C},\ Z \in \mathbb{C}^3 . \tag{13}$$

($x = z + Z$ being a realization of the splitting (1), [62].) We shall see, however, that the representation of G_{SM}, obtained by restriction from $Spin(9)$ only involves $SU(2)_L$-doublets, it has no room for e_R, u_R, d_R. This is, in fact, a manifestation of a general result (see, e.g. [14], Proposition 15.2 (p. 674)): the only simple compact gauge groups allowing to accommodate chiral fermions are $SU(n)$, $n \geq 3$, $Spin(4n + 2)$ and E_6.

2 Triality Realization of $Spin(8)$; $C\ell_{-6}$

2.1 *The Action of Octonions on Themselves. $Spin(8)$ as a Subgroup of $SO(8) \times SO(8) \times SO(8)$*

The group $Spin(8)$, the double cover of the orthogonal group $SO(8) = SO(\mathbb{O})$, can be defined (see [9, 64]) as the set of triples $(g_1, g_2, g_3) \in SO(8) \times SO(8) \times SO(8)$ such that

$$g_2(xy) = g_1(x)\, g_3(y) \text{ for any } x, y \in \mathbb{O} . \tag{14}$$

If u is a unit octonion, $u^*u = 1$, then the left and right multiplications by u are examples of isometries of $\mathbb{O}$

$$|L_u\, x|^2 = \langle ux, ux\rangle = \langle x, x\rangle,\ |R_u\, x|^2 = \langle xu, xu\rangle = \langle x, x\rangle \text{ for } \langle u, u\rangle = 1 . \tag{15}$$

Using the *Moufang identity*[6]

$$u(xy)u = (ux)(yu) \text{ for any } x, y, u \in \mathbb{O}\,, \tag{16}$$

one verifies that the triple $g_1 = L_u$, $g_2 = L_u R_u$, $g_3 = R_u$ satisfies (1) and hence belongs to $Spin(8)$. It turns out that triples of this type generate $Spin(8)$ (see [9] or Yokota's book [64] for more details).

The mappings $x \to L_x$ and $x \to R_x$ are, of course, not algebra homomorphisms as L_x and R_y generate each an associative algebra while the algebra of octonions is non-associative. They do preserve, however, the quadratic relation $xy^* + yx^* = 2\langle x, y\rangle 1\!\!\mathrm{I}$:

$$L_x L_{y^*} + L_y L_{x^*} = 2\langle x, y\rangle 1\!\!\mathrm{I} = R_x R_{y^*} + R_y R_{x^*}\,. \tag{17}$$

Equation (17), applied to the span of the first six imaginary octonion units e_j, $j = 1, \dots, 6$, setting $L_{e_j} =: L_j$, $R_{e_j} =: R_j$ becomes the defining relation of the Clifford algebra $C\ell_{-6}$:

$$L_j L_k + L_k L_j = -2\delta_{jk} = R_j R_k + R_k R_j,\ \ j, k = 1, \dots, 6\,. \tag{18}$$

In general, $L_x L_y \neq L_{xy}$ (and similarly for R), but remarkably, as noted in [31], the relation $(e_1(e_2(e_3(e_4(e_5(e_6\, e_a)))))) = e_7\, e_a$ is satisfied for all $a = 1, \dots, 8$, so that

$$L_1 L_2 \cdots L_6 = L_{e_7} =: L_7,\ \ R_1 R_2 \cdots R_6 = R_{e_7} =: R_7\,. \tag{19}$$

While $L_a R_a = R_a L_a$ (for $a \in \mathbb{O}$) the non-associativity of the algebra of octonions is reflected in the fact that for $x \neq y$, L_x and R_y, in general, do not commute.

2.2 $C\ell_{-6}$ as a Generating Algebra of $\mathbb{O}$ and of $so(\mathbb{O})$

The Lie algebra $so(8)$ is spanned by the elements of negative square of $C\ell_{-6}$. If we denote the exterior algebra on the span of $L_1, \dots, L_6$ by

$$\Lambda^* \equiv \Lambda^* C\ell_{-6} = \Lambda^0 + \Lambda^1 + \cdots + \Lambda^6 \left(\Lambda^1 = \underset{1\le j\le 6}{\mathrm{Span}}\, L_j,\ \ \Lambda^6 = \{\mathbb{R}\, L_7\}\right)$$

then $so(8) = \Lambda^1 + \Lambda^2 + \Lambda^5 + \Lambda^6$. A basis of the Lie algebra, given by

$$L_{\alpha 8} = \tfrac{1}{2}\, L_\alpha,\ \ L_{\alpha\beta} = -\tfrac{1}{4}\,[L_\alpha, L_\beta],\ \ \alpha, \beta = 1, \dots, 7 \tag{20}$$

obeys the standard commutation relations (CRs)

[6] See [54] for a reader friendly review of Moufang loops and for a glimpse of the personality of Ruth Moufang (1905–1971).

$$[L_{ab}, L_{cd}] = \delta_{bc}\, L_{ad} - \delta_{bd}\, L_{ac} + \delta_{ad}\, L_{bc} - \delta_{ac}\, L_{bd}\,,$$
$$L_{ab} = \tfrac{1}{4}(L_a L_b^* - L_b L_a^*),\ a, b, c, d = 1, 2, \ldots, 8 \tag{21}$$

(and similarly for R_{ab}). Each element of $so(8)$ of square -1 defines a *complex structure*. (For a review of this notion in the context of Clifford algebras and spinors—see [22].) Following [35] we shall single out the *Clifford pseudoscalars* L_7 and R_7 (19) (called *volume forms* in the highly informative lectures [45] and Coxeter elements in [58]). We shall use the (mod 7) multiplication rules of [1] for the imaginary octonion units

$$L_i e_j (= e_i e_j) = -\delta_{ij} + f_{ijk}\, e_k,\ f_{ijk} = 1$$

$$\text{for } (i, j, k) = (1, 2, 4)(2, 3, 5)(3, 4, 6)(4, 5, 7)(5, 6, 1)(6, 7, 2)(7, 1, 3) \tag{22}$$

and f_{ijk} is fully antisymmetric within each of the above seven triples. The Clifford pseudoscalar is naturally associated with the Cartan subalgebra of $so(6)$ spanned by

$$(L_{13}, L_{26}, L_{45}) \text{ as } L_7(e_1, e_2, e_4) = (e_3, e_6, e_5)\,. \tag{23}$$

We can write

$$L_7 = 2^3 L_{13} L_{26} L_{45} \text{ (as } 2L_{13} = L_1 L_3^* = -L_1 L_3 \text{ etc.)} \tag{24}$$

The infinitesimal counterpart of (14) reads

$$T_\alpha(x, y) = (L_\alpha\, x) y + x(R_\alpha\, y) \text{ for } \alpha, x, y \in \mathbb{O},\ \alpha^* = \alpha\,,$$
$$\text{i.e.} \qquad T_\alpha = L_\alpha + R_\alpha\,. \tag{25}$$

There is an involutive outer automorphism π of the Lie algebra $so(8)$ such that

$$\pi(L_\alpha) = T_\alpha,\ \pi(R_\alpha) = -R_\alpha,\ \pi(T_\alpha) = L_\alpha\ (\pi^2 = id)\,. \tag{26}$$

As proven in Appendix A

$$\pi(L_{ab}) = E_{ab} \text{ where } E_{ab}\, e_c = \delta_{bc}\, e_a - \delta_{ac}\, e_b\ (a, b, c = 1, 2, \ldots, 8,\ e_8 = 1) \tag{27}$$

(L_{ab}), (E_{ab}) and (R_{ab}) provide three bases of $so(8)$, each obeying the CRs (21). They are expressed by each other in terms of the involution π:

$$L_{ab} = \pi(E_{ab}),\ E_{\alpha 8} = L_{\alpha 8} + R_{\alpha 8},\ \alpha = 1, \ldots, 7\,. \tag{28}$$

We find, in particular—see Appendix A:

$$L_7 = 2L_{78} = E_{78} - E_{13} - E_{26} - E_{45}\,,$$

$$R_7 = 2R_{78} = E_{78} + E_{13} + E_{26} + E_{45} = -L_{78} - L_{13} - L_{26} - L_{45} \,. \tag{29}$$

While $L_{78} = 4L_{13}L_{26}L_{45}$ (24) commutes with the entire Lie algebra $spin(6) = su(4)$ the $u(1)$ generator

$$C_1 = L_{13} + L_{26} + L_{45} \text{ centralizes } u(3) = u(1) \oplus su(3) \subset su(4) \tag{30}$$

(that is the unbroken part of the gauge Lie algebra of the SM). The reader may verify the identity $R_7^2 = -1$ for the right hand side of (29) using the relations

$$L_{jk}^2 = -\frac{1}{4}, \; C_1^2 = -\frac{3}{4} + 2C_2, \; -C_1 L_{07} = C_2 := L_{13}L_{26} + L_{13}L_{45} + L_{26}L_{45} \,. \tag{31}$$

The above relations will be useful for the study of higher Clifford and Lie algebras that involve $so(8)$ (expressed in terms of L_{ab} or R_{ab}) as a subalgebra. We shall apply them in the next section to the chain of nested Clifford algebras and their derivation (Lie) algebras

$$(C\ell_{-6} \subset)\, C\ell_8 \subset C\ell_9 \subset C\ell_{10} \leftrightarrow so(8) \subset so(9) \subset so(10) \,. \tag{32}$$

In order to accommodate the duality between antihermitian symmetry generators (of a compact gauge group) and the corresponding conserved hermitian observables within the same (internal space counterpart of) Haag's [39] field algebra we need multiplication by an imaginary unit. Thus the algebraic counterpart of Nœther's theorem (cf. [2]) requires a complexification of the algebras (32). In particular, the Cartan subalgebra of $so(8)$ singled out by the complex structure L_7 is spanned by the four commuting hermitian elements

$$2i\, L_{78}, \; 2i\, L_{j\,3j\,(\mathrm{mod}\,7)} = 2i(L_{13}, L_{26}, L_{45}) \; (j = 1, 2, 4) \tag{33}$$

of square one, where the complex imaginary unit i ($i^2 = -1$) commutes with the octonion units e_α. We shall single out the $u(3)$ Lie subalgebra of the derivation algebra $su(4) = so(6)$ that contains the colour $su(3)$ by identifying its centralizer $u(1)$ with the sum of the operator $2i\, L_{j3j}$ (33). It is a multiple of the observable

$$B - L = \frac{2i}{3}\,(L_{13} + L_{26} + L_{45}) \,, \tag{34}$$

the difference between the baryon and the lepton numbers. $B - L$ takes eigenvalues $\pm\frac{1}{3}$ for (anti)quarks and ∓ 1 for (anti)leptons so that

$$[(B - L)^2 - 1][9(B - L)^2 - 1] = 0 \,. \tag{35}$$

3 $C\ell_{10} = C\ell_4 \hat{\otimes} C\ell_6$ as Internal Space Algebra

3.1 Equivalence Class of Lorentz Like Clifford Algebras

Nature appears to select real Clifford algebras $C\ell(s,t)$ of the equivalence class of $C\ell(3,1)$ (with Lorentz signature in four dimensions) in Elie Cartan's classification[7]:

$$C\ell(s,t) = \mathbb{R}[2^n], \text{ for } s-t = 2 (\text{mod}\, 8),\ s+t = 2n\,. \tag{36}$$

They act on $2n$ dimensional *Majorana spinors* that transform irreducibly under the *real* 2^n dimensional representation of the spin group $Spin(s,t)$. If $\gamma_1, \ldots, \gamma_{2n}$ is an orthonormal basis of the underlying vector space $\mathbb{R}^{s,t}$ then the Clifford pseudoscalar defines a complex structure

$$\omega_{s,t} = \gamma_1 \cdots \gamma_{2n},\ 2n = s+t,\ \omega_{s,t}^2 = -1\,, \tag{37}$$

which commutes with the action of $Spin(s,t)$. Upon complexification the resulting *Dirac spinor* splits into two *inequivalent* 2^{n-1}-dimensional complex *Weyl* (or *chiral*) *spinor* representations irreducible *over* $\mathbb{C}$ under $Spin(s,t)$. The corresponding projectors Π_L and Π_R on left and right spinors are given in terms of the chirality χ which involves the imaginary unit i:

$$\begin{gathered}\Pi_L = \tfrac{1}{2}(1-\chi),\ \Pi_R = \tfrac{1}{2}(1+\chi),\ \chi = i\omega_{s,t}\,,\\ \chi^2 = 1\!\!\mathrm{I} \Leftrightarrow \Pi_L^2 = \Pi_L,\ \Pi_R^2 = \Pi_R,\ \Pi_L\Pi_R = 0,\ \Pi_L + \Pi_R = 1\!\!\mathrm{I}\,.\end{gathered} \tag{38}$$

Another interesting example of the same equivalence class (also with indefinite metric) is the *conformal Clifford algebra* $C\ell(4,2)$ (with isometry group $O(4,2)$). We shall demonstrate that just as $C\ell_{-6}$ was viewed (in Sect. 2) as the *Clifford algebra of the octonions*, $C\ell(4,2)$ plays the role of the *Clifford algebra of the split octonions* (cf. (6)):

$$\begin{gathered}x = v + V + \ell(w+W),\ v,w \in \mathbb{R},\ V = iV_1 + jV_2 + kV_3,\ W = iW_1 + jW_2 + kW_3\\ i^2 = j^2 = k^2 = ijk = -1,\ \ell^2 = 1,\ V\ell = -\ell V\,.\end{gathered} \tag{39}$$

Indeed, defining the mapping

$$i \to \gamma_{-1},\ j \to \gamma_0,\ \ell \to \gamma_1,\ j\ell \to \gamma_2,\ \ell k \to \gamma_3,\ \ell i \to \gamma_4$$

$$[\gamma_\mu, \gamma_\nu]_+ = 2\eta_{\mu\nu} 1\!\!\mathrm{I},\ \eta_{11} = \eta_{22} = \eta_{33} = \eta_{44} = 1 = -\eta_{-1,-1} = -\eta_{00} \tag{40}$$

[7] For any associative ring $\mathbb{K}$, in particular, for the division rings $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we denote by $\mathbb{K}[m]$ the algebra of $m \times m$ matrices with entries in $\mathbb{K}$.

we find that the missing split-octonion (originally, quaternion) imaginary unit k $(= ij = -ji)$ can be identified with the $C\ell(4,2)$ pseudoscalar:

$$\omega_{4,2} = \gamma_{-1}\,\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3\,\gamma_4 \to k,\ \omega_{4,2}^2 = -1,\ [w_{4,2}, \gamma_\nu]_+ = 0\,. \tag{41}$$

The conjugate to the split octonion x (39) and its norm are

$$x^* = v - V - \ell(w + W),\ N(x) = xx^* = v^2 + V^2 - w^2 - W^2$$

so that the isometry group of $\widetilde{\mathbb{O}}$ is $O(4,4)$.

As we are interested in the geometry of the internal space of the SM, acted upon by a compact gauge group we shall work with (positive or negative) definite Clifford algebras $C\ell_{2\ell}$, $\ell = 1(\mathrm{mod}\,4)$. The algebra $C\ell_{-6}$, considered in Sect. 2, belongs to this family (with $\ell = -3$). For $\ell = 1$ we obtain the Clifford algebra of 2-dimensional conformal field theory; the 1-dimensional Weyl spinors correspond to analytic and antianalytic functions. Here we shall argue that for the next allowed value, $\ell = 5$, the algebra $C\ell_{10} = C\ell_4 \,\widehat{\otimes}\, C\ell_6$ (9), fits beautifully the internal space of the SM, if we associate the two factors to colour and flavour degrees of freedom, respectively. We shall strongly restrict the physical interpretation of the generators $\gamma_{ab}\,\big(= \frac{1}{2}\,[\gamma_a, \gamma_b],\ a, b = 1, \ldots, 10\big)$ of the derivations of $C\ell_{10}$ by demanding that the splitting (9) of $C\ell_{10}$ into $C\ell_4$ and $C\ell_6$ is preserved. This reflects the demand of preserving the lepton-quark splitting (1) and amounts to select a first step of symmetry breakings of the GUT group $Spin(10)$ leading to the semisimple Pati-Salam group $(Spin(4) \times Spin(6))/\mathbb{Z}_2$ (8). Furthermore, recalling the discussion of Sect. 2, we identify the first seven γ_α with multiples of the left imaginary units L_α.

3.2 Realization in Terms of Fermi Oscillators

We start with a basis of γ-matrices adapted to the chain of subalgebras (32):

$$\gamma_\alpha = \sigma_0 \otimes \epsilon \otimes L_\alpha,\ \sigma_0 = 1\!\mathrm{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \epsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \alpha = 1, \ldots, 7,$$
$$\gamma_8 = \sigma_0 \otimes \sigma_1 \otimes 1\!\mathrm{I}_8,\ \gamma_9 = \sigma_2 \otimes \sigma_3 \otimes 1\!\mathrm{I}_8,\ \gamma_{10} = \sigma_1 \otimes \sigma_3 \otimes 1\!\mathrm{I}_8, \tag{42}$$

σ_k being the 2×2 hermitian Pauli matrices. The internal space algebra $C\ell_4 \,\widehat{\otimes}\, C\ell_6$ is most suggestively expressed in terms of Fermi oscillators [F] setting (in the notation of [60]):

$$\tfrac{1}{2}\,(\gamma_1 + i\gamma_3) = b_1,\ (\tfrac{1}{2}(\gamma_1 - i\gamma_3) = b_1^*)\,,$$
$$\tfrac{1}{2}\,(\gamma_2 + i\gamma_6) = b_2,\ \tfrac{1}{2}(\gamma_4 + i\gamma_5) = b_3$$

$$\Longrightarrow i\gamma_{13} = [b_1^*, b_1],\ i\gamma_{26} = [b_2^*, b_2],\ i\gamma_{45} = [b_3^*, b_3]\ \Big(\gamma_{jk} = \frac{1}{2}[\gamma_j, \gamma_k]\Big)\,; \tag{43}$$

$$\gamma_7 = a_2 + a_2^*,\ i\gamma_8 = a_2 - a_2^*\,;\ \gamma_9 = a_1 + a_1^*,\ i\gamma_{10} = a_1 - a_1^*\,;$$
$$[a_\alpha,\ a_\beta^*]_+ = \delta_{\alpha\beta},\ [b_j, b_k^*]_+ = \delta_{jk},\ [a_\alpha^{(*)}, b_j^{(*)}]_+ = 0\,. \tag{44}$$

We shall use five pairs of commuting orthogonal projections:

$$\pi_\alpha = a_\alpha a_\alpha^*,\ \pi'_\alpha = a_\alpha^* a_\alpha = 1 - \pi_\alpha,\ \alpha = 1, 2;\ p_j = b_j b_j^* = 1 - p'_j,\ j = 1, 2, 3, \tag{45}$$

α (= 1, 2) and j (= 1, 2, 3) playing the role (roughly) of flavour and colour indices, respectively. In fact, the weak hypercharge Y involves both:

$$\frac{1}{2}Y = \frac{1}{3}\sum_{j=1}^{3} b_j^* b_j - \frac{1}{2}\sum_{\alpha=1}^{2} a_\alpha^* a_\alpha = \frac{1}{3}(p'_1 + p'_2 + p'_3) - \frac{1}{2}(\pi'_1 + \pi'_2) =$$
$$= \frac{1}{2}(\pi_1 + \pi_2) - \frac{1}{3}(p_1 + p_2 + p_3)\,. \tag{46}$$

The left and right chiral (weak) isospin components are expressed entirely in terms of $a_\alpha^{(*)}$:

$$I_+^L = a_1^*\, a_2,\ I_-^L = a_2^*\, a_1,\ [I_+^L, I_-^L] = 2I_3^L = \pi'_1\pi_2 - \pi_1\pi'_2 = \pi'_1 - \pi'_2\,;$$

$$I_+^R = a_2\, a_1,\ I_-^R = a_1^*\, a_2^*,\ [I_+^R, I_-^R] = 2I_3^R = \pi_1\pi_2 - \pi'_1\pi'_2 = \pi_2 - \pi'_1\,. \tag{47}$$

We note that the projection on non-zero left and right isospin are mutually orthogonal:

$$P_1 := (2I_3^L)^2 = \pi'_1\pi_2 + \pi_1\pi'_2 (= P_1^2),\ P'_1 := (2I_3^R)^2 = \pi_1\pi_2 + \pi'_1\pi'_2 (= (P'_1)^2)\,,$$
$$P_1 P'_1 = 0,\ P_1 + P'_1 = 1\,. \tag{48}$$

The generators of $su(3)_c$, on the other hand, are written in terms of $b_j^{(*)}$:

$$T_a = \tfrac{1}{2} b^* \lambda_a\, b,\ \lambda_a \in \mathcal{H}_3(\mathbb{C}),\ \mathrm{tr}\,\lambda = 0,\ \mathrm{tr}\,\lambda_a\,\lambda_b = 2\delta_{ab},\ a, b = 1, \ldots, 8\,. \tag{49}$$

The $u(1)$ generator (corresponding to C_1 (30)) is a multiple of $B - L$ (34)

$$B - L = \frac{i}{3}(\gamma_{13} + \gamma_{26} + \gamma_{45}) = \frac{1}{3}\sum_{j=1}^{3}[b_j^*, b_j] = \frac{1}{3}\sum_j (p'_j - p_j)\,. \tag{50}$$

The states of the fundamental (anti)fermions are given by the primitive idempotents of $C\ell_{10}$, represented by the $2^5 = 32$ different products of the five pairs of basic projectors $\pi_\alpha^{(')}$, $p_j^{(')}$ (45). All but two of them are labelled by the eigenvalues of the weak hypercharge $Y = B - L + 2I_3^R$ (46) and the electric charge

$$Q = \frac{1}{2}Y + I_3^L = \frac{1}{3}\sum_{j=1}^{3} b_j^* b_j - a_2^* a_2 = \frac{1}{3}(p_1' + p_2' + p_3') - \pi_2' . \quad (51)$$

Setting $|Q, Y\rangle$ and $\langle Q, Y|$ for the corresponding ket and bra vectors we find:

$$\begin{aligned}
&(\nu_L) = \ell\, \pi_1' \pi_2 = |0, -1\rangle\langle 0, -1| = |\nu_L\rangle\langle\nu_L| ,\\
&(e_L) = \ell\, \pi_1 \pi_2' = |-1, -1\rangle\langle -1, -1| = |e_L\rangle\langle e_L| , \ \ell := p_1\, p_2\, p_3 ;\\
&(e_R) = \ell\, \pi_1' \pi_2' = |-1, -2\rangle\langle -1, -2| = |e_R\rangle\langle e_R| ;
\end{aligned} \quad (52)$$

$$\begin{aligned}
&(u_L^j) = q_j\, \pi_1' \pi_2 = |\tfrac{2}{3}, \tfrac{1}{3}\rangle\langle\tfrac{2}{3}, \tfrac{1}{3}| = |u_L^j\rangle\langle u_L^j| , \ q_1 = p_1\, p_2'\, p_3' (= p_1\, p_3'\, p_2') \text{ etc.}\\
&(d_L^j) = q_j\, \pi_1 \pi_2' = |-\tfrac{1}{3}, \tfrac{1}{3}\rangle\langle -\tfrac{1}{3}, \tfrac{1}{3}| = |d_L^j\rangle\langle d_L^j| ; \ j = 1, 2, 3,\\
&(u_R^j) = q_j\, \pi_1 \pi_2 = |\tfrac{2}{3}, \tfrac{4}{3}\rangle\langle\tfrac{2}{3}, \tfrac{4}{3}| = |u_R^j\rangle\langle u_R^j| ,\\
&(d_R^j) = q_j\, \pi_1' \pi_2' = |-\tfrac{1}{3}, -\tfrac{2}{3}\rangle\langle -\tfrac{1}{3}, -\tfrac{2}{3}| = |d_R^j\rangle\langle d_R^j| ,\\
&\quad q_j = p_j\, p_k'\, p_\ell' \text{ for } (j, k, \ell) \in \mathrm{Perm}(1, 2, 3) ,
\end{aligned} \quad (53)$$

where j stands for the colour label. (As the colour is unobservable we do not bother to assign to it eigenvalues of the diagonal operators $i\gamma_{13}, i\gamma_{26}, i\gamma_{45}$.)

Remark—The factorisation of the primitive idempotents (52) (53) into bra and kets include choices. We demand, following [60], that they are hermitian conjugate elements of $C\ell_{10}$, homogeneous in $a_\alpha^{(*)}$ and $b_j^{(*)}$ such that the kets corresponding to a left(right)chiral *particle* contains an odd (respectively even) number of factors. The result is:

$$\begin{aligned}
&|\nu_R\rangle = \ell\, \pi_1 \pi_2 (= \langle\nu_R| = (\nu_R)) , \ |\nu_L\rangle = a_1^* |\nu_R\rangle = a_1^* \pi_2\, \ell ,\\
&|e_L\rangle = I_-^L |\nu_L\rangle = \pi_1 a_2^*\, \ell , \ |e_R\rangle = a_1^* |e_L\rangle = a_1^* a_2^*\, \ell ;\\
&|d_L^j\rangle = \pi_1 a_2^*\, q_j , \ |u_L^j\rangle = I_+^L |d_L^j\rangle = a_1^* \pi_2\, q_j ,\\
&|d_R^j\rangle = a_1^* |d_L^j\rangle = a_1^* a_2^*\, q_j , \ u_R^j = a_1 |u_L^j\rangle = \pi_1 \pi_2\, q_j ,
\end{aligned} \quad (54)$$

$q_j = p_j\, p_k'\, p_\ell'$, $j, k, \ell \in \mathrm{Perm}(1, 2, 3)$, i.e. $q_1 = p_1\, p_2'\, p_3' = p_1\, p_3'\, p_2'$ etc. We note that all above kets as well as all primitive idempotents (53) obey a system of 5 equations (specific for each particle), $a_\alpha|\nu_R\rangle = 0 = b_j|\nu_R\rangle$, $a_1^*|\nu_L\rangle = a_2|\nu_L\rangle = 0 = b_j|\nu_L\rangle$, $\alpha = 1, 2$, $j = 1, 2, 3$, etc. so that they are minimal right ideals in accord with the philosophy of Furey [31].

The exceptional pair consists of the right handed sterile neutrino ν_R and its antiparticle $\overline{\nu}_L$, both with $Q = 0 = Y$. They could be distinguished by introducing a third quantum number, I_3^R or $B - L$,

$$2I_3^R = L - B \ (= 1 \text{ for } \nu_R \text{ and } -1 \text{ for } \overline{\nu}_L).$$

It is argued in [60] that, if the generator of the centre $\frac{1}{2}Y$ (46) of the gauge Lie algebra of the SM is superselected, [63], chiral particles and antiparticles are mandatory separated iff $Y \neq 0$. The sterile neutrino and its antiparticle (both with $Y = 0$) can mix (as they do in the popular theory of neutrino oscillations) into a Majorana neutrino. We shall return to the implications of this assumption in Sect. 4 below. Here we shall stay with the majority's convention and include the right handed (sterile) neutrino ν_R, such that

$$(2I_3^R - 1)|\nu_R\rangle = 0\,(= Y|\nu_R\rangle = Q|\nu_R\rangle)\,, \tag{55}$$

in the list of 16 particle states. The corresponding list of antiparticle projectors is obtained by exchanging primed and unprimed π_α and p_j, reversing the signs of Q, Y (and I_3^R) and exchanging left and right. The sum of four flavours (52) and (55) of leptons and (53) of quarks gives the 4-dimensional projector ℓ on leptons and the 12 dimensional projector q on coloured quarks:

$$\ell = (\nu_L) + (e_L) + (\nu_R) + (e_R) = p_1\, p_2\, p_3\,,\ \ell^2 = \ell,\ \mathrm{tr}\,\ell = 4\,; \tag{56}$$

$$q_j = (u_L^j) + (d_L^j) + (u_R^j) + (d_R^j) = p_j\, p_k'\, p_\ell',\ q_i\, q_j = \delta_{ij} q_j,\ \mathrm{tr}\, q_j = 4\,;$$
$$(j, k, \ell) \in \mathrm{Perm}(1, 2, 3),\ q = q_1 + q_2 + q_3 = q^2,\ \mathrm{tr}\, q = 12\,. \tag{57}$$

3.3 Expressing the $C\ell_6$ Pseudoscalar in Terms of (anti)particle Projectors

We now proceed to displaying a remarkable relation between the total particle and antiparticle projectors

$$\mathcal{P} = \ell + q,\ \mathcal{P}' = \ell' + q' \quad \mathcal{P}^{(\prime)2} = \mathcal{P}^{(\prime)}, \mathcal{P}\mathcal{P}' = 0,\ \mathcal{P} + \mathcal{P}' = 1\!\!1_{32}$$
$$\ell' = p_1'\, p_2'\, p_3'\,,\ q' = p_1'\, p_2\, p_3 + p_1\, p_2'\, p_3 + p_1\, p_2\, p_3'\,, \tag{58}$$

and the $C\ell_6$ counterpart of the complex structure L_7 (24), proposed as a first step in the sequence of symmetry breakings of the $Spin(10)$ GUT in [35].

We define the $C\ell_6$ pseudoscalar in the graded tensor product (9) by

$$\omega_6 = \gamma_1\gamma_2 \cdots \gamma_6 = -\gamma_{13}\,\gamma_{26}\,\gamma_{45} = \sigma_0 \otimes \epsilon^6 \otimes L_7 = -1\!\!1_4 \otimes L_7$$
$$\gamma_{jk} = \tfrac{1}{2}[\gamma_j, \gamma_k]\,,\ L_7 = L_1 \cdots L_6\,, \tag{59}$$

implying (in view of (43))

$$i\omega_6 = (p_1' - p_1)(p_2' - p_2)(p_3' - p_3) = \mathcal{P}' - \mathcal{P}((\mathcal{P}' - \mathcal{P})^2 = \mathcal{P}' + \mathcal{P} = 1\!\!1_{32})\,. \tag{60}$$

We thus find that the $C\ell_6$ pseudoscalar complex structure ω_6 gives rise to the projector

$$\mathcal{P} = \frac{1 - i\omega_6}{2} \quad (\mathcal{P}^2 = \mathcal{P},\ tr\,\mathcal{P} = 16) \tag{61}$$

on the particle subspace, invariant under the Pati-Salam group $G_{\rm PS}$ (8), which preserves the splitting (9).

If we omit the first factor σ_0 (the 2×2 unit matrix) from γ_a for $a = 1, \ldots, 8$, (37), we obtain an irreducible representation of $C\ell_8$. We keep the same Fermi oscillator realization (43) for the $C\ell_8$ γ-matrices, so that, in particular

$$i\gamma_{13} = [b_1^*, b_1] = p_1' - p_1,\ i\gamma_{26} = [b_2^*, b_2] = p_2' - p_2,\ i\gamma_{45} = [b_3^*, b_3] = p_3' - p_3\,. \tag{62}$$

Thus $i\omega_6$ is given by the same expression (55) for $C\ell_8$ (but with $tr\,\mathcal{P} = 8$) and for $C\ell_9$ but has a smaller invariance Lie algebra

$$u(4) = su(4) \oplus u(1) \subset so(8) \text{ for } C\ell_8\,; \quad su(4) \oplus su(2) \subset so(9) \text{ for } C\ell_9\,. \tag{63}$$

Inspired by [35, 44] we shall display in both cases the complex structure given by the Clifford pseudoscalar corresponding to the right action of the octonions:

$$\omega_6^R = \gamma_1^R \cdots \gamma_6^R \text{ for } \gamma_\alpha^R = \epsilon \otimes R_\alpha \quad \alpha = 1, \ldots, 7\,. \tag{64}$$

We shall view, following [35], its invariance group, $G_{\rm LR}$, as the second of the nested subgroups of $Spin(10)$: $(Spin(10) \supset)\, G_{\rm PS} \supset G_{\rm LR} \cdots \supset G_{\rm SM} \cdots$ in the sequence of consecutive symmetry breakings. Written in terms of the colour projectors p_j and p_j' the hermitian pseudoscalar $i\omega_6^R$ assumes the form:

$$i\omega_6^R = \tfrac{1}{2}(\mathcal{P}' - \mathcal{P} - 3(B - L)) = \ell + q' - \ell' - q\,, \tag{65}$$

since

$$L = \ell - \ell'\,,\ 3B = q - q'\,. \tag{66}$$

While the term $\mathcal{P}' - \mathcal{P}$ (60) commutes with the entire derivation algebra $spin(6) = su(4)$ of $C\ell_6$ the centralizer of $B - L$ in $su(4)$ is $u(3)$—see Proposition A2 in Appendix A. It follows that the commutant of ω_6^R in $so(8)$ is $u(3) \oplus u(1)$ while its centralizer in $so(9)$ is the gauge Lie algebra $\mathcal{G}_{\rm SM} = su(3) + su(2) + u(1)$ of the SM; finally, in $so(10)$, ω_6^R is invariant under the left-right symmetric extension of $\mathcal{G}_{\rm SM}$:

$$\mathcal{G}_{\rm LR} = su(3)_c \oplus su(2)_L \oplus su(2)_R \oplus u(1)_{B-L}\,. \tag{67}$$

Furthermore, as proven in [43], the subgroup of $Spin(9)$ that leaves ω_6^R invariant is precisely the gauge group[8] $G_{\text{SM}} = S(U(2) \times U(3))$ (10) of the SM (with the appropriate $\mathbb{Z}_6$ factored out). One is then tempted to assume that $C\ell_9$, the associative envelope of the Jordan algebra $J_2^8 = \mathcal{H}_2(\mathbb{O})$, may play the role of the internal algebra of the SM, corresponding to one generation of fundamental fermions, with $Spin(9)$ as a GUT group [26, 61]. We shall demonstrate that although G_{SM} appears as a subgroup of $Spin(9)$ its representation, obtained by restricting the (unique) spinor irreducible representation (IR) **16** of $Spin(9)$ to $S(U(2) \times U(3))$ only involves $SU(2)$ doublets, so it has no room for (e_R), (u_R), (d_R) (52) (53). We shall see how this comes about when restricting the realization (47) of $\mathbf{I}^L$ and $\mathbf{I}^R$ to $Spin(9) \subset C\ell_9$. It is clear from (44) that only the sum $a_1 + a_1^* = \gamma_9$ (not a_1 and a_1^* separately) belongs to $C\ell_9$. So the $su(2)$ subalgebra of $spin(9)$ corresponds to the diagonal embedding $su(2) \hookrightarrow su(2)_L \oplus su(2)_R$:

$$I_+ = I_+^L + I_+^R = (a_1^* + a_1)\, a_2 = \gamma_9\, a_2,\;\; I_- = I_-^L + I_-^R = a_2^* \gamma_9$$
$$2I_3 = 2I_3^L + 2I_3^R = [a_2, a_2^*] = \pi_2 - \pi_2' \,. \tag{68}$$

In other words the spinorial IR **16** of $Spin(9)$ is an eigensubspace of the projector $P_1 = (2I_3^L)^2$. It consists of four $SU(2)_L$ particle doublets and of their right chiral antiparticles. More generally, as recalled in the introduction the only simple orthogonal groups with a pair of inequivalent complex conjugate fundamental IRs, are $Spin(4n+2)$. They include $Spin(10)$ but not $Spin(9)$.

There is one more pseudoscalar, ω_4, associated with the first factor, $C\ell_4$, of the tensor product (9):

$$\omega_4 = \gamma_7\, \gamma_8\, \gamma_9\, \gamma_{10} = [a_1, a_1^*][a_2^*, a_2] = P_1 - P_1' \,, \tag{69}$$

$P_1 = \pi_1' \pi_2 + \pi_1 \pi_2'$ is the projector (48) on the subspace with $(2I_3^L)^2 = 1$ and $P_1' = \pi_1 \pi_2 + \pi_1' \pi_2'$ is its orthogonal complement. (We have $\omega_4^2 = 1$; such a ω_4 is called a pseudo complex structure.)

The $C\ell_{10}$ pseudoscalar $\omega_{10} = \omega_6\, \omega_4$ defines the ($spin(10)$ invariant) *chirality*

$$\chi = i\omega_{10} = i\omega_6\, \omega_4 = (\mathcal{P}' - \mathcal{P})(P_1 - P_1') = \Pi_R - \Pi_L \,. \tag{70}$$

It gives rise to the projector

$$\Pi_L = \frac{1-\chi}{2} = \mathcal{P} P_1 + \mathcal{P}' P_1^1 \tag{71}$$

on the left chiral particles (four $SU(2)_L$ doublets) and the 8 antiparticles (the conjugates to the eight right chiral $SU(2)_L$-singlets).

[8] As noted in the introduction the correct G_{SM} was earlier obtained as the stabilizer of the automorphism ω of order 3 (see (12), (13)).

A direct description of the IR $\mathbf{16}_L$ of $Spin(10)$ acting on $\mathbb{CH} \otimes \mathbb{CO}$ is given in [34]. (Here $\mathbb{CH}$ and $\mathbb{CO}$ are a short hand for the complexified quaternions and octonions: $\mathbb{CH} := \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$.) The right action of $\mathbb{CH}$ on elements of $\mathbb{CH} \otimes \mathbb{CO}$ which commutes with the left acting $spin(10)$, is interpreted in [34] as Lorentz ($SL(2, \mathbb{C})$) transformation of (unconstrained) 2-component Weyl spinors.

The left-right symmetric extension $\mathcal{G}_{\text{LR}}$ (67) of $\mathcal{G}_{\text{SM}}$ has a long history, starting with [47] and vividly (with an admitted bias) told in [53]. It has been recently invigorated in [20, 40]. The group G_{LR} was derived by Boyle [7] starting with the automorphism group E_6 of the complexified exceptional Jordan algebra $\mathbb{C}J_3^8$ and following the procedure of [62].

4 Particle Subspace and the Higgs Field

4.1 Particle Projection and Chirality

Theories whose field algebra is a tensor product of a Dirac spinor bundle on a space-time manifold with a finite dimensional "quantum" internal space usually encounter the problem of fermion doubling [36] (still discussed over 20 years later, [6]). It was proposed in [27] as a remedy to consider the algebra $\mathcal{P}C\ell_{10}\mathcal{P}$ where $\mathcal{P}$ is the projector (53) on the 16 dimensional particle subspace (including the hypothetical right-handed sterile neutrino). The resulting subspace is $\mathbb{Z}_2$ graded by the chirality operator separating left and right chiral *particles* (with antiparticles projected out):

$$\chi_{\mathcal{P}} = i\omega_{10}\,\mathcal{P} = \mathcal{P}(\Pi_R - \Pi_L)\,, \quad \mathcal{P}\Pi_L = \mathcal{P}P_1\,, \tag{72}$$

where P_1 (48) projects on $SU(2)_L$ doublets. The Dirac operator $\not{D} = \gamma^\mu D_\mu$ ($D_\mu = \partial_\mu + A_\mu$) anticommutes with space-time chirality $\gamma_5 = i\,\gamma^1\gamma^2\gamma^3\gamma^0$ and hence intertwines—like the Higgs field—left and right chiral spinors. This has motivated Connes and coworkers [11, 15, 16] to introduce an internal space Dirac operator in the framework of noncommutative (almost commutative) geometry that involves the Higgs field. Following the pioneering work of Ne'eman and Fairlie [28, 48], Thierry-Mieg and Ne'eman [55] developed effectively a superconnection approach to the SM, prior to its introduction (and naming) in mathematics [51]. (For later reviews and more references—see [5, 52, 56].) The Clifford algebra approach with the chirality operator $\chi_{\mathcal{P}}$ (72), developed in [27] appears to be ideally suited for a geometric interpretation of the Higgs field. (An alternative approach to internal space connection involving scalar fields is been pursued by Dubois-Violette and coworkers for over thirty years [22, 24, 25].) It turns out that there is another unanticipated benefit in introducing the projector $\mathcal{P}$: it kills odd polynomials of colour carrying Fermi operators:

$$\mathcal{P}b^{(*)}\mathcal{P} = 0\;(= \mathcal{P}C\ell_6^1\mathcal{P}) \quad \text{for} \quad \omega_6\, C\ell_6^1 = -C\ell_6^1\,\omega^6 \tag{73}$$

while projecting a_α^* into non-zero odd elements:

$$\mathcal{P}a_\alpha^{(*)}\mathcal{P} = \mathcal{P}a_\alpha^{(*)} = a_\alpha^{(*)}\mathcal{P}\,,\ [\mathcal{P}a_\alpha, \mathcal{P}a_\beta^*]_+ = \delta_{\alpha\beta}\,\mathcal{P}\,. \tag{74}$$

One may thus place the Higgs field in the odd part, $C\ell_4^1$, of the first factor $C\ell_4$ of the product (4) and hence mediate the breaking of the electroweak flavour symmetry without affecting the quark colour $SU(3)_c$ symmetry which is known to be exact. While the odd part $C\ell_6^1$ of $C\ell_6$ maps the particle subspace into its orthogonal complement the $u(3)$ generators $\frac{1}{2}[b_j^*, b_k] \in C\ell_6^0$ are projected onto non-zero elements of $C\ell_6^0$ obeying the same CRs; in particular, for (j, k, ℓ) a permutation of $(1, 2, 3)$ we have

$$\mathcal{P}\, b_j^*\, b_k\, \mathcal{P} = q_k\, b_j^*\, b_k\, q_j = b_j^*\, b_k\, p'_\ell =: B_{jk} \Rightarrow [B_{jk}, B_{k\ell}] = B_{j\ell}\,. \tag{75}$$

4.2 *The Higgs as a Scalar Part of a Superconnection*

Let D be the Yang-Mills connection 1-form of the SM,

$$D = dx^\mu(\partial_\mu + A_\mu(x))\,,$$

$$i\,A_\mu = W_\mu^+ I_+^L + W_\mu^- I_-^L + W_\mu^3 I_3^L + \frac{N}{2} Y B_\mu + G_\mu^a\, T_a\,, \tag{76}$$

where Y, $\mathbf{I}^L$ and T_a are given by (46), (47) and (49), respectively, G_μ^a is the gluon field, $\mathbf{W}_\mu$ and B_μ provide an orthonormal basis of electroweak gauge bosons. Then one defines a superconnection $\mathbb{D}$ by

$$\mathbb{D} = \chi D + \Phi\,, \quad \Phi = \sum_\alpha (\phi_\alpha\, a_\alpha^* - \overline{\phi}_\alpha\, a_\alpha)\,. \tag{77}$$

(We omit, for the time being, the projector $\mathcal{P}$ in A_μ and Φ.) The factor χ (first introduced in this context in [56]) insures the anticommutativity of Φ and $\chi\mathcal{D}$ without changing the Yang-Mills curvature $D^2 = (\chi D)^2$.

The projector $\mathcal{P}$ (58) on the 16 dimensional particle subspace that includes the hypothetical right chiral neutrino (and is implicit in (77)) was adopted in [27]. By contrast, particles are only distinguished from antiparticles in [60] if they have different quantum numbers with respect to the Lie algebra of the SM. In fact, $\mathcal{G}_{\rm SM} = s(u(2) \oplus u(3))$ is precisely the Lie subalgebra of $\mathcal{G}_{\rm LR}$ (64) which annihilates the sterile (anti)neutrino:

$$\mathcal{G}_{\rm SM} = \{\alpha \in \mathcal{G}_{\rm LR};\, \alpha(\nu_R) = 0 = \alpha(\overline{\nu}_L)(= \alpha(a_1 a_2\, b_1 b_2\, b_3 + b_3^*\, b_2^*\, b_1^* a_2^*\, a_1^*))\,. \tag{78}$$

Thus, in [60] $\mathcal{P}$ is restricted to the 15-dimensional projector $\mathcal{P}_r$ on the *restricted particle space*:

$$\mathcal{P}_r = \mathcal{P} - (\nu_R) = q + \ell_r\,, \quad \ell_r = \ell(1 - \pi_1\pi_2)\,. \tag{79}$$

The projected odd operators $a_\alpha^{(*)}$ in the lepton sector,

$$\ell_r\, a_\alpha\, \ell_r = \ell(1 - \pi_1\pi_2)\, a_\alpha,\;\; \ell_r\, a_\alpha^*\, \ell_r = \ell\, a_\alpha^*(1 - \pi_1\pi_2) \Rightarrow$$

$$\ell_r\, a_1\, \ell_r = \ell\, a_1\pi_2',\;\; \ell_r\, a_2\, \ell_r = \ell\, a_2\, \pi_1',\;\; \ell_r\, a_1^*\, \ell_r = \ell\, a_1^*\pi_2',\;\; \ell_r\, a_2^*\, \ell_r = \ell\, a_2^*\, \pi_1'\,, \tag{80}$$

have modified anticommutation relations. In fact, they provide a realization of the four odd elements of the 8-dimensional simple Lie superalgebra $s\ell(2|1)$ whose even part is the 4-dimensional Lie algebra $u(2)$ of the Weinberg-Salam model of the electroweak interactions (see [60] for details). It is precisely the Lie superalgebra proposed in 1979 independently by Ne'eman and by Fairlie [28, 48] (and denoted by them $su(2|1)$) in their attempt to unify $su(2)_L$ with $u(1)_Y$ (and explain the spectrum of the weak hypercharge). Let us stress that the representation space of $s\ell(2|1)$ consists of the observed left and right chiral leptons (rather than of bosons and fermions like in the popular speculative theories in which the superpartners are hypothetical). Note in passing that the trace of Y on negative chirality leptons (ν_L, e_L) is equal to its eigenvalue on the unique positive chirality (e_R) (equal to -2) so that only the supertrace of Y vanishes on the lepton (as well as on the quark) space. This observation is useful in the treatment of anomaly cancellation (cf. [57]).

We shall sketch the main steps in the application of the superconnection (77) to the bosonic sector of the SM emphasizing specific additional hypotheses used on the way (for a detailed treatment see [60]).

The canonical curvature form

$$\mathbb{D}^2 = D^2 + \chi[D, \Phi] + \Phi^2,\;\; [D, \Phi] = dx^\mu(\partial_\mu\Phi^*[A_\mu, \Phi]) \tag{81}$$

satisfies the *Bianchi identity*

$$\mathbb{D}\mathbb{D}^2 = \mathbb{D}^2\mathbb{D}\; (\Rightarrow \chi(d\Phi^2 + [A, \phi^2] + [\Phi, D\Phi]_+) = 0)\,, \tag{82}$$

equivalent to the (super) Jacobi identity of our Lie superalgebra. It is important that the Bianchi identity, needed for the consistency of the theory still holds if we add to $\mathbb{D}^2$ a constant matrix term with a similar structure. Without such a term the Higgs potential would be a multiple of $\mathrm{Tr}\,\Phi^4$ and would only have a trivial minimum at $\Phi = 0$ yielding no symmetry breaking. The projected form of Φ (77) and hence the admissible constant matrix addition to Φ^2 depends on whether we use the projector $\mathcal{P}$ (as in [27]) or P_r (as in [60]). In the first case we just replace $a_\alpha^{(*)}$ with $a_\alpha^{(*)}\mathcal{P}$. In the second, however, the odd generators for leptons and quarks differ and we set:

$$\Phi = \ell[(\phi_1 a_1^* - \overline{\phi}_1 a_1)\pi_1' + (\phi_2\, a_2^* - \overline{\phi}_2\, a_2)\pi_2'] + \rho q \sum_{\alpha=1}^{2} (\phi_\alpha\, a_\alpha^* - \overline{\phi}_\alpha\, a_\alpha)\,, \quad (83)$$

where ρ (like N in (76)) is a normalization constant that will be fixed later. Recalling that ℓ and q are mutually orthogonal ($\ell q = 0 = q\ell,\ \ell + q = \mathcal{P}$) we find

$$\begin{aligned} \Phi^2 = {} & \ell(\phi_1\overline{\phi}_2\, I_+^L + \overline{\Phi}_1\phi_2\, I_-^L - \phi_1\overline{\phi}_1\pi_2' - \phi_2\,\overline{\phi}_2\pi_1') \\ & - \rho^2 q(\phi_1\overline{\phi}_1 + \phi_2\,\overline{\phi}_2)\ (\phi_\alpha = \phi_\alpha(x))\,. \end{aligned} \quad (84)$$

This suggests defining the SM field strength (the extended curvature form) as

$$\mathbb{F} = i(\mathbb{D}^2 + \widehat{m}^2)\,, \quad \widehat{m}^2 = m^2(\ell(1 - \pi_1\,\pi_2) + \rho^2 q) \quad (85)$$

($\widehat{m}^2 = m^2\mathcal{P}$ for the 16 dimensional particle subspace of [27]).

4.3 Higgs Potential and Mass Formulas

This yields the bosonic Lagrangian

$$\mathcal{L}(x) = \mathrm{Tr}\left\{\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} - (\partial_\mu\Phi + [A_\mu, \Phi])(\partial^\mu\Phi + [A^\mu, \Phi])\right\} - V(\Phi) \quad (86)$$

where the Higgs potential $V(\Phi)$ is given by

$$V(\Phi) = \mathrm{Tr}\,(\widehat{m}^2 + \Phi^2)^2 - \tfrac{1}{4}m^4 = \tfrac{1}{2}(1 + 6\rho^4)(\phi\overline{\phi} - m^2)^2\,. \quad (87)$$

Minimizing $V(\Phi)$ gives the expectation value of the square of $\phi = (\phi_1, \phi_2)$:

$$\langle\phi\,\overline{\phi}\rangle = \phi_1^m\,\overline{\phi_1^m} + \phi_2^m\,\overline{\phi_2^m} = m^2,\ \text{for}\ \Phi^m = \sum_{\alpha=1}^{2}\phi_a^m a_\alpha^*(\ell\ell\pi_{3-\alpha}' + \rho q) + c\cdot c\,. \quad (88)$$

(The superscript m indicates that ϕ_α take constant in x values depending on the mass parameter m.) The mass spectrum of the gauge bosons is determined by the term $-\,\mathrm{Tr}\,[A_\mu, \Phi][A^\mu, \Phi]$ of the Lagrangian (86) with A_μ and Φ given by (76) and (88) for $\phi_\alpha = \phi_\alpha^m$. The gluon field G_μ does not contribute to the mass term as $C\ell_6^0$ commutes with $C\ell_4^1$. The resulting quadratic form is, in general, not degenerate, so it does not yield a massless photon. It does so however if we assume that Φ^m is electrically neutral (i.e. commutes with Q (51)):

$$[\Phi^m, Q] = 0 \Rightarrow \phi_2^m = 0\ (= \overline{\phi_2^m})\,. \quad (89)$$

The normalization constant $N(= \mathrm{tg}\,\theta_w)$ is fixed by assuming that $2I_3^L$ and NY are equally normalized:

$$N^2 = \frac{\mathrm{Tr}\,(2I_3^L)^2}{\mathrm{Tr}\,Y^2} = \frac{3}{5}\left(= (\mathrm{tg}\,\theta_w)^2 \Rightarrow \sin^2\theta_w = \frac{3}{8}\right). \tag{90}$$

As $Y(\nu_R) = 0 = I_3^L(\nu_R)$ this result for the "Weinberg angle at unification scale" is independent on whether we use $\mathcal{P}$ or $\mathcal{P}_r$. If one takes the trace over the leptonic subspace the result would have been $(\mathrm{tg}\,\theta_w)^2 = \frac{1}{3}(\Rightarrow \sin\theta_w = \frac{1}{2}$, [28]) closer to the measured low energy value.

Demanding, similarly, that the leptonic contribution to Φ^2 is the same as that for a coloured quark (which gives $\rho = 1$ for the unrestricted projector $\mathcal{P}$) we find

$$\rho^2 = \frac{\mathrm{Tr}(\ell(1-\pi_1\pi_2)\Phi^2)}{\mathrm{Tr}\,q_j\,\Phi^2} = \frac{Tr(\pi_1'\pi_2'\,\phi\,\overline{\phi} + \pi_1'\pi_2\,\phi_2\,\overline{\phi}_2 + \pi_1\pi_2'\,\phi_1\,\overline{\phi}_1)}{4\,\phi\,\overline{\phi}} = \frac{1}{2}\,. \tag{91}$$

The ratio $\frac{m_H^2}{m_W^2}$, on the other hand is found to be

$$\frac{m_H^2}{m_W^2} = 4\,\frac{1+6\rho^4}{1+6\rho^2} = \begin{cases} 4 \mathrm{for} \rho^2 = 1([N], [DT20]) \\ \frac{5}{2} \mathrm{for} \rho^2 = \frac{1}{2}([T21]) \end{cases} \tag{92}$$

The result of [60], much closer to the observed value, can also be written in the form $m_H^2 = 4\cos^2\theta_W\,m_H^2$, where θ_W is the theoretical Weinberg angle (90).

5 Outlook

5.1 Coming to $C\ell_{10}$

The search for an appropriate choice of a finite dimensional algebra suited to represent the internal space $\mathcal{F}$ of the SM is still going on. Our road to the choice of $C\ell_{10}$, adopted in this survey, has been convoluted.

In view of the lepton-quark correspondence which is embodied in the splitting (1) of the normed division algebra $\mathbb{O}$ of the octonions, the choice of Dubois-Violette [23] of the exceptional Jordan algebra $\mathcal{F} = \mathcal{H}_3(\mathbb{O})$ (7) looked particularly attractive. We realized [61, 62] that the simpler to work with subalgebra

$$J_2^8 = \mathcal{H}_2(\mathbb{O}) \subset \mathcal{H}_3(\mathbb{O}) = J_3^8 \tag{93}$$

corresponds to the observables of one generation of fundamental fermions. The associative envelope of J_2^8 is $C\ell_9 = \mathbb{R}[16] \oplus \mathbb{R}[16]$ with associated symmetry group $Spin(9)$. It was proven in [62] that the SM gauge group G_{SM} (10) is the intersection of $Spin(9)$ with the subgroup F_4^{ω} (13) of the automorphism group F_4 of J_3^8 that preserves the splitting (1) of $\mathbb{O}$, yielding (12).

So we were inclined to identify $Spin(9)$ as a most economic GUT group. As demonstrated in Sect. 3.3, however, the restriction of the spinor IR **16** of $Spin(9)$ to its subgroup G_{SM} gives room to only half of the fundamental fermions: the $SU(2)_L$ doublets; the right chiral singlets, e_R, u_R, d_R, are left out. It was thus recognized that the Clifford algebra $C\ell_{10}$ (which also involves the octonions) does the job.

After a synopsis of the triality realization of $Spin(8)$ on the octonions (Sect. 2) the present survey starts directly with the (complexified) Clifford algebra $C\ell_{10}$ displaying in Sect. 3.1 its salient features which place it in the same equivalence family under the Cartan classification as the Lorentzian Clifford algebra $C\ell(3,1)$. The particle interpretation of $C\ell_{10}$ is dictated by the choice of a (maximal) set of five commuting operators in the derivation algebra $so(10)$ of $C\ell_{10}$. It follows the presentation of $C\ell_{10}$ by the $\mathbb{Z}_2$ graded tensor product (9),

$$C\ell_{10} = C\ell_6 \,\widehat{\otimes}\, C\ell_4 \,, \tag{94}$$

which is preserved by the Pati-Salam subgroup G_{PS} (8) of $Spin(10)$. This led us to presenting all chiral leptons and quarks of one generation as mutually orthogonal idempotents (52) (53).

Furay [32] arrived (back in 2018) at the tensor product (94) following the $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ road. In fact, Clifford algebras have arisen as an outgrow of Grassmann algebras and the quaternions.[9] The 32 products $e_a\varepsilon_\nu (= \varepsilon_\nu e_a)$, $a = 1, \ldots, 8$ ($e_8 = 1\!\!1$), $\nu = 0, 1, 2, 3$ of octonion and quaternion units may serve as components of a $Spin(10)$ Dirac (bi)spinor, acted upon by $C\ell_{10}$ (with generators (42) involving the operators L_α of left multiplication by octonion units)—cf. [34].

5.2 Two Ways to Avoid Fermion Doubling

There are two inequivalent possibilities to avoid fermion doubling within $C\ell_{10}$. One, adopted in [27, 60] and in Sect. 3 of the present survey consists in projecting on the particle subspace, which incorporates four $SU(2)_L$ doublets and eight $SU(2)_L$ (right chiral) singlets, with projector

$$\mathcal{P} = \ell + q = \frac{1 - i\omega_6}{2}\,, \; \ell = p_1\, p_2\, p_3\,, \; q = q_1 + q_2 + q_3 \tag{95}$$

[9] The Dublin Professor of Astronomy William Rowan Hamilton (1805–1865) and the Stettin Gymnasium teacher Hermann Günter Grassmann (1809–1877) published their papers, on quaternions and on "extensive algebras", respectively, in the same year 1844. William Kingdom Clifford (1845–1879) combined the two in a "geometric algebra" in 1878, a year before his death, aged 33, referring to both of them.

(see (57), (58) and (60)). Here ω_6 is the $C\ell_6$ pseudoscalar, the distinguished complex structure, used in [35] as a first step in the "cascade of symmetry breakings". The particle projector (95) is only invariant under the Pati-Salam subgroup (8) of $Spin(10)$. The more popular alternative, adopted in [34], projects on left chiral fermions (4 particle doublets and 8 antiparticle singlets) with projector (71), defined in terms of the $C\ell_{10}$ chirality $\chi = i\omega_{10}$:

$$\Pi_L = \frac{1-\chi}{2} = \mathcal{P}P_1 + \mathcal{P}'P_1', \quad (\mathcal{P} + \mathcal{P}' = 1 = P_1 + P_1'), \tag{96}$$

where P_1 projects on $SU(2)_L$ doublets, invariant under the entire $Spin(10)$. The components of the resulting $\mathbf{16}_L$ are viewed in [34] as Weyl spinors; the right action of (complexified) quaternions (which commutes with the left $spin(10)$ action) is interpreted as an $s\ell(2,\mathbb{C})$ (Lorentz) transformation.

The difference of the two approaches which can be labeled by the projectors $\mathcal{P}$ and Π_L (on left and right particles and on left particles and antiparticles, respectively) has implications in the treatment of generalized connection (including the Higgs) and anomalies. Thus, for the Π_L (anti)leptons $(\nu_L, e_L), \overline{e}_L, \overline{\nu}_L$ we have vanishing trace of the hypercharge, $\mathrm{tr}\,\Pi_L Y = 0$. For $\mathcal{P}$ leptons, $(\nu_L, e_L), \nu_R, e_R$, the traces of the left and right chiral hypercharge are equal: $\mathrm{tr}(\mathcal{P}\Pi_L Y) = -2 = \mathrm{tr}(\mathcal{P}\Pi_R Y)$, so that, as noted in Sect. 4.2, only the supertrace vanishes in this case. The associated Lie superalgebra fits ideally Quillen's notion of super connection. A real "physical difference" only appears under the assumption that the electroweak hypercharge is superselected and the particle projector is restricted to the projector $\mathcal{P}_r$ on the 15-dimensional particle subspace (with the sterile neutrino ν_R, with vanishing hypercharge, excluded). Then the leptonic (electroweak) part of the SM is governed by the Lie superalgebra $s\ell(2|1)$, whose four odd generators are given by third degree monomials in $a_\alpha^{(*)}$, the $C\ell_4$ Fermi oscillators. The replacement of $\mathcal{P}$ by $\mathcal{P}_r$ breaks the quark-lepton symmetry: while each coloured quark q_j appears in four flavours, the colourless leptons are just three. This yields a relative normalization factor between the quark and leptonic projection of the Higgs field and allows to derive (in [60]) the relation (see (92))

$$m_H^2 = \tfrac{5}{2}\, m_W^2 = 4\cos^2\theta_{\mathrm{th}}\, m_W^2\,, \tag{97}$$

where θ_{th} is the *theoretical* Weinberg angle, such that $\mathrm{tg}^2\,\theta_W = \frac{3}{5}$. The relation (97) is satisfied within 1% accuracy by the observed Higgs and $W^{\pm}$ masses.

5.3 A Challenge

What is missing for completing the "Algebraic Design of Physics"—to quote from the title of the 1994 book by Geoffrey Dixon—is a true understanding of the *three generations* of fundamental fermions. None of the attempts in this direction [7, 23, 29, 59] has brought a clear success so far. The exceptional Jordan algebra $J_3^8 = \mathcal{H}_3(\mathbb{O})$

(7) with its built in triality was first proposed to this end in [23] (continued in [26]); in its most naive form, however, it corresponds to the triple coupling of left and right chiral spinors with a vector in internal space, rather than to three generations of fermions. As recalled in (Sect. 5.2 of) [59] any finite-dimensional unital module over $\mathcal{H}_3(\mathbb{O})$ has the (disappointingly unimaginative) form of a tensor product of $\mathcal{H}_3(\mathbb{O})$ with a finite dimensional real vector space E. It was further suggested there that the dimension of E should be divisible by 3 but the idea was not pursued any further. Boyle [7] proposed to consider the complexified exceptional Jordan algebra whose automorphism group is the compact form of E_6. This led to a promising left-right symmetric extension of the gauge group of the SM but the discussion has not yet shed new light on the 3 generation problem.

Acknowledgements Discussions with John Baez, Latham Boyle, Michel Dubois-Violette, Cohl Furey and Kirill Krasnov are gratefully acknowledged. It is a pleasure to thank Cécile Gourgues for her expert typing. The author thanks IHES for hospitality and support during the course of this work.

Appendix

Inter relations between the L, E, and R bases of $so(8)$

The imaginary octonion units $e_1, \ldots, e_7$ obey the anticommutation relations of $C\ell_{-7}$,

$$[e_\alpha, e_\beta]_+ := e_\alpha e_\beta + e_\beta e_\alpha = -2\,\delta_{\alpha\beta}\,,\ \ \alpha, \beta = 1, \ldots, 7 \tag{A.1}$$

and give rise to the seven generators $L_\alpha = L_{e_\alpha}$ of the Lie algebra $so(8)$:

$$L_{\alpha 8} := \tfrac{1}{2}\, L_\alpha =: -L_{8\alpha}\,,\ \ L_{\alpha\beta} := [L_{\alpha 8}, L_{8\beta}] \in so(7) \subset so(8)\,. \tag{A.2}$$

For $\alpha \neq \beta$ there is a unique γ such that

$$L_\alpha\, e_\beta = f_{\alpha\beta\gamma}\, e_\gamma = \pm e_\gamma\,,\ \ f_{\alpha\beta\gamma} = -f_{\beta\alpha\gamma} = f_{\gamma\alpha\beta}\,. \tag{A.3}$$

The *structure constants* $f_{\alpha\beta\gamma}$ (which only take values 0, ± 1) obey for different triples (α, β, γ) the relations

$$f_{\alpha\beta\gamma} = f_{\alpha+1\,\beta+1\,\gamma+2} = f_{2\alpha,2\beta,2\gamma} \pmod 7\,. \tag{A.4}$$

The list (22) follows from $f_{124} = 1$ and the first equation (A.4), taking into account relations like $f_{679} \equiv f_{672} \pmod 7$ etc. Note that for $f_{\alpha\beta\gamma} \neq 0$ $f_{\alpha\beta\gamma}$ are the structure constants of a (quaternionic) $su(2)$ Lie algebra. They are *not* structure constants of $so(7) \subset so(8)$.

Define the involutive outer automorphism π of the Lie algebra $so(8)$ by its action (26) on left and right multiplication L_α and R_α of octonions by imaginary octonions $\alpha = -\alpha^*$:

$$\pi(L_\alpha) = L_\alpha + R_\alpha =: T_\alpha\,,\ \pi(R_\alpha) = -R_\alpha \Rightarrow \pi(T_\alpha) = L_\alpha\,. \tag{A.5}$$

In the basis (A.1) (A.3) of imaginary octonion units e_α ($\alpha = 1, \dots, 7$), setting $e_8 = 1\!\!1$ and $L_{\alpha 8} = \frac{1}{2} L_\alpha$ (A.2), $R_{\alpha 8} = \frac{1}{2} R_\alpha = -R_{8\alpha}$, we define E_{ab} by the second relation (27)

$$E_{ab}\, e_c := \delta_{bc}\, e_a - \delta_{ac}\, e_b\,,\ a, b, c = 1, \dots, 8 \quad (e_8 = 1)\,. \tag{A.6}$$

Proposition A.1—Under the above assumptions/definitions we have

$$\pi(L_{ab}) = E_{ab} \quad (\text{for } L_{\alpha\beta} := [L_{\alpha 8}, L_{8\beta}]\,,\ L_{\alpha 8} = \tfrac{1}{2} L_\alpha = -L_{8\alpha})\,. \tag{A.7}$$

Proof—From the first equation (A.5) and from (A.1) (A.2) and (A.6) it follows that

$$E_{\alpha 8} = L_{\alpha 8} + R_{\alpha 8} = \pi(L_{\alpha 8})\,. \tag{A.8}$$

The proposition then follows from the relations

$$L_{\alpha\beta} = [L_{\alpha 8}, L_{8\beta}]\,, \quad E_{\alpha\beta} = [E_{\alpha 8}, E_{8\beta}] \tag{A.9}$$

and from the assumption that π is a Lie algebra homomorphism.

Corollary—From (A.7) and the involutive character of π it follows that, conversely,

$$\pi(E_{ab}) = L_{ab}\,. \tag{A.10}$$

To each $\alpha = 1, \dots, 7$ there correspond 3 pairs $\beta\gamma$ such that $L_{\beta\gamma}$ and $E_{\beta\gamma}$ commute with L_α and among themselves and allow to express $L_\alpha = 2L_{\alpha 8}$ in terms of $E_{\alpha 8}$ and the corresponding $E_{\beta\gamma}$:

$$\begin{aligned}
L_1 &= 2\, L_{18} = E_{18} - E_{24} - E_{37} - E_{56}\,,\\
L_2 &= 2\, L_{28} = E_{28} + E_{14} - E_{35} - E_{67}\,,\\
L_3 &= 2\, L_{38} = E_{38} + E_{17} + E_{25} - E_{46}\,,\\
L_4 &= 2\, L_{48} = E_{48} - E_{12} + E_{36} - E_{57}\,,\\
L_5 &= 2\, L_{58} = E_{58} + E_{16} - E_{23} - E_{47}\,,\\
L_6 &= 2\, L_{68} = E_{68} - E_{15} + E_{27} - E_{34}\,,\\
L_7 &= 2\, L_{78} = E_{78} - E_{13} - E_{26} - E_{45}\,,\ \text{ or } L_\alpha = E_{\alpha 8} - \sum_{\beta<\gamma} f_{\alpha\beta\gamma}\, E_{\beta\gamma}\,.
\end{aligned} \tag{A.11}$$

Recalling that $E_{ab} = \pi(L_{ab})$ (A.8) and the fact that π is involutive, so that $\pi(E_{ab}) = L_{ab}$ (A.10) we deduce, in particular,

$$2\, E_{78} = L_{78} - L_{13} - L_{26} - L_{45}\,,$$

$$R_7 = 2\,E_{78} - 2\,L_{78} = -L_{78} - L_{13} - L_{26} - L_{45}\,, \tag{A.12}$$

thus reproducing (29).

We now proceed to displaying the commutant of $i\omega_6$ and $i\omega_6^R$ in $so(7+j)$, $j = 1, 2, 3$.

Proposition A.2—While the Lie algebra $spin(6) = su(4)$ commutes with L_7, the commutant of R_7 (A.12) in $su(4) \subset s\ell(4, \mathbb{C})$ is $u(3)(\subset s\ell(4, \mathbb{C}))$ given by

$$u(3) = \left\{ \sum_{j,k=1}^{3} C_{jk}[b_j^*, b_k]\,;\ C_{jk} \in \mathbb{C}\,,\ C_{kj} = -\overline{C_{jk}} \right\} \tag{A.13}$$

in the fermionic oscillator relalization of $C\ell_6(\mathbb{C})$ (the bar over C_{jk} standing for complex conjugation).

Proof—The fact that $L_7 = 2\,L_{78}$ commutes with the generators $L_{\alpha\beta}$ $(\alpha, \beta = 1, \ldots, 6)$ of $so(6)$ follows from (21). To find the commutant of R_7 (A.12) it is convenient to use the fermionic realization of the complexification $s\ell(4, \mathbb{C})$ of $su(4)$ which is spanned by the 9 commutators $[b_j^*, b_k]$ in (A.13) and the 6 products

$$b_j\,b_k = -b_k\,b_j\,,\ b_j^*\,b_k^* = -b_k^*\,b_j^*\,,\ j, k = 1, 2, 3,\ j \neq k\,. \tag{A.14}$$

The sum $L_{13} + L_{26} + L_{45}$ in (A.12) is a multiple of $B - L$ (45), the hermitian generator of the centre of $s\ell(3, \mathbb{C})$,

$$B - L\left(= \frac{i}{3}(\gamma_{13} + \gamma_{26} + \gamma_{45})\right) = \frac{1}{3}\sum_{j=1}^{3}[b_j^*, b_j]\,. \tag{A.15}$$

The relations

$$[B - L, b_j^*\,b_k^*] = \tfrac{2}{3}\,b_j^*\,b_k^*\,,\ [B - L, b_j\,b_k] = -\tfrac{2}{3}\,b_j\,b_k\,,$$
$$\big[[B - L, [b_j^*, b_k]]\big] = 0\,,\ j, k = 1, 2, 3,\ j \neq k\,, \tag{A.16}$$

show that the commutant of $B - L$ (and hence of R_7) in $su(4)$ is $u(3)$.

Corollary—The commutant of ω_6^R in $so(8)$ is $u(3) \oplus u(1)$; the commutant of ω_6^R in $spin(9)$ is the gauge Lie algebra of the SM:

$$\mathcal{G}_{\text{SM}} = \{a \in spin(9)\,;\ [a, \omega_6^R] = 0\} = u(3) \oplus su(2)\,. \tag{A.17}$$

References

1. J.C. Baez, The octonions. Bull. Amer. Math. Soc. **39**(2), 145–205 (2002); Errata, *ibid* **42**, 213 (2005). math/0105155
2. J.C. Baez, Getting to the bottom of Noether theorem. arXiv:2006.14741 [hep-th]
3. J.C. Baez, Can we understand the standard model? talk at the Workshop "*Octonions and the Standard Model*" (Perimeter Institute, 2021)
4. J.C. Baez, J. Huerta, The algebra of grand unified theory. Bull. Am. Math. Soc. **47**(3), 483–552 (2010). (arXiv:0904.1556v2 [hep-th])
5. J.V. Beltrán, J. Monterde, J.A. Vallejo, Quillen superconnection and connections on supermanifolds. J. Geom. Phys. **86**, 180–193 (2014). arXiv:1305.3677
6. A. Bochniak, A. Sitarz, A spectral geometry for the standard model without fermion doubling. Phys. Rev. D **101**, 075038 (2020). arXiv:2001.02902; A. Bochniak, A. Sitarz, P. Zalecki, Spectral action of the electroweak θ-terms for the standard model without fermion doubling. JHEP **12**, 142 (2021). arXiv:2106.10890
7. L. Boyle, The standard model, the exceptional Jordan algebra and triality. arXiv:2006.16265 [hep-th]
8. L. Boyle, S. Farnsworth, The standard model, the Pati-Salam model, and "Jordan geometry". New J. Phys. **22**, 073023 (2020). arXiv:1910.11888v3
9. R.L. Bryant, Notes on spinors in low dimensions. arXiv:2011.05568
10. R. Casalbuoni, R. Gato, Unified theory of quarks and leptons based on Clifford algebras. Phys. Lett. B **90**, 81–86 (1979)
11. A.H. Chamseddine, A. Connes, Noncommutative geometry as a framework for unification of all fundamental interactions including gravity. Fortschr. Phys. **58**, 553–600 (2010). arXiv:1004.0464 [hep-th]
12. A.H. Chamseddine, A. Connes, W.D. van Suijlekom, Grand unification in the spectral Pati-Salam model. JHEP **1511**, 011 (2010). arXiv:1507.08161 [hep-ph]
13. A.H. Chamseddine, J. Iliopoulos, W.D. van Suijlekom, Spectral action in matrix form. Eur. Phys. J. C **80**, 1045 (2020). arXiv:2009.03367v2
14. R. Compoamor Strursberg, M.R. De Traubenberg, *Group Theory in Physics, A Practitioner's Guide* (World Scientific, 2019)
15. A. Connes, Essay on physics and noncommutative geometry, in *The Interface of Mathematics and Particle Physics*, vol. 24 (Oxford, 1988), Institute for Mathematics and its Applications Conference Series New Series, vol. 24 (Oxford University Press, 1990), pp. 9–48
16. A. Connes, J. Lott, Particle models and noncommutative geometry. Nucl. Phys. Proc. Suppl. B **18**, 29–47 (1990)
17. G.M. Dixon, Division algebras; spinors; idempotents; the algebraic structure of reality. arXiv:1012.1304 [hep-th]
18. G.M. Dixon, Seeable matter; unseeable antimatter. Comment. Math. Univ. Carolin. **55**(3) 381–386 (2014). arXiv:1407.4818 [physics.gen-ph]
19. C. Doran, A. Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, 2003)
20. D. Dunsky, L.J. Hall, K. Harigaya, Sterile neutrino dark matter and leptogenesis in left-right Higgs parity. JHEP **01**, 125 (2021). arXiv:2007.12711 [hep-ph]
21. M. Dubois-Violette, Complex structures and the Elie Cartan approach to the theory of spinors, in *Spinors, Twistors, Clifford Algebras and Quantum Deformations* (Kluwer Acad., 1993), pp. 17–23; hep-th/9210108
22. M. Dubois-Violette, Lectures on differential graded algebras and noncommutative geometry, in *Noncommutative Differential Geometry and Its Applications to Physics* (Kluwer Acad., 2001), pp. 245–306
23. M. Dubois-Violette, Exceptional quantum geometry and particle physics. Nucl. Phys. B **912**, 426–444 (2016). arXiv:1604.01247 [hep-th]
24. M. Dubois-Violette, Finite quantum geometry, octonions and the theory of fundamental particles, extended notes on talk at the Workshop, in *Octonions and the Standard Model* (Perimeter Institute, 2021)

25. M. Dubois-Violette, R. Kerner, J. Madore, Non-commutative differential geometry of matrix algebras, SLAC-PPF 88–45. J. Math. Phys. **31**, 316–322; Non-commutative differential geometry and new models of gauge theory. J. Math. Phys. **31**(1990), 323–329 (1990)
26. M. Dubois-Violette, I. Todorov, Exceptional quantum geometry and particle physics II. Nucl. Phys. B **938**, 751–761 (2019). arXiv:1808.08110
27. M. Dubois-Violette, I. Todorov, Superconnection in the spinfactor approach to particle physics. Nuclear Phys. **B957**, 115065 (2020). arXiv:2003.06591 [hep-th]
28. D. Fairlie, Higgs field and the determination of the Weinberg angle. Phys. Lett. B **82**, 97–100 (1979)
29. C. Furey, Generations: three prints, in colour. JHEP **10**, 046 (2014). arXiv:1405.4601 [hep-th]
30. C. Furey, Charge quantization from a number operator. Phys. Lett. **B742**, 195–199 (2015). arXiv:1603-04078 [hep-th]
31. C. Furey, Standard model physics from an algebra? arXiv:1611.09182 [hep-th]
32. C. Furey, $SU(3)c \times SU(2)L \times U(1)Y(\times U(1)X)$ as a symmetry of the division algebra ladder operators. Eur. Phys. J. **C78**, 375 (2018). arXiv:1806.00612 [hep-th]
33. C. Furey, Three generations, two unbroken gauge symmetries, and one eight-dimensional algebra. Phys. Lett. **B85**, 84–89 (2018). arXiv:1910-08395 [hep-th]
34. N. Furey, M.J. Hughes, One generation of standard model Weyl spinors as a single copy of $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$. Phys. Lett. B (to be published)
35. N. Furey, M.J. Hughes, Division algebraic symmetry breaking, talk at the Workshop, in *Octonions and the Standard Model* (PI, 2021)
36. J.M. Gracia-Bondia, B. Iochum, T. Schucker, The Standard Model in noncommutative geometry and fermion doubling. Phys. Lett. B **416**, 123 (1998). hep-th/9709145
37. M. Günaydin, F. Gürsey, Quark structure and octonions. J. Math. Phys. **14**(11), 1651–1667 (1973)
38. F. Gürsey, Color quarks and octonions, The Johns Hopkins Workshop on Current Problems in High Energy Theory, Octonionic structures in particle physics, Group Theoretical Methods in Physics, LNP 94. Springer **1979**, 508–521 (1974)
39. R. Haag, *Local Quantum Physics, Fields, Particles, Algebras* (Springer, Berlin, 1993)
40. L.J. Hall, K. Harigaya, Implications of Higgs discovery to the strong CP problem and unification. JHEP **10**, 130 (2018). 1803.08119
41. P. Jordan, Über Verallgemeinerungsmöglichkeiten des Formalismus der Quantenmechanik, Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. **I**(41), 209217 (1933)
42. P. Jordan, J. von Neumann, E. Wigner, On an algebraic generalization of the quantum mechanical formalism. Ann. Math. **36**, 29–64 (1934)
43. K. Krasnov, $SO(9)$ characterization of the standard model gauge group. J. Math. Phys. **62**(2), 021703 (2021). (arXiv:1912.11282v2 [hep-th])
44. K. Krasnov, $Spin(8, 9, 10)$, *Octonions and the Standard Model* (Octonions and the Standard Model, PI, Talk at the Workshop, 2021)
45. G.W. Moore, Quantum symmetries and compatible Hamiltonians. Notes Phys. **695**, (Rutgers University, 2013), 267 p
46. V. Matthai, D. Quillen, Superconnections, Thom classes, and covariant differential forms. Topology **25**, 85–110 (1986)
47. R.N. Mahapatra, J.C. Pati, "Natural" left-right symmetry. Phys. Rev. D **11**, 2558–2561 (1975)
48. Y. Neeman, Internal subgroup prediction of the Goldstone-Higgs particle mass. Phys. Lett. B **81**(3–4), 309–310 (1979)
49. J.D.H. van Nuland, W.D. van Suijlekom, One loop corrections to the spectral action. arXiv:2107.08485
50. J. Pati, Abdus Salam, Unified Lepton-Hadron symmetry and a gauge theory of the basic interactions. Phys. Rev. D **8**(4), 1240–1251 (1973)
51. D. Quillen, Superconnections and the Chern character. Topology **24**, 85–95 (1985)
52. G. Roepstorff, Superconnections and the Higgs field. J. Math. Phys. **40**, 2698–2715 (1999). arXiv:hep-th/9801040

53. G. Senjanović, Is left-right symmetry the key? Mod. Phys. Lett. A **32**, 173004 (2017). 1610.04209v2[hep-ph]
54. M. Stener, *Moufang Loops* General theory and visualization of non-associative Moufang loops of order 16, Uppsala University Thesis (2016)
55. J. Thierry-Mieg, Y. Ne'eman, Exterior gauging of an internal symmetry and $SU(2M)$ quantum asthenodynamics. Proc. Nat. Acad. Sci **79**, 7068–7072 (1982)
56. J. Thierry-Mieg, Chirality, the missing key to the definition of the connection and curvature of a Lie-Kac superalgebra. JHEP **01**, 111 (2021). arXiv:2003.12234
57. J. Thierry-Mieg, Scalar anomaly cancellation reveals the hidden algebraic structure of the quantum chiral $SU(2|1)$ model of leptons and quarks. JHEP **10**, 167 (2020). arXiv:2005.04754v2 [hep-th]
58. I. Todorov, Clifford algebras and spinors. Bulg. J. Phys. **58**(1), 3–28 (2011). (arXiv:1106.3197)
59. I. Todorov, Exceptional quantum algebra for the standard model of particle physics, in: *Lie Theory and Its Applications in Physics*, V.K. Dobrev (Ed.), Chpt. 3, Springer Nature, 2020; 1911.13124 [hep-th]; -, Jordan algebra approach to finite quantum geometry, *PoS* (Corfu 2019) 163
60. I. Todorov, Superselection of the weak hypercharge and the algebra of the Standard Model. JHEP **04**, 164 (2021). arXiv:2010.15621v3
61. I. Todorov, S. Drenska, Composition algebras, exceptional Jordan algebra and related groups. JGSP **46**, 59–63 (2017); I. Todorov, S. Drenska, Octonions, exceptional Jordan algebra, and the role of the group F_4 in particle physics. Adv. in Appl. Clifford Alg. **28**, 82 (2018). arXiv:1805.06739v2
62. I. Todorov, M. Dubois-Violette, Deducing the symmetry of the standard model from the automorphism and structure groups of the exceptional Jordan algebra. Int. J. Mod. Phys. A **33**, 1850118 (2018). arXiv:1806.09450
63. G.C. Wick, A.S. Wightman, E.P. Wigner, The intrinsic parity of elementary particles. Phys. Rev. **88**(1), 101–105 (1952); G.C. Wick, A.S. Wightman, E.P. Wigner, Superselection rule for charge. Phys. Rev. D **1**(12), 3267–3269 (1970)
64. I. Yokota, Exceptional Lie groups. arXiv:0902.0431 [math.DG] (204 p)

The Jacobi Sigma Model

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Abstract We review the main aspects of the recently introduced Jacobi sigma model. This is a 2-dimensional topological field theory with target space a Jacobi manifold.

Keywords Jacobi sigma model · Topological field theory · Jacobi manifold

1 Introduction

This paper is based on a lecture given at the XIV International Workshop on Lie Theory and Its Applications in Physics, which has been held in Sofia in June 2021. The main goal being here to convey the main aspects of the Jacobi sigma model, many technical details and in deep calculations are left aside and we refer to [1, 2] for an extended presentation of the results.

The Jacobi sigma model (JSM) [1–3] is a two-dimensional topological field theory defined on a source manifold with boundary, which we shall indicate with Σ. The target space is a D dimensional manifold, M, equipped with a Jacobi structure [4], namely, a twisted Poisson bi-vector field, Λ and a vector field, E, so called Reeb vector field, which satisfy the following relation

$$[\Lambda, \Lambda]_S = 2E \wedge \Lambda, \qquad [E, \Lambda]_S = 0. \tag{1}$$

The bracket on the left of the previous equations is the Schouten bracket, which is defined on the algebra of multi-vector fields $\mathcal{L}(M) = \oplus_{k=0}^{D} V^k(M)$ where $V^0(M)$ is identified with the algebra of smooth functions on the manifold, $V^1(M)$ is the algebra of vector-fields and $V^k(M), k > 1$, are k-vector fields, namely antisymmetric

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_4

k-contravariant tensor fields. The pair (Λ, E) endows the set of smooth functions on M with a Lie algebra structure provided by the Jacobi bracket [4]

$$\{f, g\}_J := \Lambda(df, dg) + f E(g) - g E(f). \tag{2}$$

The latter satisfies Jacobi identity but not the Leibniz rule which is instead replaced by the following

$$\{f, gh\}_J = \{f, g\}_J h + g\{f, h\}_J + gh\, E(f). \tag{3}$$

The model is a natural generalisation of the Poisson sigma model (PSM) [5–7]. It reduces to the latter when the Reeb vector field is zero. Moreover, it may be related to a PSM with a target space of higher dimensionality (more precisely of one dimension more), by considering the so called poissonization of the Jacobi structure (Λ, E). Noteworthy cases of Jacobi manifolds are represented by contact and locally conformally symplectic manifolds, with Poisson, symplectic and globally conformally symplectic as special cases. The model has first and second class constraints, with the former generating gauge transformations. The imposition of constraints and of gauge symmetries gives rise to a finite-dimensional reduced phase space. The dynamics which survives on the boundary is that of a classical mechanical system which represents the holographic dual of the JSM in the bulk [8], in analogy with what happens for the PSM [9]. Similarly to the PSM, upon quantization it is expected to give rise to a noncommutative quantum mechanical system.

The model provides new backgrounds for strings dynamics [3]; it naturally contains a three-form field, which in some cases is not exact and may be related to non-trivial fluxes.[1]

2 The Model

Let us first recall what is the PSM, to which the JSM is inspired. The former is a two-dimensional, topological field theory with source space a manifold with boundary, Σ, and target space a Poisson manifold (M, Π). The first order action is given in terms of the fields X, η,

$$S = \int_\Sigma \left[\eta_i \wedge dX^i + \frac{1}{2}\Pi^{ij}(X)\eta_i \wedge \eta_j\right], \quad i, j = 1, \ldots, \dim M \tag{4}$$

where $X : \Sigma \to M$, is the embedding map, while $\eta \in \Omega^1(\Sigma, X^*(T^*M))$, is a one form on Σ with values in the pull-back of the cotangent bundle T^*M. Analogously,

[1] There might be a relation with twisted Poisson sigma models [10, 11]. We thank Peter Schupp for pointing this out to us.

$dX \in \Omega^1(\Sigma, X^*(TM))$ is a one form of Σ with values in the pull-back of the tangent bundle TM. The equations of motion read

$$dX^i + \Pi^{ij}(X)\eta_j = 0 \tag{5}$$

$$d\eta_i + \frac{1}{2}\partial_i \Pi^{jk}(X)\eta_j \wedge \eta_k = 0 \tag{6}$$

and consistency of the two requires the vanishing of the Schouten bracket, $[\Pi, \Pi]_S = 0$. If the source manifold is such that $\partial\Sigma \neq 0$ boundary conditions are required for the auxiliary fields, e. g. $\eta|_{\partial\Sigma} = 0$. The latter can be integrated away, resulting in a second order action, only if the target space is a symplectic manifold, in which case $\omega = \Pi^{-1}$ and

$$S = \int_\Sigma \omega_{ij}\, dX^i \wedge dX^j. \tag{7}$$

The model is invariant under diffeomorphisms of the source space. On choosing $\Sigma = \mathbb{R} \times [0, 1]$ with $t \in \mathbb{R}$, $u \in [0, 1]$ and the notation $\beta_i = \eta_{ti}$, $\zeta_i = \eta_{ui}$, $\dot{X} = \partial_t X$, $X' = \partial_u X$, the Lagrangian and Hamiltonian of the model respectively read

$$L = \int_I du \left[-\zeta_i \dot{X}^i + \beta_i \left(X'^i + \Pi^{ij}(X)\zeta_j\right)\right] \tag{8}$$

$$H = -\int_I du\, \beta_i \left[X'^i + \Pi^{ij}(X)\zeta_j\right]. \tag{9}$$

Then one easily observes that:

X^i and $-\zeta_i$ are conjugate variables with Poisson brackets $\{X^i, \zeta_j\} = \delta^i_j \delta(u - u')$;
the conjugate momentum to β, π_β, is zero, thus yielding a primary constraint;
the conservation of the constraint along the motion implies the secondary constraint $\mathcal{G}_\beta = X'^i + \Pi^{ij}(X)\zeta_j$;
therefore the Hamiltonian itself is a pure constraint, more precisely a one-parameter family, $H_\beta \simeq 0$.

It is possible to check that the set of constraints closes under Poisson brackets, in particular,

$$\{H_\beta, H_{\tilde{\beta}}\} = H_{[\beta,\tilde{\beta}]} \tag{10}$$

with

$$[\beta, \tilde{\beta}] = d\langle\beta, \Pi(\tilde{\beta})\rangle - i_{\Pi(\beta)} d\tilde{\beta} + i_{\Pi(\tilde{\beta})} d\beta \tag{11}$$

the (extension of) the Koszul bracket of one-forms $\beta \in \Omega^1(M)$ to $\beta = \beta(u)$; $\langle\ ,\ \rangle$ is the natural pairing between T^*M and TM. Noticeably, (11) satisfies Jacobi identity provided Π is a Poisson tensor. Therefore, the map $\beta \to H_\beta$ is a Lie algebra homomorphism, the Hamiltonian constraints are first class and the Hamiltonian vector fields generate gauge transformations [12, 13].

The reduced phase space of the model is defined in the usual way as the quotient $C = P/\mathcal{G}$, where $\mathcal{G}$ is the gauge group, while P is the (infinite-dimensional) constrained phase space. It has been proven [12, 13] that the reduced phase space is a finite-dimensional manifold of dimension 2dim(M).

2.1 The Jacobi Sigma Model

Two main classes of Jacobi manifolds are locally conformally symplectic (LCS) and contact manifolds (see [14–16]), with generic ones admitting foliations in terms of contact and/or LCS leaves.

LCS manifolds are even-dimensional manifolds equipped with a one-form $\alpha \in \Omega^1(M)$ and a non-degenerate two-form $\omega \in \Omega^2(M)$, locally equivalent to a symplectic form ξ, i.e.

$$\begin{aligned}&\omega = e^{-f}\xi, \quad f \in C^\infty(U_i)\\ &d\omega = -\alpha \wedge \omega \quad (\alpha = df \text{ locally})\end{aligned}$$

The global structures (Λ, E) are uniquely defined in terms of (α, ω)

$$\iota_E\omega = -\alpha, \quad \iota_{\Lambda(\gamma)}\omega = -\gamma, \qquad \forall\gamma \in \Omega^1(M) \tag{12}$$

Globally conformal symplectic and symplectic manifolds are particular cases of LCS, respectively corresponding to the one-form α being exact or zero.

Contact manifolds are odd-dimensional manifolds, $dim M = 2n + 1$, with a contact one-form, ϑ and a volume form Ω s.t.

$$\vartheta \wedge (d\vartheta)^n = \Omega$$

The global structure (Λ, E) is uniquely fixed by

$$\iota_E\left(\vartheta \wedge (d\vartheta)^n\right) = (d\vartheta)^n \quad \iota_\Lambda\left(\vartheta \wedge (d\vartheta)^n\right) = n\vartheta \wedge (d\vartheta)^{n-1} \tag{13}$$

Finally, let us shortly mention the Poissonization procedure [4], mainly because it allows for the definition of Hamiltonian vector fields associated with the Jacobi structure $J = (\Lambda, E)$.

Given a Jacobi manifold (M, Λ, E) the manifold $M \times \mathbb{R}$ may be given a one-parameter family of homogeneous Poisson structures

$$\Pi = e^{-\tau}(\Lambda + \frac{\partial}{\partial\tau} \wedge E), \qquad \tau \in \mathbb{R}$$

(homogeneous meaning that $L_{\partial_\tau}\Pi = -\Pi$). This allows for a consistent definition of Hamiltonian vector fields associated with J as a projection:

$$X_f := \pi_*(X^{\Pi}_{e^\tau f})|_{\tau=0} \tag{14}$$

where $X^{\Pi}_{e^\tau f}$ is the Hamiltonian vector field associated with the Poisson bracket on $M \times \mathbb{R}$ and $\pi : M \times \mathbb{R}) \to M$ is the projection map. This yields, for any smooth function on M,

$$X_f = \Lambda(df, \cdot) + fE \tag{15}$$

It is important to stress that the map $f \to X_f$ is a homomorphism of Lie algebras, namely $[X_f, X_g] = X_{\{f,g\}_J}$, where the bracket $[\cdot, \cdot]$ is the standard Lie bracket of vector fields. The Jacobi bracket (2) may be rewritten as

$$\{f, g\}_J = X_f(g) - gE(f) = -X_g(f) + fE(g). \tag{16}$$

Having settled the mathematical structures we are ready for introducing the model. The Jacobi sigma model with source space a two-dimensional manifold Σ with boundary $\partial\Sigma$ and target space (M, Λ, E) is defined by the action functional [1]

$$S[X, (\eta, \lambda)] = \int_\Sigma \left[\eta_i \wedge dX^i + \frac{1}{2}\Lambda^{ij}(X)\eta_i \wedge \eta_j - E^i(X)\eta_i \wedge \lambda\right] \tag{17}$$

with boundary condition $\eta_{|\partial\Sigma} = 0$. On comparing with the action of the PSM we notice some similarities, (the action is the same when the Reeb vector field is zero, with a new auxiliary field, λ, which is necessary to pair with the Reeb vector field. Therefore, the field configurations are $(X, (\eta, \lambda))$, with

$X : \Sigma \to M$ the embedding map;
$(\eta, \lambda) \in \Omega^1(\Sigma, X^*(J^1M))$;

the pair (η, λ) represents a section of the pull-back bundle of $J^1M = T^*M \oplus \mathbb{R}$, the 1-jet bundle of real functions on M. Loosely speaking, η is a one form on Σ *and* on M, exactly as it was the case for the PSM, while λ is a one form on Σ but a scalar field on M. It is useful to remark that sections of J^1M are isomorphic to one-forms of the kind $e^\tau(\alpha + f d\tau)$ with $\alpha \in \Omega^1(M)$, $f \in C^\infty(M)$, $\tau \in \mathbb{R}$. They form a closed Lie subalgebra of $\Omega^1(M \times \mathbb{R})$, the one forms of the Poissonized manifold, with respect to the Koszul bracket. The induced bracket for the sections of J^1M [17] reads

$$\begin{aligned}[(\alpha, f), (\beta, g)] = \Big(\big(&L_{\Lambda(\alpha)}\beta - L_{\Lambda(\beta)}\alpha - d(\Lambda(\alpha, \beta) + fL_E\beta - \\ &- gL_E\alpha - \alpha(E)\beta + \beta(E)\alpha\big), \\ &\big(\{f, g\}_J - \Lambda(df - \alpha, dg - \beta)\big)\Big).\end{aligned} \tag{18}$$

The variation of the action gives the following equations of motion

$$dX^i + \Lambda^{ij}\eta_j - E^i\lambda = 0, \tag{19}$$

$$d\eta_i + \frac{1}{2}\partial_i \Lambda^{jk}\eta_j \wedge \eta_k - \partial_i E^j \eta_j \wedge \lambda = 0, \tag{20}$$

$$E^i \eta_i = 0. \tag{21}$$

Consistency of the three, together with the properties of the Jacobi structure (Λ, E), (1), yields another equation,

$$d\lambda = \frac{1}{2}\Lambda^{ij}\eta_i \wedge \eta_j. \tag{22}$$

Let us switch to the Hamiltonian approach, which is the framework adopted in [1, 2] to analyse the constraints of the theory and discuss its gauge symmetry. In local coordinates $t \in \mathbb{R}$, $u \in [0, 1]$ for $\Sigma = \mathbb{R} \times [0, 1]$, we have

$$dX = \dot{X}dt + X'du, \quad \eta = \beta dt + \zeta du, \quad \lambda = \lambda_t dt + \lambda_u du.$$

The Lagrangian and the Hamiltonian read respectively

$$L = \int_I du \left[-\dot{X}^i \zeta_i + \beta_i \left(X'^i + \Lambda^{ij}\zeta_j - E^i \lambda_u\right) + \lambda_t \left(E^i \zeta_i\right)\right] \tag{23}$$

$$H = -\int_I du\, \beta_i \left(X'^i + \Lambda^{ij}\zeta_j - E^i \lambda_u\right) + \lambda_t \left(E^i \zeta_i\right) \tag{24}$$

with λ_t, λ_u scalar fields, $\dot{X}$, X' and β, ζ carrying and extra index on (the pull-back of) M. Moreover, $-\zeta_i$, X^i are conjugate variables with canonical Poisson brackets. The b.c. $\eta_{\partial\Sigma} = 0$ results in $\beta_{\partial\Sigma} = 0$ and no b.c. for λ. From the analysis of the Hamiltonian it is possible to infer that the theory is constrained (see [1, 2] for details), with

$$\pi_{\beta_i} = \delta L/\delta\beta_i, \quad \pi_{\lambda_t} = \delta L/\delta\lambda_t, \quad \pi_{\lambda_u} = \delta L/\delta\lambda_u, \quad \text{primary constraints}$$

$$\left.\begin{array}{l} \dot{\pi}_{\beta_i} \equiv \mathcal{G}_{\beta_i} = X'^i + \Lambda^{ij}\zeta_j - E^i\lambda_u, \\ \dot{\pi}_{\lambda_t} \equiv \mathcal{G}_{\lambda_t} = E^i\zeta_i \\ \dot{\pi}_{\lambda_u} \equiv \mathcal{G}_{\lambda_u} = E^i\beta_i \end{array}\right\} \text{secondary constraints}$$

Therefore, as for the PSM, the Hamiltonian is itself a sum of constraints,

$$H = -\int du\, \left[\beta_i \mathcal{G}_{\beta_i} + \lambda_t \mathcal{G}_{\lambda_t}\right]. \tag{25}$$

Then, on indicating with χ_k a generic element of the family of constraints, we compute their Poisson brackets. To this, we recall that, according to the Dirac prescription for treating the constraints, one has to use the canonical Poisson structure of the unconstrained phase space, $\{\pi_A(u), \varphi^B(v)\} = \delta(u - v)\delta^B_A$, where $\varphi^B(v)$ indicates a generic field among those characterizing the Hamiltonian model, namely $(X^i, \beta_i, \lambda_t, \lambda_u)$, with $\pi_A(u)$ its conjugate momentum. We find that the matrix

$\{\chi_k, \chi_{k'}\}$ has finite, non-zero rank, which implies the existence of second class constraints. Moreover, the rank is not maximal (indeed it can be shown to be equal to 4, independently from the dimension of the target space), which implies that some of the constraints are first class. In order to easily isolate the second class constraints one can choose without loss of generality local coordinates over the target space adapted to the Reeb vector field, so that, for example, E has only non-zero component along the m-th direction: $E = \mathcal{E}(X)\partial/\partial X^m$. Then, with a little algebra, one finds the second class constraints to be $\pi_{\lambda_u}, \pi_{\beta_m}, \mathcal{G}_{\lambda_u}, \mathcal{G}_{\beta_m}$. The remaining constraints are first class, thus generating gauge transformations. $\Longrightarrow$ A combination of these yields gauge transformations, with generating functional

$$K(\beta, \lambda_t, a_t, a_{\beta_a}) = \int \mathrm{d}u\, \lambda_t \mathcal{G}_{\lambda_t} + \beta_a \mathcal{G}_{\beta_a} + a_t \pi_{\lambda_t} + a_{\beta_a}\pi_{\beta_a}, \quad a = 1, \ldots, m-1 \tag{26}$$

and $\beta, \lambda_t, a_t, a_{\beta_a}$ gauge parameters.

The next step is to compute the Poisson brackets of the gauge generators, in order to verify whether they close a Poisson algebra. In doing that, we observe that primary constraints may be ignored, because their Poisson brackets are strongly zero. Then, it can be checked by direct computation that, similarly to the Poisson sigma model, the algebra will only close on-shell. Indeed we find

$$\begin{aligned}
\{K(\beta, \lambda_t), K(\tilde\beta, \tilde\lambda_t)\} = \int \mathrm{d}u\mathrm{d}u' \Big[\mathcal{G}_{\beta_c}\Big(&\beta_a\tilde\beta_b\partial_c\Lambda^{ba} + \Lambda^{aj}(\tilde\beta_a\partial_j\beta_c - \beta_a\partial_j\tilde\beta_c) \\
&+ \mathcal{E}(\tilde\lambda_t\partial_m\beta_c - \lambda_t\partial_m\tilde\beta_c)\Big) \\
&+ \mathcal{G}_{\lambda_t}\Big(\beta_a\tilde\beta_b\Lambda^{ab} + \Lambda^{aj}(\tilde\beta_a\partial_j\lambda_t - \beta_a\partial_j\tilde\lambda_t) + \mathcal{E}(\tilde\lambda_t\partial_m\lambda_t - \tilde\lambda_t\partial_m\tilde\lambda_t)\Big)\Big], \\
&a, b, c = 1, \ldots, m-1
\end{aligned} \tag{27}$$

Therefore, not surprisingly, in order to obtain a closed algebra off-shell, one has to follow the same strategy as for the PSM: the gauge parameters have to be functions of the fields, namely

$$\beta_a \to \beta_a(u, X(u)), \lambda_t \to \lambda_t(u, X(u)).$$

This implies that the gauge parameters (β, λ_t) inherit the bracket of sections of the 1-jet bundle J^1M, which reads [17]

$$\begin{aligned}
[(\alpha, f), (\beta, g)] = \Big(&L_{\Lambda(\alpha)}\beta - L_{\Lambda(\beta)}\alpha - d(\Lambda(\alpha,\beta)) - d(fL_E\beta - \\
&- gL_E\alpha - \alpha(E)\beta + \beta(E)\alpha), \\
&\{f, g\}_J - \Lambda(df - \alpha, dg - \beta)\Big).
\end{aligned} \tag{28}$$

The latter satisfies Jacobi identity because (Λ, E) is a Jacobi structure, in the same way as the Koszul bracket (11) satisfies Jacobi identity provided Π is a Poisson tensor. Therefore, on extending (28) to field dependent parameters, (27) may be rearranged in the following form

$$\{K_{(\beta,\lambda_t)}, K_{(\tilde{\beta},\tilde{\lambda}_t)}\} = -K_{[(\beta,\lambda_t),(\tilde{\beta},\tilde{\lambda}_t)]} \tag{29}$$

which is what we were looking for.

3 Conclusions

To summarise, the Jacobi sigma model is a two-dimensional topological field theory, with first and second class constraints. First class constraints generate gauge transformations, which close off-shell, provided one allows for field-dependent gauge parameters. Thanks to Eq. (29), the map $(\beta, \lambda_t) \to K_{(\beta,\lambda_t)}$ is a Lie algebra homomorphism between the algebra of gauge parameters and the algebra of gauge generators, if the target space M is a Jacobi manifold.

Further results have been obtained in [1, 2]. The reduced phase space of the model has been shown to be finite-dimensional, with dimension equal to 2dimM $-$ 2. A dynamical version of the model has been proposed, with the addition of a new term to the Lagrangian, which is proportional to the metric tensor of the target manifold. This yields a Polyakov action, where the background fields are directly related to the Jacobi structure. A non-zero three form may occur, depending on the target space (for example, whether the latter is a contact or LCS manifold). Finally, the model may be related to a Poisson sigma model with target space $M \times \mathbb{R}$ within a "Poissonization" procedure. The latter approach has been explored in [3].

References

1. F. Bascone, F. Pezzella, P. Vitale, Jacobi sigma models. JHEP **03**, 110 (2021). arXiv:hep-th/2007.12543
2. F. Bascone, F. Pezzella, P. Vitale, Topological and dynamical aspects of jacobi sigma models. Symmetry **13**, 1205 (2021). arXiv:2105.09780 [hep-th]
3. A. Chatzistavrakidis, G. Simunic, Gauged sigma-models with nonclosed 3-form and twisted Jacobi structures. JHEP **11**, 173 (2020). arXiv:hep-th/2007.08951
4. A. Lichnerowicz, Les variétés de Jacobi et leurs algèbres de Lie associées. J. Math. Pures Appl. **57**, 453 (1978)
5. N. Ikeda, Two-dimensional gravity and nonlinear gauge theory. Ann. Phys. **235**, 435 (1994). arXiv:hep-th/9312059
6. P. Schaller, T. Strobl, Poisson structure induced (topological) field theories. Mod. Phys. Lett. A **09**, 3129 (1994). arXiv:hep-th/9405110
7. P. Schaller, T. Strobl, Introduction to Poisson σ-models. Lect. Notes Phys. **469**, 321 (1996). arXiv:hep-th/9507020

8. I.V. Vancea, Classical boundary field theory of Jacobi sigma models by Poissonization, in *Contribution to 4th International Conference on Holography, String Theory and Discrete Approach in Hanoi, Vietnam*. arXiv:2012.02756 [hep-th]
9. D.V. Vassilevich, Holographic duals to poisson sigma models and noncommutative quantum mechanics. Phys. Rev. D **87**(10), 104011 (2013). https://doi.org/10.1103/PhysRevD.87.104011. arXiv:1301.7029 [hep-th]
10. C. Klimcik, T. Strobl, WZW - Poisson manifold. J. Geom. Phys. **43**, 341 (2002). https://doi.org/10.1016/S0393-0440(02)00027-X. arXiv: math/0104189
11. J.-S. Park, *Topological Open P-branes, Symplectic Geometry and Mirror Symmetry (Seoul, 2000)* (World Scientific Publishing, River Edge, NJ, 2001), pp. 311–384
12. A.S. Cattaneo, G. Felder, A path integral approach to the Kontsevich quantization formula. Commun. Math. Phys. **212**, 591 (2000). arXiv:math/9902090
13. A.S. Cattaneo, G. Felder, Poisson sigma models and deformation quantization. Mod. Phys. Lett. A **16**, 179–190 (2001). arxiv:hep-th/0102208
14. I. Vaisman, A lecture on Jacobi manifolds. Selected Topics Geom. Math. Phys. **1**, 81–100 (2002)
15. C.-M. Marle, On Jacobi manifolds and Jacobi bundles, in *Symplectic Geometry, Groupoids, and Integrable Systems*, ed. by P. Dazord, A. Weinstein. Mathematical Sciences Research Institute Publications, vol. 20 (Springer, New York, NY, 1991)
16. I. Vaisman, Locally conformally symplectic manifolds. Int. J. Math. & Math. Sci. **8**, 521–536 (1985)
17. I. Vaisman, The BV-algebra of a Jacobi manifold. Ann. Pol. Math. **73**(3), 275–290 (2000)

Levi-Civita Connections on Braided Algebras

Paolo Aschieri

Abstract A braided symmetric algebra carries a representation of a triangular Hopf algebra. Its noncommutativity is captured by the universal R-matrix. Its differential geometry is canonically constructed from these data. We review the Cartan structure equations and the Bianchi identities for the curvature and torsion of arbitrary connections, i.e., not necessarily invariant under the Hopf algebra action. An arbitrary metric tensor defines a unique metric compatible and torsion free connection. The associated Ricci tensor allows to consider in vacuum noncommutative Einstein equations.

Keywords Noncommutative Levi-Civita connection · Quantum Riemannian geometry · Noncommutative Bianchi identities · Noncommutative Cartan structure equations

1 Introduction

The existence and uniqueness of the Levi-Civita connection of a metric tensor is the fundamental theorem of (pseudo)-Riemannian geometry. The study of its noncommutative analogues has been very active in the past years. Due to the variety of noncommutative algebras one can consider, and to the non uniqueness of the associated differential calculi, there are different notions of metric and of Levi-Civita connections. Examples are then a main tool to study noncommutative Riemannian geometry. The subject is also motivated by noncommutative gravity. This is possibly relevant for capturing some aspects of quantum gravity, where quantization of space time itself is expected (e.g. via gedanken experiments [12]). In particular, closed strings in the presence of higher forms fluxes are conjectured to lead to gravity on

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_5

non geometric backgrounds that can be understood as noncommutative spaces (see the recent review [18]).

There are complementary approaches to Levi-Civita connections. On the one hand, on specific noncommutative algebras, there exist preferred metrics and one is interested to study the Levi-Civita connections of these distinguished metrics. For example in [16] invariant (Killing) metrics on quantum groups are considered. More in general, compatibility between the noncommutative multiplication of an algebra and its metric structure is used to strongly constrain the possible metrics and then to study their geometry. Typically the compatibility is that of requiring the metric $\mathsf{g} \in \Omega(A) \otimes_A \Omega(A)$, where $\Omega(A)$ is the space of one forms on the algebra A, to be central ($\mathsf{g}a = a\mathsf{g}$ for all $a \in A$), see e.g. [8–11, 17, 20].

On the other hand it is interesting to study Levi-Civita connections for metrics g that, up to a proper notion of symmetry, are arbitrary symmetric tensors in $\Omega(A) \otimes_A \Omega(A)$, i.e., that are not constrained by the noncommutativity of the algebra and that in general are not central. It is indeed in this framework that a metric can be studied as a dynamical field describing the gravitational degrees of freedoms. A first problem to overcome in this approach is that the metric compatibility condition $\nabla \mathsf{g} = 0$ requires the lifting of the connection ∇ on $\Omega(A)$ to the tensor product $\Omega(A) \otimes_A \Omega(A)$. While this is doable if the connection itself has some invariance or centrality properties, e.g. if it is a bimodule connection [13] (as it is the case for Levi-Civita connections of central metrics [4]), for arbitrary connections this is itself an open problem.

For a selected class of noncommutative algebras it is possible to overcome these centrality and invariance constraints on the metric and the connection. In case of $\mathbb{R}^n$ with Moyal–Weyl noncommutativity the Levi-Civita connection of an arbitrary symmetric metric was constructed in [3] using a noncommutative Koszul formula (see also [6, Sects. 3.4, 8.5]). A similar result holds on the noncommutative torus [19]. The noncommutative 3-sphere is in [1]. These studies and [4] rely on the existence of (undeformed) derivations of the noncommutative algebra A generating the A-module of vector fields (dual to that of one forms) or a submodule thereof. The Lie algebra generated by these derivations acts on A, that therefore carries a representation of the universal enveloping algebra H of this Lie algebra.

We study more in general algebras that carry an action of a triangular Hopf algebra H and with noncommutativity controlled by the universal R-matrix $\mathcal{R} \in H \otimes H$, for all $a, b \in A$, $ab = (\bar{R}^\alpha \triangleright b)(\bar{R}_\alpha \triangleright a)$, where $\mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha$. These are braided symmetric (braided commutative) algebras. For example noncommutative algebras arising from Drinfeld twist (2-cocycle) deformation of commutative algebras are of this kind, their differential geometry was studied in [5]. Another example, that does not rely on twist deformation, is given by A a cotriangular Hopf algebra.

In this general setting we develop, reviewing [2], a noncommutative coordinate free approach to Riemannian geometry that thus avoids using a preferred set of derivations of the algebra A (usually unavailable).

We begin in Sect. 2 recalling basic facts about triangular Hopf algebras and their modules and A-bimodules, like that of braided derivations (braided vector fields) and the dual module of one forms $\Omega(A)$. The differential and Cartan calculus of the exterior, inner and Lie derivatives [15, 20] is presented in Sect. 3 and is extended in

Sect. 4 to include covariant derivatives. This leads to the Cartan structure equations, which show that there is a unique notion of curvature and of torsion of a connection, independently form their realization as operators on covariant or contravariant tensors or on forms. We also see that the braiding associated with the universal $\mathcal{R}$ matrix allows to lift arbitrary connections on $\Omega(A)$ to the tensor product $\Omega(A) \otimes_A \Omega(A)$. In Sect. 5 existence and uniqueness of the Levi-Civita connection of an arbitrary braided symmetric metric is proven using a noncommutative Koszul formula. The Ricci tensor and the Einstein in vacuum equations are then presented.

In this paper we do not provide full proofs with detailed calculations, we rather streamline the exposition so to clarify the logic flow of the arguments leading to the main propositions and theorems. For a complementary detailed exposition of the material presented here we refer to [2].

2 Triangular Hopf Algebra Representations

We consider modules and algebras over a field $\Bbbk$ of characteristic zero or the ring of formal power series in a variable $\hbar$ over such field. With slight abuse of notation $\Bbbk$-modules and $\Bbbk$-module maps will simply be called linear spaces and linear maps. The tensor product over $\Bbbk$ is denoted $\otimes$. Algebras over $\Bbbk$ are assumed associative and unital. Hopf algebras are assumed with invertible antipode.

When we have a Lie group G acting on a manifold M the spaces of vector fields, one forms and their tensor products are bimodules over $A = C^\infty(M)$ and are representations of G. When A is noncommutative G is replaced by a Hopf algebra H and we consider representations of H that are also A-bimodules.

Let H be a Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S)$ an H-module is a linear space V with an H-action $\triangleright : H \otimes V \to V$. A linear map $f : V \to W$ between H-modules is H-equivariant if

$$h \triangleright f(v) = f(h \triangleright v) \;, \tag{1}$$

for all $h \in H$ and $v \in V$. We denote by ${}^H\mathscr{M}$ the category of H-modules. The tensor product $V \otimes W$ of H-modules is an H-module with action $h \triangleright (v \otimes w) := (h_{(1)} \triangleright v) \otimes (h_{(2)} \triangleright w)$, where we have used the Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ (with summation understood) for the coproduct of H.

For any V, W in ${}^H\mathscr{M}$, let $\mathrm{hom}_\Bbbk(V, W)$ in ${}^H\mathscr{M}$ be the linear space $\mathrm{Hom}_\Bbbk(V, W)$ of linear maps $L : V \to W$ equipped with the adjoint H-action

$$\triangleright : H \otimes \mathrm{hom}_\Bbbk\big(V, W\big) \to \mathrm{hom}_\Bbbk\big(V, W\big) \;, \quad h \triangleright L := h_{(1)} \triangleright \circ L \circ S(h_{(2)}) \triangleright \;, \tag{2}$$

i.e., $(h \triangleright L)(v) = h_{(1)} \triangleright (L(S(h_{(2)}) \triangleright v))$. There is another H-adjoint action on linear maps $V \to W$. We denote by ${}_\Bbbk\mathrm{hom}(V, W)$ the linear space $\mathrm{Hom}_\Bbbk(V, W)$ with H-action $\triangleright^{cop}$ defined by

$$\triangleright^{cop} : H \otimes {}_{\Bbbk}\mathrm{hom}\big(V, W\big) \to {}_{\Bbbk}\mathrm{hom}\big(V, W\big), \quad h \triangleright^{cop} \tilde{L} := h_{(2)} \triangleright \circ \tilde{L} \circ S^{-1}(h_{(1)}) \triangleright , \tag{3}$$

i.e., $(h \triangleright^{cop} \tilde{L})(v) = h_{(2)} \triangleright (\tilde{L}(S^{-1}(h_{(1)}) \triangleright v))$. While linear maps $L \in \mathrm{hom}_{\Bbbk}(V, W)$ naturally act from the left, indeed the $\triangleright$ adjoint action satisfies, for all $h \in H, v \in V$, $h \triangleright (L(v)) = (h_{(1)} \triangleright L)(h_{(2)} \triangleright v)$, linear maps $\tilde{L} \in {}_{\Bbbk}\mathrm{hom}(V, W)$ naturally act from the right, indeed the $\triangleright^{cop}$ adjoint action satisfies,

$$h \triangleright (\tilde{L}(v)) = (h_{(2)} \triangleright^{cop} \tilde{L})(h_{(1)} \triangleright v) \ , \tag{4}$$

that, evaluating $\tilde{L}$ on v from the right, reads $h \triangleright ((v)(\tilde{L})) = (h_{(1)} \triangleright v)(h_{(2)} \triangleright^{cop} \tilde{L})$.

Let now H be a triangular Hopf algebra with universal $\mathcal{R}$-matrix $\mathcal{R} \in H \otimes H$. We recall that it satisfies

$$\Delta^{cop}(h) = \mathcal{R}\Delta(h)\mathcal{R}^{-1} \text{ for all } h \in H,$$

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \ , \quad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$$

and the triangularity condition $\mathcal{R}_{21} = \mathcal{R}^{-1}$. For each H-module V, W we then have the braiding isomorphism

$$\tau_{V,W} : V \otimes W \longrightarrow W \otimes V \ , \qquad v \otimes w \longmapsto \big(\bar{R}^{\alpha} \triangleright w\big) \otimes \big(\bar{R}_{\alpha} \triangleright v\big) \tag{5}$$

where we used the notation $\mathcal{R} = R^{\alpha} \otimes R_{\alpha}$, $\mathcal{R}^{-1} = \bar{R}^{\alpha} \otimes \bar{R}_{\alpha}$. These isomorphisms provide a representation of the permutation group. With slight abuse of notation we shall frequently omit the indices in the isomorphisms $\tau_{V,W}$ and simply write τ.

A left H-module algebra A is an algebra with a compatible H-module structure,

$$h \rhd (ab) = (h_{(1)} \rhd a)(h_{(2)} \rhd b) \ , \quad h \rhd 1_A = \epsilon(h)1_A$$

for all $h \in H$ and $a, b \in A$. When H is triangular we consider A to be braided symmetric or braided commutative (also called symmetric or quasi-commutative) if, for all $a, b \in A$,

$$ab = (\bar{R}^{\alpha} \triangleright b)(\bar{R}_{\alpha} \triangleright a) \ . \tag{6}$$

Similarly, an H-equivariant A-bimodule (or relative H, A-module) V is an H-module and a compatible A-bimodule: for all $a \in A, v \in V$, $h \rhd (av) = (h_{(1)} \rhd a)(h_{(2)} \rhd v)$, $h \rhd (va) = (h_{(1)} \rhd v)(h_{(2)} \rhd a)$. It is braided symmetric if

$$av = (\bar{R}^{\alpha} \triangleright v)(\bar{R}_{\alpha} \triangleright a) \ . \tag{7}$$

We denote by ${}^{H}_{A}\mathscr{M}^{\mathrm{sym}}_{A}$ the category of braided symmetric H-equivariant A-bimodules. If V, W are modules in ${}^{H}_{A}\mathscr{M}^{\mathrm{sym}}_{A}$ the balanced tensor product $V \otimes_A W$ is in ${}^{H}_{A}\mathscr{M}^{\mathrm{sym}}_{A}$ (with obvious left and right A-actions inherited from those of V and W respectively). Furthermore, $\mathrm{hom}_{\Bbbk}(V, W)$ and ${}_{\Bbbk}\mathrm{hom}(V, W)$ are A-bimodules, the first via the left

A-module structure of V and W, the second via the right A-module structure of V and W: For all $a \in A$, $v \in V$, $L \in \hom_{\Bbbk}(V, W)$, $\tilde{L} \in {}_{\Bbbk}\hom(V, W)$,

$$(aL)(v) = a(L(v)) \ , \quad (La)(v) = L(av) \ , \tag{8}$$

$$(\tilde{L}a)(v) = \tilde{L}(v)a \ , \quad (a\tilde{L})(v) = \tilde{L}(va) \ . \tag{9}$$

Let $\hom_A(V, W) \subset \hom_{\Bbbk}(V, W)$ and ${}_A\hom(V, W) \subset {}_{\Bbbk}\hom(V, W)$ be the H-submodules of right A-linear maps: for all $a \in A$, $L(va) = L(v)a$, and left A-linear maps: for all $a \in A$, $\tilde{L}(av) = a\tilde{L}(v)$. Then $\hom_A(V, W) \subset \hom_{\Bbbk}(V, W)$ and ${}_A\hom(V, W) \subset {}_{\Bbbk}\hom(V, W)$ are A-subbimodules and are modules in ${}^H_A\mathscr{M}^{\mathrm{sym}}_A$. (Thus ${}^H_A\mathscr{M}^{\mathrm{sym}}_A$ is a braided symmetric biclosed monoidal category).

Given a braided commutative A-bimodule V in ${}^H_A\mathscr{M}^{\mathrm{sym}}_A$ the dual module ${}^*V := {}_A\hom(V, A)$ is in ${}^H_A\mathscr{M}^{\mathrm{sym}}_A$. The evaluation of elements of ${}^*V$ on elements of V is denoted as the pairing

$$\langle\ ,\ \rangle : V \otimes_A {}^*V \to A \ , \quad v \otimes_A \omega \mapsto \langle v, \omega\rangle \tag{10}$$

which is well defined on the balanced tensor product $\otimes_A$ because of the second expression in (9). It is right A-linear because of the first one in (9), left A-bilinear and H-equivariant by definition of ${}_A\hom(V, A)$. The pairing $\langle\ ,\ \rangle : V \otimes_A {}^*V \to A$ is therefore a morphism in ${}^H_A\mathscr{M}^{\mathrm{sym}}_A$.

We shall further consider modules in ${}^H_A\mathscr{M}^{\mathrm{sym}}_A$ that are finitely generated and projective. This means that the pairing (10) allows for dual bases $\{e_i\}$ and $\{\omega^i\}$ of elements $e_i \in V$, $\omega^i \in {}^*V$, $i = 1, 2 \ldots n$ with the property: for all $v \in V$, $\omega \in {}^*V$,

$$v = \langle v, \omega^i\rangle e_i \ , \quad \omega = \omega^i\langle e_i, \omega\rangle$$

(sum over i understood). Despite the name, the vectors e_i are in general not independent over A, and similarly the ω^i (unless V and ${}^*V$ are free modules).

We denote by ${}^H_A\mathscr{M}^{\mathrm{sym,fp}}_A$ the subcategory of braided commutative H-equivariant A-bimodules finitely generated and projective and consider from now on such bimodules. The spaces of vector fields on A, of one forms and their tensor products will all be examples of modules in ${}^H_A\mathscr{M}^{\mathrm{sym,fp}}_A$.

3 Differential and Cartan Calculus

Let A be a braided commutative H-module algebra, the linear space $\mathfrak{X}(A)$ of braided derivations is that of linear maps $u \in \hom_{\Bbbk}(A, A)$ that satisfy the braided Leibniz rule, for all $a, b \in A$,

$$u(ab) = u(a)b \;+\; \bar{R}^\alpha \triangleright a\,(\bar{R}_\alpha \triangleright u)(b)\;. \tag{11}$$

$\mathfrak{X}(A)$ is an H-module (submodule of $\mathrm{hom}_{\Bbbk}(A, A)$): for all $h \in H$, $u \in \mathfrak{X}(A)$, $h \triangleright u$ is still a braided derivation. As for derivations on a commutative algebra, it is not difficult to see (cf. [20, Lemma 3.1]) that the braided commutator

$$[\ ,\ ] : \mathfrak{X}(A) \otimes \mathfrak{X}(A) \to \mathfrak{X}(A)\;,\;\; u \otimes v \mapsto\;\; [u, v] := uv - \bar{R}^\alpha \triangleright v\, \bar{R}_\alpha \triangleright u\;,$$

(where composition of operators is understood) closes in $\mathfrak{X}(A)$, is an H-equivariant map (for all $h \in H$, $u, v, \in \mathfrak{X}(A)$, $h \triangleright [u, v] = [h_{(1)} \triangleright u, h_{(2)} \triangleright v]$) and structures the H-module $\mathfrak{X}(A)$ as a braided Lie algebra with respect to the triangular Hopf algebra $(H, \mathcal{R})$, i.e., we have the braided antisymmetry property and the braided Jacobi identity, for all $u, v, z \in \mathfrak{X}(A)$),

$$[u, v] = -[\bar{R}^\alpha \triangleright v, \bar{R}_\alpha \triangleright u]\;,\quad [u, [v, z]] = [[u, v], z] + [\bar{R}^\alpha \triangleright v, [\bar{R}_\alpha \triangleright u, z]]\;. \tag{12}$$

Braided derivations are furthermore a module in ${}^H_A\mathscr{M}^{\mathrm{sym}}_A$ by defining, for all $a, b \in A$,

$$(au)(b) = a\,u(b)\quad,\quad ua = (\bar{R}^\alpha \triangleright a)\bar{R}_\alpha \triangleright u\;; \tag{13}$$

(au is a braided derivation because of the braided commutative property (6) of A). We call $\mathfrak{X}(A)$ the bimodule of (braided) vector fields.

Let $\Omega(A) := {}^*\mathfrak{X}(A) = {}_A\mathrm{hom}(\mathfrak{X}(A), A)$ be the dual module of left A-linear maps $\mathfrak{X}(A) \to A$ with H-action $\triangleright^{cop}$ defined in (3) and A-bimodule structure defined in (9). Recall the pairing notation (10). The exterior derivative $\mathrm{d} : A \to \Omega(A)$ is given by

$$\langle u, \mathrm{d}a\rangle = u(a) \tag{14}$$

for all $u \in \mathfrak{X}(A)$. It is well-defined since both $\langle\ , \mathrm{d}a\rangle : \mathfrak{X}(A) \to A$ and $\hat{a} : \mathfrak{X}(A) \to A$, $u \mapsto u(a)$ are left A-linear maps. The map d is H-equivariant, indeed, for all $h \in H$ the identities

$$h \triangleright \langle u, \mathrm{d}a\rangle = \langle h_{(1)} \triangleright u, h_{(2)} \triangleright^{cop} \mathrm{d}a\rangle\;,\quad h \triangleright (u(a)) = (h_{(1)} \triangleright u)(h_{(2)} \triangleright a)$$

imply $h \triangleright^{cop} (\mathrm{d}a) = \mathrm{d}(h \triangleright a)$. Next we prove the undeformed Leibniz rule $\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b$:

$$\begin{aligned}\langle u, \mathrm{d}(ab)\rangle = u(ab) &= u(a)b + \bar{R}^\alpha \triangleright a\,(\bar{R}_\alpha \triangleright u)(b)\\ &= \langle u, (\mathrm{d}a)b\rangle + (\bar{R}^\alpha \triangleright a)\langle \bar{R}_\alpha \triangleright u, \mathrm{d}b\rangle\\ &= \langle u, (\mathrm{d}a)b\rangle + \langle u, a\mathrm{d}b\rangle\;,\end{aligned}$$

where we used (13). The module of one forms is the submodule of $\Omega(A)$ in ${}^H_A\mathscr{M}^{\mathrm{sym}}_A$ defined by $A\mathrm{d}A = \{\omega \in \Omega(A)\;;\; \omega = a^j \mathrm{d}a_j\}$ for all $a^j, a_j \in A$, with finite sum over the index j understood (the right A-action closes in $\Omega(A)$ due to the Leibniz rule).

We shall assume that the submodule $A\mathrm{d}A$ is finitely generated and projective over A in this case also $\mathfrak{X}(A) \in {}^H_A\mathscr{M}^{\mathrm{sym,fp}}_A$ and it can be proven that $A\mathrm{d}A = \Omega(A)$. Then the A-bimodule $\Omega(A)$ and the exterior derivative $d : A \to \Omega(A)$ constitute a *first order differential calculus on* A. An H-equivariant one, since A and $\Omega(A)$ are in ${}^H_A\mathscr{M}^{\mathrm{sym,fp}}_A$ and $\mathrm{d} : A \to \Omega(A)$ is H-equivariant.

Associated with $\Omega(A)$ and $\mathfrak{X}(A)$ we have the modules in ${}^H_A\mathscr{M}^{\mathrm{sym,fp}}_A$: $\mathcal{T}^{p,0} = \Omega(A)^{\otimes_A p}$ and $\mathcal{T}^{0,q} = \mathfrak{X}(A)^{\otimes_A q}$, $p, q \in \mathbb{N}$, with $\mathcal{T}^{0,0} = A$, and the graded H-module algebras of contravariant tensor fields $\mathcal{T}^{\bullet,0} = \bigoplus_{p\in\mathbb{N}} \mathcal{T}^{p,0}$ and of covariant tensor fields $\mathcal{T}^{0,\bullet} = \bigoplus_{q\in\mathbb{N}} \mathcal{T}^{0,q}$. We also have the graded H-module algebra $\mathcal{T}^{\bullet,\bullet} = \bigoplus_{p,q\in\mathbb{N}} \mathcal{T}^{p,q}$ with product that on elements of homogeneous degree is defined by,

$$\otimes_A : \mathcal{T}^{p,q} \otimes \mathcal{T}^{p',q'} \to \mathcal{T}^{p+p',q+q'},\ \theta \otimes_A \nu \otimes \theta' \otimes_A \nu' \mapsto \theta \otimes_A \bar{R}^\alpha \triangleright^{cop} \theta' \otimes_A \bar{R}_\alpha \triangleright \nu \otimes_A \nu' \tag{15}$$

where $\theta \in \mathcal{T}^{p,0}$, $\nu \in \mathcal{T}^{0,q}$, $\theta' \in \mathcal{T}^{p',0}$, $\nu' \in \mathcal{T}^{0,q'}$. The pairing $\langle\ ,\ \rangle$ can be extended to the morphism in ${}^H_A\mathscr{M}^{\mathrm{sym,fp}}_A$ defined to be trivial if $r > p$ and otherwise given by

$$\langle\ ,\ \rangle : \mathcal{T}^{0,r} \otimes_A \mathcal{T}^{p,q} \ \to\ \mathcal{T}^{p-r,q}\ ,\ \ \langle \nu, \theta \otimes_A \eta\rangle := \langle \nu, \theta\rangle\eta \tag{16}$$

for all $\nu \in \mathcal{T}^{0,r}$, $\theta \in \mathcal{T}^{r,0}$, $\eta \in \mathcal{T}^{p-r,q}$. In particular, for $r = 1$ we obtain the contraction operator

$$\mathrm{i} : \mathfrak{X}(A) \to \hom_A(\mathcal{T}^{p,q}, \mathcal{T}^{p-1,q})\ ,\quad v \mapsto \mathrm{i}_v = \langle v,\ \ \rangle\ .$$

Therefore, the evaluation in (16) is just the iteration of the contraction operator r-times: $\langle v_r \otimes_A \dots v_2 \otimes_A v_1, \eta\rangle = \mathrm{i}_{v_r} \circ \dots \mathrm{i}_{v_2} \circ \mathrm{i}_{v_1}(\eta)$.

The graded H-module algebra of exterior forms is by definition $\Omega^\bullet(A) := \bigoplus_{n\in\mathbb{N}} \Omega^n(A)$. Here $\Omega^0(A) = A$, $\Omega^1(A) = \Omega(A)$, and $\Omega^n(A)$ is the module of completely braided antisymmetric tensors in $\mathcal{T}^{n,0}$, for example $\Omega^2(A) = \Omega(A) \wedge \Omega(A) \subset \Omega(A) \otimes_A \Omega(A)$ is the image of the projector $P_A = \frac{1}{2}(\mathrm{id}^{\otimes 2} - \tau) : \Omega^{\otimes 2}(A) \to \Omega^{\otimes 2}(A)$ that is H-equivariant and A-bilinear (a morphism in ${}^H_A\mathscr{M}^{\mathrm{sym,fp}}_A$). The wedge product of one forms is then defined by

$$\omega \wedge \omega' := \omega \otimes_A \omega' - \bar{R}^\alpha \triangleright \omega' \otimes_A \bar{R}_\alpha \triangleright \omega\ , \tag{17}$$

and is a braided antisymmetric 2-form: $\omega \wedge \omega' = -\bar{R}^\alpha \triangleright \omega' \wedge \bar{R}_\alpha \triangleright \omega$. Similarly for forms of higher degree, indeed the construction is as in the classical case since the braiding τ provides a representation of the permutation group. In particular $\Omega^\bullet(A)$ is equivalently the quotient of the tensor algebra $\mathcal{T}^{\bullet,0}$ by the two-sided ideal generated by $\ker P_A$. The exterior algebra $\Omega^\bullet(A)$ is generated in degree 0 and 1 and is a graded braided symmetric H-module algebra: for all $\theta \in \Omega^p(A)$, $\theta' \in \Omega^{p'}(A)$, $\theta \wedge \theta' = (-1)^{pp'} \bar{R}^\alpha \triangleright \theta' \wedge \bar{R}_\alpha \triangleright \theta$.

The contraction operator $\mathrm{i} : \mathfrak{X}(A) \otimes_A \Omega(A)^{\otimes_A p} \to \Omega(A)^{\otimes_A p-1}$ restricts to $\mathrm{i} : \mathfrak{X}(A) \otimes_A \Omega(A)^p \to \Omega(A)^{p-1}$, giving, for all $u \in \mathfrak{X}(A)$, the (graded) braided inner derivation $\mathrm{i}_u : \Omega^\bullet(A) \to \Omega^{\bullet-1}(A)$,

$$\mathrm{i}_u(\theta \wedge \theta') = \mathrm{i}_u(\theta) \wedge \theta' + (-1)^{|\theta|}(\bar{R}^\alpha \triangleright \theta) \wedge \mathrm{i}_{\bar{R}_\alpha \triangleright u}(\theta') ,$$

where $|\theta| \in \mathbb{N}$ is the degree of the homogeneous form θ. Applying a second inner derivative we obtain that on exterior forms

$$\mathrm{i}_u \circ \mathrm{i}_v + \mathrm{i}_{\bar{R}^\alpha \triangleright v} \circ \mathrm{i}_{\bar{R}_\alpha \triangleright u} = 0 . \tag{18}$$

We now recall the action of the braided Lie algebra of vector fields $\mathfrak{X}(A)$ on tensor fields, i.e., the Lie derivative. We define $\mathscr{L} : \mathfrak{X}(A) \otimes A \to A$, $\mathscr{L}_u(a) := u(a)$ and $\mathscr{L} : \mathfrak{X}(A) \otimes \mathfrak{X}(A) \to \mathfrak{X}(A)$, $\mathscr{L}_u(v) := [u, v]$. Since $\xi \triangleright (\mathscr{L}_u(a)) = \mathscr{L}_{\xi_{(1)} \triangleright u}(\xi_{(2)} \triangleright a)$ and $\xi \triangleright (\mathscr{L}_u(v)) = \mathscr{L}_{\xi_{(1)} \triangleright u}(\xi_{(2)} \triangleright v)$, $\mathscr{L}$ is H-equivariant and compatible with the A-bimodule structure of $\mathfrak{X}(A)$; then the extension of the action of $\mathfrak{X}(A)$ to the tensor algebra $\mathcal{T}^{0,\bullet}$ is well defined by requiring $\mathscr{L}_u$ to be a braided derivation:

$$\mathscr{L}_u(\nu \otimes_A \nu') = \mathscr{L}_u(\nu) \otimes \nu' + \bar{R}^\alpha \triangleright \nu \otimes_A \mathscr{L}_{\bar{R}_\alpha \triangleright u}(\nu')$$

for all $\nu, \nu' \in \mathcal{T}_{\mathcal{R}}^{0,\bullet}$. This implies the commutativity property $\mathscr{L}_u \circ \tau = \tau \circ \mathscr{L}_u$ between the braiding and the Lie derivative operators. The Lie derivative on contravariant tensor fields is canonically defined by duality, for all $\nu \in \mathcal{T}_{\mathcal{R}}^{0,r}$ and $\theta \in \mathcal{T}_{\mathcal{R}}^{r,0}$,

$$\mathscr{L}_u\langle \nu, \theta \rangle = \langle \mathscr{L}_u \nu, \theta \rangle + \langle \bar{R}^\alpha \triangleright \nu, \mathscr{L}_{\bar{R}_\alpha \triangleright u} \theta \rangle \tag{19}$$

i.e., $\langle \nu, \mathscr{L}_u \theta \rangle := \mathscr{L}_{\bar{R}^\alpha \triangleright u}\langle \bar{R}_\alpha \triangleright \nu, \theta \rangle - \langle \mathscr{L}_{\bar{R}^\alpha \triangleright u} \bar{R}_\alpha \triangleright \nu, \theta \rangle$. It follows that vector fields acts on the tensor algebra $\mathcal{T}^{\bullet,\bullet}$ as braided derivations. On tensor fields $\mathcal{T}^{\bullet,\bullet}$ we have

$$\mathscr{L}_u \circ \mathscr{L}_v - \mathscr{L}_{\bar{R}^\alpha \triangleright v} \circ \mathscr{L}_{\bar{R}_\alpha \triangleright u} = \mathscr{L}_{[u,v]} ; \tag{20}$$

this follows from the braided Jacobi identity in (12), the braided Leibniz rule and (19). Equation (20) shows that the Lie derivative $\mathscr{L} : \mathfrak{X}(A) \otimes \mathcal{T}^{\bullet,\bullet} \to \mathcal{T}^{\bullet,\bullet}$ is an action of the braided Lie algebra of derivations $\mathfrak{X}(A)$ on $\mathcal{T}^{\bullet,\bullet}$.

The commutativity of the Lie derivative with the braiding, $\mathscr{L}_u \circ \tau = \tau \circ \mathscr{L}_u$, and the definition of the wedge product in terms of braided antisymmetric tensor products imply that vector fields also act as braided derivations on the exterior algebra $\Omega^\bullet(A)$. From (19) it is immediate to compute, for all $u, v \in \mathfrak{X}(A)$, $\omega \in \Omega(A)$,

$$(\mathscr{L}_u \circ \mathrm{i}_v - \mathrm{i}_{\bar{R}^\alpha \triangleright v} \circ \mathscr{L}_{\bar{R}_\alpha \triangleright u})\omega = \mathrm{i}_{[u,v]}\omega ; \tag{21}$$

since both left hand side and right hand side are braided derivations on $\Omega^\bullet(A)$, this relation extends to arbitrary exterior forms.

The Lie derivative commutes with the exterior derivative on A, for all $a \in A$, $v \in \mathfrak{X}(A)$, $\mathscr{L}_v \mathrm{d}a = \mathrm{d}\mathscr{L}_v a$, indeed, for all $u \in \mathfrak{X}(A)$,

$$\begin{aligned}\langle u, \mathscr{L}_v \mathrm{d}a\rangle &= \mathscr{L}_{\bar{R}^\alpha \triangleright v}\langle \bar{R}_\alpha \triangleright u, \mathrm{d}a\rangle - \langle [\bar{R}^\alpha \triangleright v, \bar{R}_\alpha \triangleright u], \mathrm{d}a\rangle \\ &= \mathscr{L}_{\bar{R}^\alpha \triangleright v}\mathscr{L}_{\bar{R}_{\alpha \triangleright u}} a - \mathscr{L}_{[\bar{R}^\alpha \triangleright v, \bar{R}_\alpha \triangleright u]} a \\ &= \mathscr{L}_u \mathscr{L}_v a = \langle u, \mathrm{d}\mathscr{L}_v a\rangle \ .\end{aligned}$$

Using induction on the form degree we have that $\mathrm{d}\mathscr{L}_v \theta = \mathscr{L}_v \mathrm{d}\theta$ for any $\theta \in \Omega^\bullet(A)$.

Similarly, for all $v \in \mathfrak{X}(A)$,

$$\mathscr{L}_v = \mathrm{i}_v \circ \mathrm{d} + \mathrm{d} \circ \mathrm{i}_v$$

trivially holds on A and by induction on the form degree it holds on $\Omega^\bullet(A)$ since both the right hand side and the left hand side are braided derivations of $\Omega^\bullet(A)$.

The equations $\mathscr{L}_z \circ \mathrm{d} = \mathrm{d} \circ \mathscr{L}_z$, $\mathscr{L}_v = \mathrm{i}_v \circ \mathrm{d} + \mathrm{d} \circ \mathrm{i}_v$ and (18), (20) (restricted to exterior forms), (21), $\mathrm{d}^2 = 0$, constitute the Cartan calculus of the exterior, Lie and inner derivatives [15, 20] that we summarize in the following

Theorem 1 (Braided Cartan calculus) *Let A be a braided commutative left H-module algebra and consider the associated braided differential algebra* $(\Omega^\bullet(A), \wedge, \mathrm{d})$. *The exterior derivative, the Lie derivative and inner derivative along vector fields $u, v \in \mathfrak{X}(A)$ are graded braided derivations of $\Omega^\bullet(A)$ (respectively of degree* $1, 0, -1$*) that satisfy*

$$\begin{aligned}[\mathscr{L}_u, \mathscr{L}_v] &= \mathscr{L}_{[u,v]}, & [\mathrm{i}_u, \mathrm{i}_v] &= 0, \\ [\mathscr{L}_u, \mathrm{i}_v] &= \mathrm{i}_{[u,v]}, & [\mathrm{i}_u, \mathrm{d}] &= \mathscr{L}_u, \\ [\mathscr{L}_u, \mathrm{d}] &= 0, & [\mathrm{d}, \mathrm{d}] &= 0,\end{aligned}$$

where$[L, L'] = L \circ L' - (-1)^{|L||L'|}\bar{R}^\alpha \triangleright L' \circ \bar{R}_\alpha \triangleright L$ *is the graded braided commutator of linear maps L, L' on $\Omega^\bullet(A)$ of degree $|L|$ and $|L'|$.*

Examples of differential and Cartan calculi according to this construction are those on a cotriangular Hopf algebra K (dual to H), in this case the calculus is a bicovariant differential calculus à la Woronowicz [21]. It is fixed by the triangular structure of K and turns out to be that defined in [14, Sect. 4.3]. Another class of examples [5, 20] arises via Drinfeld twist deformations of the differential and Cartan calculus on G-manifolds M, with G a Lie group. The Drinfeld two cocycle is associated with the universal enveloping algebra of the Lie algebra of G.

4 Braided Differential Geometry

We extend the noncommutative Cartan calculus to connections presenting their Cartan formula with the inner derivatives. This allows to unify different definitions of curvature and torsion. Considering connections on dual modules we further relate curvature and torsion on vector fields to curvature and torsion on forms via the Cartan structure equations. The Bianchi identities are obtained.

4.1 Connections

Let H be a triangular Hopf algebra, A be a braided commutative H-module algebra and $\big(\Omega^\bullet(A), \wedge, \mathrm{d}\big)$ the associated braided differential graded algebra (differential calculus) constructed in Sect. 3. A *right connection* on a module Γ in ${}^H_A\mathscr{M}^{\mathrm{sym}}_A$ is a $\Bbbk$-linear map

$$\nabla : \Gamma \to \Gamma \otimes_A \Omega(A) \tag{22}$$

in $\hom_\Bbbk(\Gamma, \Gamma \otimes_A \Omega(A))$, which satisfies the Leibniz rule, for all $s \in \Gamma$, $a \in A$,

$$\nabla(sa) = \nabla(s)a + s \otimes_A \mathrm{d}a \ . \tag{23}$$

A *left connection* on Γ is a $\Bbbk$-linear map

$$\nabla : \Gamma \to \Omega(A) \otimes_A \Gamma \tag{24}$$

in ${}_\Bbbk\hom(\Gamma, \Omega(A) \otimes_A \Gamma)$, which satisfies the Leibniz rule,

$$\nabla(as) = \mathrm{d}a \otimes_A s + a\nabla(s) \ . \tag{25}$$

We denote by $\mathrm{Con}_A(\Gamma)$ and ${}_A\mathrm{Con}(\Gamma)$ the set of all right, respectively left connections.

The H-adjoint action (2) on $\nabla \in \mathrm{Con}_A(\Gamma) \subset \hom_\Bbbk(\Gamma, \Gamma \otimes_A \Omega(A))$, reads, for all $h \in H$, $h \triangleright \nabla := h_1 \triangleright \circ \nabla \circ S(h_2)\triangleright$. This linear map is easily seen to satisfy (cf. [7, Sect. 6.2]), for all $s \in \Gamma$ and $a \in A$,

$$(h \triangleright \nabla)(sa) = (h \triangleright \nabla)(s)a \ + \ s \otimes_A \varepsilon(h)\mathrm{d}a \ . \tag{26}$$

In particular we see that if $\varepsilon(h) = 0$ then $h \triangleright \nabla \in \hom_A(\Gamma, \Gamma \otimes_A \Omega(A))$, while if $\varepsilon(h) = 1$ then $h \triangleright \nabla \in \mathrm{Con}_A(\Gamma)$. Similarly for $\nabla \in {}_A\mathrm{Con}(\Gamma)$. Using this action and the braided commutativity of the A-bimodule Γ, a right connection ∇ on Γ is shown to be also a braided left connection, cf. [7, Proposition 6.8], and similarly a left connection ∇ on Γ is also a braided right connection, for all $a \in A$, $s \in \Gamma$,

$$\begin{aligned}\nabla(as) &= (\bar{R}^\alpha \triangleright a)(\bar{R}_\alpha \triangleright \nabla)(s) + \bar{R}^\alpha \triangleright s \otimes_A \bar{R}_\alpha \triangleright \mathrm{d}a\ ,\\ \nabla(sa) &= (\bar{R}^\alpha \triangleright^{cop} \nabla)(s)(\bar{R}_\alpha \triangleright a) + \bar{R}^\alpha \triangleright \mathrm{d}a \otimes_A \bar{R}_\alpha \triangleright s\ .\end{aligned} \tag{27}$$

If ∇ is H-equivariant we have $\nabla(as) = a\nabla(s) + \bar{R}^\alpha \triangleright s \otimes_A \bar{R}_\alpha \triangleright \mathrm{d}a$, and thus recover the notion of bimodule connection studied in [13].

A connection ∇ on Γ defines a connection on $\Gamma \otimes_A \Gamma$ via the braided Leibniz rule

$$\begin{aligned}&\nabla : \Gamma \otimes_A \Gamma \to \Omega(A) \otimes_A \Gamma \otimes_A \Gamma\ ,\\ &\qquad s \otimes_A \hat{s} \mapsto \nabla(s \otimes_A \hat{s}) := (\bar{R}^\alpha \triangleright^{cop} \nabla)(s) \otimes_A (\bar{R}_\alpha \triangleright \hat{s}) + \tau_{12} \circ (s \otimes_A \widehat{\nabla}\hat{s})\ ,\end{aligned} \tag{28}$$

where $\tau_{12} : \Gamma \otimes_A \Omega(A) \otimes_A \Gamma \to \Omega(A) \otimes_A \Gamma \otimes_A \Gamma$ is the braiding isomorphisms. Due to the braided property (27) this definition is well defined, i.e., it is independent from the representative chosen in $\Gamma \otimes \Gamma$ for the balanced tensor product $\Gamma \otimes_A \Gamma$ (e.g. $sa \otimes \hat{s}$ versus $s \otimes a\hat{s}$). The map in (28) transforms according to the H-adjoint action $\triangleright^{cop}$ so that it is in ${}_{\Bbbk}\mathrm{hom}(\Gamma \otimes_A \Gamma, \Omega(A) \otimes_A \Gamma \otimes_A \Gamma)$. Finally it is a connection because it satisfies the Leibniz rule. A similar expression holds also for right connections.

The connections $\nabla \in \mathrm{Con}_A(\Gamma)$ and $\nabla \in {}_A\mathrm{Con}(\Gamma)$ also extend to connections $\mathrm{d}_\nabla \in \mathrm{hom}_{\Bbbk}(\Gamma \otimes_A \Omega^\bullet(A), \Gamma \otimes_A \Omega^{\bullet+1}(A))$ and $\mathrm{d}_\nabla \in {}_{\Bbbk}\mathrm{hom}(\Omega^\bullet(A) \otimes_A \Gamma, \Omega^{\bullet+1}(A) \otimes_A \Gamma)$ well-defined by

$$\mathrm{d}_\nabla : \Gamma \otimes_A \Omega^\bullet(A) \longrightarrow \Gamma \otimes_A \Omega^{\bullet+1}(A)\ ,\ \ \mathrm{d}_\nabla(s \otimes_A \theta) := \nabla(s) \wedge \theta + s \otimes_A \mathrm{d}\theta\ , \tag{29}$$

and, for all $k \in \mathbb{N}$,

$$\mathrm{d}_\nabla : \Omega^k(A) \otimes_A \Gamma \longrightarrow \Omega^{k+1}(A) \otimes_A \Gamma\ ,\quad \mathrm{d}_\nabla(\theta \otimes_A s) := \mathrm{d}\theta \otimes_A s + (-1)^k \theta \wedge \nabla(s)\ . \tag{30}$$

The H-action reads, for all $h' \in H$, $h' \triangleright \mathrm{d}_\nabla = \mathrm{d}_{h' \triangleright \nabla}$, $h' \triangleright^{cop} \mathrm{d}_\nabla = \mathrm{d}_{h' \triangleright^{cop} \nabla}$, so that we have the Leibniz rule, for all $\varsigma \in \Gamma \otimes_A \Omega^k(A)$, $\vartheta \in \Omega^\bullet(A)$ and for all $\sigma \in \Omega^\bullet(A) \otimes_A \Gamma$ and $\theta \in \Omega^k(A)$,

$$\mathrm{d}_\nabla(\varsigma \wedge \vartheta) = \mathrm{d}_\nabla \varsigma \wedge \vartheta + (-1)^k \varsigma \wedge \mathrm{d}\vartheta\ \ ,\quad \mathrm{d}_\nabla(\theta \wedge \sigma) = \mathrm{d}\theta \wedge \sigma + (-1)^k \theta \wedge \mathrm{d}_\nabla \sigma\ . \tag{31}$$

The definitions in (29), (30) are well-defined because independent from the representative chosen for the balanced tensor product over A; for the proof one can use for example Eqs. (23), (25), (26).

We extend the inner derivative $\mathrm{i}_u : \Omega^\bullet(A) \to \Omega^{\bullet-1}(A))$ to $\Omega^\bullet(A) \otimes_A \Gamma$ by, for all $u \in \mathfrak{X}(A)$,

$$\mathrm{i}_u : \Omega^\bullet(A) \otimes_A \Gamma \to \Omega^{\bullet-1}(A) \otimes_A \Gamma\ ,\ \ \theta \otimes_A s \mapsto \mathrm{i}_u(\theta \otimes_A s) = \mathrm{i}_u(\theta) \otimes_A s\ . \tag{32}$$

This allows to define the *covariant derivative* of a left connection along a vector field $u \in \mathfrak{X}(A)$. It is the linear operator of zero degree $\mathrm{d}_{\nabla_u} : \Omega^\bullet(A) \otimes_A \Gamma \to \Omega^\bullet(A) \otimes_A$

Γ defined by

$$\mathrm{d}_{\nabla_u} := \mathrm{i}_u \circ \mathrm{d}_\nabla + \mathrm{d}_\nabla \circ \mathrm{i}_u \ , \tag{33}$$

in particular on Γ we have $\mathrm{d}_{\nabla_u} = \mathrm{i}_u \circ \nabla$ that, as usual, we denote by ∇_u. The key property of ∇_u and d_{∇_u} is that they are composition of a linear map ∇ acting from the right and a linear map i_u acting from the left. This implies the braided Leibniz rule, for all $\theta \in \Omega^\bullet(A)$, $\sigma \in \Omega^\bullet(A) \otimes_A \Gamma$

$$\mathrm{d}_{\nabla_u}(\theta \wedge \sigma) = \mathscr{L}_u(\theta) \wedge \sigma + (\bar{R}^\alpha \triangleright^{cop} \theta) \wedge \mathrm{d}_{\nabla_{\bar{R}_\alpha \triangleright u}}(\sigma) \ . \tag{34}$$

In turn this and the braided Cartan relation $[\mathscr{L}_u, \mathrm{i}_v] = \mathrm{i}_{[u,v]}$, i.e., $[\mathrm{i}_u, \mathscr{L}_v] = \mathrm{i}_{[u,v]}$ imply the braided Cartan relation

$$\mathrm{d}_{\nabla_u} \circ \mathrm{i}_v - \mathrm{i}_{\bar{R}^\alpha \triangleright v} \circ \mathrm{d}_{\nabla_{\bar{R}_\alpha \triangleright u}} = \mathrm{i}_{[u,v]} \quad \text{i.e.,} \quad \mathrm{i}_u \circ \mathrm{d}_{\nabla_v} - \mathrm{d}_{\nabla_{\bar{R}^\alpha \triangleright v}} \circ \mathrm{i}_{\bar{R}_\alpha \triangleright u} = \mathrm{i}_{[u,v]} \tag{35}$$

where, despite the connection d_∇ is not H-equivariant, but since it is a left connection, the braiding acts nontrivially only the vector fields u and v, as in $[\mathscr{L}_u, \mathrm{i}_v] = \mathrm{i}_{[u,v]}$.

4.2 Curvature

The curvature of the connection $\nabla \in {}_A\mathrm{Con}(\Gamma)$ is defined by

$$\mathrm{d}^2_\nabla = \mathrm{d}_\nabla \circ \mathrm{d}_\nabla \ . \tag{36}$$

This is a left $\Omega^\bullet(A)$-linear map in ${}_{\Omega^\bullet(A)}\mathrm{hom}(\Omega^\bullet(A) \otimes_A \Gamma, \Omega^{\bullet+2} \otimes_A \Gamma)$. For example, to prove that for all $\theta \in \Omega^k(A)$, $\sigma \in \Omega^\bullet(A) \otimes_A \Gamma$, $\mathrm{d}^2_\nabla(\theta \wedge \sigma) = \theta \wedge \mathrm{d}^2_\nabla \sigma$ just use twice (31). We have a second definition of curvature of a left connection. As in [5] we define the curvature R_∇ to be the linear map $R_\nabla : \mathfrak{X}(A) \otimes \mathfrak{X}(A) \otimes \Gamma \to \Gamma$,

$$R_\nabla(u, v, s) := (\nabla_u \circ \nabla_v - \nabla_{\bar{R}^\alpha \triangleright v} \circ \nabla_{\bar{R}_\alpha \triangleright u} - \nabla_{[u,v]})(s) \ . \tag{37}$$

It satisfies, for all $u, v \in \mathfrak{X}(A)$ and $s \in \Gamma$,

$$R_\nabla(u, v, s) = -\,\mathrm{i}_u \circ \mathrm{i}_v \circ \mathrm{d}^2_\nabla(s) \ . \tag{38}$$

This implies that R_∇ is a tensor field in ${}_A\mathrm{hom}(\mathfrak{X}^2(A) \otimes_A \Gamma, \Gamma)$, where $\mathfrak{X}^2(A) = \mathfrak{X}(A) \wedge \mathfrak{X}(A) = {}^*\Omega^2(A)$. These two definitions of curvature are equivalent since ${}_A\mathrm{hom}(\Gamma, \Omega^2(A) \otimes_A \Gamma) \simeq {}_A\mathrm{hom}(\mathfrak{X}^2(A) \otimes_A \Gamma, \Gamma)$ as finitely generated projective modules in ${}^H_A\mathscr{M}_A^{\mathrm{sym,fp}}$.

4.3 Torsion

Since $\Gamma = \mathfrak{X}(A)$ is a finitely generated and projective A-bimodule we consider the canonical H-equivariant element $I := \omega^j \otimes_A e_j \in \Omega(A) \otimes \mathfrak{X}(A)$, where $\Omega(A) = {}^*\mathfrak{X}(A) = {}_A\mathrm{hom}(\mathfrak{X}(A), A)$ is the right dual module of $\mathfrak{X}(A)$ and $\{\omega^j\}$, $\{e_j\}$ a pair of dual bases. The torsion 2-form of a connection ∇ is then the tensor field

$$\mathrm{d}_{\nabla}(I) \in \Omega^2(A) \otimes_A \mathfrak{X}(A) .$$

We also define the torsion T_{∇} as the linear map $T_{\nabla} : \mathfrak{X}(A) \otimes \mathfrak{X}(A) \to \mathfrak{X}(A)$,

$$T_{\nabla}(u, v) := \nabla_u v - \nabla_{\bar{R}^\alpha \triangleright v} \bar{R}_\alpha \triangleright u - [u, v] . \tag{39}$$

Use of the Cartan identity (35) shows that it satisfies

$$T_{\nabla}(u, v) = -\,\mathrm{i}_u \circ \mathrm{i}_v \circ \mathrm{d}_{\nabla}(I) . \tag{40}$$

This implies that T_{∇} is a left A-linear map $\mathfrak{X}^2(A) \to \mathfrak{X}(A)$.

Analogous results holds for curvatures and torsions of a right connection.

4.4 Dual Connections, Cartan Structure Equations and Bianchi Identities

The dual of a left connection $\nabla \in {}_A\mathrm{Con}(\Gamma)$ is the right connection ${}^*\nabla \in \mathrm{Con}_A({}^*\Gamma)$ defined by, for all $s \in \Gamma$, ${}^*\!s \in {}^*\Gamma$,

$$\mathrm{d}\langle s, {}^*\!s\rangle = \langle \nabla s, {}^*\!s\rangle + \langle s, {}^*\nabla\, {}^*\!s\rangle . \tag{41}$$

The right connection property is indeed easily proven. Notice the absence of a braided Leibniz rule, cf. (28), which is compensated by considering a left and a right connection. This definition implies that the dual of the left connection d_{∇} is the right connection $\mathrm{d}_{{}^*\nabla}$: for all $\sigma \in \Omega^\bullet(A) \otimes_A \Gamma$ of homogeneous form degree $|\sigma|$ and ${}^*\sigma \in {}^*\Gamma \otimes_A \Omega^\bullet(A)$,

$$\mathrm{d}\langle \sigma, {}^*\sigma\rangle = \langle \mathrm{d}_{\nabla}\sigma, {}^*\sigma\rangle + (-1)^{|\sigma|}\langle \sigma, \mathrm{d}_{{}^*\nabla}{}^*\sigma\rangle ; \tag{42}$$

here we use the pairing $\langle\ ,\ \rangle : \Omega^\bullet(A) \otimes_A \Gamma \otimes_A {}^*\Gamma \otimes_A \Omega^\bullet(A) \to \Omega^\bullet(A)$, which is $\Omega^\bullet(A)$-bilinear.

The Cartan structure equations for curvature and torsion relate R_{∇} and T_{∇} on vectors fields to the curvature $\mathrm{d}_{{}^*\nabla}^{\,2}$ and the torsion $\mathrm{d} + \wedge \circ {}^*\nabla$ on forms. This last expression comes from pairing a one form θ with the torsion $\mathrm{d}_{\nabla}(I)$ and using (42):

$$\langle \mathrm{d}_{\nabla}(I), \theta \rangle = \mathrm{d}\langle I, \theta \rangle + \langle I, {}^*\nabla\, \theta \rangle = (\mathrm{d} + \wedge \circ {}^*\nabla)\theta \ .$$

Here too we used the above $\Omega^\bullet(A)$-bilinear pairing $\langle\ ,\ \rangle$ with now $\mathfrak{X}(A) = \Gamma$ and $\Omega(A) = {}^*\Gamma$.

Theorem 2 (Cartan structure equations) *For all* $u, v, z \in \mathfrak{X}(A)$*,* $\theta \in \Omega(A)$ *we have*

$$\langle R_{\nabla}(u, v, z), \theta \rangle = \langle u \otimes_A v \otimes_A z, d_{{}^*\nabla}^{\,2}\, \theta \rangle \ ,$$

$$\langle T_{\nabla}(u, v), \theta \rangle = -\langle u \otimes_A v, (d + \wedge \circ {}^*\nabla\,)\, \theta \rangle \ .$$

Proof We prove the second equation: $\langle T_{\nabla}(u, v), \theta \rangle = -\langle\, \mathrm{i}_u \circ \mathrm{i}_v \circ \mathrm{d}_{\nabla}(I), \theta \rangle = -\,\mathrm{i}_u \circ \mathrm{i}_v \langle \mathrm{d}_{\nabla}(I), \theta \rangle = -\,\mathrm{i}_u \circ \mathrm{i}_v (\mathrm{d}\theta + \wedge \circ {}^*\nabla\, \theta) = -\langle u \otimes_A v, \mathrm{d}\theta + \wedge \circ {}^*\nabla\, \theta \rangle$. □

Using a dual basis $\{e_i, \}$, $\{\omega^i\}$ of $\mathfrak{X}(A)$ and $\Omega(A)$, we define the coefficients one forms of the connection ${}^*\nabla \in \mathrm{Con}_A(\Omega(A))$, dual to $\nabla \in {}_A\mathrm{Con}(\mathfrak{X}(A))$,

$$\omega_k{}^l := \langle e_k, {}^*\nabla \omega^l \rangle \ ,$$

so that, since $\omega^k \otimes_A \langle e_k,\ \rangle$ is the identity map on $\Omega(A)$, ${}^*\nabla\omega^l = \omega^k \otimes_A \omega_k{}^l$. In terms of these coefficients we obtain

$$\mathrm{d}_{{}^*\nabla}^{\,2}\, \omega^l = \omega^k \otimes_A (\mathrm{d}\omega_k{}^l + \omega_k{}^j \wedge \omega_j{}^l) := \omega^k \otimes_A (-\mathsf{R}_k{}^l)$$

$$(\mathrm{d} + \wedge \circ {}^*\nabla\,)\omega^l = \mathrm{d}\omega^l + \omega^j \wedge \omega_j{}^l := \mathsf{T}^l$$

where we have defined the curvature and torsion coefficients 2-forms $\mathsf{R}_k{}^l$ and T^l. As in commutative differential geometry, applying $\mathrm{id}_{\Omega(A)} \otimes_A \mathrm{d}$ to the first equation and differentiating the second we readily obtain the Bianchi identities,

$$\begin{aligned} &\omega^k \otimes_A (\mathrm{d}\mathsf{R}_k{}^l + \omega_k{}^j \wedge \mathsf{R}_j{}^l - \mathsf{R}_k{}^j \wedge \omega_j{}^l) = 0 \ , \\ &\mathrm{d}\mathsf{T}^l - \mathsf{T}^j \wedge \omega_j{}^l = \omega^j \wedge \mathsf{R}_j{}^l \ . \end{aligned} \tag{43}$$

Notice that the commutator $[\omega, \mathsf{R}]_k^{\,l} := \omega_k{}^j \wedge \mathsf{R}_j{}^l - \mathsf{R}_k{}^j \wedge \omega_j{}^l$ in the first identity is not a braided commutator.

5 Braided Riemannian Geometry

Since $\Omega(A) = {}^*\mathfrak{X}(A)$ is a finitely generated A-bimodule (a module in ${}^H_A\mathscr{M}_A^{\mathrm{sym,fp}}$) we have the isomorphism

$$\Omega(A) \otimes_A \Omega(A) \xrightarrow{\flat} {}_A\mathrm{hom}(\mathfrak{X}(A), \Omega(A)) \ , \ \ \mathsf{g} = \mathsf{g}^a \otimes_A \mathsf{g}_a \mapsto \ \mathsf{g}^\flat := \langle\ , \mathsf{g}^a \rangle \otimes_A \mathsf{g}_a$$

where $\mathsf{g}^\flat(u) = \langle u, \mathsf{g}^a\rangle\, \mathsf{g}_a$ for all $u \in \mathfrak{X}(A)$. We say that an element $\mathsf{g} \in \Omega(A) \otimes_A \Omega(A)$ is *braided symmetric* if $\mathsf{g} = \tau(\mathsf{g})$.

A *pseudo-Riemannian metric* $\mathsf{g} \in \Omega(A) \otimes_A \Omega(A)$ is a braided symmetric element such that $\mathsf{g}^\flat \in {}_A\mathrm{hom}(\mathfrak{X}(A), \Omega(A))$ is invertible.

Recall that a connection ∇ on $\mathfrak{X}(A)$ lifts, via Eq. (28), to a connection on $\mathfrak{X}(A) \otimes_A \mathfrak{X}(A)$ and, considering the dual connection, to a connection ${}^*\nabla$ on $\Omega(A) \otimes_A \Omega(A)$.

Definition 1 A connection $\nabla \in {}_A\mathrm{Con}(\mathfrak{X}(A))$ is *metric compatible* if ${}^*\nabla \mathsf{g} = 0$ where ${}^*\nabla$ is the dual connection on $\Omega(A) \otimes_A \Omega(A)$. A Levi-Civita connection $\nabla \in {}_A\mathrm{Con}(\mathfrak{X}(A))$ is a metric compatible and torsion free connection.

We can now state the main result.

Theorem 3 (Levi-Civita connection) *Let H be a triangular Hopf algebra, A a braided commutative H-module algebra and the associated module of one forms be finitely generated and projective. For any metric g there is a unique Levi-Civita connection.*

Sketch of the proof Assume $\nabla \in {}_A\mathrm{Con}(\mathfrak{X}(A))$ is a torsion free metric compatible connection. Applying i_u to the identity $\mathrm{d}\langle v \otimes_A z, \mathsf{g}\rangle = \langle \nabla(v \otimes_A z), \mathsf{g}\rangle + \langle v \otimes_A z, {}^*\nabla \mathsf{g}\rangle = \langle \nabla(v \otimes_A z), \mathsf{g}\rangle$ we obtain $\mathscr{L}_u\langle v \otimes_A z, \mathsf{g}\rangle = \langle \nabla_u(v \otimes_A z), \mathsf{g}\rangle$. We use the braided Leibniz rule in (28) in order to compute $\nabla_u = \mathrm{i}_u \circ \nabla$ on v and z. Then, as in the commutative case we use the torsion free condition $T(u, v) = \nabla_u v - \nabla_{{}^\eta v}\, {}_\eta u - [u, v] = 0$ and the braiding properties to obtain

$$\mathscr{L}_u\langle v \otimes_A z, \mathsf{g}\rangle = \langle {}^{\beta\gamma}z \otimes_A \nabla_{{}_\beta{}^\eta v}\, {}_{\gamma\eta}u\,, \mathsf{g}\rangle \,+\, \langle [u, v] \otimes_A z\,, \mathsf{g}\rangle \,+\, \langle {}^\alpha v \otimes_A \nabla_{{}_\alpha u} z\,, \mathsf{g}\rangle \tag{44}$$

where we used the notation ${}^\alpha v \otimes {}_\alpha u = \bar{R}^\alpha \triangleright v \otimes \bar{R}_\alpha \triangleright u$ and similarly ${}^{\beta\gamma}z \otimes {}_\beta{}^\eta v \otimes {}_{\gamma\eta}u = \bar{R}^\beta \bar{R}^\gamma \triangleright z \otimes \bar{R}_\beta \bar{R}^\eta \triangleright v \otimes \bar{R}_\gamma \bar{R}_\eta \triangleright v$. We rewrite this identity for the cyclically permuted elements $u \otimes v \otimes z \mapsto {}^{\alpha\beta}z \otimes {}_\alpha u \otimes {}_\beta v$ and $u \otimes v \otimes z \mapsto {}^\eta v \otimes {}^\gamma z \otimes {}_{\gamma\eta}u$; then subtract the second from the first and add the third thus obtaining (after using the Yang–Baxter equation, the braided symmetry of the metric and the braided antisymmetry of the braided Lie bracket of vector fields)

$$\begin{aligned} 2\langle {}^\alpha v \otimes_A \nabla_{{}_\alpha u} z\,, \mathsf{g}\rangle &= \mathscr{L}_u\langle v \otimes_A z, \mathsf{g}\rangle - \mathscr{L}_{{}^\alpha v}\langle {}_\alpha u \otimes_A z, \mathsf{g}\rangle + \mathscr{L}_{{}^{\alpha\beta}z}\langle {}_\alpha u \otimes_A {}_\beta v, \mathsf{g}\rangle \\ &\quad - \langle [u, v] \otimes_A z\,, \mathsf{g}\rangle + \langle u \otimes_A [v, z]\,, \mathsf{g}\rangle + \langle [u, {}^\beta z] \otimes_A {}_\beta v\,, \mathsf{g}\rangle\,. \end{aligned} \tag{45}$$

The right hand side of this identity uniquely determines the left hand side. Since $\mathsf{g}^\flat$ is invertible it determines the unique covariant derivative $\nabla_u : \mathfrak{X}(A) \to \mathfrak{X}(A)$ for all $u \in \mathfrak{X}(A)$ and hence uniqueness of the metric compatible and torsion free connection ∇. We now do not assume existence of the Levi-Civita connection and define ∇ using (45). The properties of the right hand side of (45) then show that ∇ is a connection. □

We observe that the Koszul formula (45) generalizes to the case of arbitrary metric tensors the Koszul formula obtained in [20] for the case of H-equivariant metrics: $h \triangleright \langle u \otimes_A v, \mathsf{g}\rangle = \langle h \triangleright (u \otimes_A v), \mathsf{g}\rangle$, for all $h \in H, u, v, \in \mathfrak{X}(A)$.

The Ricci tensor is the trace of the Riemann tensor, using a pair of dual basis $\{\omega^j\}$, $\{e_j\}$ it is given by, for all $u, v \in \mathfrak{X}(A)$,

$$Ric(u, v) = \langle \omega^i, R_{\nabla}(e_i, u, v)\rangle' \ , \tag{46}$$

where

$$\langle\ ,\ \rangle' := \langle\ ,\ \rangle \circ \tau : \Omega(A) \otimes_A \mathfrak{X}(A) \to A \ , \quad \langle \theta, u\rangle' = \langle \bar{R}^\alpha \triangleright u, \bar{R}_\alpha \triangleright^{cop} \theta\rangle$$

is the pairing with forms on the left of vector fields. Since $I = \omega^i \otimes_A e_i \in \Omega \otimes_A \mathfrak{X}(A)$ commutes with the elements of A and since the curvature R_{∇} is left A-linear, Ric is a well defined left A-linear tensor in ${}_A\mathrm{hom}(\mathfrak{X}(A) \otimes_A \mathfrak{X}(A), A)$. We can then define an Einstein metric on A to be a metric g proportional (via a coefficient λ) to the Ricci tensor of its Levi-Civita connection, for all $u, v \in \mathfrak{X}(A)$,

$$Ric(u, v) = \lambda \langle u \otimes_A v, \mathsf{g}\rangle \ .$$

Here both Ric and $\langle\ \ , \mathsf{g}\rangle$ are left A-linear tensors in ${}_A\mathrm{hom}(\mathfrak{X}(A) \otimes_A \mathfrak{X}(A), A)$. Noncommutative Einstein spaces arise as solutions of this equation.

Acknowledgements The author would like to thank the organizers of the XIV International Workshop *Lie Theory and its applications in physics*, where this research was presented. The author acknowledges partial support from INFN, CSN4, Iniziativa Specifica GSS. This research has a financial support from Università del Piemonte Orientale. The author is affiliated to INdAM-GNFM.

References

1. J. Arnlind, M. Wilson, Riemannian curvature of the noncommutative 3-sphere. J. Noncommut. Geom. **11**, 507–536 (2017)
2. P. Aschieri, Cartan structure equations and Levi-Civita connection in braided geometry. arXiv: 2006.02761 [math.QA]
3. P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp, J. Wess, A gravity theory on noncommutative spaces. Class. Quant. Grav. **22**, 3511–3532 (2005)
4. P. Aschieri, L. Castellani, Noncommutative gravity solutions. J. Geom. Phys. **60**, 375–393 (2010)
5. P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess, Noncommutative geometry and gravity. Class. Quantum Gravity **23**, 1883–1911 (2006)
6. P. Aschieri, M. Dimitrijevic, P. Kulish, F. Lizzi, J. Wess, Noncommutative spacetimes: symmetries in noncommutative geometry and field theory. Springer Lect. Notes Phys. 774 (2009)
7. P. Aschieri, A. Schenkel, Noncommutative connections on bimodules and Drinfeld twist deformation. Adv. Theor. Math. Phys. **18**, 513–612 (2014)
8. E.J. Beggs, S. Majid, *-compatible connections in noncommutative Riemannian geometry. J. Geom. Phys. **61**, 95–124 (2011)

9. E.J. Beggs, S. Majid, *Quantum Riemannian Geometry* (Springer, 2020)
10. J. Bhowmick, D. Goswami, G. Landi, The Koszul formula in noncommutative geometry. Rev. Math. Phys. **32**(10), 2050032 (2020)
11. J. Bhowmick, D. Goswami, G. Landi, Levi-Civita connections and vector fields for noncommutative differential calculi. Int. J. Math. **31**, 2050065 (2020)
12. S. Doplicher, K. Fredenhagen, J.E. Roberts, The quantum structure of spacetime at the planck scale and quantum fields. Commun. Math. Phys. **172**, 187–220 (1995)
13. M. Dubois-Violette, P.W. Michor, Connections on central bimodules in noncommutative differential geometry. J. Geom. Phys. **20**, 218–232 (1996)
14. X. Gomez, S. Majid, Braided Lie algebras and bicovariant differential calculi over coquasitriangular Hopf algebras. J. Algebr. **261**, 334–388 (2003)
15. D. Gurevich, A. Radul, V. Rubtsov, Noncommutative differential geometry related to the Young-Baxter equation. J. Math. Sci. **77**, 3051–3062 (1995)
16. I. Heckenberger, K. Schmüdgen *Levi-Civita Connections on the Quantum Groups $SL_q(N)$, $O_q(N)$ and $Sp_q(N)$*, Commun. Math. Phys. 185 (1997) 177 - 196
17. J. Madore, *An Introduction to Noncommutative Differential Geometry and its Physical Applications* (London Mathematical Society, 1999)
18. E. Plauschinn, Non-geometric backgrounds in string theory. Phys. Rept. **798**, 1–122 (2019)
19. J. Rosenberg, Levi-Civita's theorem for noncommutative tori. SIGMA **9**, 071 (2013)
20. T. Weber, Braided Cartan calculi and submanifold algebras. J. Geom. Phys. **150**, 103612 (2020)
21. S.L. Woronowicz, Differential calculus on compact matrix pseudogroups (quantum groups). Comm. Math. Phys. **122**, 125–170 (1989)

Notes on AdS_4 Holography and Higher-Derivative Supergravity

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Abstract I summarize recent results on higher-derivative conformal supergravity which find important applications in AdS/CFT and black hole physics. In particular, I show how to derive the higher-derivative action of four-dimensional minimal gauged supergravity and use it in conjunction with holography and supersymmetric localization to derive new results for the large N partition functions of three-dimensional supersymmetric matter coupled Chern–Simons theories. In addition, these methods can be used to derive the leading corrections to the Bekenstein–Hawking entropy of general four-dimensional AdS black holes.

Keywords Supergravity · AdS/CFT · Black holes · String theory

1 Introduction

The gravitational side of the AdS/CFT correspondence is under good calculational control in the classical supergravity limit of string and M-theory which allows for explicit calculations of physical observables in the planar limit of the dual gauge theory. Going beyond this approximation requires calculating higher-derivative corrections to ten- or eleven-dimensional supergravity and understanding their effects on holographic observables. This is technically challenging and there are few explicit results available in the literature. The goal of this note is to summarize recent results which bypass some of these difficulties by eschewing the need to work in ten or eleven dimensions and study higher-derivative corrections directly in four-dimensional gauged supergravity [1–3].

The simplest setup in this context is given by the gravity multiplet of four-dimensional $\mathcal{N} = 2$ gauged supergravity which captures the universal dynamics of the energy-momentum multiplet in the dual three-dimensional $\mathcal{N} = 2$ SCFT. The two-derivative action for the bosonic fields is the Einstein–Maxwell action with a

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_6

negative cosmological constant. The four-derivative action for this model can be studied using techniques from conformal supergravity. It was shown in [1, 3] that there are only two supersymmetric four-derivative terms in the action which have arbitrary real coefficients, c_1 and c_2. Despite the presence of non-trivial corrections to the Lagrangian of the theory every solution of the two-derivative equations of motion also solves the four-derivative equations. Moreover the amount of supersymmetry preserved by a given solution is not affected by the four-derivative corrections.

These supergravity results prove powerful in the context of holography. In particular they allow for an explicit evaluation of the regularized on-shell action of any solution to the four-derivative supergravity theory which in turn captures the path integral of the dual SCFT. In addition, the presence of the higher-derivative corrections modifies the thermodynamics of black hole solutions in the theory. The main results for the on-shell action and the black hole entropy in this context can be found in (10) and (22) below, respectively.

The constants $c_{1,2}$ are free parameters in four-dimensional supergravity but they should be uniquely fixed by the embedding of the model in string or M-theory. In the absence of such an explicit embedding one can appeal to the holographically dual field theory and study its path integral to subleading order in the planar limit. Indeed, this proves to be a fruitful strategy in the context of three-dimensional SCFTs realized on the worldvolume of N M2-branes which are dual to orbifolds of the $\mathrm{AdS}_4 \times S^7$ background of M-theory. Combining the results for the higher-derivative on-shell action with supersymmetric localization results in the large N limit one finds that the partition function of the SCFTs on various compact manifolds can be computed explicitly to order $N^{\frac{1}{2}}$, see (30) below.

2 Minimal Gauged Supergravity

We start with a short summary of the conformal supergravity formalism, see [4] for a review and further references. We are interested in the four-dimensional minimal gauged $\mathcal{N} = 2$ supergravity specified by the Weyl multiplet, an auxiliary vector multiplet and an auxiliary hypermultiplet. We will mostly work in Euclidean signature in view of the applications of our results to evaluating on-shell actions and comparing them to the dual CFT partition functions evaluated by supersymmetric localization. In Euclidean signature, the vector multiplet is related to the reducible combination of a real chiral multiplet $\mathcal{X}_+$ and a real anti-chiral multiplet $\mathcal{X}_-$, see [5]. The quadratic terms $(\mathcal{X}_\pm)^2$ can be used to construct a supersymmetric Lagrangian by using (anti-)chiral superspace integrals [6],

$$\mathcal{L}_{\mathrm{V}} = \tfrac{1}{2} \int d^4\theta \, \mathcal{E}_+ \, (\mathcal{X}_+)^2 + \tfrac{1}{2} \int d^4\bar{\theta} \, \mathcal{E}_- \, (\mathcal{X}_-)^2 \, , \tag{1}$$

where $\mathcal{E}_\pm$ is the (anti-)chiral superspace measure. The two-derivative action of minimal gauged supergravity is obtained by adding to (1) the Lagrangian for the hypermultiplet as described in [5]. The hypermultiplet implements the gauging in the supergravity theory since it transforms locally under a U(1) subgroup of the SU(2) R-symmetry.

Our goal is to study two supersymmetric Lagrangian densities containing higher-derivative couplings. One of them is constructed from the Weyl multiplet, related to a chiral and an anti-chiral tensor multiplet $\mathcal{W}_{\pm}^{ab}$, which is squared to arrive at the superspace integrals [7]

$$\mathcal{L}_{\mathrm{W}^2} = -\tfrac{1}{64} \int d^4\theta \, \mathcal{E}_+ \, (\mathcal{W}_+^{ab})^2 - \tfrac{1}{64} \int d^4\bar{\theta} \, \mathcal{E}_- \, (\mathcal{W}_-^{ab})^2 \,. \tag{2}$$

The other is built from the $\mathbb{T}$-log multiplet and contains the supersymmetric completion of the Gauss–Bonnet term [8]. In superspace notation, it can be written as

$$\mathcal{L}_{\mathbb{T}\log} = -\tfrac{1}{2} \int d^4\theta \, \mathcal{E}_+ \, \Phi'_+ \overline{\nabla}^4 \ln \Phi_- + \text{anti-chiral} \,. \tag{3}$$

Here Φ'_+ is a chiral multiplet and Φ_- is an anti-chiral multiplet and, as discussed in [8], when Φ'_+ is a constant multiplet, Φ_- can be identified with $\mathcal{X}_-$. Note that, in minimal gauged supergravity, identifying Φ'_+ with a composite chiral multiplet, which needs to carry zero Weyl weight [8], leads to terms in the Lagrangian (3) with at least six derivatives. We do not consider such higher-order terms here and therefore set $\Phi'_+ = 1$.

One can study other supersymmetric R^2-invariants constructed from tensor multiplets, see [9, 10]. However the effects of the gauging on these invariants has not been studied in detail in the literature. Moreover as discussed recently in [3] the addition of these other supersymmetric invariants is either not allowed or does not change the results presented below.

The Lagrangians in (1), (2), and (3) are superconformally invariant by construction. The coefficient of the two-derivative Lagrangian can be set to unity by simple field redefinitions, and therefore we find that there are two arbitrary real coefficients, c_1 and c_2, which determine the four-derivative Lagrangian. Moreover, one can show that the bosonic terms in (2) and (3) are related by $\mathcal{L}_{\mathrm{W}^2} + \mathcal{L}_{\mathbb{T}\log} = \mathcal{L}_{\mathrm{GB}}$, where $\mathcal{L}_{\mathrm{GB}}$ is the Gauss–Bonnet density [8]. We can thus eliminate the $\mathbb{T}$-log Lagrangian in favor of the Weyl-squared and Gauss–Bonnet terms and arrive at the following superconformal higher-derivative Lagrangian

$$\mathcal{L}_{\mathrm{HD}} = \mathcal{L}_{2\partial} + (c_1 - c_2) \, \mathcal{L}_{\mathrm{W}^2} + c_2 \, \mathcal{L}_{\mathrm{GB}} \,. \tag{4}$$

Starting from (4), one can obtain the Lagrangian density in the Poincaré frame by gauge-fixing the extra symmetries and eliminating the auxiliary fields that ensure off-shell closure of the superconformal algebra. This procedure results in an action that involves only the dynamical fields of minimal gauged supergravity. This calculation is straightforward but tedious and is discussed in more details in [3]. An important ingredient in this technical analysis is the observation that, upon choosing convenient gauge-fixing conditions for the superconformal symmetries, the extra superconformal fields can be eliminated from (4) using their two-derivative solutions, even in the presence of the higher-derivative couplings. The result of these conformal supergravity calculations is the following four-derivative bosonic Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L}_{2\partial} &= -(16\pi\, G_N)^{-1}\big[R + 6\,L^{-2} - \tfrac{1}{4}\,F_{ab}F^{ab}\big]\,, \\ e^{-1}\mathcal{L}_{\mathrm{W}^2} &= \big(C_{ab}{}^{cd}\big)^2 - L^{-2}F_{ab}F^{ab} + \tfrac{1}{2}\big(F^{+}_{ab}\big)^2\big(F^{-}_{cd}\big)^2 \\ &\qquad - 4\,F^{-}_{ab}R^{ac}F_c^{+b} + 8\,\big(\nabla^a F^{-}_{ab}\big)\big(\nabla^c F_c^{+b}\big)\,, \\ e^{-1}\mathcal{L}_{\mathrm{GB}} &= R^{abcd}\,R_{abcd} - 4\,R^{ab}R_{ab} + R^2\,, \end{aligned} \tag{5}$$

where G_N is the Newton constant, $C_{ab}{}^{cd}$ is the Weyl tensor, F_{ab} is the graviphoton field strength, and L determines the cosmological constant.

A somewhat unexpected property of the Lagrangian (5) was observed in [1], namely it was shown in [1, 3] that the solutions of the two-derivative equations of motion also solve the equations of motion derived from (5). Similar results have been shown also for ungauged $\mathcal{N} = 2$ supergravity, see [11], as well as for non-supersymmetric gravitational theories [12]. In addition to that, it can be shown that the supersymmetry variations of the gravitini are not modified by the presence of the higher-derivative couplings. This implies that any solution of the two-derivative Poincaré action of four-dimensional minimal gauged supergravity is also a solution of the higher-derivative action (4), and it preserves the same amount of supersymmetry. From now on we focus solely on solutions of the two-derivative equations of motion.

3 On-Shell Action

A central observable in holography is the appropriately regularized value of the on-shell gravitational action on asymptotically AdS solutions. In order to evaluate the action (4) on solutions of the two-derivative EoM one can use (5) to derive the following identity for the on-shell values of the three actions:

$$I_{\mathrm{W}^2} = I_{\mathrm{GB}} - \frac{64\pi\, G_N}{L^2}\, I_{2\partial}\,. \tag{6}$$

The divergences in the on-shell actions on the right-hand side of (6) can be removed via holographic renormalization using the following counterterms, see [3] for more details:

$$\begin{aligned} I^{\mathrm{CT}}_{2\partial} &= \frac{1}{8\pi\, G_N}\int d^3x\sqrt{h}\,\big(-K + \tfrac{1}{2}\,L\,\mathcal{R} + 2\,L^{-1}\big)\,, \\ I^{\mathrm{CT}}_{\mathrm{GB}} &= 4\int d^3x\sqrt{h}\,\big(\mathcal{J} - 2\,\mathcal{G}_{ab}\,K^{ab}\big)\,, \end{aligned} \tag{7}$$

where h_{ab} is the induced metric on the boundary, K_{ab} is the extrinsic curvature, $\mathcal{R}$ and $\mathcal{G}_{ab}$ are the boundary Ricci scalar and Einstein tensor, respectively, and $\mathcal{J}$ is defined in term of the extrinsic boundary curvature and its trace as

$$\mathcal{J} = \tfrac{1}{3}\big(3K(K_{ab})^2 - 2(K_{ab})^3 - K^3\big)\,. \tag{8}$$

Using (5) and (7), one finds the following regularized on-shell actions

$$I_{2\partial} + I_{2\partial}^{\text{CT}} = \frac{\pi L^2}{2G_N}\mathcal{F}\,, \qquad I_{\text{GB}} + I_{\text{GB}}^{\text{CT}} = 32\pi^2\chi\,. \tag{9}$$

Here $\mathcal{F}$ depends on the two-derivative solution $\mathcal{M}_4$, and χ is the Euler characteristic of $\mathcal{M}_4$. Combining the results in (4), (6), and (9), one arrives at the following universal formula for the regularized four-derivative on-shell action in minimal gauged supergravity:

$$I_{\text{HD}} = \Big[1 + \frac{64\pi G_N}{L^2}(c_2 - c_1)\Big]\frac{\pi L^2}{2G_N}\mathcal{F} + 32\pi^2 c_1\,\chi\,. \tag{10}$$

This simple formula expresses the full four-derivative on-shell action in terms of the regularized two-derivative result, determined by $\mathcal{F}$, together with the topological invariant χ. We emphasize that this result is valid for *all* solutions of the two-derivative EoM and is independent of supersymmetry.

In the context of holography the on-shell action in (10) is dual to the logarithm of the partition function of a three-dimensional $\mathcal{N} = 2$ SCFT defined on the boundary of $\mathcal{M}_4$. Two important examples for $\mathcal{M}_4$ include Euclidean AdS$_4$ solutions with squashed S^3 boundary as well as Euclidean black hole solutions with $S^1 \times \Sigma_{\mathfrak{g}}$ boundary. We now present some more details on these two solutions.

We start by discussing an Euclidean $\frac{1}{4}$-BPS solution which can be obtained from the Plebanski–Demianski solutions of the Einstein–Maxwell theory. This solution is holographically dual to a 3d SCFT placed on the squashed S^3 background with $U(1) \times U(1)$ invariance studied in [13]. The Euclidean supersymmetric gravity solution of interest is studied in some detail in [14] and can be written as

$$\begin{aligned} ds^2 &= f_1(x,y)^2 dx^2 + f_2(x,y)^2 dy^2 + \frac{(d\Psi + y^2 d\Phi)^2}{f_1(x,y)^2} + \frac{(d\Psi + x^2 d\Phi)^2}{f_2(x,y)^2}\,, \\ A &= \frac{b^4 - 1}{L(x+y)}(d\Psi - xy\,d\Phi)\,, \end{aligned} \tag{11}$$

where the metric functions f_1 and f_2 are

$$f_1(x,y)^2 = L^2\frac{y^2 - x^2}{(x^2-1)(b^4-x^2)}\,, \qquad f_2(x,y)^2 = L^2\frac{y^2 - x^2}{(y^2-1)(y^2-b^4)}\,. \tag{12}$$

The solution depends on a single real parameter $b > 0$. In the limit $b \to 1$ the gauge field vanishes and we recover the Euclidean AdS_4 solution. The coordinate ranges for the noncompact coordinates are $x \in [1, b^2]$, $y \in [b^2, \infty)$. The conformal boundary is approached as $y \to \infty$, and thus in order to evaluate the on-shell action we introduce a cut-off on the y range of integration at a finite value y_b. Using the counterterms discussed above and evaluating explicitly this on-shell action we find

$$\mathcal{F} = \frac{1}{4}\left(b + \frac{1}{b}\right) , \qquad \chi = 1 . \tag{13}$$

To obtain the full result for the on-shell action these values should be plugged in the general formula (10).

Another important solution of the two-derivative equations of motion is given by a supersymmetric Euclidean version of the dyonic Reissner–Nordström black hole. We refer to this background as the Euclidean Romans solution, see for instance [15, 16]. The solution takes the explicit form

$$\begin{aligned} ds^2 &= U(r)d\tau^2 + \frac{dr^2}{U(r)} + r^2 ds^2_{\Sigma_\mathfrak{g}} , \\ U(r) &= \left(\frac{r}{L} + \frac{\kappa L}{2r}\right)^2 - \frac{Q^2}{4r^2} , \qquad F = \pm \kappa L\, V_{\Sigma_\mathfrak{g}} + \frac{Q}{r^2} d\tau \wedge dr . \end{aligned} \tag{14}$$

With $ds^2_{\Sigma_\mathfrak{g}}$ we denote the metric on a constant curvature Riemann surface of genus $\mathfrak{g}$ with normalization chosen such that the curvature κ is given by $\kappa = 1$, $\kappa = 0$, and $\kappa = -1$ for genus $\mathfrak{g} = 0$, $\mathfrak{g} = 1$, and $\mathfrak{g} > 1$, respectively. Note that supersymmetry requires the magnetic flux P across the Riemann surface to have magnitude $|P| = |\kappa| L$. The electric charge Q on the other hand is a free parameter and is not restricted by supersymmetry. We denote the volume form on the Riemann surface by $V_{\Sigma_\mathfrak{g}}$, and define $\omega_{\Sigma_\mathfrak{g}}$ to be the one-form potential for this volume form such that $d\omega_{\Sigma_\mathfrak{g}} = V_{\Sigma_\mathfrak{g}}$. Integrating the volume form yields:

$$\mathrm{Vol}(\Sigma_\mathfrak{g}) = \int_{\Sigma_\mathfrak{g}} V_{\Sigma_\mathfrak{g}} = 2\pi\eta , \tag{15}$$

where $\eta = 2|\mathfrak{g} - 1|$ if $\mathfrak{g} \neq 1$ and $\eta = 1$ if $\mathfrak{g} = 1$. The metric function $U(r)$ has two zeroes $r_\pm$, given by

$$r_\pm = L\sqrt{-\frac{\kappa}{2} \pm \frac{|Q|}{2L}} . \tag{16}$$

We impose that the outer radius r_+ is real in order for the spacetime to cap off at a real value of the coordinate r. Additionally, we need to ensure that $r_+ > 0$ to avoid a naked singularity where the Riemann surface shrinks down to zero size. We therefore have to demand that

$$|Q| > \kappa L . \tag{17}$$

This imposes the constraint $|Q| > L$ for $\mathfrak{g} = 0$, $|Q| > 0$ for $\mathfrak{g} = 1$, while imposing no constraint for a higher-genus Riemann surface. Thus, for $\kappa = 0, 1$, we cannot take the $Q \to 0$ limit if we insist on having a non-singular and real metric.

Assuming that the above conditions are satisfied, then as $r \to r_+$, the metric becomes locally $\mathbb{R}^2 \times \Sigma_{\mathfrak{g}}$. The $\mathbb{R}^2$ is written in polar coordinates (r, τ) where the τ coordinate has periodicity

$$\beta = \frac{2\pi L r_+}{|Q|} = \frac{2\pi L^2}{|Q|}\sqrt{-\frac{\kappa}{2} + \frac{|Q|}{2L}} \,. \tag{18}$$

To compute the on-shell action of this solution we integrate the radial coordinate from $r = r_+$ to a cut-off at $r = r_b$, and the time coordinate from $\tau = 0$ to $\tau = \beta$. Using the counterterms described above and taking the cutoff to infinity we find

$$\mathcal{F} = 1 - \mathfrak{g} \,, \qquad \chi = 2(1 - \mathfrak{g}) \,. \tag{19}$$

All dependence on the electric charge Q drops out of the final on-shell action, and so once we fix the genus $\mathfrak{g}$ of the Riemann surface we have a one-parameter family of solutions (labelled by the charge Q) all with the same on-shell action. This independence of the on-shell action on Q was discussed in detail in [16] and here we have shown how the result extends in the presence of higher derivative corrections. We note that using the relation (18) the independence of the on-shell action on Q implies that it is independent of the periodicity of the Euclidean time coordinate τ.

Our results can be related to an important observable in AdS$_4$ holography: the coefficient, C_T, of the two-point function of the energy momentum tensor in the dual SCFT. Using the four-derivative action in (5) and the results in [17] we find

$$C_T = \frac{32L^2}{\pi G_N} + 2048(c_2 - c_1) \,. \tag{20}$$

This result is valid for all three-dimensional holographic SCFTs captured by our minimal supergravity setup. Another way to derive the same result for C_T is to use a Ward identity that relates it to the second derivative of the squashed sphere partition function, see [18]. One can indeed use (10) and (13) to confirm that the Ward identity $C_T = \frac{32}{\pi^2}\frac{\partial^2 I_{S^3_b}}{\partial b^2}|_{b=1}$ is obeyed. This constitutes a non-trivial consistency check of our results.

4 Black Hole Thermodynamics

The thermodynamics of black holes is modified by the four-derivative terms in (5). To study this we consider a stationary black hole solution to the two-derivative equations of motion and work in Lorentzian signature implemented via a Wick-rotation of the Lagrangian in (5).

In a higher-derivative theory of gravity, the black hole entropy can be computed using the Wald formalism [19]:

$$S = -2\pi \int_H E^{abcd}\, \varepsilon_{ab}\, \varepsilon_{cd}\,, \tag{21}$$

where the integral is over the two-dimensional horizon H, E^{abcd} is the variation of the Lagrangian with respect to the Riemann tensor, and ε_{ab} is the unit binormal to the horizon. Using the Lagrangian (5) and the equations of motion, we find that the entropy is

$$S = (1+\alpha)\,\frac{A_H}{4G_N} - 32\pi^2 c_1\, \chi(H)\,, \tag{22}$$

where A_H is the area and $\chi(H)$ the Euler characteristic of the horizon and we defined $\alpha := \frac{64\pi G_N}{L^2}(c_2 - c_1)$. We find two modifications to the entropy: a topological term independent of the charges of the black hole, accompanied by an overall rescaling of the Bekenstein–Hawking area law.

The four-derivative terms in the action also modify the conserved quantities associated with Killing vectors of the spacetime. To study this take Σ to be a time-like boundary at spatial infinity. The conserved charge $\mathcal{Q}$ associated with a Killing vector K can be computed by the Komar integral

$$\mathcal{Q}[K] = \int_{\partial\Sigma} d^2x\, \sqrt{\gamma}\, n^a K^b \tau_{ab}\,, \quad \tau_{ab} := \frac{2}{\sqrt{h}}\frac{\delta \mathcal{L}_{\mathrm{HD}}}{\delta h^{ab}}\,, \tag{23}$$

with γ the induced metric on the boundary surface $\partial\Sigma$, n^a the unit normal to $\partial\Sigma$, and τ_{ab} the boundary stress tensor [20]. Using (6) and (7), we find that for any solution of the two-derivative EoM the boundary stress tensor takes the universal form

$$\tau^{ab} = (1+\alpha)\tau^{ab}_{2\partial} - c_1\, \tau^{ab}_{\mathrm{GB}}\,, \tag{24}$$

where $\tau^{ab}_{2\partial}$ is the boundary stress-tensor associated with $\mathcal{L}_{2\partial}$ and τ^{ab}_{GB} is the boundary stress tensor associated with $\mathcal{L}_{\mathrm{GB}}$ in (5). The topological nature of the Gauss–Bonnet term ensures that τ^{ab}_{GB} gives no contribution to the Komar integral [21]. This implies that the four-derivative terms in (5) simply rescale the Komar charges of the original two-derivative solution. In particular, the mass and angular momentum of the black hole are $M = (1+\alpha)M_{2\partial}$ and $J = (1+\alpha)J_{2\partial}$, respectively.

To study the electromagnetic charges of the black hole we note that the Maxwell equations can be written as $dG = dF = 0$, where F is the two-form graviphoton field strength and G is the two-form defined by

$$(\star G)_{\mu\nu} = 32\pi\, G_N\, \frac{\delta \mathcal{L}_{\mathrm{HD}}}{\delta F^{\mu\nu}}\,. \tag{25}$$

The electric and magnetic charges Q and P are defined by integrating G and F over $\partial\Sigma$:

$$Q = \int_{\partial\Sigma} G \,, \qquad P = \int_{\partial\Sigma} F \,. \tag{26}$$

The field strength F, and therefore the magnetic charge, is unaffected by the higher-derivative terms. However, the latter modify G, which in turn modifies the electric charge as $Q = (1+\alpha)Q_{2\partial}$.

As a consistency check of our results we consider the quantum statistical relation between the thermodynamic quantities of a black hole and its Euclidean on-shell action [22]

$$I = \beta\,(M - TS - \Phi Q - \omega J) \,, \tag{27}$$

where $T = \beta^{-1}$ is the temperature, Φ is the electric potential, and ω is the angular velocity of the black hole. These intensive quantities are fully determined by the two-derivative solution and are therefore not modified since the black hole background is not affected by the four-derivative terms in the action. The same is not true for the extensive quantities I, S, M, Q, and J computed above. Comparing (10) to (22), we find that if the quantum statistical relation is satisfied in the two-derivative theory then it is also satisfied in the four-derivative theory provided that the Euler characteristics of the full Euclidean solution and the horizon are equal, $\chi(\mathcal{M}_4) = \chi(H)$. This relation can been shown to hold in general, see [3], and we have checked it explicitly for all known asymptotically AdS$_4$ stationary black holes.

Our results imply that the ratio Q/M for extremal black holes is not affected by the four-derivative terms in (5) and thus the corrections to the black hole entropy in (22) have no relation to the extremality bound. Moreover, the black hole entropy corrections do not have a definite sign and therefore do not necessarily lead to an increase in the entropy for all black holes. These results are in conflict with some of the claims in the literature about the weak gravity conjecture implying positivity of entropy corrections and are discussed further in [3].

5 Field Theory and Holography

To make a connection between the results above and holography we now assume that the four-dimensional supergravity action in (4) arises as a consistent truncation of M-theory on an orbifold of S^7. This consistent truncation has been established at the two-derivative level and in the absence of orbifold singularities in [23] and we will assume that these results extend also to our setup.[1] We consider two classes of smooth orbifolds for which the low energy dynamics of N M2-branes is captured by

[1] It will be very interesting to extend this result to include higher-derivative terms and to study potential subtleties arising from orbifolds with fixed points.

Table 1 The constants in (30) for two classes of SCFTs

Theory	A	B	C
ABJM at level k	$\frac{\sqrt{2k}}{3}$	$-\frac{k^2+8}{24\sqrt{2k}}$	$-\frac{1}{\sqrt{2k}}$
$\mathcal{N}=4$ SYM w. N_f fund.	$\frac{\sqrt{2N_f}}{3}$	$\frac{N_f^2-4}{8\sqrt{2N_f}}$	$-\frac{N_f^2+5}{6\sqrt{2N_f}}$

the $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$ ABJM theory [24] or a $\mathrm{U}(N)$ $\mathcal{N}=4$ SYM theory with one adjoint and N_f fundamental hypermultiplets [25].

For M-theory constructions arising from N M2-branes it is expected that the dimensionless ratio $\frac{L^2}{2G_N}$ scales as $N^{\frac{3}{2}}$ while the four-derivative coefficients $c_{1,2}$ scale as $N^{\frac{1}{2}}$, see for example [26]. We expect that the coefficients of the six- and higher-derivative terms in the four-dimensional supergravity Lagrangian are more subleading in the large N limit. To implement this scaling it is convenient to define the constants $v_i := 32\pi\, c_i\, N^{-\frac{1}{2}}$. In addition, we allow for an $N^{\frac{1}{2}}$ correction to $\frac{L^2}{2G_N}$ by defining

$$\frac{L^2}{2G_N} = A\, N^{\frac{3}{2}} + a\, N^{\frac{1}{2}}\,. \tag{28}$$

With this at hand the on-shell action in (10) becomes

$$I_{\rm HD} = \pi\, \mathcal{F}\left[A\, N^{\frac{3}{2}} + B\, N^{\frac{1}{2}}\right] - \pi\, (\mathcal{F}-\chi)\, C\, N^{\frac{1}{2}}\,, \tag{29}$$

where $B := a + v_2$ and $C := v_1$. To determine the constants (A, B, C) we can use supersymmetric localization results on the squashed sphere S^3_b discussed above. In particular for the round sphere at $b=1$ the free energy for the ABJM theory and the $\mathrm{U}(N)$ $\mathcal{N}=4$ SYM was computed in [27, 28] and [25], respectively. These results allow us to determine the constant A as well as the sum $B+C$ in (29). For both families of SCFTs it is also possible to compute the constant C_T [29] and one can combine this with (29) and the supergravity result in (20) to determine B and C individually. The outcome of these calculations is summarized in Table 1 below. Note that these results unambiguously fix the coefficient c_1 in (4), while c_2 cannot be fully determined due to the shift by the constant a in (28).[2]

Using this amalgam of four-derivative supergravity and supersymmetric localization results we arrive at the general form of the partition function for these two classes of SCFTs (30)

$$-\log Z = \pi\, \mathcal{F}\left[A\, N^{\frac{3}{2}} + B\, N^{\frac{1}{2}}\right] - \pi\, (\mathcal{F}-\chi)\, C\, N^{\frac{1}{2}}\,. \tag{30}$$

[2] In [3] it was shown that under some very plausible assumptions about the STU model in gauged supergravity one can also determine the constants c_2 and a individually for the ABJM theory.

This is our central result in the holographic context. It amounts to a prediction for the leading and subleading terms in the large N expansion of the partition function of these two classes of SCFTs when placed on general compact Euclidean three-manifolds that admit Killing spinors. We discuss two explicit examples in more detail below.

As a consistency check we note that our results for the ABJM theory at level $k = 1$ and the $\mathrm{U}(N)$ $\mathcal{N} = 4$ SYM theory for general N_f agree with [30] where the squashed sphere partition function was computed for $b^2 = 3$. For more general values of the squashing parameter we obtain the following result for the ABJM free energy, $F := -\log Z$:

$$F_{S_b^3} = \tfrac{\pi\sqrt{2k}}{12}\left[\left(b+\tfrac{1}{b}\right)^2\left(N^{\frac{3}{2}}+\left(\tfrac{1}{k}-\tfrac{k}{16}\right)N^{\frac{1}{2}}\right)-\tfrac{6}{k}N^{\frac{1}{2}}\right]. \tag{31}$$

This result was also recently confirmed in [31] by a perturbative analysis of the matrix model arising from supersymmetric localization to fifth order in an expansion around $b = 1$.

The result in (30) allows also for the calculation of the leading correction to the large N results for the topologically twisted index on $S^1 \times \Sigma_{\mathfrak{g}}$ for the so-called universal twist [15, 32]. For the ABJM theory we find

$$-\log Z_{S^1\times\Sigma_{\mathfrak{g}}} = (1-\mathfrak{g})\tfrac{\pi\sqrt{2k}}{3}\left(N^{\frac{3}{2}}-\tfrac{32+k^2}{16k}N^{\frac{1}{2}}\right). \tag{32}$$

This agrees with the result from supersymmetric localization for $\mathfrak{g} = 0$ in [33]. Finally, we note that using the explicit results for (A, B, C) in Table 1 and the result in (22) we can compute the leading correction to the entropy of any asymptotically AdS$_4 \times S^7$ black hole. In particular, as shown in [3], this amounts to a prediction for the leading correction to the Bekenstein–Hawking entropy of the AdS-Schwarschild black hole.

6 Outlook and Further Developments

We studied four-derivative corrections to minimal $\mathcal{N} = 2$ gauged supergravity and analyzed their effects on black hole thermodynamics and holography. These results can be extended and generalized in several ways and below we list some of the recent developments in this area as well as a few open problems

- When deriving the four-derivative action in (5) certain reality condition on the matter fields and the coefficients in the action were assumed. These assumptions can be relaxed and a more general four-derivative action can be derived, see [3]. These more general actions find applications in the holographic context for 3d SCFTs of class $\mathcal{R}$ arising from M5-branes wrapped on hyperbolic three-manifolds, see [34, 35].

- The result for the on-shell action in (10) can be understood also more mathematically in terms of the topology of the Euclidean supergravity solution. As shown in [36], the two-derivative on-shell action $\mathcal{F}$ localizes on the fixed points of the preserved equivariant supercharges, and this principle allows for its explicit evaluation for generic NUT or Bolt solutions. The on-shell action (10) suggests that the higher-derivative corrections to the on-shell action can also be written purely in terms of topological fixed point data. Indeed, this was demonstrated explicitly in [37] after an appropriate generalization of the results in [36].
- As discussed above the coefficients of the higher-derivative terms in the action can be fixed by using supersymmetric localization for two classes of theories arising from M2-branes. It will certainly be very interesting to understand how to extend these results for other 3d $\mathcal{N} = 2$ SCFTs with holographic duals. Some progress in this direction was made in [2] where 3d $\mathcal{N} = 2$ SCFTs of class $\mathcal{R}$ arising from M5-branes were analyzed. There are numerous other interesting examples awaited to be explored.
- The main focus of this review was on 4d minimal gauged supergravity. It is important to understand how to calculate the higher-derivative corrections to more general matter coupled supergravity theories. For the STU model of gauged supergravity arising as a consistent truncation of 11d supergravity on S^7 this was analyzed in [3], however the general structure of such higher-derivative corrections still remains to be uncovered.
- The approach to finding higher derivative corrections to gauged supergravity reviewed here can be naturally extended to five dimension. This was recently studied in [38, 39] in the context of minimal gauged supergravity, see also [40] for earlier results in matter-coupled supergravity. Generalizing these higher-derivative results to six and seven dimensions is of great interest for holographic applications.

Acknowledgements I am grateful to Anthony Charles, Dongmin Gang, Kiril Hristov, and Valentin Reys for numerous illuminating discussions and for the enjoyable collaboration that led to the results summarized here. I am supported in part by an Odysseus grant G0F9516N from the FWO and by the KU Leuven C1 grant ZKD1118 C16/16/005.

References

1. N. Bobev, A.M. Charles, K. Hristov, V. Reys, Phys. Rev. Lett. **125**(13), 131601 (2020). https://doi.org/10.1103/PhysRevLett.125.131601. arXiv:2006.09390 [hep-th]
2. N. Bobev, A.M. Charles, D. Gang, K. Hristov, V. Reys, JHEP **04**, 058 (2021). https://doi.org/10.1007/JHEP04(2021)058. arXiv:2011.05971 [hep-th]
3. N. Bobev, A.M. Charles, K. Hristov, V. Reys, JHEP **08**, 173 (2021). https://doi.org/10.1007/JHEP08(2021)173. arXiv:2106.04581 [hep-th]
4. E. Lauria, A. Van Proeyen, Lect. Notes Phys. **966** (2020). https://doi.org/10.1007/978-3-030-33757-5. arXiv:2004.11433 [hep-th]
5. B. de Wit, V. Reys, JHEP **12**, 011 (2017). https://doi.org/10.1007/JHEP12(2017)011. arXiv:1706.04973 [hep-th]
6. D. Butter, JHEP **10**, 030 (2011). https://doi.org/10.1007/JHEP10(2011)030. arXiv:1103.5914 [hep-th]

7. E. Bergshoeff, M. de Roo, B. de Wit, Nucl. Phys. B **182**, 173–204 (1981). https://doi.org/10.1016/0550-3213(81)90465-X
8. D. Butter, B. de Wit, S.M. Kuzenko, I. Lodato, JHEP **12**, 062 (2013). https://doi.org/10.1007/JHEP12(2013)062. arXiv:1307.6546 [hep-th]
9. S.M. Kuzenko, J. Novak, Phys. Rev. D **92**(8), 085033 (2015). https://doi.org/10.1103/PhysRevD.92.085033. arXiv:1507.04922 [hep-th]
10. S. Hegde, B. Sahoo, JHEP **01**, 070 (2020). https://doi.org/10.1007/JHEP01(2020)070. arXiv:1911.09585 [hep-th]
11. A.M. Charles, F. Larsen, JHEP **10**, 142 (2016). https://doi.org/10.1007/JHEP10(2016)142. arXiv:1605.07622 [hep-th]
12. J. Smolic, M. Taylor, JHEP **06**, 096 (2013). https://doi.org/10.1007/JHEP06(2013)096. arXiv:1301.5205 [hep-th]
13. N. Hama, K. Hosomichi, S. Lee, JHEP **05**, 014 (2011). https://doi.org/10.1007/JHEP05(2011)014. arXiv:1102.4716 [hep-th]
14. D. Martelli, A. Passias, J. Sparks, Nucl. Phys. B **864**, 840–868 (2012). https://doi.org/10.1016/j.nuclphysb.2012.07.019. arXiv:1110.6400 [hep-th]
15. F. Azzurli, N. Bobev, P.M. Crichigno, V.S. Min, A. Zaffaroni, JHEP **02**, 054 (2018). https://doi.org/10.1007/JHEP02(2018)054. arXiv:1707.04257 [hep-th]
16. N. Bobev, A.M. Charles, V.S. Min, JHEP **10**, 073 (2020). https://doi.org/10.1007/JHEP10(2020)073. arXiv:2006.01148 [hep-th]
17. K. Sen, A. Sinha, JHEP **07**, 098 (2014). https://doi.org/10.1007/JHEP07(2014)098. arXiv:1405.7862 [hep-th]
18. C. Closset, T.T. Dumitrescu, G. Festuccia, Z. Komargodski, JHEP **05**, 017 (2013). https://doi.org/10.1007/JHEP05(2013)017. arXiv:1212.3388 [hep-th]
19. R.M. Wald, Phys. Rev. D **48**(8), R3427–R3431 (1993). https://doi.org/10.1103/PhysRevD.48.R3427. arXiv:gr-qc/9307038 [gr-qc]
20. V. Balasubramanian, P. Kraus, Commun. Math. Phys. **208**, 413–428 (1999). https://doi.org/10.1007/s002200050764. arXiv:hep-th/9902121 [hep-th]
21. R.C. Myers, J.Z. Simon, Phys. Rev. D **38**, 2434–2444 (1988). https://doi.org/10.1103/PhysRevD.38.2434
22. G.W. Gibbons, M.J. Perry, C.N. Pope, Class. Quant. Grav. **22**, 1503–1526 (2005). https://doi.org/10.1088/0264-9381/22/9/002. arXiv:hep-th/0408217 [hep-th]
23. J.P. Gauntlett, O. Varela, Phys. Rev. D **76**, 126007 (2007). https://doi.org/10.1103/PhysRevD.76.126007 ([arXiv:0707.2315 [hep-th]].)
24. O. Aharony, O. Bergman, D.L. Jafferis, J. Maldacena, JHEP **10**, 091 (2008). https://doi.org/10.1088/1126-6708/2008/10/091. arXiv:0806.1218 [hep-th]
25. M. Mezei, S.S. Pufu, JHEP **02**, 037 (2014). https://doi.org/10.1007/JHEP02(2014)037. arXiv:1312.0920 [hep-th]
26. X.O. Camanho, J.D. Edelstein, J. Maldacena, A. Zhiboedov, JHEP **02**, 020 (2016). https://doi.org/10.1007/JHEP02(2016)020. arXiv:1407.5597 [hep-th]
27. M. Marino, P. Putrov, J. Stat. Mech. **1203**, P03001 (2012). https://doi.org/10.1088/1742-5468/2012/03/P03001. arXiv:1110.4066 [hep-th]
28. H. Fuji, S. Hirano, S. Moriyama, JHEP **08**, 001 (2011). https://doi.org/10.1007/JHEP08(2011)001. arXiv:1106.4631 [hep-th]
29. S.M. Chester, R.R. Kalloor, A. Sharon, JHEP **07**, 041 (2020). https://doi.org/10.1007/JHEP07(2020)041. arXiv:2004.13603 [hep-th]
30. Y. Hatsuda, JHEP **07**, 026 (2016). https://doi.org/10.1007/JHEP07(2016)026. arXiv:1601.02728 [hep-th]
31. S.M. Chester, R.R. Kalloor, A. Sharon, JHEP **04**, 244 (2021). https://doi.org/10.1007/JHEP04(2021)244. arXiv:2102.05643 [hep-th]
32. F. Benini, K. Hristov, A. Zaffaroni, JHEP **05**, 054 (2016). https://doi.org/10.1007/JHEP05(2016)054. arXiv:1511.04085 [hep-th]
33. J.T. Liu, L.A. Pando Zayas, V. Rathee, W. Zhao, JHEP **01**, 026 (2018). https://doi.org/10.1007/JHEP01(2018)026. arXiv:1707.04197 [hep-th]

34. S. Choi, D. Gang, N. Kim, JHEP **06**, 078 (2021). https://doi.org/10.1007/JHEP06(2021)078. arXiv:2012.10944 [hep-th]
35. P.B. Genolini. arXiv:2110.15955 [hep-th]
36. P. Benetti Genolini, J.M. Perez Ipiña, J. Sparks, JHEP **10**, 252 (2019). https://doi.org/10.1007/JHEP10(2019)252. arXiv:1906.11249 [hep-th]
37. P.B. Genolini, P. Richmond, Phys. Rev. D **104**(6), L061902 (2021). https://doi.org/10.1103/PhysRevD.104.L061902. arXiv:2107.04590 [hep-th]
38. N. Bobev, K. Hristov, V. Reys. arXiv:2112.06961 [hep-th]
39. J.T. Liu, R.J. Saskowski. arXiv:2201.04690 [hep-th]
40. M. Baggio, N. Halmagyi, D.R. Mayerson, D. Robbins, B. Wecht, JHEP **12**, 042 (2014). https://doi.org/10.1007/JHEP12(2014)042. arXiv:1408.2538 [hep-th]

Homothetic Rota–Baxter Systems and Dyck^m-Algebras

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Abstract It is shown that generalized Rota–Baxter operators introduced in [W. A. Martinez, E. G. Reyes, M. Ronco, Int. J. Geom. Meth. Mod. Phys. **18**, 2150176 (2021)] are a special case of Rota–Baxter systems [T. Brzeziński, J. Algebra **460**, 1–25 (2016)]. The latter are enriched by homothetisms and then shown to give examples of Dyck^m-algebras.

Keywords Rota-Baxter algebra · Rota-Baxter system · Double homothetism · Dyck^m-algebra

1 Introduction

Rota–Baxter operators first appeared in [2] in analysis of differential operators on commutative Banach algebras, then were brought to combinatorics [17], and are now intensively studied e.g. in probability, renormalization of quantum field theories, the theory of operads, dialgebras, trialgebras, dendriform algebras, pre-Lie algebras etc.; see [7] for progress up to the early 2010s.

A *Rota–Baxter operator of weight* λ on an associative algebra A over a field $\mathbb{K}$ is a linear operator $R : A \to A$, such that, for all $a, b \in A$,

$$R(a)R(b) = R\left(R(a)b + aR(b) + \lambda\, ab\right), \tag{1}$$

where λ is a scalar. The pair (A, R) is often referred to as a *Rota–Baxter algebra*.

In a recently published article [14] Martinez, Reyes and Ronco introduced a generalization of Rota–Baxter operators that involves a pair of scalars (α, β). A *generalized*

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_7

Rota–Baxter operator of weights (α, β) on an associative algebra A is a linear map $\bar{R} : A \to A$, such that, for all $a, b \in A$,

$$\bar{R}(a)\bar{R}(b) = \bar{R}\left(\bar{R}(a)b + a\bar{R}(b) + \alpha\, ab\right) + \beta ab. \tag{2}$$

The authors of [14] then proceed to show that such operators of weights (3, 2) give rise to a class of Dyck^m-algebras introduced and studied in [11, 12] in context of dendriform structures.

This note has two aims. First, we show that generalized Rota–Baxter operators are examples of Rota–Baxter systems [3]. Second, we enrich Rota–Baxter systems with homothetisms [16] or self-permutable bimultiplications [13] that were introduced in studies of ring extensions and constrain them in such a way as to produce examples of Dyck^m-algebras that extend those in [14].

2 Rota–Baxter Systems, Homothetisms and Dyck^m-Algebras

2.1 *Rota–Baxter Systems and Generalized Rota–Baxter Operators*

The following notion was introduced in [3]. An associative algebra A (over a field $\mathbb{K}$) together with a pair of linear operators $R, S : A \longrightarrow A$ is called a *Rota–Baxter system* if, for all $a, b \in A$,

$$R(a)R(b) = R\left(R(a)b + aS(b)\right), \tag{3a}$$

$$S(a)S(b) = S\left(R(a)b + aS(b)\right). \tag{3b}$$

As explained in [5], conditions (3) can be recast in the form of a single Nijenhuis operator. Recall from [6] that a Nijenhuis tensor or operator on an associative algebra B is a linear function $N : B \longrightarrow B$ such that, for all $a, b \in B$,

$$N(a)N(b) = N(N(a)b + aN(b) - N(ab)). \tag{4}$$

Starting with an algebra A we can form an algebra B on the vector space $A \oplus A \oplus A$ with the product

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} aa' \\ bb' \\ ac' + cb' \end{pmatrix}$$

Then (A, R, S) is a Rota–Baxter system if and only if

$$N = \begin{pmatrix} 0 & 0 & R \\ 0 & 0 & S \\ 0 & 0 & 0 \end{pmatrix}$$

is a Nijenhuis operator on B.

If R is a Rota–Baxter operator of weight λ, then setting $S = R + \lambda\,\mathrm{id}$ one obtains a Rota–Baxter system. In a similar way, one can interpret a generalized Rota–Baxter operator of weights (α, β) as a Rota–Baxter system. More precisely,

Lemma 1 *Let $\bar{R}$ be a generalized Rota–Baxter operator of weights (α, β) and let $\lambda, \mu \in \mathbb{K}$ be such that*

$$\alpha = \lambda + \mu, \qquad \beta = \lambda\mu. \tag{5}$$

Set

$$R = \bar{R} + \lambda\,\mathrm{id}, \qquad S = \bar{R} + \mu\,\mathrm{id}. \tag{6}$$

Then (A, R, S) is a Rota–Baxter system.

This is checked by a straightforward calculation (left to the reader). As a consequence and in view of [3, Proposition 2.5], one concludes, for example that there is a dendriform algebra associated to a generalized Rota–Baxter operator $\bar{R}$ (cf. [14, Proposition 18]).

Of course, if $\mathbb{K}$ is not an algebraically closed field, equations (5) for λ and μ might not have solutions. However, the most interesting case of a generalized Rota–Baxter operator studied in [14] corresponds to the weights $(3, 2)$, hence one can take $\lambda = 1$ and $\mu = 2$ then.

2.2 Homothetisms and Homothetic Rota–Baxter Systems

The following notion originates from studies of homology of rings and ring extensions in [13, 16]. Let A be an associative algebra. By a *double operator* σ on A we mean a pair of linear operators $\sigma = (\overrightarrow{\sigma}, \overleftarrow{\sigma})$ on A,

$$\overrightarrow{\sigma}\colon A \longrightarrow A, \quad a \longmapsto \sigma a, \qquad \overleftarrow{\sigma}\colon A \longrightarrow A, \quad a \longmapsto a\sigma.$$

The somewhat unusual way of writing the argument to the left of the operator (in the definition of $\overleftarrow{\sigma}$) proves very practical and economical in expressing the action of double operators with additional properties, in particular those that we are introducing presently.

A double operator σ on A is called a *bimultiplication* [13] or a *bitranslation* [15] if, for all $a, b \in A$,

$$\sigma(ab) = (\sigma a)b, \qquad (ab)\sigma = a(b\sigma) \quad \& \quad a(\sigma b) = (a\sigma)b. \tag{7}$$

A bimultiplication σ is called a *double homothetism* [16] or is said to be *self-permutable* [13] provided that, for all $a \in A$,

$$(\sigma a)\sigma = \sigma(a\sigma). \tag{8}$$

The first two conditions in (7) mean that $\overrightarrow{\sigma}$ is a right and $\overleftarrow{\sigma}$ is a left A-module homomorphism. A bimultiplication is called simply a *multiplication* in [9]. In functional analysis, in particular in the context of C^*-algebras, bimultiplications are known as *multipliers* [4, 8]. The set of all bimultiplications is a unital algebra, known as a *multiplier algebra*. The relations (8) mean that endomorphisms $\overrightarrow{\sigma}$ and $\overleftarrow{\sigma}$ mutually commute in the endomorphism algebra of the vector space A. Put together, conditions (7) and (8) mean that one needs not put any brackets in strings of letters that involve elements of A and σ.

Any element of A, say $s \in A$, induces a double homothetism $\bar{s} = (\overrightarrow{s}, \overleftarrow{s})$ on A,

$$\overrightarrow{s} : a \longmapsto sa, \qquad \overleftarrow{s} : a \longmapsto as.$$

Such double homothetisms are said to be *inner* and they form an ideal in the multiplier algebra. In applications of bimultiplications to ring extensions, most recently in connecting extensions of integers to trusses (sets with an associative binary operation distributing over a ternary abelian heap operation) [1] the key role is played by the quotient of the multiplier algebra by the ideal of inner homothetisms; in particular in the theory of operator algebras this is known as the *corona algebra*.

The rescaling by $\lambda \in \mathbb{K}$ understood as a pair $(\lambda\mathrm{id}, \lambda\mathrm{id})$ is a double homothetism, which we will also denote by λ. If A has the identity 1, then, of course λ is an inner homothetism, $\overline{\lambda 1}$.

We are now ready to define the main notion of this note.

Definition 1 Let (A, R, S) be a Rota–Baxter system and let σ be a double homothetism on A. We say that (A, R, S, σ) is a *homothetic Rota–Baxter system* if, for all $a \in A$,

$$S(a)\sigma - \sigma R(a) = \sigma a \sigma. \tag{9}$$

If $\bar{R}$ is a generalized Rota–Baxter operator of weights (α, β) such that

$$\gamma := \pm\sqrt{\alpha^2 - 4\beta} \in \mathbb{K},$$

then (A, R, S, γ) is a homothetic Rota–Baxter system, where (A, R, S) corresponds to $\bar{R}$ through Lemma 1 (the sign depends on the choice of λ and μ, i.e. whether $\lambda < \mu$ or $\lambda > \mu$).

2.3 Dyck^m-Algebras

The combinatorial, algebraic and operadic aspects of a certain class of lattice paths counted by Fuss-Catalan numbers led López, Préville-Ratelle and Ronco in [11] to introduce the following notion.

Let m be a natural number. A $\mathbb{K}$-vector space A together with $m+1$ linear operations $*_i : A \otimes A \longrightarrow A$, $i = 0, \dots, m$ such that, for all $a, b, c \in A$

$$a *_i (b *_j c) = (a *_i b) *_j c, \qquad 0 \leq i < j \leq m, \tag{10a}$$

$$a *_0 (b *_0 c) = \left(\sum_{i=0}^{m} a *_i b\right) *_0 c, \tag{10b}$$

$$a *_m \left(\sum_{i=0}^{m} b *_i c\right) = (a *_m b) *_m c, \tag{10c}$$

$$a *_i \left(\sum_{k=0}^{i} b *_k c\right) = \left(\sum_{k=i}^{m} a *_k b\right) *_i c, \qquad 1 \leq i \leq m-1, \tag{10d}$$

is called a *Dyck^m-algebra*.

Dyck^m-algebras generalize associative algebras (the $m = 0$ case) and Loday's dendriform algebras (the $m = 1$ case) [10, Sect. 5]. In [14] it is shown that one can associate Dyck^m-algebras to any generalized Rota–Baxter operator of weights $(3, 2)$ (see Theorem 20 and Proposition 21 in [14] for explicit formulae). Aided by this observation we will associate Dyck^m-algebras to any homothetic Rota–Baxter system.

3 Dyck^m-Algebras from Homothetic Rota–Baxter Systems

The main result of this note is contained in the following theorem.

Theorem 1 *Let (A, R, S, σ) be a homothetic Rota–Baxter system and let m be any natural number. Define $m+1$ linear operations $*_i : A \otimes A \longrightarrow A$, $i = 0, \dots, m$ as follows:*

$$a *_i b = \begin{cases} R(a)b, & i = 0, \\ (-1)^{i+1} a\sigma b, & i = 1, \dots, m-1, \\ aS(b) - \frac{1+(-1)^m}{2} a\sigma b, & i = m, \end{cases} \tag{11}$$

for all $a, b \in A$. Then $\left(A, \{*_i\}_{i=0}^{m}\right)$ is a Dyck^m-algebra.

Proof We will carefully check that all the relations between different operations listed in (10) hold. Starting with (10a), if $i \neq 0$ and $j \neq m$, these equations reduce to the equality $a\sigma(b\sigma c) = (a\sigma b)\sigma c$ which follows by (7) and (8). For $i = 0$ and $j \neq m$, (10a) amounts to the equality $R(a)(b\sigma c) = (R(a)b)\sigma c$, which holds by (7). The proofs of (10a) for $j = m$, and all the remaining equalities in (10) depend on the parity of m. So, we will consider two separate cases in turn.

Assume that m is odd. Then (10a) with $i = 0$ and $j = m$ follows immediately by the associativity of A, while $i \neq 0$ and $j = m$ amounts to the equality $a(b\sigma S(c)) = (ab)\sigma S(c)$, which holds by (7).

Since m is odd (10b) reduces to

$$a *_0 (b *_0 c) = (a *_0 b + a *_m b) *_0 c,$$

that is

$$R(a)R(b)c = R(R(a)b + aS(b))c,$$

and this immediately follows by (3a). In a similar way (10c) follows by (3b).

We split checking (10d) into two cases. If i is even, then (10d) reduces to

$$a *_i (b *_0 c) = (a *_m b + a *_{m-1} b) *_i c,$$

which amounts to the equality

$$a\sigma R(b)c = (aS(b) - a\sigma b)\sigma c, \tag{12}$$

that follows by the constraint (9). If i is odd, then (10d) is equivalent to

$$a *_i (b *_0 c + b *_1 c) = (a *_m b) *_i c,$$

that is,

$$R(a)(R(b)c + b\sigma c) = (aS(b))\sigma c \tag{13}$$

and thus again follows by the definition of a double homothetism and (9). This completes the proof of the theorem for m odd.

Assume now that m is even. We look back at two remaining cases in (10a). If $i = 0$ and $j = m$, then

$$\begin{aligned} a *_0 (b *_m c) &= R(a)bS(c) - R(a)(b\sigma c) \\ &= R(a)bS(c) - (R(a)b)\sigma c = (a *_0 b) *_m c, \end{aligned}$$

by the associativity of A and (7). In a similar way, if $i \neq 0$ and $j = m$, (10a) is equivalent to the equality

$$a\sigma(bS(c)) - a\sigma(b\sigma c) = (a\sigma b)S(c) - (a\sigma b)\sigma c,$$

which follows by (7) and (8).

Since m is even (10b) reduces to

$$a *_0 (b *_0 c) = (a *_0 b + a *_1 b + a *_m b) *_0 c,$$

and, in view of the definition of $*_m$, this immediately follows by (3a). In a similar way (10c) follows by (3b).

As for the m-odd case, we split checking (10d) into two cases. If i is even, then (10d) reduces to

$$a *_i (b *_0 c) = (a *_m b) *_i c,$$

which is the same as (12), while, for i odd, (10d) is equivalent to

$$a *_i (b *_0 c + b *_1 c) = (a *_m b + a *_{m-1} b) *_i c,$$

that is already proven equality (12). This completes the proof of the theorem. □

If $\bar{R}$ is a generalized Rota–Baxter operator of weights $(3, 2)$, then the corresponding Rota–Baxter system $R = \bar{R} + \mathrm{id}$, $S = \bar{R} + 2\mathrm{id}$ is constrained by the homothetism 1, induced by the rescaling by the identity in $\mathbb{K}$, and thus Theorem 1 implies [14, Theorem 20 and Proposition 21].

Acknowledgements The research is partially supported by the National Science Centre, Poland, grant no. 2019/35/B/ST1/01115.

References

1. R.R. Andruszkiewicz, T. Brzeziński, B. Rybołowicz, Ideal ring extensions and trusses. J. Algebra **600**, 237–278 (2022)
2. G. Baxter, An analytic problem whose solution follows from a simple algebraic identity. Pacific J. Math. **10**, 731–742 (1960)
3. T. Brzeziński, Rota-Baxter systems, dendriform algebras and covariant bialgebras. J. Algebra **460**, 1–25 (2016)
4. R.B. Busby, Double centralizers and extensions of C^*-algebras. Trans. Amer. Math. Soc. **132**, 79–99 (1968)
5. A. Das, Generalized Rota-Baxter systems (2020). arXiv:2007.13652
6. J.F. Cariñena, J. Grabowski, G. Marmo, Quantum bi-Hamiltonian systems. Int. J. Modern Phys. A **15**, 4797–4810 (2000)
7. L. Guo, *An Introduction to Rota-Baxter Algebra* (International Press, Somerville, 2012). xii+226 pp
8. S. Helgason, Multipliers of Banach algebras. Ann. Math. **64**, 240–254 (1956)
9. G. Hochschild, Cohomology and representations of associative algebras. Duke Math. J. **14**, 921–948 (1947)
10. J.-L. Loday, Dialgebras, in *Dialgebras and Related Operads*, Lecture Notes in Mathematics, vol. 1763 (Springer, Berlin, 2001), pp. 7–66

11. D. López, L.-F. Préville-Ratelle, M. Ronco, Algebraic structures defined on m-Dyck paths (2015) arXiv:1508.01252
12. D. López, L.-F. Préville-Ratelle, M. Ronco, A simplicial complex splitting associativity. J. Pure Appl. Algebra **224**, 106222 (2019)
13. S. Mac Lane, Extensions and obstructions for rings. Illinois J. Math. **2**, 316–345 (1958)
14. W.A. Martinez, E.G. Reyes, M. Ronco, Generalizing dendriform algebras: Dyck^m-algebras, Rota^m-algebras, and Rota-Baxter operators. Int. J. Geom. Meth. Mod. Phys. **18**, 2150176 (2021)
15. M. Petrich, Ideal extensions of rings. Acta Math. Hung. **45**, 263–283 (1985)
16. L. Redei, Die Verallgemeinerung der Schreierschen Erweiterungstheorie. Acta Sci. Math. Szeged **14**, 252–273 (1952)
17. G.-C. Rota, Baxter algebras and combinatorial identities. I. Bull. Amer. Math. Soc. **75**, 325–329 (1969)

Quantum Dynamics Far from Equilibrium: A Case Study in the Spherical Model

Malte Henkel

Abstract The application of quantum Langevin equations for the study of non-equilibrium relaxations is illustrated in the exactly solved quantum spherical model. A tutorial on the physical background of non-Markovian quantum noise and the spherical model quantum phase transition is followed by a review of the solution of the non-Markovian time-dependent spherical constraint and the consequences for quantum ageing at zero temperature, after a quantum quench.

Keywords Non-equilibrium quantum dynamics · Open quantum system · Quantum spherical model · Quantum ageing

1 What is a Quantum Langevin Equation?

The description of quantum-mechanical many-body problems far from equilibrium presents conceptual difficulties which go beyond those present in classical systems [7, 8, 11, 14, 23, 35, 44, 49]. Physically, one distinguishes *closed* systems, which are isolated and *open* systems, which are coupled to one or several external baths. For closed systems, the Heisenberg equations of motion are a convenient starting point.

For open quantum systems, a large variety of theoretical descriptions has been considered. Here, we shall concentrate on quantum Langevin equations, where the 'noises' must be chosen as to (i) maintain the quantum coherence of the system and (ii) to describe the interaction with the external baths. We shall begin with a tutorial for the formulation of dissipative quantum dynamics of open system and shall use

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_8

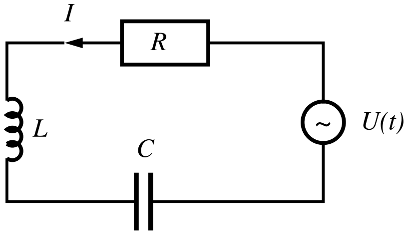

Fig. 1 Schematic LRC circuit

the quantum spherical model for a case study.[1] In this Sect. 1, we recall the generic formulation of quantum Langevin equations, in Sect. 2 we give a brief introduction to the quantum spherical model and its quantum phase transition, in Sect. 3 the quantum dynamics of this model is formulated in a way to facilitate the extraction of the long-time behaviour of physical observables and in Sect. 4, we recall the main ingredients to describe the physical ageing expected in relaxational dynamics after a quantum quench. The later sections review the results of a detailed analysis in the quantum spherical model, at temperature $T = 0$. Section 5 discusses the solution of the non-linear integral equation derived from spherical constraint. Sections 6 and 7 review the results after quenches to either the disordered phase or else onto the critical point or into the ordered phase. We conclude in Sect. 8.

Inspired by a proposal of Bedeaux and Mazur [4, 5], we consider quantum Langevin equations in the following form, for simplicity formulated for a single quantum variable s and its canonically conjugate momentum p,

$$\partial_t s = \frac{\mathrm{i}}{\hbar}\big[H, s\big] + \eta^{(s)} \quad ; \quad \partial_t p = \frac{\mathrm{i}}{\hbar}\big[H, p\big] - \gamma p + \eta^{(p)} \tag{1}$$

where H is the hamiltonian of the system and the damping constant $\gamma > 0$ describes the dissipative part of the dynamics. The moments of the noise operators $\eta^{(s)}, \eta^{(p)}$ must be specified as to maintain the required quantum properties of the dynamics.

That the noise structure in (1) is quite natural can be seen from the example of a LRC electric circuit [3], see Fig. 1. According to Kirchhoff, one has $U(t) = U_R + U_L + U_C$ and furthermore $U_R = RI$ and $U_L = L\dot{I}$, where $I = I(t)$ is the current and R, L are the resistance and the inductivity, respectively. Their combined noises are modeled by setting $U(t) = L\eta_U$. In addition, the (noisy) voltage fluctuations at the capacity C are described by $\dot{U}_C = \frac{1}{C}I + \eta_I$. This leads to

$$\partial_t U_C = \frac{1}{C}I + \eta_I \quad ; \quad \partial_t I = -\frac{1}{L}U_C - \frac{R}{L}I + \eta_U \tag{2}$$

and with the correspondences $s \leftrightarrow U_C$, $p \leftrightarrow I$ and $\eta^{(s)} \leftrightarrow \eta_I$, $\eta^{(p)} \leftrightarrow \eta_U$, (2) is identified with (1), if H describes a harmonic oscillator. Apparently the noises η_U, η_I describe the 'rough' fluctuations around the smooth averages $I = I(t)$ and $U_C =$

[1] For studies of the quantum dynamics in closed spherical models, see [2, 9, 12, 13, 25, 31].

$U_C(t)$. This example also suggests that in the context of nano-electronics, quantum noise effects might become of relevance.

Returning to (1), it remains to specify the noise correlators. For definiteness, it is assumed that any deterministic term is included into the hamiltonian H, so that $\langle \eta^{(s)} \rangle = \langle \eta^{(p)} \rangle = 0$. The non-vanishing second moments are, at temperature $T > 0$

$$\left\langle \left\{ \eta^{(s)}(t), \eta^{(p)}(t') \right\} \right\rangle = \gamma T \coth\left(\frac{\pi}{\hbar} T \big(t - t'\big) \right) \quad ; \quad \left\langle \left[\eta^{(s)}(t), \eta^{(p)}(t') \right] \right\rangle = \mathrm{i}\hbar\gamma\, \delta(t - t') \tag{3}$$

Clearly, quantum noise is explicitly non-Markovian.[2] Anti-commutators $\{.,.\}$ and commutators $[.,.]$ were used. Equation (3) can be derived in two distinct ways:

1. The classical approach of Ford, Kac and Mazur [17–19] considers explicitly the coupling of the system to an external bath, with the total hamiltonian $H_{\rm tot} = H + H_{\rm int} + H_{\rm bath}$. The composite object described by $H_{\rm tot}$ is considered as a closed system. Using for the bath hamiltonian $H_{\rm bath}$ a large ensemble of harmonic oscillators, along with a bi-linear coupling $H_{\rm int}$ to the system, the Heisenberg equations of the bath degrees of freedom can be formally solved. An average over the initial positions and momenta of the bath then leads to (3). Herein, the system is a single degree of freedom s in some external potential $V = V(s)$.
2. A phenomenological derivation [47] considers the 'desirable' physical properties of dissipative quantum dynamics which any choice of the quantum noises should keep. These are

   a. canonical equal-time commutator $\langle [s(t), p(t)] \rangle = \mathrm{i}\hbar$.
   b. the Kubo formula of linear response theory [11, 36].
   c. the virial theorem (for selecting equilibrium stationary states) [16, 41].
   d. the quantum fluctuation-dissipation theorem (QFDT) (to distinguish quantum and classical equilibrium states) [20, 26].

   While the QFDT is habitually formulated in frequency space, for any temperature $T > 0$ a mathematically equivalent statement is the Kubo–Martin–Schwinger relation [11, 36, 47]

$$C\left(t - t' + \frac{\mathrm{i}\hbar}{2T}\right) - C\left(t - t' - \frac{\mathrm{i}\hbar}{2T}\right) = \frac{\hbar}{2\mathrm{i}}\left[R\left(t - t' + \frac{\mathrm{i}\hbar}{2T}\right) + R\left(t - t' - \frac{\mathrm{i}\hbar}{2T}\right)\right] \tag{4}$$

   where $C(t - t') = \left\langle \left\{ s(t), s(t') \right\} \right\rangle$ is the stationary correlator and $R(t - t') = \left. \frac{\delta \langle s(t) \rangle}{\delta h(t')} \right|_{h=0}$ is the stationary linear response with respect to the conjugate field h. The QFDT follows from a Fourier transformation of (4) with respect to $t - t'$. If the chosen system is a harmonic oscillator, its dynamics can be formally solved which provides a relation between the noise correlators and the four physical cri-

[2] In the classical limit $\hbar \to 0$, Markovian white-noise correlators are recovered from (3) [47].

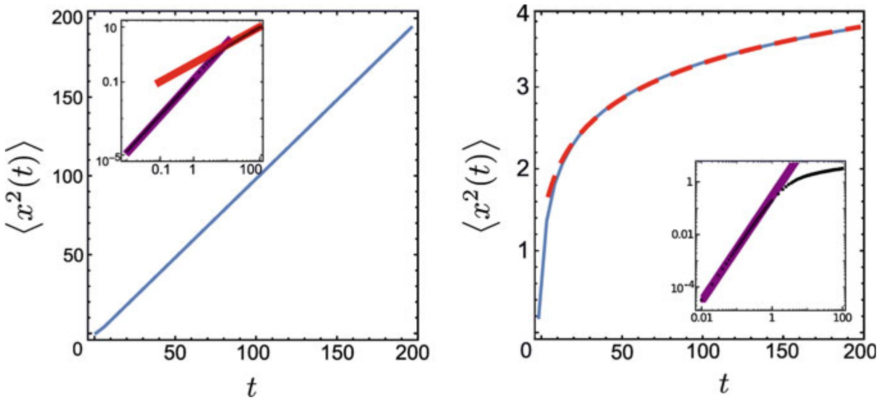

Fig. 2 Variance $\langle x^2(t) \rangle$ of a brownian particle for classical white noise with $T > 0$ (left panel) and for quantum noise (3) at $T = 0$ (right panel). Insets: cross-over from initial ballistic motion

teria raised above. Postulating that the noise correlators should not contain any system-specific parameter, Eq. (3) follows [47].

That these distinct approaches lead to the same noise correlators (3) also clarifies important physical properties of this choice of dynamics. The only physical parameters are the dissipation constant γ, the bath temperature T and Planck's constant $\hbar$. The validity of (3) does not depend on the implicit auxiliary assumptions contained in either approach. Equations (1) and (3) are the *quantum Langevin equations*, to be used in what follows.

The qualitative differences between classical and quantum noises can be illustrated through the motion of a free $1D$ Brownian particle. Figure 2 [47] compares the variance $\langle x^2(t) \rangle \sim t$ of the position $x(t)$ for classical white noise at finite temperature $T > 0$ (left panel) with the quantum result $\langle x^2(t) \rangle \sim \ln t$ obtained at $T = 0$ (right panel). Consequently, quantum diffusion is 'more weak' than classical diffusion since it needs considerably more time to homogenise a system.

Excellent reviews of dissipative quantum dynamics include [8, 23, 49].

2 What is the Quantum Spherical Model?

The *spherical model* is a simple, yet non-trivial, and exactly solvable model for the study of phase transitions [6]. Its classical version is defined in terms of a continuous spin variable $s_{\boldsymbol{n}} \in \mathbb{R}$ attached to the sites $\boldsymbol{n} \in \mathscr{L}$ of a d-dimensional hyper-cubic lattice $\mathscr{L} \subset \mathbb{Z}^d$. In its most simple variant, one uses nearest-neighbour interactions $\mathscr{H} = -\sum_{(\boldsymbol{n},\boldsymbol{m})} s_{\boldsymbol{n}} s_{\boldsymbol{m}} + \frac{\mu}{2} \sum_{\boldsymbol{n}} s_{\boldsymbol{n}}^2$, where the Lagrange multiplier μ is fixed from the (mean) *spherical constraint* [6, 30]

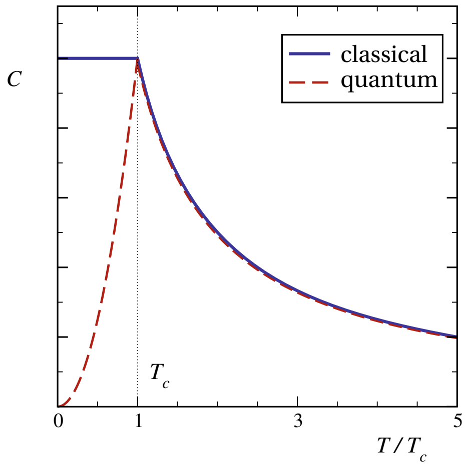

Fig. 3 Specific heat C in the classical and quantum spherical models

$$\left\langle \sum_{\boldsymbol{n}\in\mathscr{L}} s_{\boldsymbol{n}}^2 \right\rangle = \mathscr{N} \tag{5}$$

where $\mathscr{N} = |\mathscr{L}|$ is the number of sites of the lattice $\mathscr{L}$ and $\langle \cdot \rangle$ denotes the thermodynamic average. This condition was originally motivated by a comparison with the Ising model, with discrete 'Ising spins' $s_{\boldsymbol{n}} = \pm 1$ and which naturally obey (5) [6]. The spherical model is solvable since in Fourier space, the degrees of freedom decouple, but some interactions do remain because of the constraint (5). At equilibrium, the spherical model has a critical point $T_c > 0$ for any spatial dimension $d > 2$ and the universality class of the model is distinct from mean-field theory if $2 < d < 4$. The values of the critical exponents are different from those found in the Ising model, e.g. [44].

A serious physical short-coming of the classical spherical model is its low-temperature behaviour [6], see Fig. 3 for dimensions $2 < d < 4$. The cusp of the specific heat C at $T \simeq T_c$ (rather than a jump) is a manifestation of non-mean-field criticality. But for all temperatures $T \leq T_c$, C is constant! This violates the third fundamental theorem of thermodynamics [3], which requests that $C(T) \to 0$ as $T \to 0$.

The quantum spherical model corrects this deficiency. In terms of spin operators $s_{\boldsymbol{n}}$ and conjugate momenta $p_{\boldsymbol{m}}$, which obey $\left[s_{\boldsymbol{n}}, p_{\boldsymbol{m}}\right] = \mathrm{i}\hbar$, let [27, 32, 33, 45]

$$H = \frac{1}{2} \sum_{\boldsymbol{n}\in\mathscr{L}} \left[p_{\boldsymbol{n}}^2 + (r+d) s_{\boldsymbol{n}}^2 - \sum_{(\boldsymbol{n},\boldsymbol{m})} s_{\boldsymbol{n}} s_{\boldsymbol{m}} \right] \quad ; \quad \sum_{\boldsymbol{n}\in\mathscr{L}} \langle s_{\boldsymbol{n}}^2 \rangle = \frac{\mathscr{N}}{\lambda} \tag{6}$$

(after several re-scalings) and the Lagrange multiplier is now $r := \mu\lambda - d$. The quantum hamiltonian H arises from its classical counterpart $\mathscr{H}$ by adding a kinetic energy term $\sim \sum_{\boldsymbol{n}} p_{\boldsymbol{n}}^2$ and λ controls the relative importance of this term. Schematically,

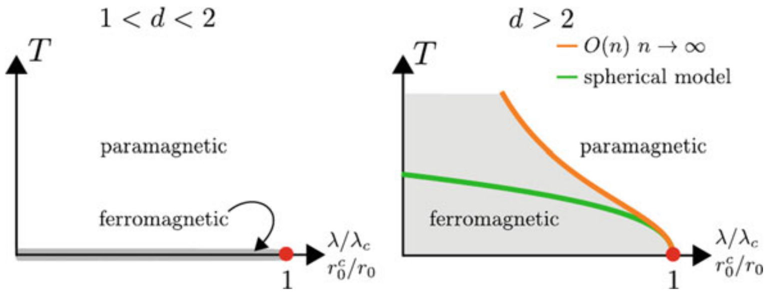

Fig. 4 Quantum and classical phase transitions in the spherical model

the behaviour of the quantum model is illustrated in Fig. 3. Indeed, for $d > 2$ there exists a critical temperature $T_c > 0$ such that the high-temperature behaviour of the quantum model is analogous to the classical one. Especially, the type of singularity of C around $T \simeq T_c$ is precisely the same [32, 34]. On the other hand, for $T < T_c$, the behaviour of the quantum model is different from the one of its classical variant and one finds $C(T) \to 0$ as $T \to 0$, as expected from the third fundamental theorem of thermodynamics.

In addition, at zero temperature $T = 0$, the quantum spherical model has a *quantum phase transition*, where λ acts as thermodynamic parameter [27, 32, 34, 45, 46]. The schematic phase diagrams at equilibrium are shown in Fig. 4 [48]. For dimensions $1 < d < 2$, there is only a quantum phase transition at $T = 0$ and at $\lambda = \lambda_c > 0$. In the ordered ferromagnetic phase, the system acquires a non-vanishing spontaneous magnetisation, but in the disordered paramagnetic phase, it remains non-magnetic. On the other hand, for $d > 2$, there exists not only a quantum phase transition at $T = 0$ and $\lambda = \lambda_c > 0$, but also a classical finite-temperature phase transition at some $T_c = T_c(\lambda) > 0$ if $\lambda < \lambda_c$. The universal properties of the quantum phase transition, notably the values of the critical exponents around $\lambda \simeq \lambda_c$, of the d-dimensional quantum model are the same as the ones of the $(d + 1)$-dimensional classical model around $T \simeq T_c(\lambda)$ [27, 29, 32, 34, 40, 45, 46].

The existence and nature of phase transitions are often expressed via upper and lower critical dimensions. The *lower critical dimension* d_ℓ is defined such that for $d < d_\ell$, no phase transition exists. The *upper critical dimension* d_u is defined such that for $d > d_u$, the critical behaviour of the model is identical to the one of mean-field theory. At equilibrium, one has for the (short-ranged) spherical model

$$\begin{cases} d_\ell = 2 \; ; \; d_u = 4 & \text{classical} \\ d_\ell = 1 \; ; \; d_u = 3 & \text{quantum} \end{cases} \tag{7}$$

These values will be needed below when discussing the relaxational dynamics of the quantum spherical model.

3 How to Formulate Quantum Dynamics of the Spherical Model?

In order to extract the long-time and large-distance properties of the relaxational dynamics of the spherical model, we first carry out a continuum limit[3] and let $s_{\boldsymbol{n}} \mapsto \phi(t, \boldsymbol{x})$ and $p_{\boldsymbol{n}} \mapsto \pi(t, \boldsymbol{x})$. The spherical model degrees of freedom decouple in Fourier space; hence we set

$$\phi_{\boldsymbol{k}}(t) := \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{x}\ \phi(t, \boldsymbol{x})\, e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \tag{8}$$

such that the quantum Langevin equations (1) and (3) become in Fourier space

$$\partial_t \phi_{\boldsymbol{k}} = \frac{\mathrm{i}}{\hbar}\big[H, \phi_{\boldsymbol{k}}\big] + \eta_{\boldsymbol{k}}^{(\phi)} \;\;;\;\; \partial_t \pi_{\boldsymbol{k}} = \frac{\mathrm{i}}{\hbar}\big[H, \pi_{\boldsymbol{k}}\big] - \gamma\pi_{\boldsymbol{k}} + \eta_{\boldsymbol{k}}^{(\pi)} \tag{9}$$

together with the non-vanishing moments

$$\left\langle\left\{\eta_{\boldsymbol{k}}^{(\phi)}(t), \eta_{\boldsymbol{k}'}^{(\pi)}(t')\right\}\right\rangle = \gamma T \coth\left(\frac{\pi T}{\hbar}(t-t')\right)\delta(\boldsymbol{k}+\boldsymbol{k}') \tag{10a}$$

$$\left\langle\left[\eta_{\boldsymbol{k}}^{(\phi)}(t), \eta_{\boldsymbol{k}'}^{(\pi)}(t')\right]\right\rangle = \mathrm{i}\hbar\gamma\,\delta(t-t')\,\delta(\boldsymbol{k}+\boldsymbol{k}') \tag{10b}$$

These describe a set of quantum harmonic oscillators which are only coupled through the spherical constraint (6). Since the Lagrange multiplier $r = r(t)$ is now time-dependent, the explicit solution of Eqs. (9) and (10) becomes very cumbersome.

But since we shall be mainly interested in the long-time dynamics of the model, we shall project onto this long-time regime by carrying out a scaling transformation. For notational simplicity, we give the procedure for a single degree of freedom [47, 48] and shall write for a moment $\phi_{\boldsymbol{k}}(t) \mapsto \phi(t)$ and so on. Consider

$$\tilde{t} := \lambda t \;\;;\;\; \phi(t) = \lambda\widetilde{\phi}(\tilde{t}) \;\;;\;\; r(t) = r(\tilde{t}) \tag{11}$$

Then the quantum Langevin equation becomes

$$\lambda^2\partial_{\tilde{t}}^2\widetilde{\phi}(\tilde{t}) = -r(\tilde{t})\widetilde{\phi}(\tilde{t}) - \tilde{\gamma}\partial_{\tilde{t}}\widetilde{\phi}(\tilde{t}) + \widetilde{\xi}(\tilde{t}) \tag{12a}$$

$$\widetilde{\xi}(\tilde{t}) := \widetilde{\eta}^{(\pi)}(\tilde{t}) + \tilde{\gamma}\lambda^{-2}\widetilde{\eta}^{(\phi)}(\tilde{t}) + \partial_{\tilde{t}}\widetilde{\eta}^{(\phi)}(\tilde{t}) \tag{12b}$$

where we set $\tilde{\gamma} := \lambda\gamma$. In the long-time scaling limit

$$t \to \infty \;\;,\;\; \lambda \to 0 \;\;,\quad \text{such that } \tilde{t} = \lambda t \text{ is kept fixed} \tag{13}$$

[3] The form (6) of H is chosen to facilitate taking this limit.

the quantum equation of motion (12a) reduces to an over-damped Langevin equation

$$\tilde{\gamma}\,\partial_{\tilde{t}}\widetilde{\phi}(\tilde{t}\,) = -r(\tilde{t})\widetilde{\phi}(\tilde{t}\,) + \widetilde{\xi}(\tilde{t}\,) \tag{14a}$$

with the noise correlators

$$\left\langle\left\{\widetilde{\xi}(\tilde{t}\,), \widetilde{\xi}(\tilde{t}'\,)\right\}\right\rangle = \frac{\hbar\tilde{\gamma}}{\pi} I\left(\frac{\hbar}{2\tilde{T}}, \tilde{t} - \tilde{t}'\right) \;\;;\;\; \left\langle\left[\widetilde{\xi}(\tilde{t}\,), \widetilde{\xi}(\tilde{t}'\,)\right]\right\rangle = 2\mathrm{i}\hbar\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\delta\bigl(\tilde{t} - \tilde{t}'\bigr) \tag{14b}$$

where we also re-scaled $\tilde{T} := T/\lambda$ and the distribution I has a known integral representation (see [23, 47–49] and (16a) below). The reduced description (14) does not apply to the early-time regime, but since the properties of that regime are non-universal anyway, one cannot hope to study them through the perspective of extremely simplified models such as considered here.

Dropping the tildes throughout and focussing on the leading low-momentum behaviour, we have found that the long-time behaviour of the quantum Langevin equations (9) and (10) simply follows from the over-damped Langevin equation ($k = |\boldsymbol{k}|$)

$$\gamma\,\partial_t\phi_{\boldsymbol{k}}(t) + \bigl(r(t) + k^2\bigr)\phi_{\boldsymbol{k}}(t) = \xi_{\boldsymbol{k}}(t) \tag{15}$$

and that the physical nature of the dynamics will be determined by the form of the noise correlators involving the $\xi_{\boldsymbol{k}}(t)$. In addition, since we aim at an understanding of the relevance of quantum noise for the long-time dynamics, we shall from now on concentrate on the zero-temperature limit[4] and let $T \to 0$. A major qualitative difference between quantum and classical dynamics is the non-Markovianity of quantum noise. Another important difference comes from different scalings. In order to elucidate their respective relevance, we shall consider the following three types of noise correlators [48]:

1. *quantum noise*, at temperature $T = 0$ and a regulator $\mathfrak{t}_0 \sim 1/\gamma$, is defined by

$$\begin{aligned}
\left\langle\left\{\xi_{\boldsymbol{k}}(t), \xi_{\boldsymbol{k}}(t)\right\}\right\rangle &= \frac{\gamma\hbar}{\pi}\int_{\mathbb{R}}\mathrm{d}\omega\;|\omega|e^{\mathrm{i}\omega(t-t')}e^{-\mathfrak{t}_0|\omega|}\;\delta(\boldsymbol{k}+\boldsymbol{k}')\\
&= \gamma\hbar\pi\,\frac{\mathfrak{t}_0^2 - (t-t')^2}{[\mathfrak{t}_0^2 + (t-t')^2]^2}\;\delta(\boldsymbol{k}+\boldsymbol{k}')
\end{aligned} \tag{16a}$$

$$\left\langle\left[\xi_{\boldsymbol{k}}(t), \xi_{\boldsymbol{k}}(t)\right]\right\rangle = 2\mathrm{i}\hbar\gamma\left(\frac{\mathrm{d}}{\mathrm{d}t}\delta(t-t')\right)\delta(\boldsymbol{k}+\boldsymbol{k}') \tag{16b}$$

These noise correlators follow from the quantum correlators (10) and (14b). The regularisation merely permits a finite expression of the associated distribution. Non-universal quantities such as critical exponents should turn out to be independent of $\mathfrak{t}_0$, whereas non-universal quantities such as the location of the critical

[4] If $T > 0$, expect after a finite time loss of quantum coherence and hence classical dynamics.

point may depend on it. At the end, one should strive at taking $t_0 \to 0$, consistently with the over-damped limit $\gamma \to \infty$ implicit in the over-damped Eq. (15).

2. *effective noise* is defined by the correlators

$$\left\langle\left\{\xi_{\boldsymbol{k}}(t), \xi_{\boldsymbol{k}}(t)\right\}\right\rangle = \mu|\boldsymbol{k}|^2\delta(t-t')\,\delta(\boldsymbol{k}+\boldsymbol{k}') \quad ; \quad \left\langle\left[\xi_{\boldsymbol{k}}(t), \xi_{\boldsymbol{k}}(t)\right]\right\rangle = 0 \tag{17}$$

Herein, the scaling $\sim \frac{1}{(t-t')^2}$ (in the $t_0 \to 0$ limit) is replaced by an equivalent scaling in the momentum $|\boldsymbol{k}|$ and μ serves as a control parameter. Hence quantum and effective noises have the same scaling properties, but effective noise is Markovian while quantum noise is not. Comparison of the results of both will permit to appreciate the importance of non-Markovian effects for the long-time dynamics.

3. *classical white noise* is of course defined by

$$\left\langle\left\{\xi_{\boldsymbol{k}}(t), \xi_{\boldsymbol{k}}(t)\right\}\right\rangle = 4\gamma\, T\,\delta(t-t')\,\delta(\boldsymbol{k}+\boldsymbol{k}') \quad ; \quad \left\langle\left[\xi_{\boldsymbol{k}}(t), \xi_{\boldsymbol{k}}(t)\right]\right\rangle = 0 \tag{18}$$

and differs in its scaling from effective noise. It is clearly Markovian.

In what follows, we shall compare the behaviour of three distinct types of dynamics:

1. *quantum dynamics*, given by (15) and (16).
2. *effective dynamics*, given by (15) and (17).
3. *classical dynamics*, given by (15) and (18).

The results will be interpreted in the context of physical ageing.

4 What is Physical Ageing?

From the solution $\phi_{\boldsymbol{k}}(t)$ of the equation of motion (15), we define the observables:

1. *equal-time correlations* $C_{\boldsymbol{k}}(t)$ are obtained as

$$\delta(\boldsymbol{k}+\boldsymbol{k}')\,C_{\boldsymbol{k}}(t) := \left\langle\left\{\phi_{\boldsymbol{k}}(t), \phi_{\boldsymbol{k}'}(t)\right\}\right\rangle \tag{19}$$

to be studied in the long-time scaling limit (hence the dynamical exponent $z = 2$)

$$t \to \infty \quad , \quad k = |\boldsymbol{k}| \to 0 \quad , \quad \text{such that } \rho := k^2 t/\gamma \text{ is kept fixed} \tag{20}$$

2. *two-time correlations* $C_{\boldsymbol{k}}(t,s)$ are obtained as

$$\delta(\boldsymbol{k}+\boldsymbol{k}')\,C_{\boldsymbol{k}}(t,s) := \left\langle\left\{\phi_{\boldsymbol{k}}(t), \phi_{\boldsymbol{k}'}(s)\right\}\right\rangle \tag{21}$$

The *auto-correlator* is $C(t,s) := \int_{\boldsymbol{k},(\Lambda)} C_{\boldsymbol{k}}(t,s)$, denoting $\int_{\boldsymbol{k},(\Lambda)} := \int_0^{\Lambda}\int_{S^d}\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d}$ such that Λ describes an UV-cutoff. Any quantity depending explicitly on Λ cannot be universal.

3. *two-time responses* $R_{\boldsymbol{k}}(t,s)$ are obtained as

$$R_{\boldsymbol{k}}(t,s) := \left.\frac{\delta\langle\phi_{\boldsymbol{k}}(t)\rangle}{\delta h_{\boldsymbol{k}}(s)}\right|_{h=0} \tag{22}$$

   where h is the magnetic field conjugate to the magnetisation $\langle\phi_{\boldsymbol{k}}(t)\rangle$. The *auto-response* is $R(t,s) := \int_{\boldsymbol{k},(\Lambda)} R_{\boldsymbol{k}}(t,s)$. Herein, t is called the *observation time* and s the *waiting time*.

Physical ageing was originally observed in the slow dynamics of glasses after a quench from a melt to below the glass-transition temperature [43]. Here, we shall characterise the initial state through a vanishing magnetisation and the initial equal-time correlator

$$\langle\phi_{\boldsymbol{k}}(0)\rangle = 0 \quad ; \quad C_{\boldsymbol{k}}(0) = c_0 + c_\alpha |\boldsymbol{k}|^\alpha \tag{23}$$

which in direct space means $C(0,\boldsymbol{x}) \sim |\boldsymbol{x}|^{-d-\alpha}$ such that for $\alpha \geq 0$ the initial correlations are short-ranged and for $\alpha < 0$ they are long-ranged. Numerous studies in classical systems lead to the following expectations, see [11, 24, 28]:

1. for a quench into the disordered phase $T > T_c$ (or $\lambda > \lambda_c$) the systems rapidly become time-translation-invariant

$$C_{\boldsymbol{k}}(t,s) = C_{\boldsymbol{k}}(t-s) = C_{\boldsymbol{k}}(\tau) \quad ; \quad R_{\boldsymbol{k}}(t,s) = R_{\boldsymbol{k}}(t-s) = R_{\boldsymbol{k}}(\tau) \tag{24}$$

   and $C_{\boldsymbol{k}}(\tau)$ and $R_{\boldsymbol{k}}(\tau)$ should decay exponentially fast with τ. Since there is a single stationary (equilibrium) state, the (quantum) fluctuation-dissipation theorem should hold but no ageing is expected.
2. for a quench onto criticality or into the ordered phase $T \leq T_c$ (or $\lambda \leq \lambda_c$), there is no time-translation-invariance. If the dynamics can be described in terms of a single length scale $L(t) \sim t^{1/z}$, one finds dynamical scaling for $t \gg \tau_{\rm micro}$, $s \gg \tau_{\rm micro}$ and $t-s \gg \tau_{\rm micro}$ ($\tau_{\rm mirco}$ is a microscopic reference time-scale)

$$C(t,s) = s^{-b} f_C\left(\frac{t}{s}\right) \quad ; \quad R(t,s) = s^{-1-a} f_R\left(\frac{t}{s}\right) \tag{25}$$

   with the asymptotic behaviour $f_{C,R}(y) \sim y^{-\lambda_{C,R}/z}$ with defines[5] the *autocorrelation exponent* λ_C and the *auto-response exponent* λ_R. The exponents a, b are called *ageing exponents*. Strong fluctuations lead to a breaking of the fluctuation-dissipation theorem. If this holds true, the three defining properties of *physical ageing* [28], namely (i) slow dynamics, (ii) breaking of time-translation-invariance and (iii) dynamical scaling are satisfied.

[5] Please do not confuse the autocorrelation exponent λ_C with the critical point λ_c.

In the classical spherical model, these expectations are fully borne out [10, 24, 39]. For the initial conditions (23), the exact values of all exponents are known for the spherical model [37], see Table 2 below. For detailed reviews, see [11, 28].

Are these classically motivated expectations verified in quenched quantum dynamics ? It is often thought that the answer should be affirmative: *". . . a large class of coarsening systems (classical, quantum, pure and disordered) should be characterised by the same scaling functions."* [1]. Is this always so ? In what follows, we shall study this question for the quantum spherical model, at temperature $T = 0$.

5 The Spherical Constraint

Having brought together in Sects. 1–4 the physical background for studying non-equilibrium quantum dynamics and ageing, we now turn to the exact solution of the quantum spherical model at $T = 0$ and describe the results [48].

The formal solution of the equation of motion (15) is

$$\phi_{\boldsymbol{k}}(t) = \frac{\exp\left(-k^2 t/\gamma\right)}{\sqrt{g(t)}}\left[\phi_{\boldsymbol{k}}(0) + \frac{1}{\gamma}\int_0^t \mathrm{d}t'\ \sqrt{g(t')}\ \exp\left(k^2 t'/\gamma\right)\xi_{\boldsymbol{k}}(t')\right] \tag{26}$$

with the important auxiliary function $g(t) := \exp\left(\frac{2}{\gamma}\int_0^t \mathrm{d}t'\ r(t')\right)$. In order to re-write the spherical constraint (6) as an equation for $g(t)$, we first define two further supplementary functions

$$A(t) = c_\alpha A_\alpha(t) := \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{k}\ \exp\left(-2\frac{k^2 t}{\gamma}\right) c_\alpha k^\alpha \tag{27a}$$

$$F(t,s) := \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{k}\ \exp\left(-\frac{k^2(t+s)}{\gamma}\right)\left\langle\left\{\xi_{\boldsymbol{k}}(t), \xi_{-\boldsymbol{k}}(s)\right\}\right\rangle \tag{27b}$$

and also $g_2(t,s) := \sqrt{g(t)g(s)}$. The spherical constraint (6) then becomes, using the definition (19) and the solution (26)

$$\frac{1}{\lambda} \stackrel{!}{=} C(t,t) = \frac{1}{g(t)}\left[A(t) + \left(g_2 * * F\right)(t,t)\right] \tag{28}$$

where $\left(h_1 * * h_2\right)(t,s) := \int_0^t \mathrm{d}x \int_0^s \mathrm{d}y\ h_1(x,y)h_2(t-x,s-y)$ is the two-dimensional convolution. It follows that the spherical constraint fixes the function $g(t)$

$$\frac{1}{\lambda}g(t) = A(t) + \left(g_2 * * F\right)(t,t) \tag{29}$$

5.1 Markovian Case

If the noise correlator $\left\langle\left\{\xi_k(t),\xi_k(t)\right\}\right\rangle \sim \delta(t-t')$ is Markovian, the constraint (29) turns into a linear Volterra equation (here for effective noise)

$$\frac{1}{\lambda}g(t) = A(t) + \frac{\mu}{\gamma^2}\big(g * A_2\big)(t) \implies \overline{g}(p) = \frac{c_\alpha \overline{A}_\alpha(p)}{1/\lambda - \mu/\gamma^2\,\overline{A}_2(p)} \tag{30}$$

which is formally solved by a Laplace transformation $\overline{h}(p) = \int_0^\infty \mathrm{d}t\; e^{-pt}h(t)$. The remainder of the procedure is now standard. Tauberian theorems [15] state that the long-time behaviour of $g(t)$ for $t\to\infty$ is related to the one of $\overline{g}(p)$ for $p\to 0$. The critical point λ_c is given by the smallest pôle of $\overline{g}(p)$. Expanding $\overline{A}_\alpha(p)$ around $p=0$ it follows that for quenches to $\lambda>\lambda_c$, one has $g(t)\sim e^{t/\tau_r}$ and for quenches to $\lambda\leq\lambda_c$, one finds $g(t)\sim t^{F}$, with the values of F listed in Table 1.

5.2 Non-Markovian Case

In the non-Markovian case, (29) is a non-linear integral equation for $g(t)$. Progress can be made by considering instead the symmetric function $G(t,s)=G(s,t)$ which satisfies the equation

$$\frac{1}{\lambda}G(t,s) = A\left(\frac{t+s}{2}\right) + \big(G**F\big)(t,s) \tag{31}$$

which reduces to (29) in the limit $s\to t$, hence $g(t)=G(t,t)$ (although $G(t,s)\neq g_2(t,s)$). Denote by $\overline{\overline{h}}(p,q) = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\; e^{-px-qy}h(x,y)$ the two-dimensional Laplace transform. Then the formal solution of (31) is

$$\overline{\overline{G}}(p,q) = \frac{\overline{\overline{A}}(p,q)}{1/\lambda - \overline{\overline{F}}(p,q)} \tag{32}$$

The interpretation of this result is again via a Tauberian theorem.

Lemma: [48] *For a homogeneous function $f(x,y)=y^{-\alpha}\phi(x/y)$ with $\phi(0)$ finite and asymptotically $\phi(u)\stackrel{u\gg 1}{\simeq}\phi_\infty\, u^{-\lambda}$, one has the scaling form*

$$\overline{\overline{f}}(p,q) = p^{\alpha-2}\Phi(q/p) \;\;,\;\; \Phi(u) = \Gamma(2-\alpha)u^{\alpha-1}\int_0^\infty \mathrm{d}\xi\; \phi(\xi u)(\xi+1)^{\alpha-2} \tag{33a}$$

If $n<\lambda<n+1$ with $n\in\mathbb{N}$, one has asymptotically for $u\to\infty$

Table 1 Non-equilibrium exponents of the quantum spherical model for $\lambda \leq \lambda_c$ at $T = 0$ [48]

Quantum region		F	λ_C	λ_R	a	b
$\lambda = \lambda_c$	**I** $0 < d < 2$	$-\frac{\alpha}{2}$	$d + \frac{\alpha}{2}$	$d - \frac{\alpha}{2}$	$\frac{d}{2} - 1$	$\frac{d}{2}$
	II $2 < d, d + \alpha < 2$	$1 - \frac{d+\alpha}{2}$	$1 + \frac{d+\alpha}{2}$	$\frac{d-\alpha}{2} + 1$	$\frac{d}{2} - 1$	1
	V $2 < d, d + \alpha > 2$	0	$d + \alpha$	d	$\frac{d}{2} - 1$	$\frac{d+\alpha}{2}$
$\lambda < \lambda_c$		$-\frac{d+\alpha}{2}$	$\frac{d+\alpha}{2}$	$\frac{d-\alpha}{2}$	$\frac{d}{2} - 1$	0

$$\Phi(u) \simeq \phi^{(1)} u^{\alpha-2} + \cdots + \phi^{(n)} u^{\alpha-1-n} + \Phi_\infty\, u^{\alpha-1-\lambda} \tag{33b}$$
$$\Phi_\infty = \phi_\infty \frac{\Gamma(1-\lambda)}{\Gamma(1+\lambda-\alpha)} \quad , \quad \phi^{(m)} = (-1)^{m-1} \frac{\Gamma(m+1-\alpha)}{(m-1)!} \int_0^\infty \mathrm{d}u\; u^{m-1} \phi(u)$$

The critical point is found from the smallest pôle of $\overline{\overline{G}}(p,q)$. Expanding $\overline{\overline{F}}(p,q)$, this gives $\frac{1}{\lambda_c} = \overline{\overline{F}}(0,0)$. Explicitly ($\Omega_d = |S^d|$, $C_E = 0.5772\ldots$ is Euler's constant)

$$\frac{1}{\lambda_c} = \begin{cases} \frac{\mu}{\gamma} \frac{\Omega_d}{(2\pi)^d} \frac{\Lambda^d}{d} & ;\ \text{effective noise} \\ -\frac{4\hbar}{\pi\gamma} \frac{\Omega_d}{(2\pi)^d} \left\{ \frac{\Lambda^d}{d} \left[\ln\left(\Lambda^2 \frac{\mathrm{t}_0}{\gamma} \right) + C_E - \frac{2}{d} \right] + \mathrm{O}(\mathrm{t}_0) \right\} & ;\ \text{quantum noise} \end{cases} \tag{34}$$

which is finite for all $d > 0$. Hence $d_\ell = 0$ for both quantum dynamics and effective dynamics which is different from the equilibrium values of d_ℓ quoted in (7). Hence *the stationary state of the* $T = 0$ *quantum dynamics cannot be an equilibrium state* ! This is even more surprising since the single-particle dynamics constructed in section 1 should for any $T > 0$ relax to the unique equilibrium state.

Qualitatively, the results for $g(t)$ of non-Markovian quantum dynamics are analogous to the ones of effective dynamics. For quenches to $\lambda > \lambda_c$, $g(t) \sim e^{t/\tau_r}$ is exponential, with $\tau_r \sim \left(\lambda - \lambda_c\right)^{-2/d}$. For quenches to $\lambda \leq \lambda_c$, we read off $\overline{\overline{G}}(p,q) = p^{-F-2}\mathbb{G}(q/p)$, hence $G(t,s) = s^F \mathscr{G}(t/s)$ by the Lemma. It follows that $g(t) = G(t,t) = t^F \mathscr{G}(1)$ and the values of F are listed in Table 1. They are the same as for effective dynamics. The constant $\mathscr{G}(1)$ will not be needed in the leading terms of the observables.

6 Quench into the Disordered Phase

We now review results for a quantum quench with $\lambda > \lambda_c$ [48], where $g(t) \sim e^{t/\tau_r}$. For the stationary single-time correlator $C_{\boldsymbol{k}}(\infty)$, we find

$$C_{\boldsymbol{k}}(\infty) \simeq \begin{cases} \frac{\mu}{\gamma^2} \frac{k^2}{1/\tau_\mathrm{r} + 2k^2/\gamma} & ;\ \text{effective noise} \\ \frac{\hbar}{\pi\gamma} g_{\mathrm{AS}}\left(\mathrm{t}_0 \left(k^2/\gamma + (2\tau_\mathrm{r})^{-1} \right) \right) & ;\ \text{quantum noise} \end{cases} \tag{35}$$

where $g_{\rm AS}(x) := \int_0^\infty \mathrm{d}t\, \frac{\cos t}{t+x} \simeq x^{-2}$ for $x \gg 1$. They are quite distinct, but the result of quantum noise is qualitatively very similar to the classical Ornstein–Zernicke form.

Next, the two-time correlators $C_{\boldsymbol{k}}(s+\tau, s)$ do indeed satisfy time-translation-invariance for $s \gg \tau_{\rm micro}$, as expected

$$C_{\boldsymbol{k}}(s+\tau, s) \simeq \begin{cases} \frac{\mu k^2}{\gamma^2} \frac{1}{\frac{1}{\tau_{\rm r}}+2\frac{k^2}{\gamma}} \exp\left(-\left(\frac{1}{2\tau_{\rm r}}+\frac{k^2}{\gamma}\right)\tau\right) & ;\ \text{effective noise} \\ -\frac{2\hbar}{\pi\gamma} \frac{1}{\left[(2\tau_{\rm r})^{-1}+k^2/\gamma\right]^2} \frac{1}{\tau^2} & ;\ \text{quantum noise} \end{cases} \tag{36}$$

but their functional forms are very different. Analogously, the two-time response $R_{\boldsymbol{k}}(s+\tau, s)$ is time-translation-invariant for $s \gg \tau_{\rm micro}$, with the same form for both effective and quantum noises

$$R_{\boldsymbol{k}}(s+\tau, s) \simeq \frac{1}{\gamma} \exp\left(-\left(\frac{1}{2\tau_{\rm r}}+\frac{k^2}{\gamma}\right)\tau\right) \tag{37}$$

For effective noise, correlators and responses decay exponentially with τ. Empirically, one might say that they satisfy an 'effective fluctuation-dissipation theorem'

$$\frac{\partial C_{\boldsymbol{k}}(\tau)}{\partial\tau} = -\frac{\mu}{2\gamma}\left|\boldsymbol{k}\right|^2 R_{\boldsymbol{k}}(\tau) = -\frac{T_{\rm eff}(k)}{\gamma} R_{\boldsymbol{k}}(\tau)$$

but therein $T_{\rm eff}(k)$ is distinct from the bath temperature $T=0$. So that relation is rather *ad hoc*. For quantum noise, the different forms of $C_{\boldsymbol{k}}(\tau)$ and $R_{\boldsymbol{k}}(\tau)$ exclude the validity of any fluctuation-dissipation theorem.

In conclusion, although there is a single stationary state of the dynamics, *this stationary state cannot be an equilibrium state,* neither for effective nor for quantum noise, since the QFDT does not hold.

7 Quench onto Criticality or Into the Ordered Phase

We now review results for a quantum quench with $\lambda \le \lambda_c$ [48], where throughout $g(t) \sim t^F$. For a quenched into the ordered phase with $\lambda < \lambda_c$, the dynamics is the same as in the classical case and only depends on the initial correlations (23). This is expected, since the same already occurs for classical white noise dynamics [37] and quantum noise is more weak than classical noise, see Fig. 2.

For critical quenches to $\lambda = \lambda_c$, one obtains several scaling regions, as shown in Fig. 5 in dependence of the values of the dimension d and the parameter α of the initial correlations (23). The regions are the same for quantum and effective dynamics. The dotted horizontal line indicates the upper critical dimension d_u, and we read off[6]

[6] In agreement with the results of Keldysch field-theory [21, 22].

Table 2 Non-equilibrium exponents of the classical spherical model for $T \le T_c$ [37]

Classical region			F	λ_C	λ_R	a	b
$T = T_c$	$\mathbf{I}_c$	$2 < d < 4$, $0 < d + \alpha < 2$	$-1 - \frac{\alpha}{2}$	$d + \frac{\alpha}{2} - 1$	$d - \frac{\alpha}{2} - 1$	$\frac{d}{2} - 1$	$\frac{d}{2} - 1$
	$\mathbf{II}_c$	$4 < d$, $0 < d + \alpha < 2$	$1 - \frac{d+\alpha}{2}$	$1 + \frac{d+\alpha}{2}$	$\frac{d-\alpha}{2} + 1$	$\frac{d}{2} - 1$	1
	$\mathbf{III}_c$	$2 < d < 4$, $d + \alpha > 2$	$\frac{d}{2} - 2$	$\frac{3}{2}d - 2$	$\frac{3}{2}d - 2$	$\frac{d}{2} - 1$	$\frac{d}{2} - 1$
	$\mathbf{IV}_c$	$4 < d$, $d + \alpha > 2$, $\alpha > -2$	0	d	d	$\frac{d}{2} - 1$	$\frac{d}{2} - 1$
	$\mathbf{V}_c$	$4 < d$, $d + \alpha > 2$, $\alpha < -2$	0	$d + \alpha$	d	$\frac{d}{2} - 1$	$\frac{d+\alpha}{2}$
$T < T_c$		$2 < d$	$-\frac{d+\alpha}{2}$	$\frac{d+\alpha}{2}$	$\frac{d-\alpha}{2}$	$\frac{d}{2} - 1$	0

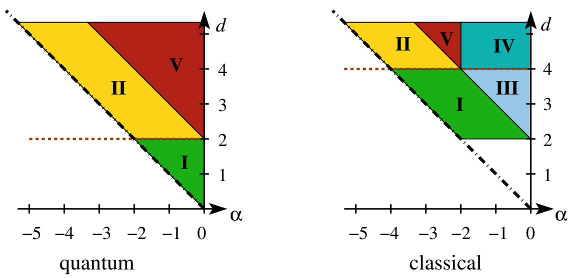

Fig. 5 Critical scaling regions for quantum and classical dynamics

$d_u^{(\mathrm{qu})} = 2$ and $d_u^{(\mathrm{cl})} = 4$. In the quantum case, this is different from the equilibrium values of d_u quoted in (7). The regions are characterised as follows:

I. both bath and initial fluctuations are relevant.
II. only initial long-ranged fluctuations are relevant.
V. no relevant fluctuations at all, long-ranged initial correlations.

For classical dynamics, two more regions exist, without a quantum counterpart (for short-ranged initial correlations with $\alpha = 0$ these are the main cases for study):

III. thermal bath fluctuations are relevant.
IV. no relevant fluctuations at all, short-ranged initial correlations.

The exact values of the non-equilibrium exponents are listed in Table 1 for quantum and effective dynamics and in Table 2 for classical dynamics. We see that in region **I** there is a shift $d \mapsto d - 1$ in λ_C and λ_R when going from quantum to classical. Otherwise, the exponents are identical (the admissible values of d and α can be different). Although the exponents are the same for quantum and effective dynamics, the scaling functions can be different. This is shown in Fig. 6 [48] for the equal-time

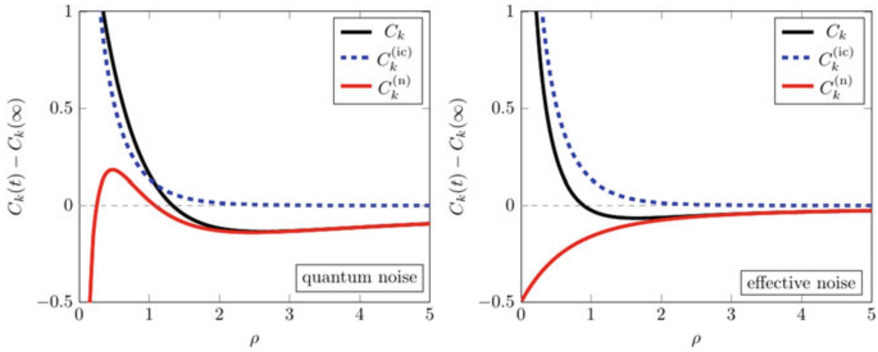

Fig. 6 Critical equal-time correlator $C_{\boldsymbol{k}}(t) - C_{\boldsymbol{k}}(\infty)$ as a function of $\rho = k^2 t/\gamma$ in region **I**

correlator $C_{\boldsymbol{k}}(t) = C_{\boldsymbol{k}}^{(ic)}(t) + C_{\boldsymbol{k}}^{(n)}(t)$ in region **I**. The contributions $C_{\boldsymbol{k}}^{(n)}(t)$ of quantum and effective noise are different, while the initial contribution $C_{\boldsymbol{k}}^{(ic)}(t)$ obviously is the same. In regions **II** and **V**, only $C_{\boldsymbol{k}}^{(ic)}(t)$ is relevant. All scaling functions are known analytically [48]. Analogous statements hold true for the two-time correlator $C_{\boldsymbol{k}}(t, s)$ while the form of the two-response $R_{\boldsymbol{k}}(t, s)$ is noise-independent.

8 Conclusions

Several surprises arise in the $T = 0$ quantum dynamics of the spherical model:

1. the stationary state is not a quantum equilibrium state, not even for $\lambda > \lambda_c$
2. the non-equilibrium exponents for $\lambda \leq \lambda_c$ are insensitive to non-Markovianity
3. non-Markovian noise is important for equal-time correlators

Figure 5 shows the correspondence of the critical scaling regimes for quantum and classical dynamics. The qualitative scenario of physical ageing (Sect. 4) is confirmed, but a comparison of Tables 1 and 2 shows that the values of the exponents are different.

Turning to possible dynamical symmetries, the underlying Eq. (15) has $z = 2$ and does admit a dynamical Schrödinger symmetry if $r(t) \sim t^{-1}$ [28, 38, 42]. This ansatz for $r(t)$ does hold true for classical dynamics at $T \leq T_c$. For quantum dynamics at $T = 0$ and $\lambda \leq \lambda_c$, $r(t) = \frac{\gamma}{2}\partial_t \ln g(t) \simeq \frac{\gamma}{2}\frac{F}{t}$. In region **I**, since $F = -\frac{\alpha}{2} \to 0$ in the limit of short-ranged initial conditions $\alpha = 0$, the ansatz[7] $r(t) \sim t^{-1}$ no longer applies. New representations for a dynamical symmetry must be sought.

Acknowledgements It is a pleasure to thank R. Araújo, A. Gambassi, M. Hase, G. T. Landi and especially and foremost S. Wald for year-long exchanges and collaborations on the quantum dynamics of the spherical model. Financial support by PHC Rila (KP-06-RILA/7) is gratefully acknowledged.

[7] It is also used in Keldysch field theory with disordered initial state [22].

References

1. C. Aron, G. Biroli, L.F. Cugliandolo, Driven quantum coarsening. Phys. Rev. Lett. **102**, 050404 (2009). arxiv:0809.0590
2. D. Barbier, L.F. Cugliandolo, G.S. Lozano, N. Nessi, M. Picco, A. Tartaglia, Pre-asymptotic dynamics of the infinite-size Neumann ($p = 2$ spherical) model. J. Phys. A **52**, 454002 (2019). arXiv:1902.06516
3. R. Becker, *Theorie der Wärme*, 2nd ed. (Springer, Heidelberg, 1978)
4. D. Bedeaux, P. Mazur, Mesoscopic non-equilibrium thermodynamics for quantum systems. Phys. A **298**, 81 (2001)
5. D. Bedeaux, Non-equilibrium thermodynamics for quantum systems. Techn. Mechanik **22**, 89 (2002)
6. T.H. Berlin, M. Kac, Spherical model of a ferromagnet. Phys. Rev. **86**, 821 (1952)
7. H.-P. Breuer, F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002)
8. A.O. Caldeira, *An Introduction to Mascroscopic Quantum Phenomena and Quantum Dissipation* (Cambridge University Press, Cambridge, 2014)
9. A. Chandran, A. Nanduri, S.S. Gubser, S.L. Sondhi, Equilibration and coarsening in the quantum O(N) model at infinite N. Phys. Rev. B **88**, 024306 (2013). arxiv:1304.2402
10. L.F. Cugliandolo, J. Kurchan, G. Parisi, Off-equilibrium dynamics and aging in unfrustrated systems. J. Physique **I4**, 1641 (1994). arXiv:cond-mat/9406053
11. L.F. Cugliandolo, in *Slow Relaxations and Non-equilibrium Dynamics in Condensed Matter*, ed. by J.-L. Barrat, M. Feiglman, J. Kurchan, J. Dalibard, Les Houches LXXVII (Springer, Heidelberg, 2003), pp. 367–521. arxiv:cond-mat/0210312
12. L.F. Cugliandolo, G.S. Lozano, N. Nessi, M. Picco, A. Tartaglia, Quenched dynamics of classical isolated systems: the spherical model with two-body random interactions or the Neuman integrable model. arxiv:1712.07688
13. L.F. Cugliandolo, G.S. Lozano, N. Nessi, M. Picco, A. Tartaglia, Quenched dynamics of classical isolated systems: the spherical spin model with two-body random interactions or the Neumann integrable model. J. Stat. Mech. 063206 (2018). arXiv:1712.07688
14. B.-G. Englert, G. Morigi, in *Coherent Evolution in Noisy Environments*, ed. by A. Buchleitner, K. Hornberger, Springer Lecture Notes in Physics, vol. 611 (Springer, Heidelberg, 2002), pp. 55–106. arxiv:quant-ph/0206116
15. W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 2, 2nd edn. (Wiley, New York, 1971)
16. V. Fock, Bemerkung zum Virialsatz. Z. Physik **63**, 855 (1930)
17. G.W. Ford, M. Kac, P. Mazur, Statistical mechanics of assemblies of coupled oscillators. J. Math. Phys. **6**, 504 (1965)
18. G.W. Ford, M. Kac, On the quantum Langevin equation. J. Stat. Phys. **46**, 803 (1987)
19. G.W. Ford, J.T. Lewis, R.F. O'Connell, Quantum Langevin equation. Phys. Rev. A **37**, 4419 (1988)
20. G.W. Ford, The fluctuation-dissipation theorem. Contemp. Phys. **58**, 244 (2017)
21. P. Gagel, P.P. Orth, J. Schmalian, Universal postquench prethermalization at a quantum critical point. Phys. Rev. Lett. **113**, 220401 (2014). arxiv:1406.6387
22. P. Gagel, P.P. Orth, J. Schmalian, Universal postquench coarsening and aging at a quantum critical point. Phys. Rev. B **92**, 115121 (2015). arxiv:1507.05821
23. C. Gardiner, P. Zoller, *Quantum Noise*, 3rd edn. (Springer, Heidelberg, 2004)
24. C. Godrèche, J.-M. Luck, Response of non-equilibrium systems at criticality: ferromagnetic models in dimension two and above. J. Phys. A **33**, 9141 (2000). arxiv:cond-mat/0001264
25. J.D. Halimeh, M.F. Maghrebi, Quantum ageing and dynamical universality in the long-range O($N \to \infty$) model. Phys. Rev. E **103**, 052142 (2021). arxiv:2008.08583
26. P. Hänggi, G.-L. Ingold, Fundamental aspects of quantum Brownian motion. Chaos **15**, 026105 (2005). arXiv:quant-ph/0412052

27. M. Henkel, C. Hoeger, Hamiltonian formulation of the spherical model in $d = r + 1$ dimensions. Z. Physik B **55**, 67 (1984)
28. M. Henkel, M. Pleimling, *Non-equilibrium phase transitions*, vol. 2 (Springer, Heidelberg, 2010)
29. J.B. Kogut, An introduction to lattice gauge theory and spin systems. Rev. Mod. Phys. **51**, 659 (1979)
30. H.W. Lewis, G.H. Wannier, Spherical model of a ferromagnet. Phys. Rev. **88**, 682 (1952); Erratum **90**, 1131 (1953)
31. A. Maraga, A. Chiocchetta, A. Mitra, A. Gambassi, Ageing and coarsening in isolated quantum systems after a quench: exact results for the quantum O(N)-model with $N \to \infty$. Phys. Rev. E **92**, 042151 (2015) arXiv:1506.04528
32. Th.M. Nieuwenhuizen, Quantum description of spherical spins. Phys. Rev. Lett. **74**, 4293 (1995). arxiv:cond-mat/9408056
33. G. Obermair, in *Dynamical Aspects of Critical Phenomena*,ed. by J.I. Budnick, M.P. Kawars (Gordon and Breach, New York, 1972), p. 137
34. M.H. Oliveira, E.P. Raposo, M.D. Coutinho-Filho, Quantum spherical spin model on hypercubic lattices. Phys. Rev. B **74**, 184101 (2006)
35. J.M. de Oliveira, Quantum Langevin equation. J. Stat. Mech. 023106 (2020). arxiv:1912.12063
36. G. Parisi, *Statistical Field Theory* (Addison Wesley, Reading, 1988)
37. A. Picone, M. Henkel, Response of non-equilibrium systems with long-range initial correlations. J. Phys. A **35**, 5575 (2002). arxiv:cond-mat/0203411
38. A. Picone, M. Henkel, Local scale-invariance and ageing in noisy systems. Nucl. Phys. B **688**, 217 (2004). arxiv:cond-mat/0402196
39. G. Ronca, Dynamics of ferromagnets: Langevin approach to the mean spherical model. J. Chem. Phys. **68**, 3737 (1978)
40. S. Sachdev, *Quantum Phase Transitions*, 2nd edn. (Cambridge University Press, Cambridge, 2011)
41. P. Schneider, *Extragalaktische Astronomie und Kosmologie* (Springer, Heidelberg, 2008); *Extragalactic astronomy and cosmology*, 2nd edn. (Springer, Heidelberg, 2015)
42. S. Stoimenov, M. Henkel, Meta-Schrödinger transformations, these proceedings (2022), pp. 411–421
43. L.C.E. Struik, *Physical Ageing in Amorphous Polymers and Other Materials* (Elsevier, Amsterdam, 1978)
44. U.C. Täuber, *Critical Dynamics* (Cambridge University Press, Cambridge, 2014)
45. T. Vojta, Quantum version of a spherical model: crossover from quantum to classical critical behaviour. Phys. Rev. B **53**, 710 (1996)
46. S. Wald, M. Henkel, Quantum phase transition in the spin-anisotropic quantum spherical model. J. Stat. Mech. P07006 (2015). arXiv:1503.06713
47. R. Araújo, S. Wald, M. Henkel, Axiomatic construction of quantum Langevin equations. J. Stat. Mech. 053101 (2019). arXiv:1809.08975
48. S. Wald, M. Henkel, A. Gambassi, Non-equilibrium dynamics of the open quantum O(n)-model with non-Markovian noise: exact results. J. Stat. Mech. 103105 (2021). arXiv:2106.08237
49. U. Weiss, *Quantum Dissipative Systems*, 5th edn. (World Scientific, Singapore, 2021)

On First Extensions in $\mathcal{S}$-Subcategories of $\mathcal{O}$

Hankyung Ko and Volodymyr Mazorchuk

Abstract We compute the first extension group from a simple object to a proper standard object and, in some cases, the first extension group from a simple object to a standard object in the principal block of an $\mathcal{S}$-subcategory of the BGG category $\mathcal{O}$ associated to a triangular decomposition of a semi-simple finite dimensional complex Lie algebra.

Keywords Category $\mathcal{O}$ · S-subcategory · Properly stratified algebra · Proper standard module · Extension

1 Introduction and Description of the Results

Bernstein–Gelfand–Gelfand category $\mathcal{O}$ associated to a triangular decomposition of a semi-simple finite dimensional complex Lie algebra $\mathfrak{g}$ is about half a century old, it originates from the classical papers [3, 4]. However, it remains an important and intensively studied object in modern representation theory, see [6, 7, 15, 18–21] for details. Category $\mathcal{O}$ has numerous analogues and generalizations, which include:

- parabolic category $\mathcal{O}$, see [31],
- $\mathcal{S}$-subcategories in $\mathcal{O}$, see [12, 27].

Homological invariants of the above categories carry essential information about both the structure and the properties of these categories. For category $\mathcal{O}$, many homological invariants are explicitly known, see [6, 7, 18–21, 23, 24] and references therein. Using these results, many homological invariants for $\mathcal{S}$-subcategories in $\mathcal{O}$, for example, various projective dimensions, can be computed using the approach of [26, Sect. 4], especially using [26, Theorem 15]. In the present paper, inspired by

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_9

the recent results from [20, 21], we take a closer look at the first extension space between certain classes of structural object in $\mathcal{S}$-subcategories of $\mathcal{O}$.

We completely determine, in type A, the first extension space from a simple object to a proper standard object in the regular block of an $\mathcal{S}$-subcategory of $\mathcal{O}$ in Theorems 5 and 6. In many special cases (notably both for the dominant and the antidominant standard objects), we completely determine the first extension space from a simple object to a standard object in the regular block of an $\mathcal{S}$-subcategory of $\mathcal{O}$ in Proposition 9. We also obtain some general results which reduce the problem of computation of the first extension space from a simple object to a standard object in an $\mathcal{S}$-subcategory of $\mathcal{O}$ to a similar problem for certain objects in category $\mathcal{O}$, see Proposition 8.

The paper is organized as follows: Sect. 2 contains preliminaries on category $\mathcal{O}$ and its combinatorics. In Sect. 3 we survey some of the recent results of [20, 21] which describe extensions from a simple highest weight module to a Verma module in category $\mathcal{O}$. In Sect. 4 we recall the definition and basic properties of $\mathcal{S}$-subcategories in $\mathcal{O}$. Section 5 is devoted to explicit description of the first extensions space from a simple to a proper standard object in $\mathcal{S}$-subcategories in $\mathcal{O}$ in type A. We also formulate a number of general results which hold in all types. In Sect. 6 we similarly look at the first extensions space from a simple to a standard object. We complete the paper with some examples in Sect. 7. This includes a detailed $\mathfrak{sl}_3$-example (for a rank one parabolic) as well as various examples of non-trivial extension from a simple to a proper standard object for the algebra $\mathfrak{sl}_4$.

2 Preliminaries on Category $\mathcal{O}$

2.1 Category $\mathcal{O}$

Let $\mathfrak{g}$ be a semi-simple finite dimensional complex Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, see [15, 30] for details. Associated to this datum, we have the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ defined as the full subcategory of the category of all finitely generated $\mathfrak{g}$-modules, consisting of all $\mathfrak{h}$-diagonalizable and locally $U(\mathfrak{n}_+)$-finite modules, cf. [3, 4, 15, 30].

Simple modules in $\mathcal{O}$ are exactly the simple highest weight modules $L(\lambda)$, where $\lambda \in \mathfrak{h}^*$, see [8, Chap. 7] for details. For each such λ, we also have in $\mathcal{O}$ the corresponding

- Verma module $\Delta(\lambda)$,
- dual Verma module $\nabla(\lambda)$,
- indecomposable projective module $P(\lambda)$,
- indecomposable injective module $I(\lambda)$,
- indecomposable tilting module $T(\lambda)$.

Consider the principal block $\mathcal{O}_0$ of $\mathcal{O}$, which is defined as the indecomposable direct summand containing the trivial $\mathfrak{g}$-module $L(0)$. Simple modules in $\mathcal{O}_0$ are indexed by the elements of the Weyl group W of $\mathfrak{g}$. For $w \in W$, we have the corresponding simple module $L_w := L(w \cdot 0)$, where $w \cdot \_$ denotes the usual dot-action of the Weyl group on $\mathfrak{h}^*$. We will similarly denote by Δ_w, ∇_w, P_w, I_w and T_w the other structural modules corresponding to L_w.

We will use Ext and Hom to denote extensions and homomorphisms in $\mathcal{O}$, respectively. The simple preserving duality on $\mathcal{O}$ is denoted by $\star$.

2.2 Graded Category $\mathcal{O}$

The category $\mathcal{O}_0$ admits a $\mathbb{Z}$-graded lift $\mathcal{O}_0^{\mathbb{Z}}$, see [33, 34]. All structural modules in $\mathcal{O}_0$ admit graded lifts (unique up to isomorphism and shift of grading). We will use the same notation as for ungraded modules to denote the following graded lifts of the structural modules in $\mathcal{O}_0^{\mathbb{Z}}$:

- L_w denotes the graded simple object concentrated in degree 0,
- Δ_w denotes the graded Verma module with top in degree 0,
- ∇_w denotes the graded dual Verma module with socle in degree 0,
- P_w is the graded indecomposable projective module with top in degree 0,
- I_w is the graded indecomposable injective modules with socle in degree 0,
- T_w is the graded indecomposable tilting module having the unique L_w subquotient in degree 0.

For $k \in \mathbb{Z}$, we denote by $\langle k \rangle$ the functor which shifts the grading by k, with the convention that $\langle 1 \rangle$ maps degree 0 to degree -1. We will use ext and hom to denote extensions and homomorphisms in $\mathcal{O}_0^{\mathbb{Z}}$, respectively. Note that, for any $k \geq 0$ and any two structural modules M and N with fixed graded lifts $\mathtt{M}$ and $\mathtt{N}$, we have

$$\mathrm{Ext}^k(M, N) \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{ext}^k(\mathtt{M}, \mathtt{N}\langle i \rangle).$$

2.3 Combinatorics of Category $\mathcal{O}_0^{\mathbb{Z}}$

Let $\mathbf{H}$ denote the Hecke algebra of W over $\mathbb{Z}[v, v^{-1}]$ in the normalization of [33]. It has the standard basis $\{H_w : w \in W\}$ and the Kazhdan–Lusztig basis $\{\underline{H}_w : w \in W\}$. The Kazhdan–Lusztig polynomials $\{p_{x,y} : x, y \in W\}$ are the entries of the transformation matrix between these two bases, that is

$$\underline{H}_y = \sum_{x \in W} p_{x,y} H_x, \text{ for all } y \in W.$$

Taking the Grothendieck group gives rise to an isomorphism of $\mathbb{Z}[v, v^{-1}]$-modules as follows:

$$\mathrm{Gr}(\mathcal{O}_0^{\mathbb{Z}}) \cong \mathbf{H}, \qquad [\Delta_w] \mapsto H_w, \text{ for } w \in W.$$

Here the $\mathbb{Z}[v, v^{-1}]$-module structure on $\mathrm{Gr}(\mathcal{O}_0^{\mathbb{Z}})$ is given be letting the element v act as $\langle -1 \rangle$. This isomorphism maps $\mathtt{P}_w$ to $\underline{H}_w$, for $w \in W$.

2.4 Kazhdan–Lusztig Orders and Cells

Following [16], for $x, y \in W$, we write $x \geq_L y$ provided that there is $w \in W$ such that $\underline{H}_x$ appears with a non-zero coefficient in $\underline{H}_w \underline{H}_y$. This defines the *left pre-order* on W. The equivalence classes with respect to this pre-order are called *left cells* and the corresponding equivalence relation is denoted $\sim_L$.

Similarly, for $x, y \in W$, we write $x \geq_R y$ provided that there is $w \in W$ such that $\underline{H}_x$ appears with a non-zero coefficient in $\underline{H}_y \underline{H}_w$. This defines the *right pre-order* on W. The equivalence classes with respect to this pre-order are called *right cells* and the corresponding equivalence relation is denoted $\sim_R$.

Finally, for $x, y \in W$, we write $x \geq_J y$ provided that there are $w, w' \in W$ such that $\underline{H}_x$ appears with a non-zero coefficient in $\underline{H}_w \underline{H}_y \underline{H}_{w'}$. This defines the *two-sided pre-order* on W. The equivalence classes with respect to this pre-order are called *two-sided cells* and the corresponding equivalence relation is denoted $\sim_J$.

The two-sided pre-order induces a partial order on the set of the two-sided cells. The maps $w \mapsto w_0 w$ and $w \mapsto w w_0$ induce anti-involution on the poset of two-sided cells, see [5, Chap. 6]. In particular, the poset of two-sided cells has the minimum element $\{e\}$ and the maximum element $\{w_0\}$. In type A_1, there is nothing else. Outside type A_1, removing these two extreme cells, we again get a poset with the minimum and the maximum element. The new minimum element is the cell containing all simple reflections, called the *small cell* (see [17]), while the new maximum element is the image of the small cell under the $w \mapsto w_0 w$ anti-involution (note that the two new extreme cells coincide in rank 2). We call this new maximum cell the *penultimate cell* and denote it by $\mathcal{J}$.

3 First Extension from a Simple to a Verma Module in Category $\mathcal{O}$

In this section, we briefly summarize the results from [20, 21, 23] which describe the first extension from a simple module to a Verma module.

3.1 First Extension to a Verma from the Anti-dominant Simple

We have the usual length function ℓ on the Weyl group W considered as a Coxeter group with respect to the simple reflections determined by our fixed triangular decomposition of $\mathfrak{g}$. For $w \in W$, the value $\ell(w)$ is the length of a reduced expression of w. We also have the content function $\mathbf{c} : W \to \mathbb{Z}_{\geq 0}$. For $w \in W$, the value $\mathbf{c}(w)$ is the number of *different* simple reflections which appear in a reduced expression of w (from the Coxeter relations it follows that this number does not depend on the choice of a reduced expression).

As usual, we denote by w_0 the longest element of W. The following result is [23, Theorem 32].

Theorem 1 *For $w \in W$ and $i \in \mathbb{Z}$, we have*

$$\dim \operatorname{ext}^1(\Delta_{w_0}, \Delta_w\langle i\rangle) = \begin{cases} \mathbf{c}(w_0 w), & i = \ell(w_0) - \ell(w) - 2; \\ 0, & \textit{otherwise}. \end{cases}$$

3.2 First Extension to a Verma Module from Other Simple Modules and Inclusion of Verma Modules

Recall the following properties of (graded) Verma modules:

- every non-zero map between two Verma modules is injective;
- for $x, y \in W$, we have $\hom(\Delta_x, \Delta_y\langle d\rangle) \neq 0$ if and only if $x \geq y$ in the Bruhat order and $d = \ell(x) - \ell(y)$;
- $\dim \operatorname{Hom}(\Delta_x, \Delta_y) \leq 1$, for all $x, y \in W$.

The ungraded versions of these properties can be found in [8, Chap. 7]. The graded version of the second property follows by matching the degrees using standard arguments, see e.g. [34].

In particular, each $\Delta_x\langle -\ell(x)\rangle$ injects to Δ_e, and the cokernel $\Delta_e/(\Delta_x\langle -\ell(x)\rangle)$ belongs to $\mathcal{O}_0^{\mathbb{Z}}$. To ease the notation, we denote the latter by Δ_e/Δ_x. These cokernels control the first extension from non-anti-dominant simple modules to Verma modules in the following way, as observed in [20].

Proposition 1 *For each $x, w \in W$, with $x \neq w_0$, we have*

$$\dim \operatorname{ext}^1(L_x\langle d\rangle, \Delta_w\langle -\ell(w)\rangle) = [\operatorname{soc} \Delta_e/\Delta_w : L_x\langle d\rangle].$$

The proof is similar to the second part of [20, Proof of Corollary 2]. A similar argument will also be given in Proposition 4.

The rest of this section describes the cokernels Δ_e/Δ_w, for $w \in W$. To do this, we need to dive into poset-theoretic properties of the Bruhat order. An element $w \in W$ is called *join-irreducible* provided that it is not a join (supremum) of other elements, that is, there is no $U \subset W$ with $w \notin U$ such that $w = \bigvee U$. The set of all join-irreducible elements, denoted by $\mathbf{B}$, is called the *base* of the poset W.

3.3 Cokernel of Inclusion Between Verma Modules in Type A

In a few coming subsections we restrict to the case of type A. The join-irreducible elements in W of type A are explicitly identified in [22] as the bigrassmannian elements. An element $w \in W$ is called *bigrassmannian* provided that there is a unique simple reflection s such that $\ell(sw) < \ell(w)$ and there is a unique simple reflection t such that $\ell(w) < \ell(wt)$. In type A, the base $\mathbf{B}$ agrees with the set of bigrassmannian elements in W.

The Kazhdan–Lusztig two-sided order is also easier in type A. The classical Robinson-Schensted correspondence

$$\mathrm{RS} : S_n \longrightarrow \coprod_{\lambda \vdash n} \mathbf{SYT}_\lambda \times \mathbf{SYT}_\lambda$$

assigns to $w \in W$ a pair $\mathrm{RS}(w) = (p_w, q_w)$ of standard Young tableaux of shape $\lambda =: \mathrm{sh}(w)$, where λ is a partition of n, see [32, Sect. 3.1]. By [16, Theorem 1.4], we have

- $x \sim_L y$ if and only if $q_x = q_y$;
- $x \sim_R y$ if and only if $p_x = p_y$;
- $x \sim_J y$ if and only if $\mathrm{sh}(x) = \mathrm{sh}(y)$.

The poset of all two-sided cells with respect to the two-sided order is isomorphic to the poset of all partitions of n with respect to the dominance order, see [14].

Recall that $\mathcal{J}$ denotes the penultimate cell with respect to the two-sided order. In type A, the elements in $\mathcal{J}$ are naturally indexed by pairs of simple reflections in W: for any pair (s, t) of simple reflections in W, there is a unique element $w_{s,t} \in \mathcal{J}$ such that $w_0 = sw_{s,t} = w_{s,t}t$.

We now formulate the main result of [20], that is [20, Theorem 1].

Theorem 2 *(i) For $w \in S_n$, the module Δ_e/Δ_w has simple socle if and only if $w \in \mathbf{B}$.*

(ii) The map $\mathbf{B} \ni w \mapsto \mathrm{soc}(\Delta_e/\Delta_w)$ induces a bijection between $\mathbf{B}$ and simple subquotients of Δ_e of the form L_x, where $x \in \mathcal{J}$.

(iii) For $w \in S_n$, the simple subquotients of Δ_e/Δ_w of the form L_x, where $x \in \mathcal{J}$, correspond, under the bijection from ii, to $y \in \mathbf{B}$ such that $y \leq w$.

(iv) For $w \in S_n$, the socle of Δ_e/Δ_w consists of all L_x, where $x \in \mathcal{J}$, which correspond, under the bijection from (ii), to the Bruhat maximal elements in the set $\{y \in \mathbf{B} : y \leq w\}$.

Motivated by the last claim, we denote $\mathbf{BM}(w) := \max\{y \in \mathbf{B} : y \leq w\}$.

The socle of the cokernel of an inclusion between two arbitrary Verma modules can be described using Theorem 2. The following corollary is [20, Corollary 23].

Corollary 1 *Let $v, w \in S_n$ be such that $v < w$.*

(i) The bijection from Theorem 2 (ii) induces a bijection between simple subquotients of Δ_v/Δ_w of the form L_x, where $x \in \mathcal{J}$, and $y \in \mathbf{B}$ such that $y \leq w$ and $y \nleq v$.

(ii) The socle of Δ_v/Δ_w consists of all L_x, where x corresponds to an element in $\mathbf{BM}(w) \setminus \mathbf{BM}(v)$.

3.4 First Extension to a Verma from Other Simples in Type A

Let $w \in \mathbf{B}$ be such that $\ell(sw) < \ell(w)$ and $\ell(wt) < \ell(w)$, for two simple reflections s and t. Denote by $\Phi : \mathbf{B} \to \mathcal{J}$ the map which sends such w to $w_{s,t}$. Theorem 2 and Proposition 1 has the following consequence:

Corollary 2 *Let $x, y \in S_n$ with $x \neq w_0$. Then we have*

$$\dim \operatorname{Ext}^1(L_x, \Delta_y) = \dim \operatorname{Ext}^1(\nabla_y, L_x) = \begin{cases} 1, & x \in \Phi(\mathbf{BM}(y)); \\ 0, & \textit{otherwise}. \end{cases}$$

3.5 Extensions in Singular Blocks in Type A

Let λ be a dominant integral weight and $\mathcal{O}_\lambda$ the indecomposable summand of $\mathcal{O}$ containing λ. If λ is regular, then $\mathcal{O}_\lambda$ is equivalent to $\mathcal{O}_0$. In the general case, denote by W^λ the stabilizer of λ with respect to the dot action of W. Simple objects in $\mathcal{O}_\lambda$ are then in a natural bijection with the cosets in $W/_{W^\lambda}$.

For $w \in W$ denote by $\overline{w}$ the unique longest element in wW^λ. Also, denote by $\underline{w}$ the unique shortest element in wW^λ. The following claim is [20, Theorem 16]:

Theorem 3 *Let $x, y \in S_n$ and let μ be an integral, dominant weight. Then we have*

$$\dim \operatorname{Ext}^1_{\mathcal{O}}(L(x \cdot \lambda), \Delta(y \cdot \lambda)) = \begin{cases} \mathbf{c}(\overline{x}\underline{y}) - \operatorname{rank}(W^\lambda), & \overline{x} = w_0; \\ 1, & \overline{x} \in \Phi(\mathbf{BM}(\underline{y})); \\ 0, & \textit{otherwise}. \end{cases}$$

3.6 The Graded Picture in Type A

Corollary 2 admits a graded lift. Let $s_1, \ldots, s_{n-1}$ be the simple reflections in S_n such that the corresponding Dynkin diagram is

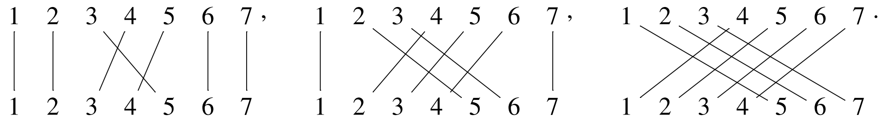
$$s_1 \;\text{———}\; s_2 \;\text{———}\; \ldots \;\text{———}\; s_{n-1}$$

For $i, j \in \{1, 2, \ldots, n-1\}$, let

$${}_i\mathbf{B}_j := \{w \in \mathbf{B} \,:\, \ell(s_i w) < \ell(w) \text{ and } \ell(w s_j) < \ell(w)\}.$$

The set ${}_i\mathbf{B}_j$ consists of $\min\{i, j, n-i, n-j\}$ elements which can be described very explicitly, see [20, Sect. 4.2]. For example, here are the three elements of ${}_4\mathbf{B}_3$ in S_7 and their graphs:

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,2\,5\,3\,4\,6\,7 \end{pmatrix}, \quad \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,5\,6\,2\,3\,4\,7 \end{pmatrix}, \quad \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 5\,6\,7\,1\,2\,3\,4 \end{pmatrix},$$



The elements of ${}_i\mathbf{B}_j$ form naturally a chain with respect to the Bruhat order on S_n. This allows us to index the elements of the set ${}_i\mathbf{B}_j$ via the tuples (i, j, k), where $0 \leq k \leq \min\{i, j, n-i, n-j\} - 1$, increasingly along the Bruhat order. Since Theorem 2 is gradable, see [20, Proposition 22], we can lift Corollary 2 to the graded setup by Proposition 1.

Proposition 2 *Let $y \in S_n$ and $x = \Phi((i, j, k))$, where $(i, j, k) \in \mathbf{BM}(y)$. Then the unique degree $m \in \mathbb{Z}$ for which* $\dim \mathrm{ext}^1(L_x\langle -m\rangle, \Delta_y\langle -\ell(y)\rangle) = 1$ *is*

$$m = \frac{(n-1)(n-2)}{2} + |i - j| + 2k.$$

Similarly, Theorem 3 can also be graded.

Proposition 3 *Let $y = \underline{y} \in S_n$ and $x = \overline{x} = \Phi((i, j, k))$, where $(i, j, k) \in$ $\mathbf{BM}(y)$. Then the unique degree $m \in \mathbb{Z}$ for which* $\dim \mathrm{ext}^1(L(x \cdot \lambda)\langle -m\rangle, \Delta(y \cdot \lambda)\langle -\ell(y)\rangle) = 1$ *is*

$$m = \frac{(n-1)(n-2)}{2} + |i - j| + 2k.$$

3.7 First Extension to a Verma from Other Simples in Other Types

By Proposition 1, the problem again reduces to determining the (socles of) Δ_e/Δ_w, for $w \in W$. However, the latter does not seem to follow a uniformly describable pattern, in general. In particular, it is shown in [21] that none of the statements in Theorem 2 is true, in general, in other types.

What remains to be true is that, for $x, w \in W$ with $x \neq w_0$, we have $\mathrm{Ext}^1(L_x, \Delta_w) = 0$, unless $x \in \mathcal{J}$. Another partial result is an upper bound. Let

$$_s\mathbf{BM}_t(w) = \{z \in \mathbf{BM}(w) \mid \; sz < z \text{ and } zt < z\},$$

for $w \in W$ and s, t simple reflections. The following is Theorem F(c) in [21].

Theorem 4 *Let $w \in W$ and $x \in \mathcal{J}$. If s, t are simple reflections in W such that $sx > x$ and $xt > x$, then*

$$\dim \mathrm{ext}^1(L_x, \Delta_w\langle d\rangle) \leq \dim \mathrm{ext}^1(L_x, \Delta_b\langle d\rangle), \tag{1}$$

for all $d \in \mathbb{Z}$, where b is the join of $_s\mathbf{BM}_t(w)$. The right hand side of (1) is again bounded by

$$\dim \mathrm{Ext}^1(L_x, \Delta_b) \leq |_s\mathbf{BM}_t(w)|. \tag{2}$$

In particular, $\mathrm{Ext}^1(L_x, \Delta_w) = 0$*, if* $_s\mathbf{BM}_t(w) = \emptyset$.

Using further computation, it is determined in [21] that

$$\dim \mathrm{ext}^1(L_x, \Delta_b\langle d\rangle)$$

is bounded by 1 in type B, by 2 in types DF, and by 3, 4, and 6 in types E_6, E_7 and E_8, respectively.

The paper [21] develops several techniques to compute specific Δ_e/Δ_w. Thus for a given $w \in W$, it is often possible to determine $\mathrm{ext}^1(L_x, \Delta_w\langle d\rangle)$, for all $x \in W$ and $d \in \mathbb{Z}$. See [21, Sect. 5] for details.

4 $\mathcal{S}$-Subcategories in $\mathcal{O}$

In this subsection we recall the definition and basic properties of $\mathcal{S}$-subcategories in $\mathcal{O}$ from [12, 27].

4.1 Definition

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h} \oplus \mathfrak{n}_+$. We denote by $W^{\mathfrak{p}}$ the corresponding parabolic subgroup of W and by $w_0^{\mathfrak{p}}$ the longest element of $W^{\mathfrak{p}}$. Denote by $\mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$ and $\mathrm{X}_{\mathfrak{p}}^{\mathrm{short}}$ the sets of the longest and the shortest representatives in the $W^{\mathfrak{p}}$-cosets from ${}_{W^{\mathfrak{p}}}\backslash W$, respectively. The map $w_0^{\mathfrak{p}} \cdot \_ : \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}} \to \mathrm{X}_{\mathfrak{p}}^{\mathrm{short}}$ is a bijection with inverse $w_0^{\mathfrak{p}} \cdot \_ : \mathrm{X}_{\mathfrak{p}}^{\mathrm{short}} \to \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$.

Recall that the parabolic category $\mathcal{O}_0^{\mathfrak{p}}$ is defined in [31] as the Serre subcategory of $\mathcal{O}_0$ generated by all L_w, where $w \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{short}}$.

We define the $\mathcal{S}$-subcategory $\mathcal{S}_0^{\mathfrak{p}}$ of $\mathcal{O}_0$ as the quotient of $\mathcal{O}_0$ modulo the Serre subcategory $\mathcal{Q}_{\mathfrak{p}}$ generated by all L_w, where $w \notin \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$. We denote by $\pi_{\mathfrak{p}} : \mathcal{O}_0 \to \mathcal{S}_0$ the Serre quotient functor.

The category $\mathcal{S}_0^{\mathfrak{p}}$ admits various realizations as a full subcategory of $\mathcal{O}_0$. For example, $\mathcal{S}_0^{\mathfrak{p}}$ is equivalent to the full subcategory of $\mathcal{O}_0$ consisting of all M which have a projective presentation of the form

$$X \to Y \to M \to 0,$$

such that, for each P_w appearing as a summand of X or Y, we have $w \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$. Alternatively, $\mathcal{S}_0^{\mathfrak{p}}$ is equivalent to the full subcategory of $\mathcal{O}_0$ consisting of all N which have an injective copresentation of the form

$$0 \to N \to X \to Y$$

such that, for each I_w appearing as a summand of X or Y, we have $w \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$. By abstract nonsense, see [2], $\mathcal{S}_0^{\mathfrak{p}}$ is equivalent to the module category over the endomorphism algebra $A^{\mathfrak{p}}$ of the direct sum of all P_w, where $w \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$.

We also note that, in the case $W^{\mathfrak{p}}$ is of type A_1, the category $\mathcal{S}_0^{\mathfrak{p}}$ is the Serre quotient of $\mathcal{O}_0$ by $\mathcal{O}_0^{\mathfrak{p}}$. In this case $W = \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}} \bigcup \mathrm{X}_{\mathfrak{p}}^{\mathrm{short}}$.

The graded version $(\mathcal{S}_0^{\mathfrak{p}})^{\mathbb{Z}}$ of $\mathcal{S}_0^{\mathfrak{p}}$ is similarly defined as the Serre quotient of $\mathcal{O}_0^{\mathbb{Z}}$ by the Serre subcategory of the latter category generated by all $L_w\langle i\rangle$, where $w \notin \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$ and $i \in \mathbb{Z}$. We use the same notation $\pi_{\mathfrak{p}}$ for the graded Serre quotient functor. The above alternative descriptions have the obvious graded analogues. For example, $(\mathcal{S}_0^{\mathfrak{p}})^{\mathbb{Z}}$ is equivalent to the full subcategory of $\mathcal{O}_0^{\mathbb{Z}}$ consisting of all objects which have a projective presentation as above with indecomposable summands of the form $P_w\langle i\rangle$, where $w \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$ and $i \in \mathbb{Z}$. Similarly for the injective copresentation.

4.2 Origins and Motivation

$\mathcal{S}$-subcategories in $\mathcal{O}$ were formally defined in [12]. They provide a uniform description for a number of generalizations of category $\mathcal{O}$ in [10, 11, 13, 25, 29]. Notably, these include various categories of Gelfand–Zeitlin module, see [25], and Whittaker modules, see [29].

The realization of the $\mathcal{S}$-subcategories in $\mathcal{O}$ as projectively presentable modules in $\mathcal{O}$ was studied in [27]. In particular, in [27] it was shown that the action of projective functors on $\mathcal{S}_0$ categorifies the permutation W-module for $W^{\mathfrak{p}}$, i.e., the W-module obtained by inducing the trivial $W^{\mathfrak{p}}$-module up to W (see also [28] for further details).

4.3 Stratified Structure

Here we recall some structural properties of $\mathcal{S}_0^{\mathfrak{p}}$ established in [12, 27].

For $w \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$, denote by

- $L_w^{\mathfrak{p}}$ the object $\pi_{\mathfrak{p}}(L_w)$ in $\mathcal{S}_0^{\mathfrak{p}}$;
- $P_w^{\mathfrak{p}}$ the object $\pi_{\mathfrak{p}}(P_w)$ in $\mathcal{S}_0^{\mathfrak{p}}$;
- $I_w^{\mathfrak{p}}$ the object $\pi_{\mathfrak{p}}(I_w)$ in $\mathcal{S}_0^{\mathfrak{p}}$;
- $T_w^{\mathfrak{p}}$ the object $\pi_{\mathfrak{p}}(T_{w_0^{\mathfrak{p}}w}\langle -\ell(w_0^{\mathfrak{p}})\rangle)$ in $\mathcal{S}_0^{\mathfrak{p}}$.

By construction, $L_w^{\mathfrak{p}}$ is simple and $\{L_w^{\mathfrak{p}} : w \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}\}$ is a complete and irredundant list of representatives of simple objects in $\mathcal{S}_0^{\mathfrak{p}}$. The objects $P_w^{\mathfrak{p}}$ and $I_w^{\mathfrak{p}}$ are the corresponding indecomposble projectives and injectives in $\mathcal{S}_0^{\mathfrak{p}}$, respectively. For structural modules, we will use the same notation for the ungraded versions of the modules and for their graded versions. The latter are obtained by applying the graded version of $\pi_{\mathfrak{g}}$ to the standard graded lifts of structural modules.

For $w \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$, denote by $\overline{\Delta}_w^{\mathfrak{p}}$ the object $\pi_{\mathfrak{p}}(\Delta_w)$ in $\mathcal{S}_0^{\mathfrak{p}}$. Then $\overline{\Delta}_w^{\mathfrak{p}} \cong \pi_{\mathfrak{p}}(\Delta_{xw}\langle \ell(x)\rangle)$, for all $x \in W^{\mathfrak{p}}$. The object $\overline{\Delta}_w^{\mathfrak{p}}$ is called the *proper standard object* corresponding to the element w.

Further, for $w \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$, let $Q_w \in \mathcal{O}_0$ denote the quotient of P_w modulo the trace in P_w of all P_y, where $y \in W$ is such that $y < w$ with respect to the Bruhat order and $y \neq xw$, for any $x \in W^{\mathfrak{p}}$. Denote by $\Delta_w^{\mathfrak{p}}$ the object $\pi_{\mathfrak{p}}(Q_w)$ in $\mathcal{S}_0^{\mathfrak{p}}$. The object $\Delta_w^{\mathfrak{p}}$ is called the *standard object* corresponding to w.

The object $\Delta_w^{\mathfrak{p}}$ has a filtration with subquotients $\overline{\Delta}_w^{\mathfrak{p}}$ (up to graded shift). The length of this filtration is $|W^{\mathfrak{p}}|$. With more details for the graded version: for $i \in \mathbb{Z}$, the multiplicity of $\overline{\Delta}_w^{\mathfrak{p}}\langle -2i\rangle$ as a subquotient of a (graded) proper standard filtration of $\Delta_w^{\mathfrak{p}}$ equals the cardinality of the set $\{w \in W^{\mathfrak{p}} : \ell(w) = i\}$. Furthermore, each projective object in $\mathcal{S}_0^{\mathfrak{p}}$ has a filtration with standard subquotients.

The simple preserving duality $\star$ on $\mathcal{O}_0$ induces a simple preserving duality on $\mathcal{S}_0^{\mathfrak{p}}$ which we will denote by the same symbol, see [27, Lemma 2.12].

The above means that the underlying algebra $A^{\mathfrak{p}}$ of the category $\mathcal{S}_0^{\mathfrak{p}}$ is properly stratified in the sense of [9]. The objects $T_w^{\mathfrak{p}}$ are tilting with respect to this structure, in the sense of [1]. An additional property of the algebra $A^{\mathfrak{p}}$ is that each $T_w^{\mathfrak{p}}$ is also cotilting. This follows from the description of tilting modules for $A^{\mathfrak{p}}$ in [12, Sect. 6] and the fact that these modules are self-dual.

5 First Extension from a Simple to a Proper Standard Module in $\mathcal{S}^{\mathfrak{p}}$

5.1 First Extension from the Antidominant Simple

Similarly as in $\mathcal{O}$, it is easy to separately treat the following special case.

Theorem 5 *For $y \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$ and $i \in \mathbb{Z}$, we have*

$$\mathrm{Ext}^1_{\mathcal{S}}(L^{\mathfrak{p}}_{w_0}, \overline{\Delta}^{\mathfrak{p}}_y) = \dim \mathrm{ext}^1_{\mathcal{S}}\left(L^{\mathfrak{p}}_{w_0}\langle -\ell(w_0)+2\rangle, \overline{\Delta}^{\mathfrak{p}}_y\langle -\ell(y)\rangle\right) = \mathbf{c}(w_0 w_0^{\mathfrak{p}} y).$$

Proof Note that $L^{\mathfrak{p}}_{w_0} \cong \overline{\Delta}^{\mathfrak{p}}_{w_0}$ is a proper standard object. The object $T^{\mathfrak{p}}_y$ is both tilting and cotilting. In particular, it is the cotilting envelope of $\overline{\Delta}^{\mathfrak{p}}_y$. Let Q be such that the following sequence is short exact in $(\mathcal{S}^{\mathfrak{p}}_0)^{\mathbb{Z}}$:

$$0 \to \overline{\Delta}^{\mathfrak{p}}_y\langle -\ell(y)\rangle \to T^{\mathfrak{p}}_y\langle 2\ell(w_0^{\mathfrak{p}}) - \ell(y)\rangle \to Q \to 0. \tag{3}$$

Set $a := 2\ell(w_0^{\mathfrak{p}}) - \ell(y)$. As proper standard and costandard (and hence also cotiltitng) objects are homologically orthogonal, it follows that

$$\dim \mathrm{ext}^1_{\mathcal{S}}(L^{\mathfrak{p}}_{w_0}\langle i\rangle, \overline{\Delta}^{\mathfrak{p}}_y\langle -\ell(y)\rangle) = \dim \hom_{\mathcal{S}}(L^{\mathfrak{p}}_{w_0}\langle i\rangle, Q) - \dim \hom_{\mathcal{S}}(L^{\mathfrak{p}}_{w_0}\langle i\rangle, T^{\mathfrak{p}}_y\langle a\rangle) + 1.$$

At the same time, we have $w_0^{\mathfrak{p}} y \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{short}}$. Therefore, $\overline{\Delta}^{\mathfrak{p}}_y \cong \pi_{\mathfrak{p}}(\Delta_{w_0^{\mathfrak{p}} y}\langle \ell(w_0^{\mathfrak{p}})\rangle)$ and $T^{\mathfrak{p}}_y \cong \pi_{\mathfrak{p}}(T_{w_0^{\mathfrak{p}} y}\langle -\ell(w_0^{\mathfrak{p}})\rangle)$. It follows that the sequence given by Formula (3) is obtained by applying $\pi_{\mathfrak{p}}$ to the following short exact sequence in $\mathcal{O}$:

$$0 \to \Delta_{w_0^{\mathfrak{p}} y}\langle -\ell(w_0^{\mathfrak{p}} y)\rangle \to T_{w_0^{\mathfrak{p}} y}\langle -\ell(w_0^{\mathfrak{p}} y)\rangle \to Q' \to 0.$$

Since Q' has a Verma flag, the socle of Q' is a direct sum of copies of shifts of Δ_{w_0}, and $w_0 \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$. Consequently, $\pi_{\mathfrak{p}}$ induces isomorphisms

$$\hom_{\mathcal{S}}(L^{\mathfrak{p}}_{w_0}\langle i\rangle, Q) = \hom_{\mathcal{O}}(L_{w_0}\langle i\rangle, Q')$$

and

$$\hom_{\mathcal{S}}(L^{\mathfrak{p}}_{w_0}\langle i\rangle, T^{\mathfrak{p}}_y\langle -\ell(w_0^{\mathfrak{p}} y)\rangle) = \hom_{\mathcal{O}}(L_{w_0}\langle i\rangle, T_{w_0^{\mathfrak{p}} y}\langle a\rangle).$$

This implies that

$$\mathrm{ext}^1_{\mathcal{S}}(L^{\mathfrak{p}}_{w_0}\langle i\rangle, \overline{\Delta}^{\mathfrak{p}}_y\langle -\ell(y)\rangle) = \mathrm{ext}^1_{\mathcal{O}}(L_{w_0}\langle i\rangle, \Delta_{w_0^{\mathfrak{p}} y}\langle -\ell(w_0^{\mathfrak{p}} y)\rangle)$$

and the claim of the theorem now follows from Theorem 1.

5.2 Inclusions Between Proper Standard Modules

Recall from Sect. 3.2 the properties of homomorphisms between Verma modules in $\mathcal{O}$. Applying the functor $\pi_{\mathfrak{p}}$ gives:

- every non-zero map between two proper standard objects in $\mathcal{S}_0^{\mathfrak{p}}$ is injective;
- for $x, y \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$, we have $\hom_{\mathcal{S}_0}(\overline{\Delta}_x^{\mathfrak{p}}, \overline{\Delta}_y^{\mathfrak{p}}\langle d\rangle) \neq 0$ if and only if $x \geq y$ and $d = \ell(y) - \ell(x)$;
- $\dim \mathrm{Hom}_{\mathcal{S}_0}(\overline{\Delta}_x^{\mathfrak{p}}, \overline{\Delta}_y^{\mathfrak{p}}) \leq 1$, for all $x, y \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$.

We thus obtain the canonical quotients $\overline{\Delta}_y^{\mathfrak{p}}/\overline{\Delta}_x^{\mathfrak{p}} := \overline{\Delta}_y^{\mathfrak{p}}/(\overline{\Delta}_x^{\mathfrak{p}}\langle \ell(y) - \ell(x)\rangle)$. The following analogue of Proposition 1 relates these quotients to extensions from simple to proper standard objects in $\mathcal{S}_0^{\mathfrak{p}}$.

Proposition 4 *For each $x, y \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$ with $x \neq w_0$, we have*

$$\dim \mathrm{ext}^1(L_x^{\mathfrak{p}}\langle d\rangle, \overline{\Delta}_y^{\mathfrak{p}}\langle -\ell(y)\rangle) = [\mathrm{soc}\, \overline{\Delta}_e^{\mathfrak{p}}/\overline{\Delta}_y^{\mathfrak{p}} : L_x^{\mathfrak{p}}\langle d\rangle].$$

Proof Let $L := (L_x^{\mathfrak{p}}\langle d\rangle)^{\oplus m}$ and suppose we have a short exact sequence

$$0 \to \overline{\Delta}_y^{\mathfrak{p}}\langle -\ell(y)\rangle \to M \to L \to 0 \tag{4}$$

such that M is indecomposable. Since L is semisimple, we have

$$\mathrm{soc}\, M = \mathrm{soc}\, \overline{\Delta}_y^{\mathfrak{p}}\langle -\ell(y)\rangle = L_{w_0}^{\mathfrak{p}}\langle -\ell(w_0)\rangle.$$

Thus, the injective covers of $\overline{\Delta}_y^{\mathfrak{p}}$ and of M coincide and are isomorphic to $I_{w_0}^{\mathfrak{p}}\langle -\ell(w_0)\rangle$. The latter is also isomorphic to a shift of $P_{w_0}^{\mathfrak{p}}$.

Being both a tilting and a cotilting object, $P_{w_0}^{\mathfrak{p}}$ has a proper standard filtration which starts with a submodule isomorphic to $\overline{\Delta}_{w_0^{\mathfrak{p}}}^{\mathfrak{p}}$, up to shift. In particular, the cokernel of the inclusion

$$0 \to \overline{\Delta}_{w_0^{\mathfrak{p}}}^{\mathfrak{p}}\langle -\ell(w_0^{\mathfrak{p}})\rangle \to I_{w_0}^{\mathfrak{p}}\langle -\ell(w_0)\rangle$$

has a proper standard filtration.

As the socle of each proper standard module is a shift of $L_{w_0}^{\mathfrak{p}}$, we have

$$\mathrm{Ext}_{\mathcal{S}}^1(L_w^{\mathfrak{p}}, \overline{\Delta}_{w_0^{\mathfrak{p}}}^{\mathfrak{p}}) = 0,$$

for any $w \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$ such that $w \neq w_0$. This implies that M must be a submodule of $\overline{\Delta}_{w_0^{\mathfrak{p}}}^{\mathfrak{p}}\langle -\ell(w_0^{\mathfrak{p}})\rangle$. In other words, L should be a summand of the socle of the cokernel of the canonical inclusion $\overline{\Delta}_y^{\mathfrak{p}}\langle -\ell(y)\rangle \subset \overline{\Delta}_{w_0^{\mathfrak{p}}}^{\mathfrak{p}}\langle -\ell(w_0^{\mathfrak{p}})\rangle$.

On the other hand, any summand of this socle gives rise to a non-split short exact sequence as in Formula (4) (since in that case M obviously has simple socle). The claim follows.

5.3 Cokernel of Inclusion of Proper Standard Modules

Lemma 1 *Let $x, y \in \mathtt{X}_{\mathfrak{p}}^{\text{short}}$ be such that $x \geq y$. Let $z \in W$ be such that L_z appears in the socle of Δ_y/Δ_x. Then $z \in \mathtt{X}_{\mathfrak{p}}^{\text{long}}$.*

Proof Note that $x \in \mathtt{X}_{\mathfrak{p}}^{\text{short}}$ is equivalent to $sx > x$, for each $s \in S \cap W^{\mathfrak{p}}$. Thus, if L_z appears in the socle of Δ_y/Δ_x, (and thus in the socle of Δ_e/Δ_x,) then, by [20, Proposition 6], we have $sz < z$, for each $s \in S \cap W^{\mathfrak{p}}$. The latter is equivalent to $z \in \mathtt{X}_{\mathfrak{p}}^{\text{long}}$, as desired.

Proposition 5 *For $x, y \in \mathtt{X}_{\mathfrak{p}}^{\text{long}}$ such that $x \geq y$, we have*

$$\mathrm{soc}(\overline{\Delta}_y^{\mathfrak{p}}/\overline{\Delta}_x^{\mathfrak{p}}) \cong \pi_{\mathfrak{p}}(\mathrm{soc}\, \Delta_{w_0^{\mathfrak{p}}y}/\Delta_{w_0^{\mathfrak{p}}x}).$$

This isomorphism holds as well for graded modules with the standard shifts, that is, if we shift each $\overline{\Delta}_w^{\mathfrak{p}}$ or Δ_w by $\langle -\ell(w)\rangle$.

Proof As mentioned in Sect. 4.3, we have the isomorphisms

$$\overline{\Delta}_x^{\mathfrak{p}} \cong \pi_{\pi}(\Delta_{w_0^{\mathfrak{p}}x})\langle \ell(w_0^{\mathfrak{p}})\rangle \quad \text{and} \quad \overline{\Delta}_y^{\mathfrak{p}} \cong \pi_{\pi}(\Delta_{w_0^{\mathfrak{p}}y})\langle \ell(w_0^{\mathfrak{p}})\rangle.$$

Note that $w_0^{\mathfrak{p}}x, w_0^{\mathfrak{p}}y \in \mathtt{X}_{\mathfrak{p}}^{\text{short}}$. Therefore we may apply Lemma 1 to conclude that the socle of the cokernel of the inclusion $\Delta_{w_0^{\mathfrak{p}}x} \subset \Delta_{w_0^{\mathfrak{p}}y}$ contains only L_z such that $z \in \mathtt{X}_{\mathfrak{p}}^{\text{long}}$. Now the claim of the proposition follows by applying $\pi_{\mathfrak{p}}$.

5.4 Ungraded Statements in Type A

In type A, the above results can be summarized and made more precise as follows.

Proposition 6 *In type A, for $x, y \in \mathtt{X}_{\mathfrak{p}}^{\text{long}}$ such that $x \geq y$, the cokernel of $\overline{\Delta}_x^{\mathfrak{p}} \subset \overline{\Delta}_y^{\mathfrak{p}}$ is isomorphic to the (multiplicity-free) direct sum of all simples $L_z^{\mathfrak{p}}$, where $z \in \mathtt{X}_{\mathfrak{p}}^{\text{long}}$ and $z \in \Phi(\mathbf{BM}(x) \setminus \mathbf{BM}(y))$.*

Proof This follows from Corollary 1 and Proposition 5.

Theorem 6 *In type A, let $x, y \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$. Then we have*

$$\dim \mathrm{Ext}^1_{\mathcal{S}}(L^{\mathfrak{p}}_x, \overline{\Delta}^{\mathfrak{p}}_y) = \begin{cases} \mathbf{c}(w_0 w_0^{\mathfrak{p}} y), & x = w_0; \\ 1, & x \in \Phi(\mathbf{BM}(y)); \\ 0, & \textit{otherwise}. \end{cases} \tag{5}$$

Proof The case $x = w_0$ is covered by Theorem 5. For $x \neq w_0$, Formula (5) follows from Propositions 6 and 4.

5.5 Graded Statement in Type A

We can also explicitly determine the degree shifts for the graded non-zero extensions in Theorem 6.

Proposition 7 *Assume we are in type A. Let $y \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$ and $x = \Phi((i, j, k))$, for some $(i, j, k) \in \mathbf{BM}(y) \cap \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$. Then the unique degree $m \in \mathbb{Z}$ for which* $\dim \mathrm{ext}^1(L^{\mathfrak{p}}_x\langle -m\rangle, \overline{\Delta}^{\mathfrak{p}}_y\langle -\ell(y)\rangle) = 1$ *is*

$$m = \frac{(n-1)(n-2)}{2} + |i - j| + 2k.$$

Proof This follows from Propositions 5, 4 and 2.

5.6 First Extension from Other Simples to Proper Standard Modules in Other Types

Propositions 4 and 5 translate all graded and ungraded results from [19] to the corresponding statements on the first extension spaces. In particular, we have

- for $x, y \in \mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$, we have $\mathrm{Ext}^1(L^{\mathfrak{p}}_x, \overline{\Delta}^{\mathfrak{p}}_w) = 0$ unless $x \in \mathcal{J}$;
- if $x \in \mathcal{J}$, we have
$$\dim \mathrm{Ext}^1(L^{\mathfrak{p}}_x, \overline{\Delta}^{\mathfrak{p}}_w) \leq |{}_s\mathbf{BM}_t(w)|,$$
  where s, t are simple reflections in W such that $sx > x$ and $xt > x$.

We emphasize that the main point of giving the above bound is to have a general statement, and that the bound $|{}_s\mathbf{BM}_t(w)|$ is a gross exaggeration in most of the cases. For computing/bounding first extension spaces between simple and proper standard modules, it is strongly recommended to ignore the bound $|{}_s\mathbf{BM}_t(w)|$ and instead look at [19, Sect. 5] (see also the discussion after [19, Theorem F]).

6 First Extension from a Simple to a Standard Module in $\mathcal{S}^{\mathfrak{p}}$

6.1 Elementary General Observations

Since w_0 corresponds to the minimum element for the partial order with respect to which $A^{\mathfrak{p}}$ is stratified, the standard object $\Delta^{\mathfrak{p}}_{w_0}$ is a tilting object. Due to the special properties of $A^{\mathfrak{p}}$ mentioned at the end of Sect. 4.3, it is also a cotilting module. The simple object $L^{\mathfrak{p}}_{w_0}$ is a proper standard module. Therefore, due to the homological orthogonality of proper standard and cotilting modules, we have

$$\mathrm{Ext}^i_{\mathcal{S}}(L^{\mathfrak{p}}_{w_0}, \Delta^{\mathfrak{p}}_{w_0}) = 0, \quad \text{for all } i > 0.$$

The projective-injective object $I^{\mathfrak{p}}_{w_0}$ is a tilting object and is thus the tilting envelope of the standard object $\Delta^{\mathfrak{p}}_{w_0^{\mathfrak{p}}}$. Therefore the cokernel of the inclusion $\Delta^{\mathfrak{p}}_{w_0^{\mathfrak{p}}} \hookrightarrow I^{\mathfrak{p}}_{w_0}$ has a standard filtration. As the socle of each standard object is isomorphic to $L^{\mathfrak{p}}_{w_0}$, it follows that the only simple object appearing in the socle of the cokernel of the above inclusion is $L^{\mathfrak{p}}_{w_0}$. Consequently,

$$\mathrm{Ext}^1_{\mathcal{S}}(L^{\mathfrak{p}}_x, \Delta^{\mathfrak{p}}_{w_0^{\mathfrak{p}}}) = 0, \quad \text{for all } x \in \mathrm{X}^{\mathrm{long}}_{\mathfrak{p}} \setminus \{w_0\}.$$

We will generalize this result below in Sect. 6.3.

6.2 Reduction to Category $\mathcal{O}$

The following statement reduces the problem of computing first extensions from simple to standard objects in $\mathcal{S}$ to the problem of computing first extensions between certain modules in $\mathcal{O}$.

Proposition 8 *For $x, y \in \mathrm{X}^{\mathrm{long}}_{\mathfrak{p}}$ and $i \in \mathbb{Z}$, we have an isomorphism*

$$\mathrm{ext}^1_{\mathcal{S}}(L^{\mathfrak{p}}_x, \Delta^{\mathfrak{p}}_y\langle i\rangle) \cong \mathrm{ext}^1(L_x, Q_y\langle i\rangle).$$

Proof The functor $\pi_{\mathfrak{p}}$ connects $\mathcal{O}_0$ and $\mathcal{S}^{\mathfrak{p}}_0$. Since $\pi_{\mathfrak{p}}(L_x) = L^{\mathfrak{p}}_x$ and $\pi_{\mathfrak{p}}(Q_y) = \Delta^{\mathfrak{p}}_y$, we need to show that the socle of the cokernel C_y of the natural inclusion $Q_y \hookrightarrow I_{w_0}$ only contains simples of the form L_z, where $z \in \mathrm{X}^{\mathrm{long}}_{\mathfrak{p}}$.

Let $\mathfrak{a}$ be the semi-simple part of $\mathfrak{p}$. For $w \in \mathrm{X}^{\mathrm{long}}_{\mathfrak{p}}$, the module Q_w is obtained by parabolic induction (from $\mathfrak{p}$ to $\mathfrak{g}$) of a projective-injective module in the category $\mathcal{O}$ for $\mathfrak{a}$, see [27, Proposition 2.9]. In particular, Q_w is an (infinite) direct sum of projective-injective modules in the category $\mathcal{O}$ for $\mathfrak{a}$. Since $I_{w_0} = P_{w_0}$ has a filtration whose subquotients are various Q_w's, the module I_{w_0} is an (infinite) direct sum of projective injective module in the category $\mathcal{O}$ for $\mathfrak{a}$. Consequently, the module C_y is an (infinite) direct sum of projective injective module in the category $\mathcal{O}$ for $\mathfrak{a}$. In particular, for any simple root α of $\mathfrak{a}$, the action of a non-zero element in $\mathfrak{a}_\alpha$ on any simple submodule L_z of C_y is injective.

This means that $sz > z$, for any simple reflection $s \in W^{\mathfrak{p}}$, and hence $z \in X_{\mathfrak{p}}^{\text{long}}$ as asserted. Now the statement of the proposition follows by comparing the long exact sequence obtained by applying $\hom^1(L_x, \_\langle i\rangle)$ to the short exact sequence

$$0 \to Q_y \to I_{w_0} \to C_y \to 0$$

with the image of this long exact sequence under $\pi_{\mathfrak{p}}$.

As a corollary, we have the following general observation:

Corollary 3 *For $x, y \in X_{\mathfrak{p}}^{\text{long}}$ and $i \in \mathbb{Z}$. If $\mathrm{ext}^1_{\mathcal{S}}(L^{\mathfrak{p}}_x, \Delta^{\mathfrak{p}}_y\langle i\rangle) \neq 0$, then $x \in \mathcal{J} \cup \{w_0\}$.*

Proof By Proposition 8, we need to show that $\mathrm{ext}^1(L_x, Q_y\langle i\rangle) \neq 0$ implies $x \in \mathcal{J} \cup \{w_0\}$. The module Q_w has a Verma flag, by construction. From [20, Proposition 3] it follows that $\mathrm{ext}^1(L_x, \Delta_w\langle i\rangle) \neq 0$, for $w \in W$, implies $x \in \mathcal{J} \cup \{w_0\}$. As any non-zero extension from L_x to $Q_y\langle i\rangle$ must induce a non-zero extension from L_x to one of the Verma subquotients of $Q_y\langle i\rangle$, the claim of the corollary follows.

6.3 *The Case of Standard Modules which Can be Obtained Using Projective Functors*

An element $w \in W$ is called ($\mathfrak{p}$-)*special* provided that the subgroup $w^{-1}W^{\mathfrak{p}}w$ is parabolic, that is, there exists a parabolic subgroup $W^{\tilde{\mathfrak{p}}}$ of W such that $W^{\mathfrak{p}}w = wW^{\tilde{\mathfrak{p}}}$. For example, any $w \in W^{\mathfrak{p}}$, in particular $w_0^{\mathfrak{p}}$, is special. Also, w_0 is special, for we can choose $W^{\tilde{\mathfrak{p}}} = w_0 W^{\mathfrak{p}} w_0$.

Proposition 9 *Let $x, y \in X_{\mathfrak{p}}^{\text{long}}$ and assume that y is special.*

(i) We have

$$\mathrm{Ext}^1_{\mathcal{S}}(L^{\mathfrak{p}}_x, \Delta^{\mathfrak{p}}_y) \cong \mathrm{Ext}^1_{\mathcal{O}}(L(x\cdot\lambda), \Delta(y\cdot\lambda)),$$

where λ is an integral dominant weight which has the dot-stabilizer $W^{\tilde{\mathfrak{p}}}$.

(ii) Under the additional assumption $x \neq w_0$, we have

$$\dim \mathrm{Ext}^1_{\mathcal{S}}(L^{\mathfrak{p}}_x, \Delta^{\mathfrak{p}}_y) = \dim \mathrm{Ext}^1_{\mathcal{O}}(L(x\cdot\lambda), \Delta(y\cdot\lambda)) = [\mathrm{soc}\, \Delta_e/\Delta_{w_0^{\mathfrak{p}}y} : L_x].$$

Proof Let x, y be as above and let $\tilde{\mathfrak{p}}$ be such that $W^{\mathfrak{p}}y = yW^{\tilde{\mathfrak{p}}}$. Let $\tilde{w}_0$ be the longest element in $W^{\tilde{\mathfrak{p}}}$. We have $Q_w \cong \theta_{\tilde{w}_0}\Delta_w$, since both sides are characterized as the quotient of P_w with a filtration where the factors are exactly $\Delta_z\langle -\ell(w) + \ell(z)\rangle$

for $z \in W^{\mathfrak{p}}w = wW^{\tilde{\mathfrak{p}}}$ (with multiplicity one). Let λ be a dominant integral weight for which $W^{\tilde{\mathfrak{p}}}$ is the dot-stabilizer. Let $\mathcal{O}_\lambda$ be the corresponding block of $\mathcal{O}$. Consider the corresponding projective functors

$$\theta^{\text{on}}_{\tilde{w}_0} : \mathcal{O}_0 \to \mathcal{O}_\lambda \quad \text{and} \quad \theta^{\text{out}}_{\tilde{w}_0} : \mathcal{O}_\lambda \to \mathcal{O}_0$$

of translation onto and out of the $W^{\tilde{\mathfrak{p}}}$-wall, respectively. These functors are biadjoint and $\theta_{\tilde{w}_0} \cong \theta^{\text{out}}_{\tilde{w}_0}\theta^{\text{on}}_{\tilde{w}_0}$. In particular, for $x \in \mathrm{X}^{\text{long}}_{\mathfrak{p}}$, we have

$$\mathrm{Ext}^1(L_x, Q_w) \cong \mathrm{Ext}^1(\theta^{\text{on}}_{\tilde{w}_0}L_x, \theta^{\text{on}}_{\tilde{w}_0}\Delta_w).$$

Since $x \in \mathrm{X}^{\text{long}}_{\mathfrak{p}}$, we have $\theta^{\text{on}}_{\tilde{w}_0}L_x \cong L(x \cdot \lambda)$ in $\mathcal{O}_\lambda$. We also have $\theta^{\text{on}}_{\tilde{w}_0}\Delta_w \cong \Delta(w \cdot \lambda)$ in $\mathcal{O}_\lambda$. The claimed evaluation of $\dim \mathrm{Ext}^1_{\mathcal{S}}(L^{\mathfrak{p}}_x, \Delta^{\mathfrak{p}}_y)$ now follows from Proposition 8.

Now we prove the second equality in the second statement, where the first equality is obtained from the first claim. If $x \neq w_0$ then the proof of Proposition 4 (or of Proposition 1) identifies the value $\dim \mathrm{Ext}^1_{\mathcal{O}}(L(x \cdot \lambda), \Delta(y \cdot \lambda))$ with the value $[\mathrm{soc}\, \Delta(\lambda)/\Delta(y \cdot \lambda) : L(x \cdot \lambda)]$. The latter agrees with $[\mathrm{soc}\, \Delta_e/\Delta_{w_0^{\mathfrak{p}}y} : L_x]$ by [20, Proposition 15] since $w_0^{\mathfrak{p}}y = y\tilde{w}_0$ is the shortest element in $W^{\mathfrak{p}}y = yW^{\tilde{\mathfrak{p}}}$.

6.4 A Type A Formula

By Sect. 3.5, Proposition 9 completely computes the first extension between simple and standard in $\mathcal{S}$-subcategories in type A.

Proposition 10 *Let $x, y \in \mathrm{X}^{\text{long}}_{\mathfrak{p}}$ with y special and assume we are in type A. Then*

$$\dim \mathrm{Ext}^1_{\mathcal{S}}(L^{\mathfrak{p}}_x, \Delta^{\mathfrak{p}}_y) = \begin{cases} \mathbf{c}(\overline{x}\underline{y}) - \mathrm{rank}(W^{\mathfrak{p}}), & \overline{x} = w_0; \\ 1, & \overline{x} \in \Phi(\mathbf{BM}(\underline{y})); \\ 0, & \textit{otherwise}. \end{cases}$$

The graded version of this claim is obtain in the obvious way using the shifts described in Sect. 3.6.

7 Examples

7.1 sl$_3$-*Example*

Consider the case of the Lie algebra $\mathfrak{sl}_3$. In this case we have $W = S_3 = \{e, s, t, st, ts, w_0 = sts = tst\}$. Let $\mathfrak{p}$ be such that $W^{\mathfrak{p}} = \{e, s\}$. With such a choice, we have

$$\mathrm{X}_{\mathfrak{p}}^{\mathrm{long}} = \{s, st, w_0\} \quad \text{and} \quad \mathrm{X}_{\mathfrak{p}}^{\mathrm{short}} = \{e, t, ts\}$$

and the Hasse diagrams for the (opposite of the) Bruhat order on W, $\mathrm{X}_{\mathfrak{p}}^{\mathrm{long}}$ and $\mathrm{X}_{\mathfrak{p}}^{\mathrm{short}}$ are as follows:

$$\begin{tikzcd}
& e \arrow[dl, dash] \arrow[dr, dash] & \\
s \arrow[d, dash] \arrow[drr, dash] & & t \arrow[d, dash] \arrow[dll, dash] \\
st \arrow[dr, dash] & & ts \arrow[dl, dash] \\
& w_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
s \arrow[d, dash] \\
st \arrow[d, dash] \\
w_0
\end{tikzcd}
\qquad
\begin{tikzcd}
e \arrow[d, dash] \\
t \arrow[d, dash] \\
ts
\end{tikzcd}$$

If we denote $L_x^{\mathfrak{p}}$ simply by x, then the subquotients of the graded filtrations of the indecomposable projectives in $\mathcal{S}_0$ are as follows:

$$\begin{array}{ccc}
 & P_{w_0}^{\mathfrak{p}} & \\
 & & \\
 & w_0 & \\
 & st & \\
w_0 & s & w_0 \\
st & & st \\
w_0 & s & w_0 \\
 & st & \\
 & w_0 &
\end{array}
\qquad
\begin{array}{ccc}
 & P_{st}^{\mathfrak{p}} & \\
 & & \\
 & st & \\
w_0 & s & \\
st & st & \\
w_0 & w_0 & s \\
 & st & \\
 & w_0 &
\end{array}
\qquad
\begin{tikzcd}
& P_s^{\mathfrak{p}} & \\
& s \arrow[d, dash] & \\
& st \arrow[dl, dash] \arrow[dr, dash] & \\
w_0 \arrow[dr, dash] & & s \arrow[dl, dash] \\
& st \arrow[d, dash] & \\
& w_0 &
\end{tikzcd}$$

The (graded and unique) Loewy filtrations of the proper standard modules are as follows:

$$\begin{array}{c}
\overline{\Delta}_{w_0}^{\mathfrak{p}} \\
 \\
w_0
\end{array}
\qquad
\begin{tikzcd}
\overline{\Delta}_{st}^{\mathfrak{p}} \\
st \arrow[d, dash] \\
w_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\Delta}_{s}^{\mathfrak{p}} \\
s \arrow[d, dash] \\
st \arrow[d, dash] \\
w_0
\end{tikzcd}$$

We note that all proper standard modules are multiplicity-free and hence the corresponding module diagrams are well-defined. This is not the case for the indecomposable projectives $P_{w_0}^{\mathfrak{p}}$ and $P_{st}^{\mathfrak{p}}$ which are not even graded multiplicity-free. The projective $P_s^{\mathfrak{p}}$ is not multiplicity-free but it is graded multiplicity-free and hence its module diagram is well-defined as well as the algebra $A^{\mathfrak{p}}$ is positively graded.

The following table contains information on the dimension and the degree shift for the extension spaces from a simple object to a proper standard object in the format (d, m) for the formula $\dim \operatorname{ext}^1_{\mathcal{S}_0}(L^{\mathfrak{p}}_x, \overline{\Delta}^{\mathfrak{p}}_y\langle m\rangle)$:

$x \setminus y$	s	st	w_0
s	–	$(1, -1)$	–
st	–	–	$(1, -1)$
w_0	$(2, 0)$	$(2, -1)$	$(1, -2)$

Note that s and w_0 are special while st is not. The following table contains information on the dimension and the degree shift for the extension spaces from a simple object to a standard object in the format (d, m) for the formula $\dim \operatorname{ext}^1_{\mathcal{S}_0}(L^{\mathfrak{p}}_x, \Delta^{\mathfrak{p}}_y\langle m\rangle)$:

$x \setminus y$	s	st	w_0
s	–	$(1, 1)$	–
st	–	–	$(1, 1)$
w_0	$(1, 2)$	$(1, 1)$	–

7.2 $\mathbf{sl}_4$*-Example*

The Lie algebra $\mathbf{sl}_4$ is the smallest Lie algebra for which there are non-trivial Kazhdan–Lusztig polynomials. These non-trivial KL-polynomials also contribute to a non-trivial extension from a simple module to a Verma module.

We have $W = S_4$ and let s_1, s_2 and s_3 be the simple reflections with the corresponding Dynkin diagram

$$s_1 \text{———} s_2 \text{———} s_3 \ .$$

As pointed out in [20, Sect. 1.3], we have the following fact (which we present here in the graded version):

$$\operatorname{ext}^1(L_{s_2 w_0}\langle -3\rangle, \Delta_{s_2}\langle -1\rangle) \cong \mathbb{C}.$$

Note that $s_2 w_0$ is a longest representative in the cosets ${}_{W^{\mathfrak{p}}} \setminus W$ for the choices of a parabolic subgroups $W^{\mathfrak{p}}$ in W given by the following subsets of simple roots:

$$\varnothing, \quad \{s_1\}, \quad \{s_2\}, \quad \{s_1, s_2\}.$$

We denote the corresponding parabolic subalgebras by $\mathfrak{p}_i$, for $i = 1, 2, 3, 4$. Consequently, we have:

$$\begin{aligned}
&\mathrm{ext}^1_{\mathcal{S}^{\mathfrak{p}}_1}(L^{\mathfrak{p}_1}_{s_2w_0}\langle -3\rangle, \overline{\Delta}^{\mathfrak{p}_1}_{s_2}\langle -1\rangle) \cong \mathbb{C},\\
&\mathrm{ext}^1_{\mathcal{S}^{\mathfrak{p}}_2}(L^{\mathfrak{p}_2}_{s_2w_0}\langle -3\rangle, \overline{\Delta}^{\mathfrak{p}_2}_{s_1s_2}\langle -2\rangle) \cong \mathbb{C},\\
&\mathrm{ext}^1_{\mathcal{S}^{\mathfrak{p}}_3}(L^{\mathfrak{p}_3}_{s_2w_0}\langle -3\rangle, \overline{\Delta}^{\mathfrak{p}_3}_{s_3s_2}\langle -2\rangle) \cong \mathbb{C},\\
&\mathrm{ext}^1_{\mathcal{S}^{\mathfrak{p}}_4}(L^{\mathfrak{p}_4}_{s_2w_0}\langle -3\rangle, \overline{\Delta}^{\mathfrak{p}_4}_{s_1s_3s_2}\langle -3\rangle) \cong \mathbb{C}.
\end{aligned}$$

Acknowledgements For the second author, this research is supported by the Swedish Research Council.

References

1. I. Ágoston, D. Happel, E. Lukács, L. Unger, Standardly stratified algebras and tilting. J. Algebra **226**(1), 144–160 (2000)
2. M. Auslander, Representation theory of Artin algebras. I, II. Comm. Algebra **1**, 177–268 (1974); ibid. **1**, 269–310 (1974)
3. I. Bernstein, I. Gelfand, S. Gelfand, Differential operators on the base affine space and a study of g-modules, in *Lie Groups and Their Representations* (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971) (Halsted, New York, 1975), pp. 21–64
4. I. Bernstein, I. Gelfand, S. Gelfand, A certain category of g-modules. (Russian) Funkcional. Anal. i Prilozen. **10**(2), 1–8 (1976)
5. A. Björner, F. Brenti, *Combinatorics of Coxeter groups*. Graduate Texts in Mathematics, vol. 231 (Springer, New York, 2005), xiv+363 pp
6. K. Coulembier, V. Mazorchuk, Some homological properties of category $\mathcal{O}$. III. Adv. Math. **283**, 204–231 (2015)
7. K. Coulembier, V. Mazorchuk, Some homological properties of category $\mathcal{O}$. IV. Forum Math. **29**(5), 1083–1124 (2017)
8. J. Dixmier, *Enveloping algebras*. Revised reprint of the 1977 translation. Graduate Studies in Mathematics, vol. 11 (American Mathematical Society, Providence, RI, 1996), xx+379 pp
9. V. Dlab, Properly stratified algebras. C. R. Acad. Sci. Paris Sér. I Math. **331**(3), 191–196 (2000)
10. V. Futorny, S. König, V. Mazorchuk, Categories of induced modules and standardly stratified algebras. Algebr. Represent. Theory **5**(3), 259–276 (2002)
11. V. Futorny, S. König, V. Mazorchuk, A combinatorial description of blocks in $\mathcal{O}(\mathcal{P}, \Lambda)$ associated with $\mathfrak{sl}(2)$-induction. J. Algebra **231**(1), 86–103 (2000)
12. V. Futorny, S. König, V. Mazorchuk, $\mathcal{S}$-subcategories in $\mathcal{O}$. Manuscripta Math. **102**(4), 487–503 (2000)
13. V. Futorny, S. König, V. Mazorchuk, Categories of induced modules for Lie algebras with triangular decomposition. Forum Math. **13**(5), 641–661 (2001)
14. M. Geck, Kazhdan-Lusztig cells and the Murphy basis. Proc. London Math. Soc. (3) **93**(3), 635–665 (2006)
15. J. Humphreys, *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*. Graduate Studies in Mathematics, vol. 94 (American Mathematical Society, Providence, RI, 2008), xvi+289 pp
16. D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras. Invent. Math. **53**(2), 165–184 (1979)
17. T. Kildetoft, M. Mackaay, V. Mazorchuk, J. Zimmermann, Simple transitive 2-representations of small quotients of Soergel bimodules. Trans. Amer. Math. Soc. **371**(8), 5551–5590 (2019)

18. H. Ko, V. Mazorchuk, R. Mrden, Some homological properties of category O. V. arXiv:2007.00342. To appear in IMRN
19. H. Ko, V. Mazorchuk, R. Mrden, Some homological properties of category O. VI. Doc. Math. **26**, 1237–1269 (2021)
20. H. Ko, V. Mazorchuk, R. Mrden, Bigrassmannian permutations and Verma modules. Selecta Math. (N.S.) **27**(4), Paper No. 55, 24 pp (2021)
21. H. Ko, V. Mazorchuk, R. Mrden, Join operation for the Bruhat order and Verma modules. arXiv:2109.01067
22. A. Lascoux, M.-P. Schützenberger, Treillis et bases des groupes de Coxeter. Electron. J. Combin. **3**(2), Research paper 27, approx. 35 pp (1996)
23. V. Mazorchuk, Some homological properties of the category $\mathcal{O}$. Pacific J. Math. **232**(2), 313–341 (2007)
24. V. Mazorchuk, Some homological properties of the category $\mathcal{O}$. II. Represent. Theory **14**, 249–263 (2010)
25. V. Mazorchuk, On categories of Gelfand-Zetlin modules, in *Noncommutative structures in mathematics and physics (Kiev, 2000)*, NATO Science Series II: Mathematics, Physics and Chemistry, **22** (Kluwer Acad. Publ., Dordrecht, 2001), pp. 299–307
26. V. Mazorchuk, E. Persson Westin, Essential orders on stratified algebras with duality and $\mathcal{S}$-subcategories in $\mathcal{O}$. arXiv:2011.11492
27. V. Mazorchuk, C. Stroppel, Translation and shuffling of projectively presentable modules and a categorification of a parabolic Hecke module. Trans. Amer. Math. Soc. **357**(7), 2939–2973 (2005)
28. V. Mazorchuk, C. Stroppel, Categorification of (induced) cell modules and the rough structure of generalised Verma modules. Adv. Math. **219**(4), 1363–1426 (2008)
29. D. Miličić, W. Soergel, The composition series of modules induced from Whittaker modules. Comment. Math. Helv. **72**(4), 503–520 (1997)
30. R. Moody, A. Pianzola, *Lie Algebras with Triangular Decompositions*. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication (Wiley, New York, 1995). xxii+685 pp
31. A. Rocha-Caridi, Splitting criteria for $\mathfrak{g}$-modules induced from a parabolic and the Bernstein-Gelfand-Gelfand resolution of a finite-dimensional, irreducible $\mathfrak{g}$-module. Trans. Amer. Math. Soc. **262**(2), 335–366 (1980)
32. B. Sagan, *The Symmetric Group. Representations, Combinatorial Algorithms, and Symmetric Functions*, 2nd edn, *Graduate Texts in Mathematics*, vol. 203 (Springer, New York, 2001), xvi+238 pp
33. W. Soergel, Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. J. Amer. Math. Soc. **3**(2), 421–445 (1990)
34. C. Stroppel, Category $\mathcal{O}$: gradings and translation functors. J. Algebra **268**(1), 301–326 (2003)

Higher Dimensional CFTs as 2D Conformally-Equivariant Topological Field Theories

Robert de Mello Koch and Sanjaye Ramgoolam

Abstract Two and three-point functions of primary fields in four dimensional CFT have simple space-time dependences factored out from the combinatoric structure which enumerates the fields and gives their couplings. This has led to the formulation of two dimensional topological field theories with $SO(4,2)$ equivariance which are conjectured to be equivalent to higher dimensional conformal field theories. We review this CFT4/TFT2 construction in the simplest possible setting of a free scalar field, which gives an algebraic construction of the correlators in terms of an infinite dimensional $so(4,2)$ equivariant algebra with finite dimensional subspaces at fixed scaling dimension. Crossing symmetry of the CFT4 is related to associativity of the algebra. This construction is then extended to describe perturbative CFT4, by making use of deformed co-products. Motivated by the Wilson-Fisher CFT we outline the construction of U(so(d,2)) equivariant TFT2 for non-integer d, in terms of diagram algebras and their representations.

Keywords Conformal fied theory · Toplogical quantum field theory · Representation theory

Written version of the talk given by SR at XIV International Workshop LIE THEORY AND ITS APPLICATIONS IN PHYSICS, Sofia.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_10

1 Introduction

Conformal Field Theories (CFTs) in $d > 2$ dimensions have been an active topic of study in recent years. In part this activity has been stimulated by the AdS/CFT correspondence, originally stated as an equivalence between $\mathcal{N} = 4$ super Yang-Mills (SYM) theory on $R^{3,1}$ with $U(N)$ gauge group and 10 dimensional string theory [1]. A key question is to understand how the higher dimensional quantum gravity emerges from local CFT operators and their correlators. Another motivation has been to gain an understanding of exotic CFTs that do not have a conventional Lagrangian description. Good examples of these theories include the Argyres-Douglas fixed points in 4D [2] as well as the (0, 2) theories in 6D [3]. In addition, promising tools with which to study higher dimensional CFTs have become available with the revival of the bootstrap program [4], which uses associativity of the operator product expansion (OPE) to determine the CFT data.

CFTs in $d = 2$ (CFT2) have been well studied since the 80s. The primary stimulus for this activity is the worldsheet dynamics of strings in critical string theory, described by a CFT2 plus ghost system. These theories have a rich structure leading to a fruitful interaction between mathematics and physics [5, 6]. A central role is played by

- Infinite dimensional Lie algebras (the Virasoro algebra and current algebras) which control their spectrum and correlators.
- The representation theory of these algebras, extended by considerations of modular transformations of characters.
- Rational conformal field theories, with finitely many primary fields for these algebras.
- Vertex operator algebras, which provide mathematical constructions for field operators and for the OPEs.

An important observation is that the mathematics of CFT2s use two kinds of algebras. First, there are the symmetry algebras given by infinite dimensional Lie algebras (the Virasoro algebra, current algebra etc.). Secondly, there is the algebra of the quantum fields themselves, formalized through vertex operator algebras. This situation is analogous to constructions in non-commutative geometry where we have a fuzzy or quantum space which is an associative coordinate algebra [7–9], as well as a Hopf algebra acting as a symmetry of the quantum space. It is natural to expect a similar structure for CFTd, except that we have a finite dimensional symmetry algebra SO(d,2) replacing the Virasoro algebra (and its generalizations) and large multiplicities of irreducible representations (irreps) coming from the fields/quantum states. This expectation is, at least partly, motivated by the operator/state correspondence of radial quantization

$$\lim_{x\to 0} \mathcal{O}_a(x)|0\rangle = |\mathcal{O}_a\rangle \tag{1}$$

which is a general property of CFTs for any d. The AdS/CFT correspondence together with the operator state correspondence implies, for example, that string states in $\mathrm{AdS}_5 \times \mathrm{S}^5$ are in correspondence with operators in $\mathcal{N} = 4$ SYM. An understanding of the quantum states (and associated physics) in quantum gravity on AdS spacetimes requires a detailed understanding of the CFT operators and associated algebraic structures.

The $\frac{1}{2}$-BPS sector is an interesting sector of $\mathcal{N} = 4$ SYM where these ideas can be developed quite explicitly. On the AdS side of the duality, there is a rich spectrum of physical states including gravitons, strings, branes and non-trivial spacetime geometries. On the CFT side of the duality, this sector is constructed from a single complex matrix Z, transforming in the adjoint representation

$$Z \to U Z U^\dagger \tag{2}$$

under the $U(N)$ gauge symmetry. The generic gauge invariant operator is a multi-trace operator. For example, the complete set of gauge invariant operators that can be constructed using three fields is given by

$$\mathrm{Tr} Z^3 \qquad \mathrm{Tr} Z^2 \mathrm{Tr} Z \qquad (\mathrm{Tr} Z)^3 \tag{3}$$

These operators have degree 3 and are in correspondence with the partitions of 3

$$3 = 3 \qquad 3 = 2 + 1 \qquad 3 = 1 + 1 + 1 \tag{4}$$

In general, operators of degree n correspond to partitions of n and they can be constructed using permutations $\sigma \in S_n$ as follows

$$\mathcal{O}_\sigma(Z) = \sum_{i_1, \cdots, i_n} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} \tag{5}$$

For example

$$\mathrm{Tr} Z^3 = \sum_{i_1, i_2, i_3} Z^{i_1}_{i_2} Z^{i_2}_{i_3} Z^{i_3}_{i_1} = \sum_{i_1, i_2, i_3} Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} Z^{i_3}_{i_{\sigma(3)}} \tag{6}$$

with $\sigma = (123)$. The mapping between permutations and gauge invariant operators is not one-to-one since

$$\mathcal{O}_\sigma(Z) = \mathcal{O}_{\gamma\sigma\gamma^{-1}}(Z) \quad \text{for all} \quad \gamma \in S_n \tag{7}$$

which implies that two permutations in the same conjugacy class define the same gauge invariant operator. This nicely explains why gauge invariant operators correspond to partitions of n. The two-point function of degree n operators is derived, as usual, by using Wick's theorem and the basic 2-point function

$$\langle Z^i_j(x_1)(Z^\dagger)^k_l(x_2)\rangle = \frac{\delta^i_l \delta^k_j}{(x_1-x_2)^2} \tag{8}$$

This allows us to express the correlator in terms of permutation group multiplications as follows [10]

$$\langle O_{\sigma_1}(Z(x_1))O_{\sigma_2}(Z^\dagger(x_2))\rangle = \frac{1}{((x_1-x_2)^2)^n}\frac{n!}{|\mathcal{C}_{p_1}|\,|C_{p_2}|}\times \sum_{\sigma_1\in\mathcal{C}_{p_1}}\sum_{\sigma_2\in\mathcal{C}_{p_2}}\sum_{\sigma_3\in S_n}\delta(\sigma_1\sigma_2\sigma_3)N^{C_{\sigma_3}} \tag{9}$$

The combinatoric part of this answer is a quantity in a 2D topological field theory (TFT2). TFT2s are equivalent to Frobenius algebras. A Frobenius algebra is an associative algebra with a non-degenerate pairing. The algebra corresponding to the combinatoric TFT2 of the $\frac{1}{2}$-BPS sector is the centre of the group algebra of the symmetric group S_n. The connection to TFT2 can be generalized beyond the $\frac{1}{2}$-BPS sector and it turns out that multi-matrix sectors of $\mathcal{N}=4$ SYM are related to other Frobenius algebras, built from permutations or associated diagram algebras such as Brauer algebras [10–16]. For a review of these ideas, the reader can consult [17].

It is natural to ask if the space-time dependence of correlators in CFT4 (and CFTd for $d>2$) can also be described using a TFT2/Frobenius algebra language. The paper [18] gives a positive answer for the case of a free 4D massless scalar field, along with the cases in which the scalar transforms in the fundamental or in the adjoint of a global symmetry. The construction uses an infinite dimensional associative algebra which reproduces free field correlators of arbitrary free field composites and is a representation of $so(4,2)$ or $Uso(4,2)$. This algebra has an so(4, 2) invariant non-degenerate pairing. In the paper [19] the algebraic structures associated with this CFT4/TFT2 construction were used to develop novel counting formulae and construction algorithms for the primary fields of free CFT4. The paper [20] describes perturbative CFTs from this algebraic point of view (equivariant algebras). Concrete examples that are described include sectors of $d=4$ $\mathcal{N}=4$ SYM at weak coupling as well as the Wilson-Fischer CFT, defined in $d=4-\epsilon$ using the ϕ^4 interaction. Novel algebraic structures needed to accomplish this include a deformed co-product for $Uso(4,2)$, the role of indecomposable representations of $Uso(4,2)$ and diagram algebras which generalize known diagram algebras appearing in the representation theory of $Uso(d)$. This work has some overlap with the paper [23].

This paper is organized as follows: Sect. 2 reviews the CFT4/TFT2 construction in the simplest possible setting of a free scalar field. The result is a $U(so(4,2))$ equivariant TFT2 with the quantum field realized as a vertex operator. Section 3 describes perturbative CFT4 by making use of deformed co-products while Sect. 4 introduces diagram algebras and their representations, motivated by the Wilson-Fisher CFT.

2 CFT4/TFT2 Construction of the Free Scalar Field

The axiomatic approach to TFT2 that we adopt associates geometrical objects to algebraic objects, following the standard discussions (see the original work [21] and textbooks such as [22]) , with appropriate adaptations to account for the infinite-dimensionality of the state spaces. For example, a vector space $\mathcal{H}$ is associated with a circle

 $\longrightarrow \quad \mathcal{H}$ (10)

while tensor products of $\mathcal{H}$ go to disjoint unions of circles

 $\longrightarrow \quad \mathcal{H} \otimes \mathcal{H}$ (11)

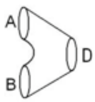
Interpolating surfaces between circles (cobordisms) are associated with linear maps between the vector spaces. For example, the map $\delta : \mathcal{H} \to \mathcal{H}$ is represented as a cylinder

$\delta_A{}^B =$  (12)

which takes circle A into circle B, while the non-degenerate pairing $\eta : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$

$\eta_{AB} =$  (13)

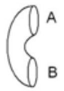
takes two circles to nothing. The product $C : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$

$C_{AB}{}^D =$  (14)

takes two circles to a circle. In the language of category theory, the circles are objects and the interpolating surfaces (cobordisms) are morphisms in a geometrical category, while the vector spaces are objects, and the linear maps are morphisms in an algebraic category. The correspondence between geometrical objects algebraic objects is a functor between the two categories. The existence of this functor requires that all relations on the geometrical side should be mirrored on the algebraic side.

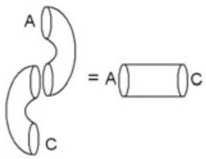
As an example, the statement that the pairing η_{AB} is non-degenerate is expressed in terms of the inverse pairing

$$\tilde{\eta}^{AB} = \tag{15}$$



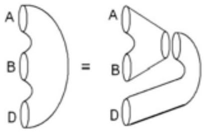
as the statement that η and $\tilde{\eta}$ glue to give the cylinder

$$\eta_{AB}\tilde{\eta}^{BC} = \delta_A{}^C \tag{16}$$



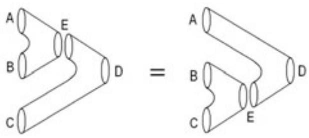
where the gluing operation is implemented by summing over the circles to be glued. Using the product $C_{AB}{}^D$ and the pairing η_{AB} we can define a new map

$$C_{ABD} = \eta_{DC} C_{AB}{}^C \tag{17}$$



which is the familiar relation between the CFT correlator (C_{ABD}) and the OPE ($C_{AB}{}^D$). Finally, associativity of the OPE is expressed as

$$C_{AB}{}^E C_{EC}{}^D = C_{BC}{}^E C_{EA}{}^D \tag{18}$$



To summarise, TFT2's correspond to commutative, associative, non-degenerate algebras known as Frobenius algebras. TFT2 with a global symmetry group G is defined by [24]. Since this will play an important role in our construction, it is worth summarizing the essential features from [24] with one important modification of the discussion due to the infinite dimensionality of the state space.

1. The state space is a representation of a group G - which will be $SO(4, 2)$ in our application.
2. The linear maps are G-equivariant linear maps.

3. The state space is infinite dimensional: amplitudes are defined for surfaces without handles. Consequently, this is a genus restricted TFT2.

The basic two point function in the CFT of a free scalar in four dimensions is

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{(x_1 - x_2)^2} \tag{19}$$

Correlators of composite operators are constructed using this contraction, according to Wick's theorem. Thus, the first step in our construction is to understand the 2-point function of the elementary field in the TFT2 language. The $so(4, 2)$ symmetry of the CFT is our starting point. The Lie algebra is spanned by the dilatation operator D which generates dilatations, the momenta P_μ which generate translations, the generators $M_{\mu\nu}$ of $so(4)$ rotations and the generators K_μ of special conformal transformations. To carry out a radial quantization of the theory we choose a point, say the origin of Euclidean $\mathbb{R}^4$. The usual state operator map associates states with the scalar field and its descendents

$$\begin{aligned} \lim_{x\to 0} \phi(x)|0\rangle &= v^+ \\ \lim_{x\to 0} \partial_\mu\phi(x)|0\rangle &= P_\mu v^+ \\ &\vdots \end{aligned} \tag{20}$$

The state v^+ is the lowest energy state in a lowest-weight representation V_+ of $so(4, 2)$

$$\begin{aligned} Dv^+ &= v^+ \\ K_\mu v^+ &= 0 \\ M_{\mu\nu} v^+ &= 0 \end{aligned} \tag{21}$$

Higher energy states are generated by $S_l^{\mu_1\mu_2\cdots\mu_l} P_{\mu_1} P_{\mu_2} \cdots P_{\mu_l} v^+$, where $S_l^{\mu_1\mu_2\cdots\mu_l}$ is a symmetric traceless tensor of $so(4)$. The index l labels a basis of linearly independent symmetric traceless tensors. There is also a dual representation V_-, which is a representation with negative scaling dimensions

$$\begin{aligned} Dv^- &= -v^- \\ P_\mu v^- &= 0 \\ M_{\mu\nu} v^- &= 0 \end{aligned} \tag{22}$$

Other states in this representation are generated by acting with $S_l^{\mu_1\mu_2\cdots\mu_l} K_{\mu_1} \cdots K_{\mu_l}$. There is a $\eta : V_+ \otimes V_- \to \mathbb{C}$, which is so(4,2) invariant

$$\eta(\mathcal{L}_a v, w) + \eta(v, \mathcal{L}_a w) = 0 \tag{23}$$

After making the choice

$$\eta(v^+, v^-) = 1 \tag{24}$$

the invariance conditions (23) and the properties of the states v^+ and v^-, determine η. For example

$$\begin{aligned}\eta(P_\mu v^+, K_\nu v^-) &= -\eta(v+, P_\mu K_\nu v^-)\\ &= \eta(v^+, (-2D\delta_{\mu\nu} + 2M_{\mu\nu})v^-) = 2\delta_{\mu\nu}\end{aligned} \tag{25}$$

Using invariance conditions one finds that $\eta(P_\mu P_\mu v^+, v^-)$ is zero. Setting $P_\mu P_\mu v^+ = 0$, which physically corresponds to imposing the equation of motion, identifies V_+ as a quotient of a bigger representation $\tilde{V}_+$. $\tilde{V}_+$ is spanned by

$$P_{\mu_1} \cdots P_{\mu_l} v^+ \tag{26}$$

i.e. it is the vector space of polynomials in P_μ. This is an indecomposable representation. After we perform the quotient by the equation of motion, we recover the irreducible representation V_+. The quotient also ensures that η is non-degenerate i.e. that there are no null vectors. So we see that η is the structure we need for the construction of a TFT2 with so(4,2) symmetry. It has both the non-degeneracy property and the invariance property. So there is an invariant in $V_+ \otimes V_-$ and thus in $V_- \otimes V_+$, but not in $V_+ \otimes V_+$ or $V_- \otimes V_-$. It is useful to introduce $V = V_+ \oplus V_-$ and define $\hat{\eta} : V \otimes V \to \mathbb{C}$

$$\hat{\eta} = \begin{pmatrix} 0 & \eta_{+-} \\ \eta_{-+} & 0 \end{pmatrix} \tag{27}$$

In V we have a state, corresponding to the quantum field, given by

$$\Phi(x) = \frac{1}{\sqrt{2}}(e^{-iP\cdot x}v^+ + x'^2 e^{iK\cdot x'} v^-), \qquad x'_\mu = \frac{x_\mu}{x^2} \tag{28}$$

and a calculation with the invariant pairing shows that

$$\eta(\Phi(x_1), \Phi(x_2)) = \frac{1}{(x_1 - x_2)^2} \tag{29}$$

This is the basic free field 2-point function, now constructed from the invariant map $\eta : V \otimes V \to \mathbb{C}$. To get all correlators, we must set up a state space, which knows about composite operators. The states obtained by the standard operator state correspondence from general local operators are of the form

$$P_{\mu_1} \cdots P_{\mu_{n_1}} \phi P_{\nu_1} \cdots P_{\nu_{n_2}} \phi \cdots P_{\tau_1} \cdots P_{\tau_{nm}} \phi \tag{30}$$

Composite operators belonging to the n field sector correspond to states in which n ϕ fields appears. Particular linear combinations of these states are primary fields, which are lowest weight states (annihilated by K_μ) that generate irreducible representations (irreps) of SO(4,2) through the action of the raising operators (P_μ). The list of primary fields in the n-field sector is obtained by decomposing the space

$$\mathrm{Proj}_{S_n \,\mathrm{invt}}(V_+^{\otimes n}) \equiv \mathrm{Sym}^n(V_+) \tag{31}$$

into SO(4,2) irreps. The symmetrization on the right hand side of (31) is needed because ϕ is a boson. We can now introduce the state space $\mathcal{H}$ of the TFT2, which we associate to a circle in TFT2. The state space consists of all possible primaries and their descendents

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(V) \tag{32}$$

where $V = V_+ \oplus V_-$. This state space is big enough to accommodate all the composite operators and it admits an invariant pairing. The state space is small enough for the invariant pairing to be non-degenerate The state space contains

$$\Phi(x) \otimes \Phi(x) \otimes \Phi \cdots \otimes \Phi(x) \tag{33}$$

which is used to construct composite operators in the TFT2 set-up. By construction, the pairing $\eta : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ reproduces all 2-point functions of arbitrary composite operators. The construction is straight forward: recall $\mathcal{H}$ is built from tensor products of V, and we have already introduced an "elementary" $\hat{\eta} : V \otimes V \to \mathbb{C}$. The construction of the complete η map is built from products of the elementary $\hat{\eta}$, using Wick contraction sums, in the obvious way. As an example, for $v_1, v_2, v_3.v_4 \in V$ we have

$$\eta(v_1 \otimes v_2, v_3 \otimes v_4) = \hat{\eta}(v_1, v_3)\hat{\eta}(v_2, v_4) + \hat{\eta}(v_1, v_4)\hat{\eta}(v_2, v_3) \tag{34}$$

We complete the definition by setting

$$\eta(v^{(n)}, v^{(m)}) \propto \delta^{mn} \tag{35}$$

where $v^{(k)} \in \mathrm{Sym}^k(V)$. This defines the pairing η_{AB} where A, B take values in the space $\mathcal{H}$ given by the sum of all n-fold symmetric products of $V = V_+ \oplus V_-$. Notice that the building blocks used in constructing η are invariant maps. The product of these invariant maps is also obviously invariant. We can also demonstrate that η is non-degenerate. The basic idea is that if you have a non-degenerate pairing $V \otimes V \to \mathbb{C}$, it extends to a non-degenerate pairing on $\mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$, by using the sum over Wick patterns. Consequently, we have

$$\eta_{AB}\tilde{\eta}^{BC} = \delta_A^C \tag{36}$$

This is the snake-cylinder equation, given in (16).

In a very similar way it is possible to define 3-point functions C_{ABC} and, in general higher point functions $C_{ABC\cdots}$ using Wick pattern products of the basic η's. By writing explicit formulae for these sums over Wick patterns, we can show that the associativity equations are satisfied. Consider Eq. (17). The C_{ABC} give 3-point functions, while the $C_{AB}^{C} = C_{ABD}\tilde{\eta}^{DC}$ give the OPE-coefficients. Through this connection, the associativity equations of the TFT2 are the crossing equations of CFT4, obtained by equating expressions for a 4-point correlator obtained by doing OPEs in two different ways. There an important property of the OPE in this language, easily illustrated by the product

$$\mathrm{Sym}^2(V) \otimes \mathrm{Sym}^2(V) \to \mathrm{Sym}^4(V) \oplus \mathrm{Sym}^2(V) \oplus \mathbb{C} \tag{37}$$

which corresponds to the free field OPE, which takes the schematic form

$$\phi^2(x)\phi^2(0) \to \phi^4 \oplus \phi^2 \oplus 1 \tag{38}$$

This demonstrates that the presence of both V_+ and V_- is needed if the TFT2 is to construct this OPE in representation theory.

The algebraic framework developed above allows us to exhibit novel ring structures in the state space of the TFT2. Further, this algebraic structure can profitably be used to give a construction of primary fields in free CFT4/CFTd [19]. The state space in radial quantization, set up around $x = 0$, is (for $x' = 0$, we would keep V_- instead)

$$\mathcal{H}_+ = \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(V_+) \tag{39}$$

The irrep V_+ is isomorphic to a space of polynomials in variables x_μ, quotiented by the ideal generated by $x_\mu x_\mu$. Taking a many-body physics view of $\mathcal{H}_+$, this is a quotient of a polynomial ring in dn variables x_μ^I. The construction of primaries, or equivalently, the problem of describing lowest weight states of irreducible representations in $\mathcal{H}_+$ is usefully done by recognizing the connection to a closely related problem about rings. It turns out that the construction of primary fields in d dimensions, and the refined counting of these primaries, according to their scaling dimension n and $so(d)$ irreps, is equivalent to studying a polynomial ring in variables the X_μ^A with $\mu \in \{1, \cdots, d\}$ and $A \in \{1, 2, \cdots, n-1\}$, under the constraints

$$\begin{aligned} A(1 - A^2)\sum_{\mu=1}^{d} X_\mu^A X_\mu^A + \sum_{B:B>A}\sum_{\mu=1}^{d} 2A(1 + A)X_\mu^A X_\mu^B \\ + \sum_{B:B<A}\sum_{\mu=1}^{d} B(1 + B)X_\mu^B X_\mu^B = 0 \end{aligned} \tag{40}$$

for $1 \le A \le (n-1)$, and

$$\sum_{A=1}^{n-1} \sum_{\mu=1}^{d} X_\mu^A X_\mu^A = 0 \tag{41}$$

The details of the derivation of these constraints are explained in [19].

3 Perturbative CFTs

Having explained the CFT4/TFT2 construction for the free scalar field, it is natural to ask about constructions for interacting theories. Towards this end, the first example theory we have in mind is the Wilson-Fischer (WF) fixed point, described by the Lagrangian

$$\int d^d x (\partial_\mu \phi \partial^\mu \phi + \frac{g}{4!} \phi^4) \tag{42}$$

together with a continuation of the Feynman rules to $d = 4 - \epsilon$ dimensions. Choosing the critical value of the coupling constant

$$g^* \sim \frac{16\pi^2}{3} \epsilon + O(\epsilon^2) \tag{43}$$

leads to a vanishing beta function and, consequently, a CFT. The fundamental field ϕ as well as composite operators (given by polynomials in derivatives of ϕ) have a modified dimension. Apart from the classical dimension, there is also an anomalous dimension, generated by loop corrections. The anomalous dimensions of the WF theory are captured by a dilatation operator. In particular, the one-loop corrections to the dimensions of composite operators

$$\partial^{k_1} \phi \partial^{k_2} \phi \cdots \partial^{k_L} \phi \tag{44}$$

are captured by a 2-body Hamiltonian

$$H = \sum_{i<j} \rho_{ij}(P_0) \tag{45}$$

where P_0 is a projector to an irrep in $V \otimes V$ with V the irrep of the scalar ϕ [25]. At order ϵ, ϕ has a vanishing anomalous dimension, while that of ϕ^2 is non-vanishing. A naive intuition informed by tensor products of representations would suggest that dimension of a composite operator is given by the sum of the dimensions of its constituents, but this is not correct. To obtain the correct dimension for ϕ^2 we need

$$D(v \otimes v) = (D \otimes 1 + 1 \otimes D)(v \otimes v) + \frac{\epsilon}{3} P_0(v \otimes v) \tag{46}$$

This motivates the definition of the deformed co-product

$$\Delta(D) = D \otimes 1 + 1 \otimes D + \frac{\epsilon}{3} P_0 \tag{47}$$

This deformation is highly reminiscent of deformations we encounter in quantum groups. For example, in $U_q(su(2))$ we have

$$\Delta(J_+) = J_+ \otimes q^H + q^{-H} \otimes J_+ \tag{48}$$

and

$$\Delta(\mathcal{L}_a) \in \mathrm{End}(V \otimes W), \qquad \Delta_\epsilon(\mathcal{L}_a) \in \mathrm{End}(V \otimes W) \tag{49}$$

with

$$\begin{aligned} \Delta(\mathcal{L}_a) &= \Delta_0(\mathcal{L}_a) + \epsilon \Delta_\epsilon(\mathcal{L}_a) \\ \Delta_0(\mathcal{L}_a) &= \mathcal{L}_a \otimes 1 + 1 \otimes \mathcal{L}_a \end{aligned} \tag{50}$$

such that

$$\begin{aligned} [\mathcal{L}_a, \mathcal{L}_b] &= f_{ab}^c \mathcal{L}_c \\ [\Delta(\mathcal{L}_a), \Delta(\mathcal{L}_b)] &= f_{ab}^c \Delta(\mathcal{L}_c) \end{aligned} \tag{51}$$

At order ϵ we have worked out the deformation needed to explain the complete spectrum of one loop anomalous dimensions. The co-products for the complete set of generators are

$$\begin{aligned} \Delta(D) &= D \otimes 1 + 1 \otimes D + \frac{\epsilon}{3} P_0 \\ \Delta(P_\mu) &= P_\mu \otimes 1 + 1 \otimes P_\mu \\ \Delta(K_\mu) &= K_\mu \otimes 1 + 1 \otimes K_\mu - \frac{\epsilon}{3} P_0 \left(\frac{\partial}{\partial P_\mu} \otimes 1 + 1 \otimes \frac{\partial}{\partial P_\mu} \right) P_0 \\ \Delta(M_{\mu\nu}) &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \end{aligned} \tag{52}$$

It is a straightforward exercise to verify that the above co-products are consistent with the commutation relations of $so(4, 2)$. For example, we have checked that

$$[\Delta(K_\mu), \Delta(P_\nu)] = 2\Delta(M_{\mu\nu}) - 2\delta_{\mu\nu}\Delta(D) \tag{53}$$

In performing this check and others like it, it is useful to note $\Delta_0(\mathcal{L}_a) P_0 = P_0 \Delta_0(\mathcal{L}_a)$ and $P_0^2 = P_0$.

The planar $SU(2)$ sector in $\mathcal{N} = 4$ SYM is another interacting CFT that has an instructive TFT2/CFT4 construction. For this example there is no need to continue to $d = 4 - \epsilon$ dimensions. In this example too, deformed co-products are needed to reproduce the one loop spectrum of anomalous dimensions. Consider the three operators

$$\mathcal{O}_z = \frac{1}{\sqrt{2}N}\mathrm{Tr}(Z^2), \qquad \mathcal{O}_y = \frac{1}{\sqrt{2}N}\mathrm{Tr}(Y^2)$$
$$\mathcal{O}_{zy} = \frac{1}{\sqrt{3}N^2}\left(\mathrm{Tr}(YZYZ) - \mathrm{Tr}(Y^2Z^2)\right) \tag{54}$$

These operators are all eigenstates of the one loop dilatation operator. $\mathcal{O}_{zy}$ has a non-zero anomalous dimension $\delta = \frac{3\lambda}{4\pi^2}$ in the planar limit [26]. The anomalous dimensions of both $\mathcal{O}_z$ and $\mathcal{O}_y$ vanish. In the free theory, the dimensions add

$$\mathrm{Dim}(\mathcal{O}_z) + \mathrm{Dim}(\mathcal{O}_y) = \mathrm{Dim}(\mathcal{O}_{zy}) \tag{55}$$

At first order in the interaction this relationship is corrected as follows

$$\mathrm{Dim}(\mathcal{O}_z) + \mathrm{Dim}(\mathcal{O}_y) = \mathrm{Dim}(\mathcal{O}_{zy}) - \delta \tag{56}$$

As the first step, consider so(4, 2) irrep generated by $\mathcal{O}_z$. In the operator-state correspondence, the operator $\mathcal{O}_z$ corresponds to a tower of operators

$$\begin{aligned} \mathcal{O}_z(0) &\to v_z \\ \partial_{\mu_1}\mathcal{O}_z(0) &\to P_{\mu_1}v_z \\ \partial_{\mu_1}\partial_{\mu_2}\mathcal{O}_z(0) &\to P_{\mu_1}P_{\mu_2}v_z \\ &\vdots \end{aligned} \tag{57}$$

The states live in a representation V_z of $so(4, 2)$. The lowest weight state v_z has the properties

$$Dv_z = 2v_z, \qquad M_{\mu\nu}v_z = 0, \qquad K_\mu v_z = 0 \tag{58}$$

At dimension $(2 + k)$ we have states

$$V_k = \mathrm{Span}\left\{P_{\mu_1}\cdots P_{\mu_k}v_z\right\} \tag{59}$$

The direct sum forms the $so(4; 2)$ irrep V_Z

$$V_Z = \bigoplus_{k=0}^{\infty V_k} \tag{60}$$

There is a similar representation V_y built on the primary $\mathcal{O}_y$. V_z and V_y are isomorphic representations of so(4, 2). We also need the representation V_{zy}, built on $\mathcal{O}_{zy}$. This representation has lowest weight state v_{zy} with properties

$$Dv_{zy} = (4 + \delta)v_{zy}$$

$$
\begin{aligned}
K_\mu v_{zy} &= 0 \\
M_{\mu\nu} v_{zy} &= 0
\end{aligned} \tag{61}
$$

States at $D = 4 + \delta + k$ are obtained by acting with k P's.

Given the non-additivity of the anomalous dimensions, we cannot model the 3-point correlator with the standard action of the Lie algebra on $V_2 \otimes V_2$. If we use the standard action, we would have

$$
\Delta_0(D)(v_z \otimes v_y) = (D \otimes 1 + 1 \otimes D)(v_z \otimes v_y) = 4 v_z \otimes v_y \tag{62}
$$

whereas the dimension of v_{zy} is $4 + \delta$. The map $f : v_{zy} \to v_z \otimes v_y$

$$
\Delta_0(D) f(v_{zy}) = f \Delta_0(D)(v_{zy}) \tag{63}
$$

can be extended to an equivariant map $V_{zy} \to V_z \otimes V_y$ at zero coupling, but cannot be so extended when we turn on δ at non-zero coupling.

Let P_4 be the projector to V_4—the so(4, 2) representation with scalar lowest weight of dimension 4 - in the standard tensor product decomposition of $V_2 \otimes V_2$. We can define a deformed co-product

$$
\begin{aligned}
\Delta(D) &= \Delta_0(D) + \delta P_4 \\
\Delta(P_\mu) &= \Delta_0(P_\mu) \\
\Delta(M_{\mu\nu}) &= \Delta_0(M_{\mu\nu}) \\
\Delta(K_\mu) &= \Delta_0(K_\mu) - \frac{\delta}{2} P_4 \Delta_0 \left(\frac{\partial}{\partial P_\mu} \right) P_4
\end{aligned} \tag{64}
$$

With the so(4, 2) action on $V_z \otimes V_y$ given by

$$
\mathcal{L}_a : v_1 \otimes v_2 \to \Delta(\mathcal{L}_a)(v_1 \otimes v_2) \tag{65}
$$

and the so(4, 2) action on V_{zy} which we will refer to as ρ_{zy}, we can extend f

$$
f : V_{zy} \to V_z \otimes V_y \tag{66}
$$

such that

$$
f \rho_{zy}(\mathcal{L}_a) = \Delta(\mathcal{L}_a) f \tag{67}
$$

Using the map f, we can construct the correlator as follows

$$
\eta((e^{-iP\cdot x_1} v^+ \otimes e^{-iP\cdot x_2} v^+), (x_3')^2 f(e^{iK\cdot x_3'} v_{zy}^+)) \tag{68}
$$

The inner product g on $V_z \otimes V_y$ is related by using the anti-automorphism on $so(4, 2)$ to the invariant pairing on

$$\eta : (V_+ \otimes V_+) \otimes (V_- \otimes V_-) \to \mathbb{C} \tag{69}$$

4 $d = 4 - \epsilon$ and Diagram Algebras

In our example of the Wilson-Fischer, we need to continue from $d = 4$ to $d = 4 - \epsilon$ dimensions in order to obtain a non-trivial CFT. Analytically continued tensor rules, and in particular the rule $\delta^\mu_\mu = 4 - \epsilon$, are needed to construct the stress tensor with the right properties. The state space V_+ used in the free scalar field theory is a quotient of a space $\tilde{V}_+$ spanned by states of the form

$$\left\{ P_{\mu_1} \cdots P_{\mu_k} v \right\} \tag{70}$$

The quotient amounts to setting to zero $P_\mu P_\mu v$. The stress tensor

$$\begin{aligned} T_{\mu\nu} = &\frac{1}{2}(P_\mu v \otimes P_\nu v + P_\nu v \otimes P_\mu v - \delta_{\mu\nu} P_\tau v \otimes P_\tau v) \\ &- \frac{\alpha}{6} \Delta (P_\mu P_\nu - P^2 \delta_{\mu\nu}) v \otimes v \end{aligned} \tag{71}$$

is a state in $\tilde{V}_+ \otimes \tilde{V}_+$. The above state is conserved and traceless upon using the interacting equation of motion, along with

$$Dv = \left(1 - \frac{\epsilon}{2}\right) v, \qquad M_{\alpha\beta} v = 0, \qquad \delta_{\mu\mu} = 4 - \epsilon \tag{72}$$

The positive part of the state space

$$\bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(V_+) \tag{73}$$

where $V_+ = \tilde{V}_+ / \{P^2 v\}$ is replaced by

$$\bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(\tilde{V}_+) \tag{74}$$

and we need to quotient by

$$P^2 v - 4 g^* v \otimes v \otimes v \tag{75}$$

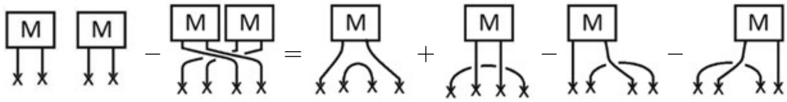
Thus, understanding the interacting equations of motion in terms a quotient space, requires working with $\tilde{V}_+$ and its tensor powers. The quotient condition relates states in $\tilde{V}_+$ to $\tilde{V}_+^{\otimes 3}$. Notice that in both the free and interacting theories, we move from V_+ to $\tilde{V}_+$ by quotienting with the equation of motion.

To make sense of the rule $\delta_{\mu\mu} = 4 - \epsilon$, we need to construct a diagram algebra, much like the Brauer algebras. In our TFT2 setting, $Uso(d)$ (and $Uso(d, 2)$) itself has to be made diagrammatic in order to give a TFT2 with conformal equivariance formulation of the perturbative correlators.

If we depict the product $M_{ij} M_{kl}$ in the universal enveloping algebra Uso(d) by juxtaposing two boxes side to side, we can express

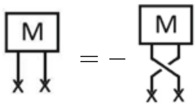
$$M_{ij} M_{kl} - M_{kl} M_{ij} = \delta_{jk} M_{il} + \delta_{il} M_{jk} - \delta_{jl} M_{ik} - \delta_{ik} M_{jl} \tag{76}$$

as a relation between diagrams as follows



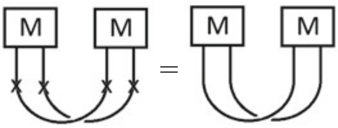
(77)

To go from the diagrammatic relation to the equation in Uso(d), we attach the labels i, j, k, l to the crosses starting with i for the left-most cross and proceeding with j, k, l as we go to the crosses towards the right. The antisymmetry can be expressed diagrammatically as follows



(78)

The quadratic Casimir $M_{ij} M_{ij}$ is associated to the diagram shown below



(79)

We will define an infinite dimensional associative algebra over $\mathbb{C}$, denoted $\mathcal{F}$, abstracted from the generators M_{ij} of Uso(d). An associative algebra is a vector space equipped with a product m

$$m : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F} \tag{80}$$

The vector space $\mathcal{F}$ is

$$\mathcal{F} = \mathbb{C} \oplus \mathrm{Span}_{\mathbb{C}}(M) \oplus \cdots \tag{81}$$

The $\cdots$ refers to subspaces which can be specified efficiently, using the oscillator construction of $Uso(d)$ and its interpretation in terms of equivariant maps and diagrams. The d-dimensional oscillator relations are

$$[a_i^\dagger, a_j] = -\delta_{ij} \tag{82}$$

and the Lie algebra generators of so(d) can be written as

$$M_{ij} = a_i^\dagger a_j - a_j^\dagger a_i \tag{83}$$

Think of this as specifying a state (which we can also call M_{ij}) using $V = \mathrm{Span}(a, a^\dagger)$ and $W = \mathrm{Span}(e_i : i \in 1, \cdots, d)$

$$M_{ij} = a^\dagger \otimes e_i \otimes a \otimes e_j - a^\dagger \otimes e_j \otimes a \otimes e_i \in V \otimes W \otimes V \otimes W \tag{84}$$

It is useful to write this as

$$M_{ij} = P_A^{W\otimes W}(a^\dagger \otimes e_i \otimes a \otimes e_j) \tag{85}$$

The number of these M_{ij} is $d(d-1)/2$. This obstructs continuing d to non-integer dimensions. In contrast to this, the space of equivariant maps

$$P_A(W \otimes W) \to P_A^{W\otimes W}(V_+ \otimes W \otimes V_- \otimes W) \tag{86}$$

is a one-dimensional vector space (for $d > 4$[1]) spanned by the map M acting as

$$M : (e_{i_1} \otimes e_{i_2} - e_{i_2} \otimes e_{i_1}) \to (a^\dagger \otimes e_{i_1} \otimes a \otimes e_{i_2} - a^\dagger \otimes e_{i_2} \otimes a \otimes e_{i_1}) \tag{87}$$

More compactly, we can write

$$M : W_2 \to (VW)_2 \tag{88}$$

where $W_2 = P_A(W \otimes W)$ and

$$(VW)_2 = P_A^{W\otimes W}(V_+ \otimes W \otimes V_- \otimes W) \tag{89}$$

Now, introduce the infinite dimensional associative algebra

$$\mathcal{F} = \oplus_{m,n=0}\mathcal{F}_{m,n} \tag{90}$$

[1] For $d = 4$ we can also use $\epsilon_{i_1 i_2 i_3 i_4}$ which gives another map, so we will use large d in the appropriate places in our definitions to keep things as simple as possible.

with

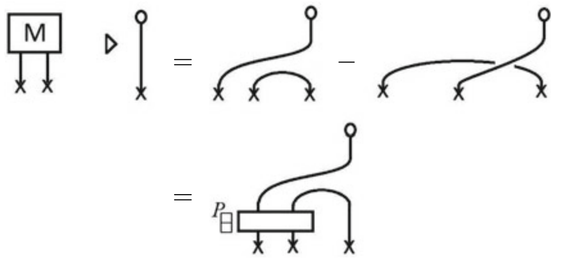
$$\mathcal{F}_{m,n} = \mathrm{Hom}_{so(d):d>2m+2n}(W_2^{\otimes m}, (VW)_2^{\otimes n}) \tag{91}$$

This algebra contains all the M-diagrams we drew earlier, and it includes diagram with arbitrarily large numbers of M-boxes. We define U_* as a quotient of this space of diagrammatic maps by the commutation relation (77).

In order to understand the representation theory of U_*, which will be a diagrammatic analog of the representation theory of Uso(d) at large d, we will start by interpreting the basic equation

$$[M_{ij}, a_k^\dagger] = \delta_{jk} a_k^\dagger - \delta_{ik} a_j^\dagger \tag{92}$$

which gives the action of $Uso(d)$ on the d-dimensional vector representation. By using labelled M-box diagrams, and associating to $a_k^\dagger$ a line joining a cross to a circle, the above equation becomes



(93)

Using the definitions from above,

$$a_i^\dagger = a^\dagger \otimes e_i \in V_+ \otimes W \qquad e_i \in W \tag{94}$$

There is an so(d) equivariant map ρ

$$\rho : W \to (V_+ \otimes W) \tag{95}$$

We can think of this as the map which attaches $e_i \in W$ to $a^\dagger$ to produce $a_i^\dagger = a^\dagger \otimes e_i$. The map commutes with so(d). This leads to the definition of a vector space of diagrams

$$V^* = \oplus_{n=0}^{\infty} \mathrm{Hom}_{so(d):d \gg n}(W_2^{\otimes n} \otimes W, V_+ \otimes W) \tag{96}$$

We can define a diagrammatic inner product for V^* (which involves loops evaluating to d) and show that $V^* \otimes V^*$ contains orthogonal subspaces corresponding to the symmetric-traceless, the trace, and the anti-symmetric. The proof proceeds by proving $B_{n=2}(d)$ commutes with U^* action on $V^* \otimes V^*$. Much as $B_2(d)$ commutes with Uso(d) on $V_d \otimes V_d$, but in the above both the algebra and the representation space are spanned by diagrams. d appears upon evaluation of Casimirs, which involve loops evaluated as d - which can then be set to $4 - \epsilon$.

There are some rather natural conjectures we can formulate about $(V^*)^{\otimes n}$. First, the action of U^* should commute with a known diagrammatic algebra, the Brauer algebra $B_d(n)$, much as $B_d(n)$ commutes with $Uso(d)$ in $V_d^{\otimes n}$. Proving this conjecture would involve generalising arguments given in [20]. These are the first steps towards a fully diagrammatic Schur-Weyl duality where U^*, with loop parameter d, acts on $V^{*\otimes n}$ and is Schur-Weyl dual to $B_d(n)$.

5 Summary and Outlook

Our key result [20] has been to define $U_{*,2}$ acting on $V^{*,2}$ as a generic d version of Uso(d,2) acting on $\tilde{V}$. To summarise, we present evidence that perturbative CFT can be formulated in terms of Uso(d,2) (for theories in integer dimensions) or $U_{*,2}$ (for theories like Wilson Fischer), using familiar constructions in algebra/representation theory, namely

- indecomposable representations,
- deformed co-products and
- diagram algebras.

Acknowledgements SR is supported by the STFC consolidated grant ST/P000754/1 "String Theory, Gauge Theory & Duality" and a Visiting Professorship at the University of the Witwatersrand. RdMK is supported by a Simons Foundation Grant Award ID 509116 and by the South African Research Chairs initiative of the Department of Science and Technology and the National Research Foundation. We thank Matt Buican and Adrian Padellaro for useful discussions on the subject of this paper.

References

1. J.M. Maldacena, The Large N limit of superconformal field theories and supergravity. Adv. Theor. Math. Phys. **2**, 231–252 (1998). https://doi.org/10.1023/A:1026654312961, arXiv:hep-th/9711200 [hep-th]
2. P.C. Argyres, M.R. Douglas, New phenomena in SU(3) supersymmetric gauge theory. Nucl. Phys. B **448**, 93–126 (1995). https://doi.org/10.1016/0550-3213(95)00281-V, arXiv:hep-th/9505062 [hep-th]
3. N. Seiberg, New theories in six-dimensions and matrix description of M theory on T**5 and T**5 / Z(2). Phys. Lett. B **408**, 98–104 (1997). https://doi.org/10.1016/S0370-2693(97)00805-8, arXiv:hep-th/9705221 [hep-th]
4. R. Rattazzi, V.S. Rychkov, E. Tonni, A. Vichi, Bounding scalar operator dimensions in 4D CFT. JHEP **12**, 031 (2008). https://doi.org/10.1088/1126-6708/2008/12/031, arXiv:0807.0004 [hep-th]
5. P. Di Francesco, P. Mathieu, D. Sénéchal, Conformal field theory, in *Graduate Texts in Contemporary Physics* (Springer, 1997)

6. I. Frenkel, J. Lepowsky, A. Meurman, *Vertex Operator Algebras and the Monster*, vol. 134, Pure and Applied Mathematics (1989)
7. J. Madore, The fuzzy sphere. Class. Quant. Grav. **9**, 69–88 (1992). https://doi.org/10.1088/0264-9381/9/1/008
8. P. Podles, Quantum spheres. Lett. Math. Phys. **14**, 193–202 (1987)
9. M. Noumi, K. Mimachi, Quantum 2-Spheres and Big q-Jacobi polynomials. Commun. Math. Phys. **128**, 521–531 (1990)
10. S. Corley, A. Jevicki, S. Ramgoolam, Exact correlators of giant gravitons from dual N = 4 SYM theory. Adv. Theor. Math. Phys. **5**, 809–839 (2002). https://doi.org/10.4310/ATMP.2001.v5.n4.a6, arXiv:hep-th/0111222 [hep-th]
11. V. Balasubramanian, D. Berenstein, B. Feng, M.X. Huang, D-branes in Yang-Mills theory and emergent gauge symmetry. JHEP **03**, 006 (2005). https://doi.org/10.1088/1126-6708/2005/03/006, arXiv:hep-th/0411205 [hep-th]
12. R. de Mello Koch, J. Smolic, M. Smolic, Giant gravitons—with strings attached (I). JHEP **06**, 074 (2007). https://doi.org/10.1088/1126-6708/2007/06/074, arXiv:hep-th/0701066 [hep-th]
13. T.W. Brown, P.J. Heslop, S. Ramgoolam, Diagonal multi-matrix correlators and BPS operators in N = 4 SYM. JHEP **02**, 030 (2008). https://doi.org/10.1088/1126-6708/2008/02/030, arXiv:0711.0176 [hep-t]
14. R. Bhattacharyya, S. Collins, R. de Mello Koch, Exact multi-matrix correlators. JHEP **03**, 044 (2008). https://doi.org/10.1088/1126-6708/2008/03/044
15. P. Mattioli, S. Ramgoolam, Quivers, words and fundamentals. JHEP **03**, 105 (2015). https://doi.org/10.1007/JHEP03(2015)105, arXiv:1412.5991 [hep-th]
16. Y. Kimura, Multi-matrix models and Noncommutative Frobenius algebras obtained from symmetric groups and Brauer algebras. Commun. Math. Phys. **337**(1), 1–40 (2015). https://doi.org/10.1007/s00220-014-2231-6, arXiv:1403.6572 [hep-th]
17. S. Ramgoolam, Permutations and the combinatorics of gauge invariants for general N. PoS **CORFU2015**, 107 (2016). https://doi.org/10.22323/1.263.0107, arXiv:1605.00843 [hep-th]
18. R. de Mello Koch, S. Ramgoolam, CFT_4 as $SO(4,2)$-invariant TFT_2. Nucl. Phys. B **890**, 302–349 (2014). https://doi.org/10.1016/j.nuclphysb.2014.11.013
19. R. de Mello Koch, S. Ramgoolam, Free field primaries in general dimensions: counting and construction with rings and modules. JHEP **08**, 088 (2018). https://doi.org/10.1007/JHEP08(2018)088, arXiv:1806.01085 [hep-th]
20. R. de Mello Koch, S. Ramgoolam, Perturbative 4D conformal field theories and representation theory of diagram algebras. JHEP **05**, 020 (2020). https://doi.org/10.1007/JHEP05(2020)020, arXiv:2003.08173 [hep-th]
21. M.F. Atiyah, Topological quantum field theory (Publications Mathématiques de l'IHÉS, 1988)
22. J. Kock, *Frobenius Algebras and 2-D Topological Quantum Field Theories* (CUP, 2010)
23. D.J. Binder, S. Rychkov, Deligne categories in lattice models and quantum field theory, or making sense of $O(N)$ symmetry with non-integer N. JHEP **04**, 117 (2020). https://doi.org/10.1007/JHEP04(2020)117, arXiv:1911.07895 [hep-th]
24. G.W. Moore, G. Segal, D-branes and K-theory in 2D topological field theory. arXiv:hep-th/0609042 [hep-th]
25. P. Liendo, Revisiting the dilatation operator of the Wilson–Fisher fixed point. Nucl. Phys. B **920**, 368–384 (2017). https://doi.org/10.1016/j.nuclphysb.2017.04.020, arXiv:1701.04830 [hep-th]
26. N. Beisert, C. Kristjansen, M. Staudacher, The Dilatation operator of conformal N=4 superYang-Mills theory. Nucl. Phys. B **664**, 131–184 (2003). https://doi.org/10.1016/S0550-3213(03)00406-1, arXiv:hep-th/0303060 [hep-th]

Reducing the $N = 1$, 10-d, E_8 Gauge Theory over a Modified Flag Manifold

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Abstract We present a split-like supersymmetric extension of the Standard Model which originates from a 10-d, $\mathcal{N} = 1$, E_8 gauge theory. The transition to four dimensions occurs after the dimensional reduction of the initial, higher-dimensional theory over the $SU(3)/U(1)^2 \times \mathbb{Z}_3$ space. Making use of the Wilson flux breaking mechanism, the resulting 4-d theory is an $\mathcal{N} = 1$ $SU(3)^3$ Grand Unified Theory. After the symmetry breaking of the $SU(3)^3$ gauge group, the model is viewed as split-like Supersymmetric Standard Model and, at the TeV scale and below, the model is treated as a two Higgs doublet model, producing the following results: top, bottom and light Higgs masses within the range given by the experiment and prediction of the unification (and first supersymmetry breaking) and (second) supersymmetry breaking scales at $\sim 10^{15}$ GeV and ~ 1.5 TeV, respectively.

Keywords Dimensional reduction · Supersymmetry · Grand unification · Small radii · Higgs

1 Introduction

Our study has come to realization due to the very fundamental and insightful works of Forgacs-Manton (F-M), the Coset Space Dimensional Reduction (CSDR) [1–3] and Scherk-Schwartz (S-S) [4], the group manifold reduction. The CSDR

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_11

mechanism took into consideration the number of dimensions and the starting gauge group, as predicted by the heterotic string [6], with those two sharing the common ground that they lead to promising Grand Unified Theories (GUTs). Moreover, it is worth noting that the higher-dimensional theory is accompanied by the unification of the gauge and scalar sectors, with the scalars being identified as the extra-dimensional components of the vector field and, if supposed supersymmetric, fermions participate in the aforementioned unification, in the sense that they consist the fermionic counterpart of the gauge fields in a vector supermultiplet. Two remarkable features of the CSDR is that the fermionic terms of the higher-dimensional action lead to 4-d Yukawa interactions and that the reduced 4-d theories can be chiral, if necessary conditions are applied on the fermionic spectrum of the higher-dimensional theory [7]. However, the most powerful property of the CSDR mechanism is that it fails to inherit the amount of supersymmetry to the 4-d theory, which means that implicitly functions as a (soft) supersymmetry breaking mechanism, which is vital for the construction of viable 4-d models [8–11] (see also [12]).

In our specific model, the initial, higher-dimensional theory is a 10-d, $\mathcal{N}=1$, E_8 gauge theory whose spectrum is minimal, consisting solely of a vector supermultiplet. The CSDR mechanism is performed over the $SU(3)/U(1)^2 \times \mathbb{Z}_3$, which is a modification of the 6-d flag manifold $SU(3)/U(1)^2$ (non-symmetric coset space), where the freely-acting $\mathbb{Z}_3$ component has been introduced to enable the triggering of the Wilson flux mechanism, which causes a diminution of the produced gauge symmetry of the reduced (grand unified) theory to the $SU(3)^3 \times U(1)^2$ [2, 8, 9, 13] (see also [14]). The produced GUT is also (softly broken) $\mathcal{N}=1$ supersymmetric.

The phenomenological part of the model is developed after the consideration of the symmetry breaking of the $SU(3)^3 \times U(1)^2$ gauge group. Assuming the coincidence of the unification and compactification scales, GUT breaking is accompanied by the supersymmetry breaking of the initial $\mathcal{N}=1$ and the supersymmetric spectrum becomes partially supermassive, which means that our model evolves into split-like supersymmetric with two Higgs doublets until the TeV scale and from the latter to the electroweak one it is treated as a two Higgs doublet model (2HDM) producing masses of the light Higgs boson and the third generation quarks within the experimental range (for the original work see [15]), passing the first tests regarding its viability.

2 Dimensional Reduction of E_8 over $SU(3)/U(1)^2$

In this section we apply directly the CSDR in our specific case, that is the 10-d, $\mathcal{N}=1$, E_8 Yang-Mills-Dirac theory with Weyl-Majorana fermions over the non-symmetric coset space $SU(3)/U(1)^2$ [2, 8, 14, 16]. The produced 4-d action is:

$$S = C \int d^4x \, \mathrm{tr}\left[-\frac{1}{8}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}(D\phi_a)^2\right] + V(\phi) + \frac{i}{2}\bar{\psi}\Gamma^{\mu}D_{\mu}\psi - \frac{i}{2}\bar{\psi}\Gamma^{a}D_{a}\psi \,, \tag{1}$$

where $V(\phi)$ is defined as:

$$V(\phi) = -\frac{1}{8} g^{ac} g^{bd} \mathrm{tr}\left(f_{ab}{}^{C} \phi_C - ig[\phi_a, \phi_b])(f_{cd}{}^{D} \phi_D - ig[\phi_c, \phi_d]\right) \tag{2}$$

and $\mathrm{tr}(T^i T^j) = 2\delta^{ij}$, where T^i are the E_8 generators. Also, g is the coupling constant, C is the coset volume, $D_\mu = \partial_\mu - igA_\mu$, D_a are the 4-d covariant derivative and the coset space covariant derivative, respectively and, last, g_{ab} is the metric of the coset space, given by $g_{\alpha\beta} = \mathrm{diag}(R_1^2, R_1^2, R_2^2, R_2^2, R_3^2, R_3^2)$. The 4-d gauge group is determined by the centralizer of $U(1) \times U(1)$ in E_8:

$$H = C_{E_8}(U(1)_A \times U(1)_B) = E_6 \times U(1)_A \times U(1)_B \,.$$

Moreover, the CSDR rules determine the representations of the particles that consist the particle spectrum of the 4-d theory (details in [2, 8, 13]). Specifically the surviving gauge fields (of $E_6 \times U(1)_A \times U(1)_B$) fall into three $\mathcal{N} = 1$ vector supermultiplets whereas the matter fields fall into six $\mathcal{N} = 1$ chiral ones, where three of the latter are E_6 singlets carrying $U(1)_A \times U(1)_B$ charges, while the rest are chiral. In particular, the matter fields transform under $E_6 \times U(1)_A \times U(1)_B$ as:

$$\alpha_i \sim 27_{(3,\frac{1}{2})},\ \beta_i \sim 27_{(-3,\frac{1}{2})},\ \gamma_i \sim 27_{(0,-1)}, \tag{3}$$

$$\alpha \sim 1_{(3,\frac{1}{2})},\ \beta \sim 1_{(-3,\frac{1}{2})},\ \gamma \sim 1_{(0,-1)} \,. \tag{4}$$

Regarding the potential of the theory, besides the terms identified as F and D-terms, the rest are interpreted as soft scalar masses and trilinear soft terms. As far as the gaugino mass is concerned, since it has geometric origin, as understood by the following relation:

$$M = (1 + 3\tau)\frac{R_1^2 + R_2^2 + R_3^2}{8\sqrt{R_1^2 R_2^2 R_3^2}} \,. \tag{5}$$

This expression implies that, in absence of torsion, the gauginos gain mass at the compactification scale [2]. This result can change in presence of torsion [8] as required for the realization of the split supersymmetry scenario, which requires a gaugino mass in the TeV scale.

3 Breaking by Wilson Flux Mechanism

In the previous section we demonstrated the case of applying the CSDR on an E_8 gauge theory over the coset space $SU(3)/U(1)^2$. Nevertheless, the resulting 4-d gauge group, $E_6 \times U(1)^2$ cannot be broken down to the gauge group of the Standard Model (SM) by the scalar Higgs accommodated in the 27 representation. Therefore, in order to end up with a different 4-d gauge group (with less symmetry), the Wilson flux breaking mechanism is introduced [17–19]. In order that the above mechanism to get induced, the coset space must be modified from simply connected, that is the default case for $SU(3)/U(1)^2$, to multiply connected. To achieve this modification, the freely-acting discrete symmetry $\mathbb{Z}_3$ is employed, therefore the space on which the reduction is performed is now the $SU(3)/U(1)^2 \times \mathbb{Z}_3$. Each $g \in \mathbb{Z}_3$ is mapped to an element $U_g \in E_6$ (a Wilson line) and the set of these elements consists the image of $\mathbb{Z}_3$, T^{E_6} in E_6. The above map turns out to be a homomorphism, and once it is determined, E_6 breaks to the centralizer $C_{E_6}(T^{E_6}) = SU(3)^3$ [27]. Also, the presence of the discrete symmetry functions as a filtering mechanism for the spectrum, i.e. only fields that are invariant under the action of $\mathbb{Z}_3$ on both their gauge and geometric indices make it through to the resulting $SU(3)^3$ gauge theory.[1] In the E_6 phase, the matter fields were belonging to the trivial or 27 representations. For the trivial case, out of the three E_6 singlets α, β, γ of Eq. (4) only one survives, specifically the $\alpha \equiv \theta_{(3,\frac{1}{2})}$. In turn, the $SU(3)^3$ representations of the non-trivial surviving matter fields are obtained by the decomposition $E_6 \supset SU(3)^3$, that is $27 = (1, 3, \bar{3}) \oplus (\bar{3}, 1, 3) \oplus (3, \bar{3}, 1)$ and are obtained to be the following:

$$\alpha_1 \equiv \Psi_1 \sim (1, 3, \bar{3})_{(3,\frac{1}{2})}, \beta_3 \equiv \Psi_2 \sim (\bar{3}, 1, 3)_{(-3,\frac{1}{2})}, \gamma_2 \equiv \Psi_3 \sim (3, \bar{3}, 1)_{(0,-1)}, \tag{6}$$

where the above are the parts of the three 27 chiral multiplets of $\alpha_i, \beta_i, \gamma_i$ of Eq. (3) and combined they form one complete generation. The reduction of the number of the generations is an unwelcome feature and in order to return to a spectrum of three ones, non-trivial monopole charges in the $U(1) \times U(1)$ part of the coset needs to be introduced, leading to three identical instances of the above fields, recovering the desired number of generations [28].

The employment of the Wilson flux breaking mechanism affects the scalar potential as well, in the sense that it can be rewritten from the E_6 language to the $SU(3)_c \times SU(3)_L \times SU(3)_R$ one as [13]:

$$V_{sc} = 3 \cdot \frac{2}{5}\left(\frac{1}{R_1^4} + \frac{1}{R_2^4} + \frac{1}{R_3^4}\right) + \sum_{l=1,2,3} V^{(l)}, \tag{7}$$

in which $V^{(l)} = V_{susy} + V_{soft} = V_D + V_F + V_{soft}$, with l being a generation index which we drop in the ensuing (unless its presence is necessary), since we focus on the third generation for our analysis and calculations. The F-terms derive from the superpotential which is given by the following expression:

$$\mathcal{W} = \sqrt{40} d_{abc} \Psi_1^a \Psi_2^b \Psi_3^c, \tag{8}$$

the various D-terms are written as:

[1] For more details on the parametrization of the filtering procedure see the original work [15] but also [20].

$$D^A = \frac{1}{\sqrt{3}}\langle\Psi_i|G^A|\Psi_i\rangle, \quad D_1 = 3\sqrt{\frac{10}{3}}(\langle\Psi_1|\Psi_1\rangle - \langle\Psi_2|\Psi_2\rangle), \tag{9}$$

$$D_2 = \sqrt{\frac{10}{3}}(\langle\Psi_1|\Psi_1\rangle + \langle\Psi_2|\Psi_2\rangle - 2\langle\Psi_3|\Psi_3\rangle - 2|\theta|^2) \tag{10}$$

and, last, the soft supersymmetry breaking terms are:

$$\begin{aligned} V_{soft} =& \left(\frac{4R_1^2}{R_2^2R_3^2} - \frac{8}{R_1^2}\right)\langle\Psi_1|\Psi_1\rangle + \left(\frac{4R_2^2}{R_1^2R_3^2} - \frac{8}{R_2^2}\right)\langle\Psi_2|\Psi_2\rangle \\ &+ \left(\frac{4R_3^2}{R_1^2R_2^2} - \frac{8}{R_3^2}\right)(\langle\Psi_3|\Psi_3\rangle + |\theta|^2) \\ &+ 80\sqrt{2}\left(\frac{R_1}{R_2R_3} + \frac{R_2}{R_1R_3} + \frac{R_3}{R_1R_2}\right)(d_{abc}\Psi_1^a\Psi_2^b\Psi_3^c + h.c) \\ =& m_1^2\langle\Psi_1|\Psi_1\rangle + m_2^2\langle\Psi_2|\Psi_2\rangle + m_3^2\Big(\langle\Psi_3|\Psi_3\rangle + |\theta|^2\Big) + (\alpha_{abc}\Psi_1^a\Psi_2^b\Psi_3^c + h.c)\,. \end{aligned} \tag{11}$$

Following [21], the multiplets of the fields found in (6) can be nicely expressed in the $SU(3)_c \times SU(3)_L \times SU(3)_R$ language as complex 3×3 matrices according to the following assignment:

$$\Psi_2 \sim (\bar{3}, 1, 3) \to (q^c)_p^{\ \alpha}, \quad \Psi_3 \sim (3, \bar{3}, 1) \to (Q_\alpha^{\ a}), \quad \Psi_1 \sim (1, 3, \bar{3}) \to L_a^{\ p}\,, \tag{12}$$

which leads to the more legible and comprehensive form of the particle content of an enhanced version of the Minimal Supersymmetric Standard Model (MSSM):

$$q^c = \begin{pmatrix} d_R^{c1} & u_R^{c1} & D_R^{c1} \\ d_R^{c2} & u_R^{c2} & D_R^{c2} \\ d_R^{c3} & u_R^{c3} & D_R^{c3} \end{pmatrix}, \quad Q = \begin{pmatrix} -d_L^1 & -d_L^2 & -d_L^3 \\ u_L^1 & u_L^2 & u_L^3 \\ D_L^1 & D_L^2 & D_L^3 \end{pmatrix}, \quad L = \begin{pmatrix} H_d^0 & H_u^+ & \nu_L \\ H_d^- & H_u^0 & e_L \\ \nu_R^c & e_R^c & S \end{pmatrix}.$$

4 Specification of Parameters and GUT Breaking

4.1 Choice of Radii

Having established the theoretical frame, in order to advance to the phenomenological part, we proceed by making two important assumptions. First, the compactification scale is considered to be high[2] and second, the compactification and unification scales

[2] Working with high compactification scale, the Kaluza-Klein excitations can be ignored. In case the compactification scale was considered at the TeV scale, then the eigenvalues of the Dirac and Laplace operators of the $SU(3)/U(1)^2$, which are not known yet, would be necessary to be included in the calculation.

coincide, $M_C = M_{GUT}$, which means that the scale of the three radii of the compactification scale is $R_l \sim \frac{1}{M_{GUT}}$, $l = 1, 2, 3$. Without any further assumption this would lead to a superheavy supersymmetric spectrum[3] (of M_{GUT} order) and soft trilinear couplings. However, we can treat one of the radii, let us call the third, to be slightly different than the others. Under this assumption, inspection of the expression of the scalar potential, (11), leads to the understanding that the supersymmetric spectrum undergoes a separation (split-like scenario), with the squarks being superheavy but the sleptons gaining mass in the TeV energy regime.

4.2 The Breaking of $SU(3)^3$

The breaking of the $SU(3)_L$ and $SU(3)_R$ parts of the gauge group can be triggered by the following vevs of the two families of L's:

$$\langle L_s^{(3)} \rangle = \text{diag}(0, 0, V)\,, \quad \langle L_s^{(2)} \rangle = \text{anti} - \text{diag}(0, 0, V)\,,$$

where the s index designates the scalar component of the supermultiplet.[4] These vevs are singlets under $SU(3)_c$, therefore they do not break the colour part of the total gauge group. Appropriate combination of the two vevs leads to the desired breaking, that is to the SM gauge group [22]:

$$SU(3)_c \times SU(3)_L \times SU(3)_R \to SU(3)_c \times SU(2)_L \times U(1)_Y\,. \tag{13}$$

According to the configuration of the scalar potential, the above breaking gives vevs to the singlet of each family (not necessarily to all three), specifically in our case, $\langle \theta^{(3)} \rangle \sim \mathcal{O}(TeV)$, $\langle \theta^{(1,2)} \rangle \sim \mathcal{O}(M_{GUT})$. As far as the two Abelian symmetries is concerned, they break due to $\langle \theta^{(1,2)} \rangle$, but their global versions remain in the theory. Last, the electroweak breaking proceeds by the following vev configuration, $\langle L_s^{(3)} \rangle = \text{diag}(v_d, v_u, 0)$ [23].

4.3 Lepton Yukawa Couplings and μ Terms

Due to the presence of the aforementioned global symmetries, invariant lepton Yukawa terms are not allowed in the Yukawa sector. Nevertheless, according to [26], the 4-dim theory can be considered as renormalizable, therefore below the unification scale an effective term can emerge in the form of higher-dimensional operator

[3] Gauginos are not taken into consideration in this reasoning, since, as stated earlier, they obtain mass in a geometric manner.

[4] There exist more vevs that can be added without affecting the breaking, see [22].

$L\bar{e}H_d\left(\frac{\overline{K}}{M}\right)^3$ [13], where $\overline{K}$ denotes the vev of the conjugate scalar component of any combination of $S^{(i)}$, $\nu_R^{(i)}$ and $\theta^{(i)}$. Similar argumentation may also allow mass terms for $S^{(i)}$ and $\nu_R^{(i)}$, ending up to be superheavy. Moreover, appropriate higher-dimensional operators can be employed for the emergence of the μ-term, one for each family $H_u H_d \overline{\theta}\frac{\overline{K}}{M}$. Due to the vev configuration, it is understood that the μ terms corresponding to the Higgs doublets of the $l = 1, 2$ generations will be supermassive, while that of the $l = 3$ generation will be at the TeV scale.

5 Phenomenological Analysis

Since the dimensional reduction led to a GUT, it is understood that all gauge couplings are equal to g at M_{GUT}. Moreover, at the higher-dimensional level, there is a single coupling, therefore the (quark) Yukawa couplings have to be equal to g at M_{GUT}. Our phenomenological analysis is performed using 1-loop β-functions for all parameters involved. Below the unification scale they run according to the RGEs of the MSSM (squarks included and gauginos excluded as they gain mass geometrically—see Eq. (5)- at the TeV scale) plus the 4 additional Higgs doublets (and their supersymmetric counterparts), down to an intermediate scale M_{int}, namely the scale below which all supermassive particles and parameters have decoupled. Below M_{int} the RGEs include only the 2 Higgs doublets that originate from the third generation, their corresponding Higgsinos and the sleptons. Last, the TeV regime, (M_{TeV}), the RGEs used are those of a non-supersymmetric 2HDM. The experimental values taken into consideration for comparison to our results are [24]:

- The strong gauge coupling:

$$a_s(M_Z) = 0.1187 \pm 0.0016\,. \tag{14}$$

- The top quark pole mass and the bottom quark mass at M_Z:

$$m_t^{\exp} = (172.4 \pm 0.7)\ \text{GeV}\,, \qquad m_b(M_Z) = 2.83 \pm 0.10\ \text{GeV}\,. \tag{15}$$

- The (SM) Higgs boson mass:

$$M_H^{\exp} = 125.10 \pm 0.14\ \text{GeV}\,. \tag{16}$$

5.1 Gauge Unification

The first test for each GUT is to produce the prediction of the unification scale, M_{GUT}. We follow the straightforward methodology, namely the $a_{1,2}$ are used for the

M_{GUT} calculation and the a_3 is used for confirmation. The 1-loop gauge β-functions are given by $2\pi\beta_i = b_i\alpha_i^2$, where the b_i coefficients vary for each of the three energy regions according to the corresponding particle spectrum [15]. Taking into account an uncertainty of 0.3% at the boundary of M_{GUT}, the various scales of our model are obtained: $M_{GUT}{\sim}1.7 \times 10^{15}$ GeV, $M_{int}{\sim}9 \times 10^{13}$ GeV, $M_{TeV}{\sim}1500$ GeV. The calculation of the α_s gives the following prediction:

$$a_s(M_Z) = 0.1218\ , \tag{17}$$

which is within 2σ of the experimental value, (14). It is remarkable that although the predicted value of the M_{GUT} is somewhat low, implying that fast proton decay would be induced and the model would be excluded already, that is not the case by virtue of the two global $U(1)$'s which forbid it.

5.2 1-Loop Results

As mentioned above, the gauge and quark Yukawa interactions share the same coupling constant g due to unification and that property was used as a boundary condition in our calculations for the M_{GUT}. However, as commented in Sect. 4.3, tau Yukawa terms are absent due to the presence of the global symmetries and that is why they were introduced through higher-dimensional operators. This means that there exists much wider freedom of the corresponding coupling constant and therefore the boundary condition is not fixed to g. This property motivated us to pick the input in our model to be the tau lepton mass [24].

Also, we take into consideration uncertainties in the two important energy scales, M_{GUT} and M_{TeV}, due to threshold corrections (for details see [25]). In the current level of our analysis, to facilitate the calculation of our first results, it was sufficient to consider all "light" supersymmetric particles on equal footing. The top and bottom Yukawa couplings come with the following uncertainties: 6% for the GUT boundary and 2% for the TeV boundary. As follows, the masses of the quarks of the third generation and that of the (light) Higgs are obtained within 2σ and 1σ of their experimental values, respectively, as given in Eqs. (15), (16):

$$m_b(M_Z) = 3.00 \text{ GeV}\ , \qquad \hat{m}_t = 171.6 \text{ GeV}\ , \qquad m_h = 125.18 \text{ GeV}\ . \tag{18}$$

As explained above, all allowed Yukawa terms share the common value of the coupling constant, g, at the unification scale. For that reason, it is necessary to consider that the model exhibits a large $tan\beta{\sim}48$ in order to recover the experimentally observed discrepancy of the fermion masses. Last, the mass of the pseudoscalar Higgs boson is considered to be $2-3$ TeV.

6 Conclusions

First we considered a 10-d, $\mathcal{N} = 1$, E_8 Yang-Mills-Dirac theory with Weyl-Majorana fermions, constructed on the compactified spacetime of the form $M_4 \times B_0/\mathbb{Z}_3$, where B_0 is the coset space $SU(3)/U(1) \times U(1)$ and $\mathbb{Z}_3$ is a discrete group which acts freely on B_0. In order to result with the promising 4-d (softly broken) $\mathcal{N} = 1$ $SU(3)^3$ GUT, we employed two mechanisms: the CSDR and the Wilson flux breaking. The GUT breaking along with the assumption of a slight discrepancy between the radii of the coset led to a split-like supersymmetric scenario where the gauginos, third generation Higgsinos and sleptons gain mass at $\sim$1.5 TeV, while the rest supersymmetric particles become supermassive ($\sim M_{GUT}$). Treating the model as a 2HDM, calculations led to valid results for the top, bottom and light Higgs masses rendering the model viable so far.

Acknowledgements This work has been supported by the Basic Research Programme, PEVE2020 of the National Technical University of Athens, Greece. GM's work is supported by the Croatian Science Foundation project IP-2019-04-4168.

References

1. P. Forgacs, N.S. Manton, Commun. Math. Phys. **72**, 15 (1980). https://doi.org/10.1007/BF01200108
2. D. Kapetanakis, G. Zoupanos, Phys. Rept. **219**, 4–76 (1992). https://doi.org/10.1016/0370-1573(92)90101-5
3. Y.A. Kubyshin, I.P. Volobuev, J.M. Mourao, G. Rudolph, Lect. Notes Phys. **349**, 1 (1990). https://doi.org/10.1007/3-540-51917-3
4. J. Scherk, J.H. Schwarz, Nucl. Phys. B **153**, 61–88 (1979). https://doi.org/10.1016/0550-3213(79)90592-3
5. M.B. Green, J.H. Schwarz, E. Witten, *Superstring Theory, vol. 1: Introduction*; Cambridge Monographs on Mathematical Physics (University Pr., Cambridge, UK, 1987), 469 p; D. Lust, S. Theisen, Lect. Notes Phys. **346**, 1–346 (1989). https://doi.org/10.1007/BFb0113507
6. D.J. Gross, J.A. Harvey, E.J. Martinec, R. Rohm, Nucl. Phys. B **256**, 253 (1985). https://doi.org/10.1016/0550-3213(85)90394-3
7. N.S. Manton, Nucl. Phys. B **193**, 502 (1981). https://doi.org/10.1016/0550-3213(81)90343-6; G. Chapline, R. Slansky, Nucl. Phys. B **209**, 461–483 (1982). https://doi.org/10.1016/0550-3213(82)90267-X
8. P. Manousselis, G. Zoupanos, Phys. Lett. B **518**, 171–180 (2001). https://doi.org/10.1016/S0370-2693(01)01040-1, arXiv:hep-ph/0106033 [hep-ph]; ibid, JHEP **03**, 002 (2002). https://doi.org/10.1088/1126-6708/2002/03/002, arXiv:hep-ph/0111125 [hep-ph]; ibid, Phys. Lett. B **504**, 122–130 (2001). https://doi.org/10.1016/S0370-2693(01)00268-4, arXiv:hep-ph/0010141 [hep-ph]; ibid, JHEP **11**, 025 (2004). https://doi.org/10.1088/1126-6708/2004/11/025,arXiv:hep-ph/0406207 [hep-ph]

9. A. Chatzistavrakidis, P. Manousselis, G. Zoupanos, Fortsch. Phys. **57**, 527–534 (2009). https://doi.org/10.1002/prop.200900012, arXiv:0811.2182 [hep-th]; A. Chatzistavrakidis, P. Manousselis, N. Prezas, G. Zoupanos, Fortsch. Phys. **56**, 389–399 (2008). https://doi.org/10.1002/prop.200710510, arXiv:0712.2717 [hep-th]; P. Manousselis, N. Prezas, G. Zoupanos, Nucl. Phys. B **739**, 85–105 (2006). https://doi.org/10.1016/j.nuclphysb.2006.01.008, arXiv:hep-th/0511122 [hep-th]
10. G. Lopes Cardoso, G. Curio, G. Dall'Agata, D. Lust, P. Manousselis, G. Zoupanos, Nucl. Phys. B **652**, 5–34 (2003). https://doi.org/10.1016/S0550-3213(03)00049-X, arXiv:hep-th/0211118 [hep-th]
11. A. Chatzistavrakidis, G. Zoupanos, JHEP **09**, 077 (2009). https://doi.org/10.1088/1126-6708/2009/09/077, arXiv:0905.2398 [hep-th]
12. M. Klaput, A. Lukas, C. Matti, JHEP **09**, 100 (2011). https://doi.org/10.1007/JHEP09(2011)100, arXiv:1107.3573 [hep-th]
13. N. Irges, G. Zoupanos, Phys. Lett. B **698**, 146–151 (2011). https://doi.org/10.1016/j.physletb.2011.03.005, arXiv:1102.2220 [hep-ph]
14. D. Lust, G. Zoupanos, Phys. Lett. B **165**, 309–314 (1985). https://doi.org/10.1016/0370-2693(85)91236-5
15. G. Manolakos, G. Patellis, G. Zoupanos, Phys. Lett. B **813**, 136031 (2021). https://doi.org/10.1016/j.physletb.2020.136031, arXiv:2009.07059 [hep-ph]
16. L. Castellani, Ann. Phys. **287**, 1–13 (2001). https://doi.org/10.1006/aphy.2000.6097, arXiv:hep-th/9912277 [hep-th]; D. Kapetanakis, G. Zoupanos, Phys. Lett. B **249**, 73–82 (1990). https://doi.org/10.1016/0370-2693(90)90529-F; ibid., Z. Phys. C **56**, 91–96 (1992). https://doi.org/10.1007/BF01589710; P. Aschieri, J. Madore, P. Manousselis, G. Zoupanos, JHEP **04**, 034 (2004). https://doi.org/10.1088/1126-6708/2004/04/034, arXiv:hep-th/0310072 [hep-th]; A. Chatzistavrakidis, P. Manousselis, N. Prezas, G. Zoupanos, Phys. Lett. B **656**, 152–157 (2007). https://doi.org/10.1016/j.physletb.2007.09.015, arXiv:0708.3222 [hep-th]
17. G. Zoupanos, Phys. Lett. B **201**, 301–305 (1988). https://doi.org/10.1016/0370-2693(88)91144-6
18. N.G. Kozimirov, V.A. Kuzmin, I.I. Tkachev, Sov. J. Nucl. Phys. **49**, 164 (1989), TPI-MINN-89-2-T.; D. Kapetanakis, G. Zoupanos, Phys. Lett. B **232**, 104–112 (1989). https://doi.org/10.1016/0370-2693(89)90565-0
19. Y. Hosotani, Phys. Lett. B **126**, 309–313 (1983). https://doi.org/10.1016/0370-2693(83)90170-3; E. Witten, Nucl. Phys. B **258**, 75 (1985). https://doi.org/10.1016/0550-3213(85)90603-0; J.D. Breit, B.A. Ovrut, G.C. Segre, Phys. Lett. B **158**, 33 (1985). https://doi.org/10.1016/0370-2693(85)90734-8; B.R. Greene, K.H. Kirklin, P.J. Miron, G.G. Ross, Nucl. Phys. B **278**, 667–693 (1986). https://doi.org/10.1016/0550-3213(86)90057-X
20. A. Chatzistavrakidis, H. Steinacker, G. Zoupanos, JHEP **05**, 100 (2010). https://doi.org/10.1007/JHEP05(2010)100, arXiv:1002.2606 [hep-th]; A. Chatzistavrakidis, G. Zoupanos, SIGMA **6**, 063 (2010). https://doi.org/10.3842/SIGMA.2010.063, arXiv:1008.2049 [hep-th]
21. T.W. Kephart, M.T. Vaughn, Ann. Phys. **145**, 162 (1983). https://doi.org/10.1016/0003-4916(83)90176-8
22. K.S. Babu, X.G. He, S. Pakvasa, Phys. Rev. D **33**, 763 (1986). https://doi.org/10.1103/PhysRevD.33.763; J. Sayre, S. Wiesenfeldt, S. Willenbrock, Phys. Rev. D **73**, 035013 (2006). https://doi.org/10.1103/PhysRevD.73.035013, arXiv:hep-ph/0601040 [hep-ph]
23. E. Ma, M. Mondragon, G. Zoupanos, JHEP **12**, 026 (2004). https://doi.org/10.1088/1126-6708/2004/12/026, arXiv:hep-ph/0407236 [hep-ph]; S. Heinemeyer, E. Ma, M. Mondragon, G. Zoupanos, AIP Conf. Proc. **1200**(1), 568–571 (2010). https://doi.org/10.1063/1.3327674, arXiv:0910.0501 [hep-ph]
24. P.A. Zyla, et al., [Particle Data Group], PTEP **2020**(8), 083C01 (2020). https://doi.org/10.1093/ptep/ptaa104
25. J. Kubo, M. Mondragon, M. Olechowski, G. Zoupanos, Nucl. Phys. B **479**, 25–45 (1996)
26. J. Polchinski, Nucl. Phys. B **231**, 269–295 (1984). https://doi.org/10.1016/0550-3213(84)90287-6

27. M.B. Green, J.H. Schwarz, E. Witten, *Superstring Theory, vol. 2: Loop Amplitudes*, Anomalies and Phenomenology (Cambridge University Press, 1987). 596 p., ISBN 0 521 32999 X (Cambridge Monographs on Mathematical Physics)
28. B.P. Dolan, JHEP **05**, 018 (2003). https://doi.org/10.1088/1126-6708/2003/05/018, arXiv:hep-th/0304037 [hep-th]

String Theories, (Super-)Gravity, Cosmology

Ramond States of the D1-D5 CFT Away from the Free Orbifold Point

Andre Alves Lima, Galen M. Sotkov, and Marian Stanishkov

Abstract The free orbifold point of the D1-D5 CFT must be deformed with a scalar marginal operator driving it to the region in moduli space where the holographic supergravity description of fuzzball microstates becomes available. We discuss the effects of the deformation operator on the twisted Ramond ground states of the CFT by computing four-point functions. One can thus extract the OPEs of the deformation operator with these Ramond fields to find the conformal dimensions of intermediate non-BPS states and the relevant structure constants. We also compute the anomalous dimensions at second order in perturbation theory, and find that individual single-cycle Ramond fields are renormalized, while the full multi-cycle ground states of the S_N orbifold remain protected at leading order in the large-N expansion.

Keywords Symmetric product orbifold of $N = 4$ SCFT · Ramond fields · Marginal deformations · Correlation functions · Renormalization

1 Introduction

In the decoupling limit, the Type IIB supergravity solution for the bound state of a large number N_1 of D1-branes wrapped around a circle $\mathbb{S}^1$, and a large number N_5 of D5-branes wrapped around $\mathbb{T}^4 \times \mathbb{S}^1$ develops a "throat" geometry; that is, it becomes $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$, where AdS_3 and $\mathbb{S}^3$ have the same (large) radius fixed by the branes' charges. The holographic CFT dual to this spacetime, called the D1-D5 CFT, is a two-

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_12

dimensional superconformal field theory with $\mathcal{N} = (4, 4)$ SUSY, and R-symmetry group $SO(4) \cong SU(2)_L \times SU(2)_R$ associated with $\mathbb{S}^3$. The superconformal algebra also has a global "SO(4)" symmetry associated with $\mathbb{T}^4$. The large central charge, fixed by the AdS_3 radius, is $c = 6N$, where $N = N_1 N_5 \gg 1$. See e.g. [3] for a review.

The D1-D5 system is a well-known laboratory for the study of microscopic properties of black holes in String Theory. The SUGRA solution (when the branes carry internal momentum along their common $\mathbb{S}^1$ direction) has a horizon whose area matches the counting of CFT states, as famously discovered by Strominger and Vafa [18]. But more than "state counting", a precise holographic dictionary exists, between states in the D1-D5 CFT and specific non-singular, horizonless 'semiclassical' SUGRA solutions [14, 17] which have the same charges as—and thus look like—the extremal supersymmetric black hole away from the would-be horizon, outside the $AdS_3 \times \mathbb{S}^3$ throat. Details of this dictionary, and the search for new and more general (less symmetric) horizonless and non-singular 'microstate geometries' are part of what has become known as the fuzzball program, an ongoing endeavor, see e.g. [2, 4, 14, 16, 17].

The D1-D5 CFT dual to gravity solutions is strongly-coupled, but conjectured to live in the moduli space of a free SCFT on the symmetric orbifold $(\mathbb{T}^4)^N / S_N$. (See [3] and references therein.) Holographic computations are usually done in this free orbifold point, where results are often exact, and then rely on the existence of non-renormalization theorems for some BPS-protected objects such as NS chiral operators, as well as on explicit matching with bulk computations, in special cases where they can also be performed in parallel with their CFT counterpart. Meanwhile, motivated by the increasing scrutiny of the relation between individual states and geometries, the effect of the deformation of the free orbifold towards the strongly-coupled CFT has been given attention over the years, see e.g. [1, 5, 6].

Here we report the results published in [7–11], on the fate of twisted Ramond ground states $|\mathscr{R}_{[g]}\rangle$, with an arbitrary twist $g \in S_N$, when the free CFT is deformed by a marginal scalar modulus $O^{(\text{int})}_{[2]}$. In terms of the action,

$$S_{\text{int}} = S_{\text{free}} + \lambda \int d^2 z \; O^{(\text{int})}_{[2]}(z, \bar{z}) \tag{1}$$

with a coupling λ. Known microstate geometries are dual to coherent superpositions of Ramond ground states, or to specific excitations thereof [4, 16, 17], providing one important motivation—besides the fact that they are naturally very elementary objects—for their study in the deformed theory. Although the actual D1-D5 CFT is strongly coupled, we work with conformal perturbation theory to order λ^2, which is the lowest possible order where one can detect lifting of dimensions of $|\mathscr{R}_{[g]}\rangle$. The analysis hinges upon us being able to compute specific four-point functions involving the Ramond states and the deformation modulus.

2 Ramond Ground States and Their Four-Point Functions with the Deformation Operator

Specifically, to assess the anomalous dimensions in second-order perturbation theory, we must compute

$$\Big\langle \mathscr{R}^{\dagger}_{[g]}(\infty,\bar{\infty})\; O^{(\mathrm{int})}_{[2]}(1,\bar{1})\; O^{(\mathrm{int})}_{[2]}(u,\bar{u})\; \mathscr{R}_{[g]}(0,\bar{0})\Big\rangle \tag{2}$$

that is a four-point function of twisted operators. The Hilbert space of the orbifold is divided into twisted sectors, created by the insertion of 'bare-twist fields' $\sigma_g(z)$, which introduces a branch cut at the point z such that, when a given field crosses it, $g \in S_N$ permutes the copies $I = \{1, \ldots, N\}$. Twisted operators are excitations of the bare twists. For example, $O^{(\mathrm{int})}_{(2)}$ is an excitation of a transposition $\sigma_{(2)}$, and in the twisted sector of the cyclic permutation $(1, \cdots, n) \in S_N$ there are two R-charged holomorphic Ramond ground states (see the Appendix for definitions)

$$R^{\pm}_{(n)}(z) = \exp\left(\pm\frac{i}{2n}\sum_{I=1}^{n}\big[\phi_{1,I} - \phi_{2,I}\big]\right)\sigma_{(1,\cdots,n)}(z) \tag{3}$$

differing by R-charge $j_{\pm} = \pm\frac{1}{2}$, and both having conformal weight $h^{\mathrm{R}}_n = \frac{6n}{24}$, appropriate for the Ramond sector of a CFT with central charge $6n$. To obtain S_N-invariant operators from (3), we sum over all elements in the conjugacy class $[n]$,

$$R^{s}_{[n]} = \frac{1}{\sqrt{N!|\mathrm{Cent}(n)|}}\sum_{h\in S_N} R^{s}_{h(n)h^{-1}}. \tag{4}$$

The normalization factor, featuring the order of the centralizer of the twist permutation, $|\mathrm{Cent}(n)| = (N-n)!n$, counts multiplicities of terms in the sum, to ensure that $R^s_{[n]}$ has the same normalization as each of its components in the r.h.s. More generally, any $g \in S_N$ can be decomposed as a product $g = \prod_n (n)^{N_n}$, of (disjoint) cyclic permutations of length n, characterized by a partition $[N_n] = \{N_n \in \mathbb{N} \mid \sum_n nN_n = N\}$, and the conjugacy class $[g]$ is the set of all permutations with the same partition $[N_n]$. Generic S_N-invariant operators for multi-cycle permutations can be constructed analogously to (4), to depend only on $[N_n]$. This applies to the Ramond ground states $\mathscr{R}_{[g]}$ of the *full* orbifold, which are *not* the fields $R^s_{[n]}$, but products

$$\mathscr{R}_{[N^{(s)}_n]} = \Big[\prod_{s,n}(R^s_{[n]})^{N^{(s)}_n}\Big] \quad \text{for a partition} \quad \sum_{n,s} N^{(s)}_n = N. \tag{5}$$

The partition takes into account an SU(2) "spin index" $s = \pm, \dot{1}, \dot{2}$ besides the twist. Any field with this structure has the correct conformal weight $h^{\mathrm{R}} = \frac{N}{4}$ for a CFT with central charge $c = 6N$, and R-charge $j = \sum_{s,n} N^{(s)}_n\, j_s$.

Connected correlators of twisted fields are associated to branched coverings of the Riemann sphere, whose genera are determined by the twist permutations via the Riemann-Hurwitz formula. It can be shown that (2) factorizes into a sum of connected functions where the Ramond fields have at most *two* cycles [10],

$$A^{s_1 s_2}_{n_1 n_2} = \Big\langle \big[R^{s_1}_{[n_1]} R^{s_2}_{[n_2]}\big]^\dagger(\infty,\bar\infty)\; O^{(\text{int})}_{[2]}(1,\bar 1)\; O^{(\text{int})}_{[2]}(u,\bar u)\; \big[R^{s_1}_{[n_1]} R^{s_2}_{[n_2]}\big](0,\bar 0)\Big\rangle, \tag{6}$$

with associated genus-zero covering surface. The covering is given by the map

$$z(t) = \left(\frac{t}{t_1}\right)^{n_1} \left(\frac{t-t_0}{t-t_\infty}\right)^{n_2} \left(\frac{t_1-t_\infty}{t_1-t_0}\right)^{n_2} \tag{7}$$

such that, respectively, the twists n_1 at $z = 0, \infty$ are lifted to $t = 0, \infty$ on the covering; the twists n_2 at $z = 0, \infty$ lift to $t = t_0, t_\infty$; and the twists 2 at $z = 1, u$ to $t = t_1, x$. The monodromies impose that t_0, t_1, t_∞ are functions of x, and $u(x) \equiv z(x)$ is

$$u(x) = x^{n_1-n_2}(x+\tfrac{n_1}{n_2})^{n_1+n_2}(x-1)^{-n_1+n_2}(x-1+\tfrac{n_1}{n_2})^{n_2-n_1}. \tag{8}$$

Following [12, 13, 15] we can use the covering surface as a tool for dealing with monodromies and compute (6). The final result is

$$\begin{aligned} A^{s_1 s_2}_{n_1 n_2}(u,\bar u) &= \frac{\varpi(n_1 n_2)}{N^2} \sum_{\mathfrak{a}=1}^{2\max(n_1,n_2)} |A^{s_1 s_2}_{n_1 n_2}(x_{\mathfrak{a}}(u))|^2 \\ A^{s_1 s_2}_{n_1 n_2}(x) &= \frac{1}{16 n_1^2}\Big[C_{s_1 s_2} + x(x-1+\tfrac{n_1}{n_2})\Big] \\ &\quad\times \frac{x^{1-n_1+n_2}(x-1)^{1+n_1+n_2}(x+\frac{n_1}{n_2})^{1-n_1-n_2}(x-1+\frac{n_1}{n_2})^{1+n_1-n_2}}{(x+\frac{n_1-n_2}{2n_2})^4} \end{aligned} \tag{9}$$

We have written the overall factor ϖ/N^2 in the large-N limit, but apart from this the result holds for finite N as well. The constant ϖ depends only on $n_1 n_2$, and $C_{s_1 s_2}$ only on n_1, n_2 and s_1, s_2. They can be found in [10].

Equation (9) is written in terms of the functions $x_{\mathfrak{a}}(u)$ that are the inverses of $u(x)$, and cannot be found in general, but can be expanded in the coincidence limits $u \to 0$ and $u \to 1$, to give the associated conformal data. For example, for $u \to 0$, we obtain the OPE

$$\begin{aligned} &O^{(\text{int})}_{(2)}(u,\bar u)\; \big[R^{s_1}_{(n_1)} R^{s_2}_{(n_2)}\big](0,\bar 0)\, |\varnothing\rangle \\ &\qquad = \frac{\langle Y^{s_1 s_2 \dagger}_{(n_1+n_2)}\, O^{(\text{int})}_{(2)}\, [R^{s_1}_{(n_1)} R^{s_2}_{(n_2)}]\rangle}{|u|^{2+\frac{n_1+n_2}{2}-\Delta^{s_1 s_2}_Y}}\, Y^{s_1 s_2}_{(n_1+n_2)}(0,\bar 0)\, |\varnothing\rangle + \cdots \end{aligned}$$

The operator $Y^{s_1 s_1}_{(n_1+n_2)}$ results from $O^{(\text{int})}_{(2)}$ joining the two single-cycle Ramond fields. Its dimension $\Delta^{s_1 s_2}_Y$, as well as the structure constant in the denominator, can be found from the expansion of (9). For instance, when $n_1 = n_2 = n$ we find

$$\Delta^{+-}_Y = \Delta^{\dot{1}\pm}_Y = \Delta^{\dot{1}\dot{2}}_Y = \Delta^{\dot{1}\dot{1}}_Y = \frac{1}{n} + n, \quad \Delta^{++}_Y = \frac{2}{n} + n.$$

In the limit $u \to 1$, we find the (symbolic) OPE $O^{(\text{int})}_{[2]} \times O^{(\text{int})}_{[2]} = [1] + [\sigma_{[3]}]$, expected from the composition of the two transpositions in the twists. The channel with the conformal family of the identity [1], is in fact used to fix the constants in (9).

The correlator (2) is a sum of functions (9) for every required pairwise combination of single-cycle Ramond fields. This sum has "symmetry factors" depending on the multiplicity of equivalent strands in the product (5), i.e. on the form of the partition $[N^{(s)}_n]$, but these factors are not dynamical and can be disregarded in the computation of the anomalous dimension below. We note that there can be contributions from functions with *single-cycle* Ramond fields and a genus-one covering surface, but we focus on the genus-zero functions here.

3 Away from the Free Orbifold Point

Conformal perturbation theory gives the dimension $\Delta^{\text{R}}_{(\text{ren})} = h^{\text{R}}_{(\text{ren})} + \tilde{h}^{\text{R}}_{(\text{ren})}$ of $\mathscr{R}_{[g]}$ at order λ^2, once we perform the integral

$$\Delta^{\text{R}}_{(\text{ren})} = \frac{N}{2} - \frac{\pi}{2}\lambda^2 \int d^2u \left\langle \mathscr{R}^{\dagger}_{[g]}(\infty, \bar{\infty})\, O^{(\text{int})}_{[2]}(1, \bar{1})\, O^{(\text{int})}_{[2]}(u, \bar{u})\, \mathscr{R}_{[g]}(0, \bar{0}) \right\rangle.$$

The integrand is a sum of functions of the type (9). With a change of variables we find, with unimportant constants $\mathscr{A}$ and $\mathscr{B}$,

$$\begin{aligned}\int d^2u\, A^{s_1 s_2}_{n_1 n_2}(u, \bar{u}) &= \int d^2x \left| u'(x)\, A^{s_1 s_2}_{n_1 n_2}(x) \right|^2 \\ &= \mathscr{A} \int d^2y\, |1-y|^{-3} + \mathscr{B} \int d^2y\, |1-y|^{-3} |y-w|^2\end{aligned} \tag{10}$$

which is divergent, but can be regularized by analytic continuation:

$$\int \frac{d^2y}{|y-1|^3} = \lim_{a\to 0} \frac{4\pi}{\Gamma(-a)} = 0, \quad \int d^2y\, \frac{|y-w|^2}{|1-y|^3} \sim \lim_{a\to 1} \frac{-16\pi}{\Gamma(-a)} = 0$$

Hence

$$\Delta^{\text{R}}_{(\text{ren})} = \tfrac{1}{2}N + \text{O}(\lambda^3) \tag{11}$$

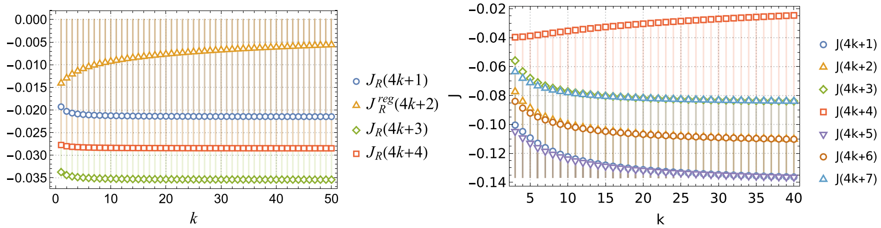

Fig. 1 Plots of the integral $J_R(n)$ in Eq. (12), for R-charged (left) and R-neutral (right) single-strand Ramond ground states of twist $n = 4k + \ell$

at least in the large-N limit, where only the genus-zero connected functions contribute to (2). The dimension of the Ramond ground states is protected.

A curious feature of the non-renormalization is that it only occurs for the total orbifold ground states with $h^{\mathrm{R}} = \frac{1}{4}N$. Individual single-cycle states with $h^{\mathrm{R}}_n = \frac{1}{4}n$ *are* renormalized. We can see that by using similar methods to compute

$$J_R = \int \mathrm{d}^2 u \left\langle R^{s\dagger}_{[n]}(\infty, \bar{\infty}) O^{(\mathrm{int})}_{[2]}(1, \bar{1}) O^{(\mathrm{int})}_{[2]}(u, \bar{u}) R^{s}_{[n]}(0, \bar{0}) \right\rangle \tag{12}$$

whose numerical values are plotted in Fig. 1. The functions in the integrand contain a leading genus-zero part coming from $O^{(\mathrm{int})}_{[2]}$ joining the field $R^s_{(n)}$ with an untwisted vacuum strand, that is absent in the decomposition of the correlator (2).

In summary, we find that the $c = 6N$ CFT Ramond ground states are protected, as it should be expected, since they are associated with SUGRA fuzzball solutions and related to BPS NS-chiral fields by spectral flow of the $c = 6N$ theory. However, the individual single-cycle components $R^s_{[n]}$ do lift, and acquire anomalous dimensions when the CFT is driven away from the free orbifold point.

Acknowledgements This work was partially supported by the Bulgarian NSF grants KP-06-H28/5 and KP-06-H38/11.

Appendix

The $(\mathbb{T}^4)^N/S_N$ orbifold has N copies of a 'seed' $\mathcal{N} = (\triangle, \triangle)$ SCFT with central charge $c_{\mathrm{seed}} = 6$. The total central charge is $c = 6N$. Each copy, labeled by an index $I = \{1, \dots, N\}$, has 4 real bosons and $(4 + \tilde{4})$ real fermions, all free, which can be gathered into SU(2) doublets. The holomorphic Ramond fields in the text are constructed from the bosonized fermions $\psi^{\alpha\dot{1}}_I(z) = [e^{-i\phi_{2,I}(z)}, e^{-i\phi_{1,I}(z)}]^T$ and $\psi^{\alpha\dot{2}}_I(z) = [e^{i\phi_{2,I}(z)}, -e^{i\phi_{1,I}(z)}]^T$. The indices $\alpha = \pm$ correspond to the holomorphic R-symmetry group $\mathrm{SU}(2)_{\mathrm{L}}$ and $\dot{A} = \dot{1}, \dot{2}$ to the factor $\mathrm{SU}(2)_2$ of the global symmetry.

References

1. S.G. Avery, B.D. Chowdhury, S.D. Mathur, JHEP **06**, 031 (2010)
2. I. Bena, S. Giusto, R. Russo, M. Shigemori, N.P. Warner, JHEP **05**, 110 (2015)
3. J.R. David, G. Mandal, S.R. Wadia, Phys. Rept. **369**, 549 (2002)
4. S. Giusto, E. Moscato, R. Russo, JHEP **11**, 004 (2015)
5. B. Guo, S.D. Mathur, JHEP **10**, 155 (2019)
6. C.A. Keller, I.G. Zadeh, J. Phys. A **53**, 095401 (2020)
7. A.A. Lima, G.M. Sotkov, M. Stanishkov, Phys. Rev. D **102**, 106004 (2020)
8. A.A. Lima, G.M. Sotkov, M. Stanishkov, Phys. Lett. B **808**, 135630 (2020)
9. A.A. Lima, G.M. Sotkov, M. Stanishkov, JHEP **7**, 211 (2021)
10. A.A. Lima, G.M. Sotkov, M. Stanishkov, JHEP **7**, 120 (2021)
11. A.A. Lima, G.M. Sotkov, M. Stanishkov, JHEP **3**, 202 (2021)
12. O. Lunin, S.D. Mathur, Commun. Math. Phys. **219**, 399 (2001)
13. O. Lunin, S.D. Mathur, Commun. Math. Phys. **227**, 385 (2002)
14. S.D. Mathur, Fortsch. Phys. **53**, 793 (2005)
15. A. Pakman, L. Rastelli, S.S. Razamat, JHEP **10**, 034 (2009)
16. S. Rawash, D. Turton, JHEP **07**, 178 (2021)
17. K. Skenderis, M. Taylor, Phys. Rept. **467**, 117 (2008)
18. A. Strominger, C. Vafa, Phys. Lett. B **379**, 99 (1996)

Primordial Black Hole Generation in a Two-Field Inflationary Model

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Abstract We summarize our work on the generation of primordial black holes in a type of two-field inflationary models. The key ingredient is a sharp turn of the background trajectory in field space. We show that certain classes of solutions to the equations of motion exhibit precisely this kind of behavior. Among them we find solutions, which describe a transition between an ultra-slow roll and a slow roll phases of inflation.

Keywords Primordial black holes · Cosmological inflation · Rapid turns · Hidden symmetry

1 Introduction

Large perturbations during cosmological inflation can seed the formation of Primordial Black Holes (PBHs). If produced with enough abundance, the latter could constitute a significant component of dark matter. This possibility has attracted a lot of attention recently, due to the observation of gravitational waves sourced by binary black hole mergers. The reason is that analysis of the observational data contains indications of primordial origin for a fraction of these black holes [1–3].

A novel mechanism for the formation of PBHs in multi-field inflation was proposed in [4, 5]. Cosmological models with multiple scalars are of particular interest in view of recent theoretical developments [6–9]. An essential difference from the single-field case is that the field-space trajectories of their background solutions can deviate from geodesics. The basic idea of [4, 5] is that a brief period of such a strongly non-geodesic motion can induce a large enhancement of the power spectrum of the curvature perturbation, thus triggering PBH generation. Studying a type of two-field

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_13

models, we showed in [10] that there are actual solutions of the background equations of motion, which behave in precisely this manner.

A key role in the considerations of [10] was played by a class of exact solutions found in [11]. The scalar field space in that case is the Poincaré disk, while the scalar potential is determined by a certain hidden symmetry. We review the investigation of [10], showing that the field-space trajectories of these solutions exhibit exactly the behavior needed for PBH generation. We also discuss how to improve the problematic behavior of the corresponding Hubble η-parameter via a suitable symmetry-breaking modification. The resulting modified solutions preserve the PBH-generating properties of the hidden symmetry ones, while describing a smooth transition between an ultra-slow roll and a slow roll inflationary phases.

2 Two-Field Inflationary Models

We will study inflationary models arising from two scalar fields $\phi^I(x^\mu)$ minimally coupled to Einstein gravity. The action describing this system is:

$$S = \int d^4x \sqrt{-\det g} \left[\frac{R}{2} - \frac{1}{2} G_{IJ} \partial_\mu \phi^I \partial^\mu \phi^J - V(\{\phi^I\}) \right] \;, \tag{1}$$

where $g_{\mu\nu}$ is the spacetime metric with $\mu, \nu = 0, 1, 2, 3$ and G_{IJ} is the scalar field-space metric with $I, J = 1, 2$. As usual in cosmology, we will assume that the background spacetime metric and scalars have the form:

$$ds_g^2 = -dt^2 + a^2(t) d\mathbf{x}^2 \quad , \quad \phi^I = \phi_0^I(t) \;, \tag{2}$$

where $a(t)$ is the scale factor. Recall that the Hubble parameter is given by $H(t) = \dot{a}/a$, where $\dot{} \equiv \partial_t$.

2.1 *Important Characteristics*

To define a number of important quantities, characterizing any inflationary model, let us introduce an orthonormal basis of tangent and normal vectors to a field-space trajectory $(\phi_0^1(t), \phi_0^2(t))$:

$$T^I = \frac{\dot{\phi}_0^I}{\dot{\phi}_0} \quad , \quad N_I = (\det G)^{1/2} \epsilon_{IJ} T^J \quad , \quad \dot{\phi}_0^2 = G_{IJ} \dot{\phi}_0^I \dot{\phi}_0^J \;. \tag{3}$$

In terms of this basis, the deviation from a geodesic is measured by the quantity [12]:

$$\Omega = -N_I D_t T^I \;, \tag{4}$$

where $D_t T^I = \dot{\phi}_0^J \nabla_J T^I$. The function $\Omega(t)$ is called the turning rate of a trajectory. On solutions of the equations of motion, which follow from (1) with (2) substituted, the expression (4) can be rewritten as:

$$\Omega = \frac{N_I V^I}{\dot{\phi}_0} \quad . \tag{5}$$

Another set of important characteristics is given by the slow roll parameters, defined in the following manner [13]:

$$\varepsilon = -\frac{\dot{H}}{H^2} \quad , \quad \eta^I = -\frac{1}{H\dot{\phi}_0} D_t \dot{\phi}_0^I \quad . \tag{6}$$

Expanding η^I in the above basis, we have:

$$\eta^I = \eta_{\parallel} T^I + \eta_{\perp} N^I \quad , \tag{7}$$

where:

$$\eta_{\parallel} = -\frac{\ddot{\phi}_0}{H\dot{\phi}_0} \quad \text{and} \quad \eta_{\perp} = \frac{\Omega}{H} \quad . \tag{8}$$

The phenomenologically-motivated slow roll conditions in the present context are $\varepsilon \ll 1$ and $|\eta_{\parallel}| \ll 1$. On the other hand, there is no restriction on the dimensionless turning rate $\eta_{\perp}$. In fact, we will see shortly that the regime of interest for PBH generation is characterized by $\eta_{\perp}^2 \gg 1$.

To explain the physical mechanism that can seed the formation of primordial black holes, we need to consider perturbations around the homogeneous background (2). In comoving gauge, the fields decompose as:

$$\begin{aligned} \phi^I(t, \mathbf{x}) &= \phi_0^I(t) + \delta\phi_{\perp} N^I \quad , \\ g_{ij}(t, \mathbf{x}) &= a^2(t)\left[(1+2\zeta)\delta_{ij} + h_{ij}\right] \quad , \end{aligned} \tag{9}$$

where $\delta\phi_{\perp}(t, \mathbf{x})$ is the entropic perturbation, $\zeta = \zeta(t, \mathbf{x})$ is the curvature one and $h_{ij}(t, \mathbf{x})$ are tensor fluctuations with $i, j = 1, 2, 3$ being spatial indices. Substituting (9) in (1), one can derive an effective action for the perturbations. The key ingredients in that action are an interaction term between ζ and the entropic perturbation, as well as a mass term for $\delta\phi_{\perp}$ of the form (see, for instance, [12]):

$$m_s^2 = N^I N^J V_{;IJ} - \Omega^2 + \varepsilon H^2 \mathcal{R} \quad , \tag{10}$$

where $V_{;IJ} = \partial_I \partial_J V - \Gamma^K_{IJ} V_K$ and $\mathcal{R}$ is the Ricci scalar of the field-space metric G_{IJ}.

The interaction term, whose strength depends on $\eta_{\perp}$, implies that $\delta\phi_{\perp}$ can affect the evolution of ζ, and thus of the density fluctuations in the Early Universe. In particular, the amplitude of the curvature perturbation can become significantly enhanced for large enough turning rate [4, 5]. The latter, however, can induce a negative entropic mass in view of (10). Thus, a brief tachyonic instability of the entropic perturbation can signify the formation of primordial black holes. Our goal will be to show that there are actual solutions to the background equations of motion, which lead to precisely this kind of behavior for $\eta_\perp(t)$ and $m_s^2(t)$.

2.2 Rotationally Invariant Field Spaces

Let us now focus on rotationally invariant scalar field spaces. Then we can write the metric G_{IJ} as:

$$ds_G^2 = d\varphi^2 + f(\varphi)d\theta^2 \quad , \tag{11}$$

where we have denoted:

$$\phi_0^1(t) \equiv \varphi(t) \qquad \text{and} \qquad \phi_0^2(t) \equiv \theta(t) \quad . \tag{12}$$

Using (11)–(12) together with (2), one finds from (1) the following equations of motion for the background:

$$\ddot{\varphi} - \frac{f'}{2}\dot{\theta}^2 + 3H\dot{\varphi} + \partial_\varphi V = 0 \quad , \quad \ddot{\theta} + \frac{f'}{f}\dot{\varphi}\dot{\theta} + 3H\dot{\theta} + \frac{1}{f}\partial_\theta V = 0 \quad , \tag{13}$$

$$\dot{\varphi}^2 + f\dot{\theta}^2 = -2\dot{H} \quad , \quad 3H^2 + \dot{H} = V \quad . \tag{14}$$

We will be interested specifically in the case with $\partial_\theta V = 0$. In that case, (5) gives for the turning rate [10]:

$$\Omega = \frac{\sqrt{f}}{\left(\dot{\varphi}^2 + f\dot{\theta}^2\right)}\,\dot{\theta}\,\partial_\varphi V \quad , \tag{15}$$

In addition, the effective entropic mass (10) acquires the form [10]:

$$m_s^2 = M_V^2 - \Omega^2 + \varepsilon H^2 \mathcal{R} \quad , \tag{16}$$

where:

$$M_V^2 \equiv \frac{f\dot{\theta}^2\partial_\varphi^2 V + \frac{f'}{2f}\dot{\varphi}^2\partial_\varphi V}{(\dot{\varphi}^2 + f\dot{\theta}^2)} \quad . \tag{17}$$

We should note that $\Omega(t)$ can be a non-trivial function, i.e. the background trajectories in field space can be genuinely non-geodesic, even though the potential does not depend on one of the two scalars, as will become clear shortly; see also [10] and references therein.

3 A Class of Exact Solutions

Now we will show that a class of exact solutions to (13)–(14), obtained in [11], leads to a brief period with large turning rate, as well as tachyonic entropic mass, as needed for PBH generation. These solutions arise from the following choices of the functions f and V:

$$f(\varphi) = \frac{8}{3}\,\sinh^2\!\left(\sqrt{\frac{3}{8}}\,\varphi\right) \quad , \tag{18}$$

$$V(\varphi,\theta) = V_0\,\cosh^2\!\left(\sqrt{\frac{3}{8}}\,\varphi\right) \quad . \tag{19}$$

Note that (18) is equivalent with taking the field-space metric (11) to be that of the Poincaré disk (with fixed Gaussian curvature). Then (19) is exactly the form of the potential required by the hidden symmetry of [11].

For the above choices of potential and field space, one can solve (13) by:

$$\begin{aligned} a(t) &= \left[u^2 - \left(v^2 + w^2\right)\right]^{1/3} \quad , \\ \varphi(t) &= \sqrt{\frac{8}{3}}\,\mathrm{arccoth}\left(\sqrt{\frac{u^2}{v^2+w^2}}\,\right) \quad , \\ \theta(t) &= \mathrm{arccot}\left(\frac{v}{w}\right) \quad , \end{aligned} \tag{20}$$

where u, v and w are the following functions:

$$\begin{aligned} &u(t) = C_1^u \sinh(\kappa\, t) + C_0^u \cosh(\kappa\, t) \quad , \quad \kappa \equiv \frac{1}{2}\sqrt{3V_0} \quad , \\ &v(t) = C_1^v\, t + C_0^v \qquad \text{and} \qquad w(t) = C_1^w\, t + C_0^w \quad , \end{aligned} \tag{21}$$

with $C_{0,1}^{u,v,w} = const$. The expressions (20)–(21) solve (14) as well, if the following relation between the integration constants is satisfied:

$$(C_1^v)^2 + (C_1^w)^2 = \kappa^2\left[(C_1^u)^2 - (C_0^u)^2\right] \quad . \tag{22}$$

Substituting (20), together with (18)–(19), inside (15) and (17) gives:

$$\Omega = \frac{3V_0}{4}\,\frac{u\,(v\dot{w} - \dot{v}w)\,\sqrt{u^2 - w^2 - v^2}}{\left[(v\dot{u} - \dot{v}u)^2 + (w\dot{u} - \dot{w}u)^2 - (v\dot{w} - \dot{v}w)^2\right]} \tag{23}$$

and

$$M_V^2 = \frac{3V_0}{4}\,\frac{\left\{u^2\left[(v\dot{u}-\dot{v}u)^2+(w\dot{u}-\dot{w}u)^2\right]-(v^2+w^2)(v\dot{w}-\dot{v}w)^2\right\}}{(u^2-v^2-w^2)\left[(v\dot{u}-\dot{v}u)^2+(w\dot{u}-\dot{w}u)^2-(v\dot{w}-\dot{v}w)^2\right]}\ , \tag{24}$$

respectively. Analyzing these expressions directly, upon substitution of (21), is rather daunting. So [10] used a combination of analytical and numerical means to understand their behavior.

For that purpose, it is very useful to introduce the canonical radial variable on the Poincaré disk, ρ, which runs in the range $0 \le \rho < 1$ and is related to the field φ via:

$$\rho = \tanh\left(\frac{1}{8}\sqrt{6}\,\varphi\right)\ . \tag{25}$$

Using (20) in (25), one finds:

$$\rho(t) = \frac{\sqrt{v^2+w^2}}{\sqrt{u^2-v^2-w^2}+\sqrt{u^2}}\ , \tag{26}$$

which implies that the extremum condition $\dot{\rho}(t) = 0$ can be written as:

$$(v^2+w^2)\,\dot{u} - (v\dot{v}+w\dot{w})\,u = 0\ . \tag{27}$$

Analyzing (27), with (21) substituted, [10] showed that the function $\rho(t)$ can have at most two local extrema. In the cases with a single extremum, the latter is always a maximum. In the cases with two extrema, the one occurring at an earlier time is a local minimum, whereas the one occurring later is a local maximum. In any case, only the local maximum corresponds to a turn of the trajectory. Furthermore, the single turn induces a peak of $\Omega(t)$, as well as a corresponding peak of $-m_s^2(t)$. We have illustrated the behavior of these functions on Fig. 1. The examples, plotted there, have been chosen for convenience.[1] It should be stressed, though, that the shape of the functions is always the same, regardless of the values of the integration constants. However, as demonstrated in [10], both the position and the height of the peak can be varied at will by choosing suitably the values of the constants. In particular, one can easily achieve numerically that $|\eta_\perp|_{t_{peak}} \approx 23$, which is necessary for triggering PBH generation according to the benchmark cases of [5]. Note that, interestingly, the rapid turn occurs during a single e-fold.

The above results show that, in principle, the exact solutions (20)–(22) are suitable for describing the generation of perturbations, large enough to seed PBH formation. Unfortunately, however, it turns out that the corresponding $\eta_\parallel$-parameter violates the slow roll approximation [10]. Since $\eta_\parallel$ equals the Hubble slow roll parameter $\eta_H \equiv -\frac{\ddot{H}}{2H\dot{H}}$ on solutions of the equations of motion, this means that the inflationary

[1] In particular, in all three examples $\varepsilon \ll 1$ (in fact, $\varepsilon(t)|_{t_{peak}} \sim 10^{-19}$) and $H = 2$ to a great degree of accuracy, implying that the number of e-folds is $N = 2t$.

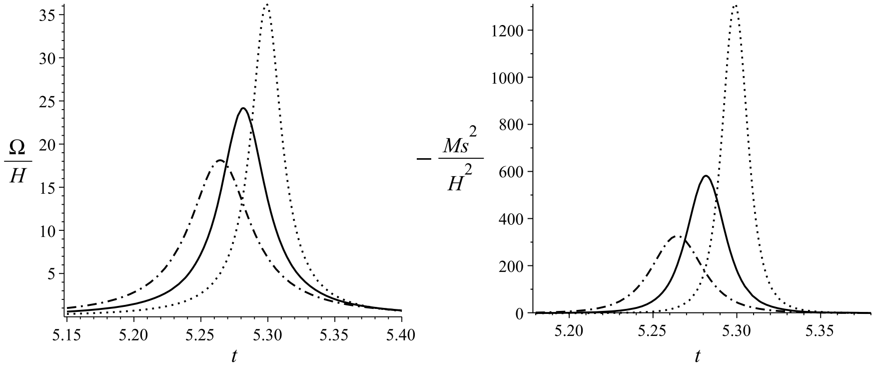

Fig. 1 Three examples of the functions Ω/H and $-m_s^2/H^2$, obtained by taking $\kappa = 3$, $C_0^u = 6$, $C_0^v = 1$, $C_1^v = -\frac{1}{5}$, $C_1^w = \frac{1}{2}$ and C_1^u - the positive root of (22), as well as $C_0^w = -2.46$ (dash-dotted curve), $C_0^w = -2.47$ (solid curve) and $C_0^w = -2.48$ (dotted curve)

regime under consideration is not viable phenomenologically. In the next Section, we will show how one can remedy the behavior of $\eta_{\|}$, while preserving the desired for PBH-generation properties of the solutions.

4 Modified Solution and PBH Generation

Our goal now is to find modified solutions of the equations of motion, which preserve the desirable behavior of $\eta_{\perp}(t)$, while improving that of $\eta_{\|}(t)$. For that purpose, let us consider the following Ansatze:

$$f(\varphi) = \frac{1}{q^2} \sinh^2(q\varphi) \ , \tag{28}$$

$$V(\varphi, \theta) = V_0 \cosh^{6p}(q\varphi) \ , \tag{29}$$

where $p, q = const$ and $p > 0$. Thus, the field space metric is still that of the Poincaré disk, although for arbitrary q the hidden symmetry of [11] is not preserved. Let us also introduce new variables $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ via the Ansatz:

$$\begin{aligned} \tilde{u} &= a^{\frac{1}{2p}} \cosh(q\varphi) \ , \\ \tilde{v} &= a^{\frac{1}{2p}} \sinh(q\varphi) \cos\theta \ , \\ \tilde{w} &= a^{\frac{1}{2p}} \sinh(q\varphi) \sin\theta \ . \end{aligned} \tag{30}$$

Notice that, taking $p = \frac{1}{3}$ and $q = \sqrt{\frac{3}{8}}$ inside (28)–(30), one recovers precisely the expressions (18)–(20), relevant for the exact solutions with hidden symmetry that we considered above. In addition, one can show that the equations of motion, resulting from (28)–(30), simplify significantly for:

$$q = \frac{1}{\sqrt{24}} \frac{1}{p} \quad . \tag{31}$$

So we will assume (31) from now on, as well.

In [10] it was argued that a phenomenologically preferable regime, ensuring that $\varepsilon \ll 1$ at early times, is obtained in the large-$\tilde{u}$ limit:

$$|\tilde{u}|, |\dot{\tilde{u}}| \gg |\tilde{v}|, |\tilde{w}|, |\dot{\tilde{v}}|, |\dot{\tilde{w}}| \quad . \tag{32}$$

In this regime, the equations of motion, that follow from (28)–(31), acquire the form:

$$24\, p\, \tilde{u}\, \ddot{\tilde{u}} + 24\, p\,(3p-1)\, \dot{\tilde{u}}^2 - 6\, V_0\, \tilde{u}^2 = 0 \ , \tag{33}$$

$$\ddot{\tilde{y}} + 2(3p-1)k_u \dot{\tilde{y}} - (3p-1)k_u^2 \tilde{y} = 0 \ , \tag{34}$$

where $\tilde{y} = \tilde{v}, \tilde{w}$. These equations can be solved respectively by:

$$\tilde{u}(t) \ = \ C_u\, e^{k_u t} \quad , \quad k_u = \sqrt{\frac{V_0}{12} \frac{1}{p}} \tag{35}$$

and

$$\tilde{v}(t) = C_v e^{k_v t} \quad , \quad \tilde{w}(t) = C_w e^{k_w t} \ , \tag{36}$$

where

$$\begin{aligned} k_v &= -k_u \left[(3p-1) + \sqrt{(3p-1)3p} \right] \ , \\ k_w &= -k_u \left[(3p-1) - \sqrt{(3p-1)3p} \right] \ . \end{aligned} \tag{37}$$

Note that substituting $p = 1/3$ inside (33)–(34) leads precisely to the hidden symmetry solutions in (21). Also, to have real $k_{v,w}$ in (37), we need $p \geq \frac{1}{3}$. In fact, we will take $p > 1$ from now on, to ensure a phenomenologically desirable behavior of $\eta_{\parallel}$ according to the discussion in [10].

Now, using (28)–(29) and (35)–(37), together with the inverse of (30), one can show that the turning rate $\Omega(t)$ in (15) has a single peak. Furthermore, one can compute analytically the position, height and width of the peak [10]. Similarly, one can show analytically that the corresponding effective entropic mass $m_s^2(t)$ in (16) has a transient tachyonic instability [10]. Specifically, we have that:

$$m_s^2|_{t=t_{peak}} = (m_V^2 - \Omega^2)|_{t=t_{peak}} = -3k_u^2 p\,(3p-2) \ , \tag{38}$$

whereas before and after the peak:

$$m_s^2 = m_V^2 - \Omega^2 \rightarrow 3k_u^2 p \quad \text{as} \quad t \rightarrow 0 \quad \text{or} \quad t \rightarrow \infty \quad . \tag{39}$$

Here we have used that $\varepsilon \ll 1$ in the entire range of validity of the new solutions (35)–(37), as well as that the Ricci scalar of the field-space metric is fixed to $\mathcal{R} = -\frac{1}{12p^2}$.

From the inverse of (30), one can also find the $\eta_\parallel$-parameter of the new solutions [10]; recall that $\eta_\parallel(t) = -\frac{\ddot{H}}{2H\dot{H}}$. At early times, i.e. for $t \approx 0$, the full analytical expression reduces to:

$$\eta_\parallel \approx \frac{(k_u - k_v)}{2pk_u} = \frac{3p + \sqrt{(3p-1)3p}}{2p} \quad . \tag{40}$$

Note that this is well-approximated numerically by $\eta_\parallel \approx 3$ for any $p > 2$. Thus, before the turn one has an ultra-slow roll inflationary phase. On the other hand, at late times, i.e. for large t, we have:

$$\eta_\parallel \approx \frac{(k_u - k_w)}{2pk_u} = \frac{3p - \sqrt{(3p-1)3p}}{2p} \quad , \tag{41}$$

which is well-approximated numerically by $\eta_\parallel \approx \frac{1}{4p}$ for any $p > 2$. Clearly, by choosing a suitably large value of p, we can ensure that the slow roll approximation $\eta_\parallel \ll 1$ is well satisfied after the turn. Hence, the modified solutions of this Section describe a smooth transition between an ultra-slow roll and a slow roll inflationary phases, for any p greater than 4 or so.[2]

We should note that the solutions (35)–(37) are approximate, since they were derived in the large-$\tilde{u}$ limit. However, they satisfy (32) more and more accurately with time. Hence, as discussed in [10], one can improve them by considering small corrections, at early times, of the form:

$$\begin{aligned} \tilde{v}(t) &= \left(C_v + C_v^{(1)}\, t + C_v^{(2)}\, t^2 + \cdots \right) e^{k_v t} \quad , \\ \tilde{w}(t) &= \left(C_w + C_w^{(1)}\, t + C_w^{(2)}\, t^2 + \cdots \right) e^{k_w t} \quad , \end{aligned} \tag{42}$$

where $C_{v,w}^{(1),(2),\ldots} = const$. Such subleading corrections in $\tilde{v}(t)$ and $\tilde{w}(t)$ leave the slow roll parameters $\varepsilon(t)$ and $\eta_\parallel(t)$ essentially unchanged, since the scale factor $a(t)$ is dominated by $\tilde{u}(t)$. However, the function $\dot{\theta}(t)$ is rather sensitive to the corrections in (42), implying that $\eta_\perp(t)$ is as well. Thus, as demonstrated in [10], the sharpness of the turn of a trajectory in field space can be significantly affected by the above subleading corrections. In view of this, it is easy to obtain any magnitude of $\eta_\perp(t_{peak})$, desirable for PBH-generation, by choosing suitably the values of the constants. Interestingly, the corresponding sharp turn can last several e-folds.

[2] This is similar to the numerical considerations of [14], where PBH generation was also triggered by a transition between two phases of inflation. However, in that work the transition was due to a separate potential term for each scalar, driving its own slow-roll expansion phase.

Finally, it is worth pointing out that, in the large $\tilde{u}$-regime, one is always near the center of field space. Specifically, our slow roll phase occurs for $\varphi \ll 1$. This is in stark contrast with the usual hyperbolic models in the literature, which rely on large field values (even close to the boundary of the Poincaré disk, which is at $\varphi \to \infty$) to achieve slow roll expansion. Thus our models provide a much more reliable effective description of the inflationary period.

Acknowledgements I have received partial support from the Bulgarian NSF grants DN 08/3 and KP-06-N38/11.

References

1. J. Garcia-Bellido, J. Siles, E. Morales, Bayesian analysis of the spin distribution of LIGO/Virgo black holes. Phys. Dark Univ. **31**, 100791 (2021). arXiv:2010.13811 [astro-ph.CO]
2. G. Franciolini, V. Baibhav, V. De Luca, K. Ng, K. Wong, E. Berti, P. Pani, A. Riotto, S. Vitale, *Quantifying the evidence for primordial black holes in LIGO/Virgo gravitational-wave data*. arXiv:2105.03349 [gr-qc]
3. E. Bagui, S. Clesse, *A boosted gravitational-wave background for primordial black holes with broad mass distributions and thermal features*. arXiv:2110.07487 [astro-ph.CO]
4. G. Palma, S. Sypsas, C. Zenteno, Seeding primordial black holes in multi-field inflation. Phys. Rev. Lett. **125**, 12 (2020). arXiv:2004.06106 [astro-ph.CO]
5. J. Fumagalli, S. Renaux-Petel, J. Ronayne, L. Witkowski, *Turning in the landscape: a new mechanism for generating Primordial Black Holes*. arXiv:2004.08369 [hep-th]
6. S. Garg, C. Krishnan, Bounds on slow roll and the de Sitter Swampland. JHEP **11**, 075 (2019). (arXiv:1807.05193 [hep-th])
7. H. Ooguri, E. Palti, G. Shiu, C. Vafa, Distance and de Sitter conjectures on the Swampland. Phys. Lett. B **788**, 180 (2019). arXiv:1810.05506 [hep-th]
8. A. Achucarro, G. Palma, The string swampland constraints require multi-field inflation. JCAP **02**, 041 (2019). arXiv:1807.04390 [hep-th]
9. R. Bravo, G. A. Palma, S. Riquelme, A tip for landscape riders: multi-field inflation can fulfill the Swampland distance conjecture. JCAP **02**, 004 (2020). arXiv:1906.05772 [hep-th]
10. L. Anguelova, On primordial black holes from rapid turns in two-field models. JCAP **06**, 004 (2021). arXiv:2012.03705 [hep-th]
11. L. Anguelova, E.M. Babalic, C. Lazaroiu, Two-field cosmological α-attractors with noether symmetry. JHEP **04**, 148 (2019). arXiv:1809.10563 [hep-th]
12. A. Achucarro, V. Atal, C. Germani, G.A. Palma, Cumulative effects in inflation with ultra-light entropy modes. JCAP **02**, 013 (2017). arXiv:1607.08609 [astro-ph.CO]
13. S. Cespedes, V. Atal, G.A. Palma, On the importance of heavy fields during inflation. JCAP **05**, 008 (2012). arXiv:1201.4848 [hep-th]
14. M. Braglia, D.K. Hazra, F. Finelli, G.F. Smoot, L. Sriramkumar, A. Starobinsky, Generating PBHs and small-scale GWs in two-field models of inflation. JCAP **08**, 001 (2020). arXiv:2005.02895 [astro-ph.CO]

Late Time Cosmic Acceleration with Uncorrelated Baryon Acoustic Oscillations

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Abstract The combination of Baryon Acoustic Oscillations (BAO), with other astrophysical probes can be used to compare different cosmological models with respect to the default ΛCDM model. In this proceeding, we summarize our recent publications in which we combined BAO dataset in the effective redshift range $z \in (0.1, 2.36)$ with the Cosmic Chronometers data, the Pantheon Type Ia supernova and the Hubble Diagram of Gamma Ray Bursts and Quasars. We tested the default Λ CDM, plus Ω_kCDM and wCDM. We found that the ΛCDM model is the best model and the Hubble constant without additional local universe priors is $H_0 = 69.85 \pm 1.27$ km/s/Mpc, with sound horizon distance: $r_d = 146.1 \pm 2.15$ Mpc. When one adds a local prior, the results are $H_0 = 71.40 \pm 0.8$ and $r_d = 143.5 \pm 2.0$.

Keywords Cosmological models · Hubble tension · Baryonic Acoustic Oscilations

1 Introduction

Baryon Acoustic Oscillations (BAO) are pressure waves oscillating in the post-inflationary universe, which freeze at the epoch of recombination and which can be seen in the mass distribution of large scale structures. As the physics of this primordial baryonic plasma waves is rather simple, it can be modelled, and thus the BAO provide a Standard ruler evolving with the Universe ever since recombination [1]. The BAO have been seen in qualitatively different objects, thus allowing us a new way to infer cosmic parameters. They have been measured in clustering of galaxies and quasars, from the correlation function of the Lyα absorption lines in the spectra of distant quasars, in cross correlation with quasar positions and galaxies. While their measurement require very precise astronomical surveys, thus can be vulnerable to

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_14

systematics, combined with other datasets, they can help us understand better the cosmological model describing the universe and to explain the so called "tensions" in cosmology [2–7]. In this proceeding, we review the results from two our articles on the topic [8, 9], in which we used BAO datasets, along with a combination of other datasets, to infer the parameters for different cosmological models. This shines new light on the H_0 tension—the 4σ difference between the measurements of the Hubble constant H_0 obtained from the late universe measurements [10] and the ones from the Cosmic Microwave Background (CMB) by Planck Collaboration [11]—and the connected to it r_d tension [12, 13]. This tension calls for a revision of the ΛCDM model or for looking for systematic error in the data gathering or analysis and it is on the forefront of modern cosmology.

2 Review of the Theory

As usual, for the ΛCDM model, we assume a Friedmann-Lemaître-Robertson-Walker metric with the scale parameter $a = 1/(1+z)$, where z is the redshift. The Friedmann equation then is:

$$E(z)^2 = \Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda, \tag{1}$$

where Ω_r, Ω_m, Ω_Λ and Ω_k are the fractional densities of radiation, matter, dark energy and the spatial curvature at redshift $z = 0$. The function $E(z)$ is the ratio $H(z)/H_0$, where $H(z) := \dot{a}/a$ is the Hubble parameter at redshift z and H_0 is the Hubble parameter today. The radiation density can be computed as: $\Omega_r = 1 - \Omega_m - \Omega_\Lambda - \Omega_k$. In order to examine a possibility for wCDM, one can generalize the Friedmann equation to $\Omega_\Lambda \to \Omega_{DE}^0(1+z)^{-3(1+w)}$.

Since observationally, one measures the distribution of the redshifts and the angles, they are converted to distances by adopting a fiducial cosmological model, and measuring the ratio of the observed BAO scale to that predicted in the fiducial model. While the transversal BAO do not depend on the fiducial cosmology [14], here we use different BAO measurements and thus we take this ratio as a free parameter.

In transverse direction, BAO measures $D_H(z)/r_d = c/H(z)r_d$, meaning, it always measures the combination $H \times r_d$. The comoving angular diameter distance is:

$$D_M = \frac{c}{H_0\sqrt{|\Omega_k|}} sinn\left(\sqrt{|\Omega_k|}\int_0^z \frac{dz'}{E(z')}\right), \tag{2}$$

where $sinn(x) = sinh(x), x, \sin(x)$ for $\Omega_k < 0, \Omega_k = 0, \Omega_k > 0$. Related to D_M are the angular diameter distance $D_A = D_M/(1+z)$ and the volume averaged distance, $D_V(z) = [zD_H(z)D_M^2(z)]^{1/3}$.

The sound horizon at drag epoch,r_d, is defined by:

$$r_d = \int_{z_d}^{\infty} \frac{c_s(z)}{H(z)} dz \tag{3}$$

where $c_s \approx c\left(3 + 9\rho_b/(4\rho_\gamma)\right)^{-0.5}$ is the speed of sound in the baryon-photon fluid with the baryon $\rho_b(z)$ and the photon $\rho_\gamma(z)$ densities respectively [15]. The drag epoch corresponds to the time when the baryons decouple from the photons at $z_d \approx 1060$. We see that r_d can be calculated in a simple way, if one makes some assumptions about the state of the pre-recombination Universe. In our works, we take it as a free parameter inferred from the data.

3 Methodology

The dataset we use consists of 17 points from different data releases (DR) of the Sloan Digital Sky Survey (SDSS), the WiggleZ Dark Energy Survey, the Dark Energy Survey (DES), the Dark Energy Camera Legacy Survey (DECaLS) and the 6dF Galaxy Survey (6dFGS) ([16–29]. To use these points in our analysis, we show that they are not strongly correlated by checking the effect of replacing the covariance matrix of uncorrelated points $C_{ii} = \sigma_i^2$ with one with random correlations $C_{ij} = 0.5\sigma_i\sigma_j$, where $\sigma_i\sigma_j$ are the 1σ errors of the data points i, j. This let us show that the effect of a correlations of up to 30% of the datapoints results in a less than 10% difference in the final values, thus our conclusions should be affected only in a minor way from such correlations.

We use a nested sampler as it is implemented within the open-source package *Polychord* [30] with the *GetDist* package [31] to present the results. The prior we choose is with a uniform distribution, where $\Omega_m \in [0.; 1.]$, $\Omega_\Lambda \in [0.; 1 - \Omega_m]$, $H_0 \in [50; 100]$ and $r_d \in [100; 200]$ Mpc. The measurement of the Hubble constant yielding $H_0 = 74.03 \pm 1.42$ (km/s)/Mpc at 68% CL by [10] has been incorporated into our analysis as an additional prior which we denote as **R19**. We use a wide prior on r_d to avoid as much as possible the bias it induces on H_0. With respect to the fiducial cosmology, we use as a prior for the ratio $r_d/r_{d,fid} \in [0.9, 1.1]$. For the Ω_k CDM, we use as priors $\Omega_k \in [-0.3; 0.3]$ and $\Omega_m \in [0.1; 1 - \Omega_\Lambda]$, while for the $wCDM$ we use as a prior $w \in [-1.25; -0.75]$

Since BAO are able to constrain only the combination $H \times r_d$, one needs complimentary datasets to be able to remove the degeneracy. Thus we use **Cosmic Chronometers (CC)** and **Standard Candles (SC)**. For the **Cosmic Chronometers (CC)** we include 30 uncorrelated CC measurements of $H(z)$ [32], for the **Standard Candles (SC)** we use uncorrelated measurements of the Pantheon Type Ia supernova dataset [33, 34]. To them we add, the measurements from Quasars [35] and Gamma Ray Bursts [36]. Over all, we have 273 datapoints and we vary 5 parameters: $\{H_0, \Omega_m, \Omega_\Lambda, r_d, r_d/r_{d,fid}\}$, to which we add w or Ω_k for the extended models.

Table 1 Constraints at 95% CL errors on the cosmological parameters for the ΛCDM, the wCDM model [8] and the $\Omega_k\Lambda$CDM model [8, 9]. The datasets are: the BAO + CC + SC combination and the same datapoints, with the R19 measurement as a Gaussian prior for H_0

Model	Parameters	BAO + CC + SC	BAO + CC + SC + R19
ΛCDM	H_0[km/s/Mpc]	69.85 ± 1.27	71.40 ± 0.89
	Ω_m	0.271 ± 0.016	0.267 ± 0.017
	Ω_Λ	0.722 ± 0.012	0.726 ± 0.012
	r_d [Mpc]	146.1 ± 2.2	143.5 ± 2.0
$\Omega_k\Lambda$CDM	H_0[km/s/Mpc]	70.48 ± 1.21	71.90 ± 0.87
	Ω_m	0.326 ± 0.026	0.326 ± 0.025
	Ω_Λ	0.765 ± 0.029	0.776 ± 0.024
	r_d [Mpc]	145.96 ± 2.4	143.45 ± 1.9
	Ω_k	-0.085 ± 0.042	-0.096 ± 0.038
wCDM	H_0[km/s/Mpc]	69.94 ± 1.08	71.65 ± 0.88
	Ω_m	0.269 ± 0.023	0.266 ± 0.022
	Ω_Λ	0.724 ± 0.019	0.727 ± 0.019
	r_d [Mpc]	146.4 ± 2.5	143.2 ± 1.9
	w	-0.989 ± 0.049	-0.989 ± 0.049

4 The MCMC Results

A summary of the numerical values of our results can be found in Table 1. To avoid repetitions, here we focus only on the combined datasets which also have smaller errors. The full analysis can be found in [8], where also the results for r_d/r_{fid} are reported. On Fig. 1 we show the posterior distribution for the two cases, and we see that the $H_0 - r_d$ distribution is the same with and without the additional H_0 prior. The Ω_m distributions are more spread out. This can be confirmed also from Fig. 2, where we plot the normalized Gaussians for the same parameters. As expected, for $r_d - H_0$ we have 2 distributions, consistent with the two different priors on H_0. When it comes to Ω_m, the distributions are no so clear cut and there is some ambiguity (i.e. here the difference is not entirely due to the H_0 prior), related to the unknown parameter-space. In this case, the higher value corresponds to the Ω_kCDM model, which is much closer to the Planck values for Ω_m.

ΛCDM: One sees that the BAO + CC + SC datasets without the R19 prior, lead to values closer to the ones announced by Planck [11], while with the R19 prior for H_0, the fit gives a result much closer to the observed in the late universe [10]. The matter energy density is smaller than the one reported by Planck ($\Omega_m^{Planck} = 0.315 \pm 0.007$), which has been also seen in [14] ($\Omega_m = 0.25 \pm 0.02$).

For the sound horizon, we get $r_d = 146.1 \pm 2.2$ Mpc for BAO + CC + SC and $r_d = 143.5 \pm 2.0$ Mpc for the R19 prior. This should be compared with the Planck measurements $\sim 147.1 \pm 0.3$ Mpc [11] and the completed SDSS lineage of exper-

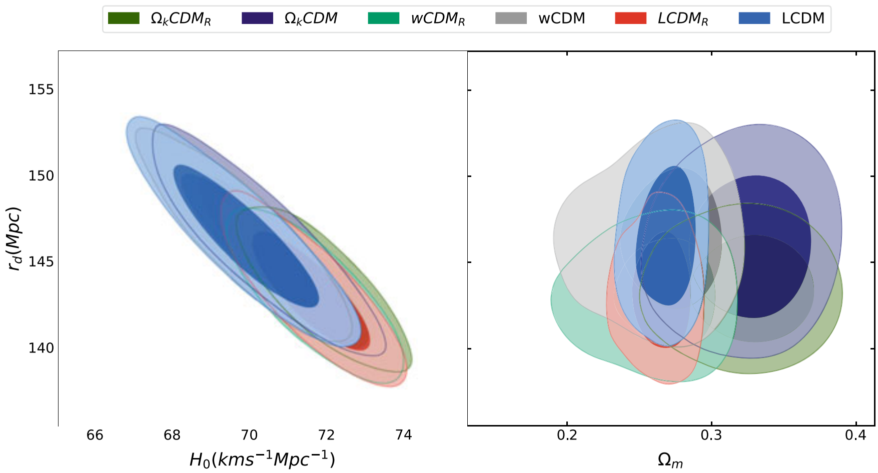

Fig. 1 The 2d posterior distribution for the parameters Ω_m, H_0 and r_d, where the index $_R$ refers to the additional R19 prior

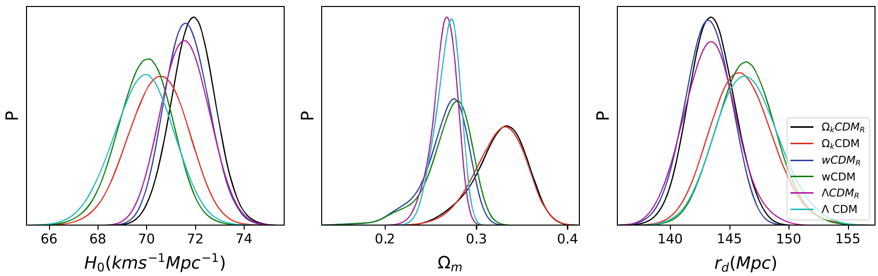

Fig. 2 The normalized Gaussians for the parameters Ω_m, H_0 and r_d, again the index $_R$ refers to the R19 prior

iments in large-scale structures measurement 149.3 ± 2.8 Mpc [37]. Using BAO, SNea, ages of early type galaxies and local determinations of the Hubble constant, [38] reports $\sim$143.9 $\pm$ 3.1 Mpc. [14] gets $\sim$ 144.1 $\pm$ 5.1 Mpc from r_d/D_M + BBN + H0LiCOW and 150.4 $\pm$ 3 Mpc from r_d/D_M + BBN + CC. For a visual comparison of different known measurements, see Fig. 1 in [9]. The discrepancy between early and late universe results has led [12] to consider it as a "tension in the $r_d - H_0$ plane" or even as a tension in the $r_d - H_0 - \Omega_m$ plane, since the value of r_d depend strongly on Ω_m. Explicitly, increasing Ω_m leads to a smaller r_d, which also affects the value of H_0, as we also have observed.

$\Omega_k \Lambda CDM$: For all the 3 samples we get a negative spatial curvature energy density ($\Omega_k < 0$) which corresponds to $k = 1$, i.e. a closed universe. This is in line with previous results obtained by the Planck 2018 collaboration [11] for CMB alone which found a preference for a closed universe at 3.4σ and also with those obtained by [39] which includes the data from CC, Pantheon and BAO measurements to conclude also negative Ω_k for relieving the H_0 tension. The numbers reported in [8, 9] exclude a flat universe and one can see a small alleviation of the H_0 tension in this case.

The wCDM model: The dark energy equation of state we obtain differs from the one obtained by the Planck collaboration 2018 [11]) which gives $w = -1.03 \pm 0.03$, i.e. it is essentially consistent with a cosmological constant. In our case, it is much closer to the analysis done in [3, 40]). The full dataset does not exclude $w = -1$.

5 Conclusion

We review the results presented in [8, 9] which include the use of uncorrelated BAO points, plus the CC and the SC datasets. We see that the addition of new datasets enables us to significantly constrain the inferred cosmological parameters. One thing that is clear from our considerations in [8], elaborated in [9] is that the connection between r_d and H_0 and Ω_m plays a significant role in understanding the H_0 tension and that the H_0 tension cannot be solved without taking into account the other two parameters. In [8] we use the reduced chi-square statistic and the Akaike information criteria to find that ΛCDM is the best model under both criteria. However, including the spatial curvature in the model, allows us to somewhat decrease the H_0 tension on the cost of having a closed Universe. Due to the higher number of parameters, this model is not statistically preferred by the measures we performed, but it raises the question about a possible non-zero curvature density of the universe. The numbers obtained by our analysis are closer to the Tip of the Red Branch measurements for H_0, showing that the BAO dataset combined with other probes can lead to consistent and interesting results and it is a powerful tool in cosmology.

Acknowledgements D.S. is thankful to Bulgarian National Science Fund for support via research grant KP-06-PN58/1.

References

1. A. Cuceu, J. Farr, P. Lemos, A. Font-Ribera, JCAP **10**, 044. arXiv:1906.11628 [astro-ph.CO]
2. W. Handley, (2019). arXiv:1908.09139 [astro-ph.CO]
3. E. Di Valentino, A. Melchiorri, J. Silk, (2020). arXiv:2003.04935 [astro-ph.CO]
4. X. Luo, Z. Huang, Q. Qian, L. Huang, (2020). arXiv:2008.00487 [astro-ph.CO]
5. D. Benisty, Phys. Dark Univ. **31**, 100766 (2021). arXiv:2005.03751 [astro-ph.CO]
6. R.C. Bernardo, J. Levi Said, JCAP **08**, 027. arXiv:2106.08688 [astro-ph.CO]
7. L.A. Anchordoqui, E. Di Valentino, S. Pan, W. Yang, JHEAp **32**, 121 (2021). (arXiv:2107.13932 [astro-ph.CO])
8. D. Benisty, D. Staicova, Astron. Astrophys. **647**, A38 (2021). (arXiv:2009.10701 [astro-ph.CO])
9. D. Staicova, in *16th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics and Relativistic Field Theories* (2021). arXiv:2111.07907 [astro-ph.CO]

10. A.G. Riess, S. Casertano, W. Yuan, L.M. Macri, D. Scolnic, Astrophys. J. **876**, 85 (2019). (arXiv:1903.07603 [astro-ph.CO])
11. N. Aghanim et al., Planck. Astron. Astrophys. **641**, A6 (2020). (arXiv:1807.06209 [astro-ph.CO])
12. L. Knox, M. Millea, Phys. Rev. D **101** (2020). https://doi.org/10.1103/physrevd.101.043533
13. G. Efstathiou, (2020). arXiv:2007.10716 [astro-ph.CO]
14. R.C. Nunes, A. Bernui, (2020). arXiv:2008.03259 [astro-ph.CO]
15. E. Aubourg et al., Phys. Rev. D **92**, 123516 (2015). arXiv:1411.1074 [astro-ph.CO]
16. F. Beutler, C. Blake, M. Colless, D.H. Jones, L. Staveley-Smith, L. Campbell, Q. Parker, W. Saunders, F. Watson, Mon. Not. Roy. Astron. Soc. **416**, 3017 (2011). (arXiv:1106.3366 [astro-ph.CO])
17. A.J. Ross, L. Samushia, C. Howlett, W.J. Percival, A. Burden, M. Manera, Mon. Not. Roy. Astron. Soc. **449**, 835 (2015). (arXiv:1409.3242 [astro-ph.CO])
18. W.J. Percival et al., SDSS. Mon. Not. Roy. Astron. Soc. **401**, 2148 (2010). (arXiv:0907.1660 [astro-ph.CO])
19. R. Tojeiro et al., Mon. Not. Roy. Astron. Soc. **440**, 2222 (2014). (arXiv:1401.1768 [astro-ph.CO])
20. C. Blake et al., Mon. Not. Roy. Astron. Soc. **425**, 405 (2012). (arXiv:1204.3674 [astro-ph.CO])
21. H.-J. Seo et al., Astrophys. J. **761**, 13 (2012). (arXiv:1201.2172 [astro-ph.CO])
22. L. Anderson et al., Mon. Not. Roy. Astron. Soc. **427**, 3435 (2013). (arXiv:1203.6594 [astro-ph.CO])
23. S. Sridhar, Y.-S. Song, A.J. Ross, R. Zhou, J.A. Newman, C.-H. Chuang, F. Prada, R. Blum, E. Gaztañaga, M. Landriau, (2020). arXiv:2005.13126 [astro-ph.CO]
24. J.E. Bautista et al., Astrophys. J. **863**, 110 (2018). (arXiv:1712.08064 [astro-ph.CO])
25. T.M.C. Abbott et al., DES. Mon. Not. Roy. Astron. Soc. **483**, 4866 (2019). (arXiv:1712.06209 [astro-ph.CO])
26. J. Hou et al., (2020). arXiv:2007.08998 [astro-ph.CO]
27. M. Ata et al., Mon. Not. Roy. Astron. Soc. **473**, 4773 (2018). (arXiv:1705.06373 [astro-ph.CO])
28. N.G. Busca et al., Astron. Astrophys. A **552**, 96 (2013). (arXiv:1211.2616 [astro-ph.CO])
29. V. de Sainte Agathe et al., Astron. Astrophys. A **629**, 85 (2019). arXiv:1904.03400 [astro-ph.CO]
30. W.J. Handley, M.P. Hobson, A.N. Lasenby, Mon. Not. Roy. Astron. Soc. L **450**, 61 (2015). (arXiv:1502.01856 [astro-ph.CO])
31. A. Lewis, (2019). arXiv:1910.13970 [astro-ph.IM]
32. M. Moresco, L. Pozzetti, A. Cimatti, R. Jimenez, C. Maraston, L. Verde, D. Thomas, A. Citro, R. Tojeiro, D. Wilkinson, JCAP **1605**, 014. arXiv:1601.01701 [astro-ph.CO]
33. D. Scolnic et al., Astrophys. J. **859**, 101 (2018). (arXiv:1710.00845 [astro-ph.CO])
34. F.K. Anagnostopoulos, S. Basilakos, E.N. Saridakis, Eur. Phys. J. C **80**, 826 (2020). (arXiv:2005.10302 [gr-qc])
35. C. Roberts, K. Horne, A.O. Hodson, A.D. Leggat, (2017). arXiv:1711.10369 [astro-ph.CO]
36. M. Demianski, E. Piedipalumbo, D. Sawant, L. Amati, Astron. Astrophys. A **598**, 112 (2017). (arXiv:1610.00854 [astro-ph.CO])
37. S. Alam et al. (eBOSS) (2020). arXiv:2007.08991 [astro-ph.CO]
38. L. Verde, J.L. Bernal, A.F. Heavens, R. Jimenez, Mon. Not. Roy. Astron. Soc. **467**, 731 (2017). (arXiv:1607.05297 [astro-ph.CO])
39. E.-K. Li, M. Du, L. Xu, Mon. Not. Roy. Astron. Soc. **491**, 4960 (2020). (arXiv:1903.11433 [astro-ph.CO])
40. S. Vagnozzi, A. Loeb, M. Moresco, (2020). arXiv:2011.11645 [astro-ph.CO]

On the Hidden Symmetries of $D = 11$ Supergravity

Lucrezia Ravera

Abstract We report on recent developments regarding the supersymmetric Free Differential Algebra describing the vacuum structure of $D = 11$ supergravity. We focus on the emergence of a hidden superalgebra underlying the theory, explaining the group-theoretical role played by the nilpotent fermionic generator naturally appearing for consistency of the construction. We also discuss the relation between this hidden superalgebra and other superalgebras of particular relevance in the context of supergravity and superstring, involving a fermionic generator with 32 components.

1 Introduction

In eleven spacetime dimensions an (almost) central extension of the supersymmetry algebra was introduced in the literature and named M-algebra [13, 19, 20, 27, 32]. Such super Lie algebra includes, besides the super-Poincaré structure, the anticommutator

$$\{Q, Q\} = -\mathrm{i}\left(C\Gamma^{a}\right)P_a - \frac{1}{2}\left(C\Gamma^{ab}\right)Z_{ab} - \frac{\mathrm{i}}{5!}\left(C\Gamma^{a_1\cdots a_5}\right)Z_{a_1\cdots a_5}\,, \tag{1}$$

where Z_{ab} and $Z_{a_1\cdots a_5}$ are Lorentz-valued almost central charges (they commute with all generators except the Lorentz one). The M-algebra is commonly considered as the Lie superalgebra underlying M-theory [16, 26, 31] (see also [17, 18, 24]) in its low-energy limit, which corresponds to $D = 11$ supergravity in the presence of non-trivial M2- or M5-brane sources [1, 5, 6, 15, 29, 30]. However, a field theory based on the M-algebra (1) is naturally defined on a superspace that is enlarged with respect to the ordinary one. Indeed, let us recall that ordinary superspace is spanned by the supervielbein $\{V^a, \Psi\}$, where V^a is the bosonic vielbein and Ψ the gravitino 1-form,

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_15

while the base space induced by the M-algebra includes also the bosonic 1-form fields B^{ab} and $B^{a_1 \cdots a_5}$, respectively dual to the generators Z_{ab} and $Z_{a_1 \cdots a_5}$. On the other hand, the low-energy limit of the M-theory, corresponding to $D = 11$ supergravity, should be based on ordinary superspace. Under this perspective, the M-algebra cannot be the final answer, as it is not sufficient to reproduce the Free Differential Algebra [28] (FDA in the following) on which $D = 11$ supergravity is based. Here let us recall that $D = 11$ supergravity [9] contains, besides the super-Poincaré fields given by the Lorentz spin connection ω^{ab} and the supervielbein $\{V^a, \Psi\}$, with $a, b, \ldots = 0, 1, \ldots, 10$, also a 3-form $A^{(3)}$, satisfying, in the superspace vacuum,

$$d A^{(3)} - \frac{1}{2} \bar{\Psi} \wedge \Gamma_{ab} \Psi \wedge V^a \wedge V^b = 0 \,, \tag{2}$$

whose closure relies on fermion 1-forms Fierz identities in superspace. As such, this theory is not based on a superalgebra, but instead on a FDA on the superspace spanned by the supervielbein.

A super Lie algebra of 1-forms leaving invariant $D = 11$ supergravity and reproducing the FDA on ordinary superspace was introduced in 1981 by D'Auria and Fré in [10] and later named D'Auria-Fré algebra (DF-algebra in the following).[1] Such Lie superalgebra, from which the M-algebra emerges as a subalgebra, includes, besides the Lorentz (J_{ab}), spacetime translations (P_a) and supersymmetry (Q_α, $\alpha = 1, \ldots, 32$) generators, and the bosonic charges Z_{ab}, $Z_{a_1 \cdots a_5}$, also a nilpotent fermionic charge Q' ($\{Q', Q'\} = 0$), such that

$$[P_a, Q] \propto \Gamma_a Q' \,, \quad [Z_{ab}, Q] \propto \Gamma_{ab} Q' \,, \quad [Z_{a_1 \cdots a_5}, Q] \propto \Gamma_{a_1 \cdots a_5} Q' \,. \tag{3}$$

The presence of the extra fermionic nilpotent charge Q' in the D'Auria-Fré construction is naturally required by supersymmetry and consistency of the theory. Let us stress that, actually, this fact is not a peculiarity of $D = 11$ supergravity, but is fully general: a hidden superalgebra underlying the supersymmetric FDA containing at least one nilpotent fermionic generator can be constructed for any supergravity theory involving antisymmetric tensor fields [2].

It was hence proven that the hidden superalgebra underlying the FDA of $D = 11$ supergravity is the DF-algebra, in the sense that it is equivalent to the FDA description of the $D = 11$ theory on ordinary superspace (and therefore to the Cremmer-Julia-Scherk theory [9]). The DF-algebra is the invariance algebra of the $D = 11$ supergravity vacuum. On the other hand, under the out-of-vacuum perspective, the DF-algebra is a local invariance of the theory (and $D = 11$ supergravity is, in fact, the local theory of the supergroup associated with the DF-algebra). The bosonic generators Z_{ab} and $Z_{a_1 \cdots a_5}$ were later understood as p-brane charges, sources of the dual potentials $A^{(3)}$ and $B^{(6)}$ appearing in the theory, and eq. (1) was interpreted as the natural generalization of the supersymmetry algebra in higher dimensions, in the

[1] The DF-algebra has recently raised a certain interest also in the Mathematical-Physicists community, due to the fact that it can be reformulated in terms of $\mathcal{L}_n \subset \mathcal{L}_\infty$ algebras, or "strong homotopy Lie algebras", see, e.g., [25].

presence of non-trivial topological extended sources (black p-branes). On the other hand, a clearer understanding of the group-theoretical and physical meaning of the (necessary) nilpotent fermionic generator Q' has been provided only rather recently, in [2]. Besides, some other issues remained open, such as: Is the DF-algebra the fully extended superalgebra underlying $D = 11$ supergravity? And would a non-Abelian charge deformation of the DF-algebra be possible? Which is the relation between the DF-algebra and the most general simple superalgebra involving a fermionic generator with 32 components, namely $\mathfrak{osp}(1|32)$? An answer to these questions was formulated in the works [2, 3], as we are going to review in the following. Before proceeding in this direction, let us briefly recap some key aspects of the geometric approach to supergravity in superspace adopted [7] and the FDA description of $D = 11$ supergravity.

2 Lie Superalgebras and Maurer-Cartan Equations

In the geometric approach to supergravity in superspace [7] the dual formulation of Lie superalgebras in terms of the associated Maurer-Cartan (MC) equations is adopted: Given a Lie superalgebra

$$[T_A, T_B\} = C_{AB}{}^C T_C \,, \tag{4}$$

where T_A are the generators in the adjoint representation of the corresponding Lie supergroup, one can introduce an equivalent description in terms of the differential 1-forms σ^A dual to the Lie superalgebra generators, $\sigma^A(T_B) = \delta^A{}_B$, obeying the MC equations

$$R^A \equiv d\sigma^A + \frac{1}{2} C_{BC}{}^A \sigma^B \wedge \sigma^C = 0 \,. \tag{5}$$

In order to describe non-trivial physical configurations, a non-vanishing right-hand side has to be switched on in (5), which corresponds to defining the supercurvatures R^A (super field-strengths). The latter are the building blocks of supergravity in the geometric approach. The MC equations $R^A = 0$ can be identified with the vacuum configuration of a supergravity theory, and their d^2-closure is equivalent to the Jacobi identities of the dual algebraic structure. Moreover, as we are dealing with the geometric formulation in superspace, let us stress that the latter is spanned by the supervielbein $\{V^a, \Psi\}$, the 1-form fields V^a and Ψ being respectively dual to the generators P_a and Q. For a detailed review of the geometric approach to supergravity in superspace we refer the reader to [12].

3 $D = 11$ Supergravity and Its Free Differential Algebra

Supergravity theories in $4 \leq D \leq 11$ spacetime dimensions have a bosonic field content that generically includes, besides the metric and a set of 1-form gauge potentials, also p-index antisymmetric tensors, and they are therefore appropriately discussed in the FDAs framework. Indeed, FDAs extend the MC equations by incorporating p-form gauge potentials. The concept of FDA was introduced by Sullivan in [28]. Subsequently, the FDA framework was applied to the study of supergravity theories by R. D'Auria and P. Fré in particular in [10], where the FDA was referred to as Cartan Integrable System (CIS), since the authors were unaware of the previous work by Sullivan [28]. Actually, FDA and CIS are equivalent concepts [11]. The latter is also known as the Chevalley-Eilenberg Lie algebras cohomology framework in supergravity (CE-cohomolgy in the following).

Let us schematically review the steps for constructing a FDA: Given a set of MC 1-forms $\{\sigma^A\}$, we can build up n-form cochains (Chevalley cochains),

$$\Omega^{(n)} = \Omega_{A_1 \cdots A_n} \sigma^{A_1} \wedge \cdots \wedge \sigma^{A_n} \,. \tag{6}$$

If a cochain is closed ($d\Omega^{(n)} = 0$) it is called cocycle, and if it is also exact it is called a coboundary. In particular, we are interested in those cocycles that are not coboundaries, which are elements of the CE-cohomology (while if the closed cocycles are also coboundaries, namely exact cochains, the cohomology class is trivial). Then, if there exists a p such that $d\Omega^{(p+1)} = 0$, i.e., a cocycle, we can introduce a p-form (gauge potential) $A^{(p)}$ such that

$$F^{(p+1)} \equiv dA^{(p)} + \Omega^{(p+1)} = 0 \,. \tag{7}$$

Consequently, we can consider $\left(\{\sigma^A\}, A^{(p)}\right)$ as new a basis of MC forms and look for new cocycles, iteratively, constructing the complete FDA.

We now turn to the FDA description of the (vacuum structure of) $D = 11$ supergravity. The theory, which in particular involves a 3-index antisymmetric tensor $A_{\mu\nu\rho}$ ($\mu, \nu, \rho, \ldots = 0, 1, \ldots, 10$), was originally built in 1978 [9] and subsequently reformulated geometrically by R. D'Auria and P. Fré in [10] in terms of a supersymmetric FDA on superspace. The latter reads as follows:

$$\begin{aligned}
\mathcal{R}^{ab} &= d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b = 0 \,, \\
R^a &\equiv \mathcal{D}V^a - \frac{\mathrm{i}}{2}\bar{\Psi} \wedge \Gamma^a \Psi = 0 \,, \\
\rho &\equiv \mathcal{D}\Psi = 0 \,, \\
F^{(4)} &\equiv dA^{(3)} - \frac{1}{2}\bar{\Psi} \wedge \Gamma_{ab}\Psi \wedge V^a \wedge V^b = 0 \,,
\end{aligned} \tag{8}$$

with $A^{(3)} = A_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho$ and where $\mathcal{D} = d - \omega$ denotes the Lorentz covariant derivative. The right-hand side of (8) defines the vacuum of the theory.

The d^2-closure of the last equation in (8) relies on the Fierz identity

$$\bar{\Psi}\Gamma_{ab}\Psi \wedge \bar{\Psi} \wedge \Gamma^a \Psi = 0\,. \tag{9}$$

Furthermore, due to another Fierz identity, namely

$$\Gamma_{[a_1a_2}\Psi \wedge \bar{\Psi} \wedge \Gamma_{a_3a_4]}\Psi + \frac{1}{3}\Gamma_{a_1\cdots a_5}\Psi \wedge \bar{\psi} \wedge \Gamma^{a_5}\Psi = 0\,, \tag{10}$$

the supersymmetric FDA also allows to include in the description

$$F^{(7)} \equiv dB^{(6)} - 15\,A^{(3)} \wedge dA^{(3)} - \frac{\mathrm{i}}{2}\bar{\Psi} \wedge \Gamma_{a_1\cdots a_5}\Psi \wedge V^{a_1} \wedge \cdots \wedge V^{a_5} = 0\,, \tag{11}$$

$F^{(7)}$ being Hodge-dual to $F^{(4)}$ on spacetime. The complete FDA is therefore defined in terms of $\left(V^a, \Psi, A^{(3)}, B^{(6)}\right)$, and it is invariant under the p-form gauge transformations

$$\delta A^{(3)} = d\Lambda^{(2)}\,, \quad \delta B^{(6)} = d\Lambda^{(5)} + 15\Lambda^{(2)} \wedge dA^{(3)}\,, \tag{12}$$

with p-form gauge parameters $\Lambda^{(2)}$ and $\Lambda^{(5)}$.

4 Hidden Superalgebra Underlying $D = 11$ Supergravity

The investigation presented in [10] proved that the FDA reported in the previous section can be traded for an ordinary Lie superalgebra. The D'Auria-Fré recipe consists of the following steps:

1. Associate to $A^{(3)}$ and $B^{(6)}$ the 1-form fields $B^{ab} = B^{[ab]}$ and $B^{a_1\cdots a_5} = B^{[a_1\cdots a_5]}$, respectively;
2. Take as basis of MC 1-forms $\sigma^A \equiv \{V^a, \Psi, \omega^{ab}, B^{ab}, B^{a_1\cdots a_5}\}$, which implies the extra MC equations

$$\mathcal{D}B^{ab} = \frac{1}{2}\bar{\Psi} \wedge \Gamma^{ab}\Psi\,, \quad \mathcal{D}B^{a_1\cdots a_5} = \frac{\mathrm{i}}{2}\bar{\psi} \wedge \Gamma^{a_1\cdots a_5}\Psi\,; \tag{13}$$

3. Assume $A^{(3)}$ to be written in terms of the 1-forms σ^A above, that is $A^{(3)} = A^{(3)}(\sigma)$, with all possible combinations,

$$\begin{aligned} A^{(3)}(\sigma) = {} & T_0 B_{ab} \wedge V^a \wedge V^b + T_1 B_{ab} \wedge B^b{}_c \wedge B^{ca} \\ & + T_2 B_{b_1a_1\cdots a_4} \wedge B^{b_1}{}_{b_2} \wedge B^{b_2a_1\cdots a_4} \\ & + T_3 \epsilon_{a_1\cdots a_5b_1\cdots b_5m} B^{a_1\cdots a_5} \wedge B^{b_1\cdots b_5} \wedge V^m \\ & + T_4 \epsilon_{m_1\cdots m_6n_1\cdots n_5} B^{m_1m_2m_3p_1p_2} \wedge B^{m_4m_5m_6}{}_{p_1p_2} \wedge B^{n_1\cdots n_5}\,; \end{aligned} \tag{14}$$

where T_0, T_1, T_2, T_3, T_4 are constant parameters;

4. Require

$$dA^{(3)}(\sigma) = \frac{1}{2}\bar{\Psi} \wedge \Gamma_{ab}\Psi \wedge V^a \wedge V^b\,, \tag{15}$$

namely that the vacuum FDA structure on ordinary superspace, spanned by the supervielbein $\{V^a, \Psi\}$, is reproduced once considering $A^{(3)} = A^{(3)}(\sigma)$.

Then, expressing the FDA with $A^{(3)} = A^{(3)}(\sigma)$ determines the latter expression. However, as shown in [10], this requires to include in the parametrization of $A^{(3)}$ in terms of 1-forms a spinor 1-form field η, such that

$$\mathcal{D}\eta = \mathrm{i}E_1\Gamma_a\Psi \wedge V^a + E_2\Gamma_{ab}\Psi \wedge B^{ab} + \mathrm{i}E_3\Gamma_{a_1\cdots a_5}\Psi \wedge B^{a_1\cdots a_5}\,, \tag{16}$$

whose d^2-closure requires $E_1 + 10E_2 - 5!E_3 = 0$, enlarging in this way the basis of MC 1-forms to $\sigma^A \equiv \{V^a, \Psi, \omega^{ab}, B^{ab}, B^{a_1\cdots a_5}, \eta\}$. We are therefore led to consider

$$\begin{aligned} A^{(3)}(\sigma) = {} & T_0 B_{ab} \wedge V^a \wedge V^b + T_1 B_{ab} \wedge B^b{}_c \wedge B^{ca} \\ & + T_2 B_{b_1 a_1\cdots a_4} \wedge B^{b_1}{}_{b_2} \wedge B^{b_2 a_1\cdots a_4} \\ & + T_3 \epsilon_{a_1\cdots a_5 b_1\cdots b_5 m} B^{a_1\cdots a_5} \wedge B^{b_1\cdots b_5} \wedge V^m \\ & + T_4 \epsilon_{m_1\cdots m_6 n_1\cdots n_5} B^{m_1 m_2 m_3 p_1 p_2} \wedge B^{m_4 m_5 m_6}{}_{p_1 p_2} \wedge B^{n_1\cdots n_5} \\ & + \mathrm{i}S_1 \bar{\Psi} \wedge \Gamma_a \eta \wedge V^a + S_2 \bar{\Psi} \wedge \Gamma_{ab}\eta \wedge B^{ab} \\ & + \mathrm{i}S_3 \bar{\Psi} \wedge \Gamma_{a_1\cdots a_5}\eta \wedge B^{a_1\cdots a_5}\,, \end{aligned} \tag{17}$$

and the requirement (15) fixes the coefficients E_i, T_j, S_k in terms of a single free parameter [4]. The dual set of generators spanning the hidden superalgebra underlying $D = 11$ supergravity (i.e., the DF-algebra) is $T_A \equiv \{P_a, Q, J_{ab}, Z_{ab}, Z_{a_1\cdots a_5}, Q'\}$. The DF-algebra includes, besides the super-Poincaré structure, the anticommutation relations (1) and $\{Q', Q'\} = 0$, and the commutators (3). As we have anticipated before, the generators Z_{ab} and $Z_{a_1\cdots a_5}$ were later understood as M-brane charges, sources of $A^{(3)}$ and $B^{(6)}$, respectively, while the role played by the necessary nilpotent fermionic charge Q' was clarified in [2] (and [3]). We report on this in the following.

4.1 Role of the Nilpotent Fermionic Generator Q'

The DF-algebra is a "spinorial central extension" of the M-algebra including Q'. In [2] it was shown that the inclusion of the spinor 1-form η (dual to Q'), whose presence in naturally required by supersymmetry in the D'Auria-Fré construction, allows to realize the M-algebra as a (hidden) symmetry of $D = 11$ supergravity. In particular, η allows a fiber bundle structure $\mathcal{G} \to$ (superspace) on the supergroup-manifold $\mathcal{G}$ generated by the M-algebra, intertwining between basis and fiber. The

p-form gauge transformations leaving invariant the supersymmetric FDA trivialized in terms of 1-forms result to be realized as diffeomorphisms in the fiber direction of the supergroup-manifold $\mathcal{G}$.

To better understand these points, let us recall here that, as the generators of the hidden Lie superalgebra span the tangent space of a supergroup-manifold, then, in the geometric approach we are adopting, the fields are naturally defined in an enlarged manifold corresponding to the supergroup-manifold, where all the invariances of the FDA are diffeomorphisms, generated by Lie derivatives. The spinor 1-form η allows the diffeomorphisms in the directions spanned by the almost central charges to be particular gauge transformations, so that one obtains the ordinary superspace as the quotient of the supergroup over the fiber subgroup of gauge transformations. More precisely, if $\eta \neq 0$, the hidden superalgebra, let us call it $\mathbb{G}$, generates a supergroup-manifold $\mathcal{G}$ with a principal fiber bundle structure $\mathcal{G} \to K$, where the base space K is superspace, spanned by $\{V^a, \Psi\} \in \mathbb{K}$, and we have

$$dA^{(3)}(\sigma) = \frac{1}{2}\bar{\Psi} \wedge \Gamma_{ab}\Psi \wedge V^a \wedge V^b \in \mathbb{K} \times \cdots \times \mathbb{K}\,, \tag{18}$$

while the fiber is generated by $\mathbb{H} = H_0 + \mathcal{H}$, where

$$\{\omega^{ab}\} \in H_0\,, \quad \{B^{ab}, B^{a_1\cdots a_5}\} \in \mathcal{H}\,. \tag{19}$$

The spinor 1-form η behaves like a cohomological "ghost" field, in the sense that it allows to realize in a "dynamical" way the gauge invariance of $A^{(3)}$, guaranteeing that only the physical degrees of freedom appear in the FDA (namely that the FDA on ordinary superspace is reproduced). On the other hand, a singular limit $\eta \to 0$ exits, where a trivialization $A^{(3)}_{\mathrm{lim}}(\sigma)$ can still be defined with the same $\mathcal{G}$ but $dA^{(3)}_{\mathrm{lim}}(\sigma) \in \mathbb{G} \times \cdots \times \mathbb{G}$, namely the FDA with $\eta \to 0$ lives in an enlarged superspace. In other words, in the singular limit $\eta \to 0$ the supersymmetric FDA parametrized in terms of 1-forms becomes ill defined: indeed, the exterior form $A^{(3)}$ is a gauge field, since it includes "longitudinal" unphysical directions corresponding to the gauge freedom $A^{(3)} \to A^{(3)} + d\Lambda^{(2)}$. In the limit $\eta \to 0$, the unphysical degrees of freedom $\Lambda^{(2)}$ get mixed with the physical directions of superspace, and all the generators of the hidden superalgebra act as generators of external diffeomorphisms. On the contrary, when $\eta \neq 0$ the hidden supergroup acquires a principal fiber bundle structure; η allows to separate the physical directions of superspace, generated by the supervielbein $\{V^a, \Psi\}$, from the other directions, belonging to the fiber of superspace, in such a way as to recover the gauge invariance of the FDA. This amounts to say that, once the superspace is enlarged, in the presence of η no explicit constraint has to be imposed on the fields, since the non-physical degrees of freedom transform into each other and do not contribute to the FDA.

Let us now discuss gauge invariance of the FDA in more detail. For $A^{(3)}(\sigma)$ the p-form gauge transformations of the FDA are realized through gauge transformations in $\mathcal{H}$,

$$\begin{cases} \delta B^{ab} = \mathcal{D}\Lambda^{ab} \,, \\ \delta B^{a_1\cdots a_5} = \mathcal{D}\Lambda^{a_1\cdots a_5} \end{cases} \quad \Rightarrow \quad \begin{cases} \delta A^{(3)} = d\Lambda^{(2)} \,, \\ \delta B^{(6)} = d\Lambda^{(5)} + 15\Lambda^{(2)} \wedge dA^{(3)} \,. \end{cases} \tag{20}$$

The gauge invariance of the FDA trivialized in terms of 1-forms requires

$$\delta_{\text{gauge}}\eta = -E_2\Lambda^{ab}\Gamma_{ab}\Psi - \mathrm{i}E_3\Lambda^{a_1\cdots a_5}\Gamma_{a_1\ldots a_5}\Psi \,. \tag{21}$$

Hence, considering the tangent vector

$$\overrightarrow{z} \equiv \Lambda^{ab}Z_{ab} + \Lambda^{a_1\cdots a_5}Z_{a_1\cdots a_5} \tag{22}$$

in $\mathcal{H} \in \mathbb{G}$, we find that there exists a $\bar{\Lambda}^{(2)} = \Lambda^{(2)}(\Lambda^{ab}, \Lambda^{a_1\cdots a_5}; \sigma) = \imath_{\overrightarrow{z}}\left(A^{(3)}(\sigma)\right)$, where $\imath$ denotes the contraction operator, such that

$$\delta_{\bar{\Lambda}}\left(A^{(3)}(\sigma)\right) = d\bar{\Lambda}^{(2)} = \ell_{\overrightarrow{z}}\left(A^{(3)}(\sigma)\right) \,, \tag{23}$$

where $\ell_{\overrightarrow{z}} = d\imath_{\overrightarrow{z}} + \imath_{\overrightarrow{z}}d$ is the Lie derivative in the direction $\overrightarrow{z}$ and where we have also used the fact that $\imath_{\overrightarrow{z}}\left(dA^{(3)}\right) = 0$. Therefore, for $\eta \neq 0$, $\delta A^{(3)}$ is genuinely realized as a diffeomorphism in the fiber direction of the supergroup-manifold $\mathcal{G}$. Let us finally mention that, as shown in [2], assuming $\bar{\Lambda}^{(5)} = \imath_{\overrightarrow{z}}\left(B^{(6)}(\sigma)\right)$ one can prove that $\delta_{\bar{\Lambda}}B^{(6)} = \ell_{\overrightarrow{z}}\left(B^{(6)}(\sigma)\right)$, whatever $B^{(6)}(\sigma)$ may be. This is particularly relevant since, even though we do not know the explicit parametrization of $B^{(6)}$ in terms of 1-form fields, at least we can say that, analogously to what happens for $A^{(3)}$, $\delta B^{(6)}$ is properly realized as a diffeomorphism in the fiber direction of $\mathcal{G}$. Remarkably, as the structure above relies on supersymmetry (and, in particular, on Fierz identities), the extension of this analysis to lower dimensions might turn out to be a useful tool in generalized geometry frameworks, such as Exceptional Field Theory (see, e.g., [21–23] and references therein), offering a dynamical way to implement the so-called section constraints.

One might ask at this point whether the DF-algebra is the fully extended superalgebra underlying $D = 11$ supergravity. Some clues to answer this question are provided to us by minimal $D = 7$ supergravity, in which case the full on-shell hidden symmetry involves two nilpotent fermionic charges, associated with the presence of two mutually dual p-forms [2]. It could therefore be conjectured that there are different spinors associated with the mutually dual p-forms even in $D = 11$, and that new extra 1-forms may be necessary to write the parameterization $B^{(6)}(\sigma)$ in such a way to reproduce the complete FDA on ordinary superspace. Under this perspective, it would be particularly useful to calculate explicitly $B^{(6)}(\sigma)$.

5 Relation Between the DF-algebra and $\mathfrak{osp}(1|32)$

To come to more realistic cases, it would be important to be able to switch on non-Abelian charges in the setup reviewed in the previous sections. Such issue could be analyzed either as in Exceptional Field Theory, by Scherk-Schwarz dimensional reduction to lower dimensions, or directly in $D = 11$. However, it is well-known that in $D = 11$ the massive theory is problematic. Nevertheless, let us have a look closer, reporting the results obtained in [3] pointing in the direction of a clearer understanding of the problem. In fact, as we have previously mentioned, the DF-algebra and the parametrization $A^{(3)}(\sigma)$ depend on a free parameter. In [3] it was shown that this dependence can be associated with an intriguing relation with $\mathfrak{osp}(1|32)$, which is the most general simple superalgebra involving a fermionic generator with 32 components and a scale parameter e. The latter has length dimension -1 and can be thought as proportional to (the square root of) a cosmological constant.

The 1-form fields appearing in the dual formulation of $\mathfrak{osp}(1|32)$ are $\{V^a, \Psi, \omega^{ab}, B^{a_1\cdots a_5}\}$. To make contact with the DF-algebra, it is first of all necessary to include a further bosonic 1-form field B^{ab}, and this was done in [8] by considering a "torsion deformation" of $\mathfrak{osp}(1|32)$, namely taking

$$\omega^{ab} \to \omega^{ab} - eB^{ab}\,, \quad \mathcal{R}^{ab} \to \mathcal{R}^{ab} - e\mathcal{D}B^{ab} + e^2 B^{ac} \wedge B_c{}^b \tag{24}$$

and then requiring a Minkowski background, $\mathcal{R}^{ab} \equiv d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b = 0$. Moreover, in order to try to make contact also with η of the DF-algebra, in [8] such torsion deformation of $\mathfrak{osp}(1|32)$ was enlarged by including a spinor 1-form η^e. The MC description of the resulting superalgebra reads

$$\begin{aligned}
\mathcal{R}^{ab} &\equiv d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b = 0\,, \\
\mathcal{D}V^a &= -eB^{ab} \wedge V_b + \frac{e}{2\cdot(5!)^2}\epsilon^{ab_1\cdots b_5 c_1\cdots c_5} B_{b_1\cdots b_5} \wedge B_{c_1\cdots c_5} \\
&\quad + \frac{\mathrm{i}}{2}\bar{\Psi} \wedge \Gamma^a \Psi\,, \\
\mathcal{D}B^{ab} &= eV^a \wedge V^b - eB^{ac} \wedge B_c{}^b + \frac{e}{24}B^{ab_1\cdots b_4} \wedge B^b{}_{b_1\cdots b_4} + \frac{1}{2}\bar{\Psi} \wedge \Gamma^{ab}\Psi\,, \\
\mathcal{D}B^{a_1\cdots a_5} &= 5eB^{m[a_1} \wedge B^{a_2\cdots a_5]}{}_m + \frac{e}{5!}\epsilon^{a_1\cdots a_5 b_1\ldots c_6} B_{b_1\cdots b_5} \wedge V_{b_6} \\
&\quad - \frac{5e}{6!}\epsilon^{a_1\cdots a_5 b_1\cdots b_6} B^{c_1c_2}{}_{b_1b_2b_3} \wedge B_{c_1c_2b_4b_5b_6} + \frac{\mathrm{i}}{2}\bar{\Psi} \wedge \Gamma^{a_1\cdots a_5}\Psi\,, \\
\mathcal{D}\Psi &= \frac{\mathrm{i}}{2}e\Gamma_a\Psi \wedge V^a + \frac{1}{4}e\Gamma_{ab}\Psi \wedge B^{ab} + \frac{\mathrm{i}}{2\cdot 5!}e\Gamma_{a_1\cdots a_5}\Psi \wedge B^{a_1\cdots a_5}\,, \\
\mathcal{D}\eta^e &= \frac{\mathrm{i}}{2}\Gamma_a\psi \wedge V^a + \frac{1}{4}\Gamma_{ab}\Psi \wedge B^{ab} + \frac{\mathrm{i}}{2\cdot 5!}\Gamma_{a_1\cdots a_5}\Psi \wedge B^{a_1\cdots a_5}\,.
\end{aligned} \tag{25}$$

In particular, we can see that $\mathcal{D}\eta^e = \frac{1}{e}\mathcal{D}\Psi$, and the MC closure does not allow any free parameters. Hence, in the $e \to 0$ limit the algebraic structure above does not reproduce the DF-algebra, as $\eta^e_{\neq}\eta$ for any value of the free parameter in the DF-algebra. On the other hand, let us observe that the reduced version of (25) without η^e in the limit $e \to 0$ gives the M-algebra.

The relation between the DF-algebra and $\mathfrak{osp}(1|32)$ was subsequently clarified in [3] under a cohomological perspective. In particular, in [3] it was found that $A^{(3)}(\sigma)$ and η admit the general decomposition

$$A^{(3)}(\sigma) = A^{(3)}_{(0)} + \alpha A^{(3)}_{(e)} , \quad \eta = \eta_{(0)} + \alpha \eta_{(e)} , \tag{26}$$

α being precisely the free parameter of the D'Auria-Fré construction. The contribution

$$A^{(3)}_{(0)} = A^{(3)}_{(0)}(V^a, \Psi, B^{ab}, \eta_{(0)}) \tag{27}$$

does not depend on $B^{a_1 \cdots a_5}$ and explicitly breaks the $\mathfrak{osp}(1|32)$ structure, while the contribution

$$A^{(3)}_{(e)} = A^{(3)}_{(e)}(V^a, \Psi, B^{ab}, B^{a_1 \cdots a_5}, \eta_{(e)}) \tag{28}$$

is covariant under the (torsion deformation of) $\mathfrak{osp}(1|32)$. In the vacuum FDA we have

$$dA^{(3)}_{(0)} = \frac{1}{2} \bar{\Psi} \wedge \Gamma_{ab} \Psi \wedge V^a \wedge V^b , \quad dA^{(3)}_{(e)} = 0 . \tag{29}$$

Therefore, it emerges that only $dA^{(3)}_{(0)}$ is responsible for the 4-form cohomology of the supersymmetric FDA, and the free parameter α parametrizes the cohomologically trivial deformation $dA^{(3)}_{(e)}$. We conclude that, as the decomposition (26) shows, $A^{(3)}(\sigma)$ is not invariant under $\mathfrak{osp}(1|32)$ (neither under its torsion deformation) because of the contribution $A^{(3)}_{(0)}$ explicitly breaking this symmetry. Such term is however the only one contributing to the vacuum 4-form cohomology in superspace.

It would be worth conducting a similar analysis for the 6-form $B^{(6)}$. Furthermore, one might then consider the out-of-vacuum FDA, where we would have

$$dA^{(3)} - \frac{1}{2} \bar{\Psi} \wedge \Gamma_{ab} \Psi \wedge V^a \wedge V^b = F^{(4)} = F^{(4)}_{(0)} + \alpha F^{(4)}_{(e)} \tag{30}$$

with

$$dA^{(3)}_{(0)} = \frac{1}{2} \bar{\Psi} \wedge \Gamma_{ab} \Psi \wedge V^a \wedge V^b + F^{(4)}_{(0)} , \quad dA^{(3)}_{(e)} = F^{(4)}_{(e)} , \tag{31}$$

and compute the charge associated with the 3-form gauge potential,

$$q = \int dA^{(3)} = q_{(0)} + \alpha q_{(e)} . \tag{32}$$

In this context, possible connections could emerge with the analysis of the 4-form cohomology of M-theory on spin manifolds [14, 34], where $dA^{(3)}_{(0)}$ might turn out to be the contribution responsible for the canonical integral class of the spin bundle of

$D = 11$ superspace. In particular, this would imply that $q_{(0)}$ could assume fractional values (in units of $q_{(e)}$).

To conclude, let us also mention that the 4-form $F^{(4)}$ appears in the topological term $A^{(3)} \wedge F^{(4)} \wedge F^{(4)}$ of the $D = 11$ supergravity Lagrangian, and it appears that the nilpotent spinor 1-form η could be an important addition towards the construction of a possible off-shell theory underlying $D = 11$ supergravity. In [19], a supersymmetric $D = 11$ Lagrangian invariant under the M-algebra and closing off-shell without requiring auxiliary fields was constructed, as a Chern-Simons form, and shown to depend on one free parameter. It would be very interesting to investigate the possible connections between this and the approach reviewed in the present report. Moreover, another aspect that deserves further investigation consists in a clearer understanding of the relation between the geometrical formulation of $D = 11$ supergravity based on the out-of-vacuum FDA structure (and its trivialization in terms of 1-form fields) and the derivation of the CJS theory from some sector of the OSp(1|32) Chern-Simons action as stated in [33].

Acknowledgements The author would like to thank the Department of Applied Science and Technology of the Polytechnic of Turin, and in particular Laura Andrianopoli and Francesco Raffa, for financial support. The work reported in this proceeding was performed in collaboration with Laura Andrianopoli and Riccardo D'Auria.

References

1. A. Achucarro, J.M. Evans, P.K. Townsend, D.L. Wiltshire, Super p-Branes. Phys. Lett. B **198**, 441 (1987). https://doi.org/10.1016/0370-2693(87)90896-3
2. L. Andrianopoli, R. D'Auria, L. Ravera, Hidden gauge structure of supersymmetric free differential algebras. JHEP **08**, 095 (2016). https://doi.org/10.1007/JHEP08(2016)095. [arXiv:1606.07328] [hep-th]
3. L. Andrianopoli, R. D'Auria, L. Ravera, More on the hidden symmetries of 11D supergravity. Phys. Lett. B **772**, 578–585 (2017). https://doi.org/10.1016/j.physletb.2017.07.016 [arXiv:1705.06251] [hep-th]
4. I.A. Bandos, J.A. de Azcarraga, J.M. Izquierdo, M. Picon, O. Varela, On the underlying gauge group structure of D = 11 supergravity. Phys. Lett. B **596**, 145–155 (2004). https://doi.org/10.1016/j.physletb.2004.06.079 [arXiv:hep-th/0406020] [hep-th]
5. E. Bergshoeff, E. Sezgin, P.K. Townsend, Supermembranes and eleven-dimensional supergravity. Phys. Lett. B **189**, 75 (1987). https://doi.org/10.1016/0370-2693(87)91272-X
6. E. Bergshoeff, E. Sezgin, P.K. Townsend, Properties of the eleven-dimensional super membrane theory. Ann. Phys. **185**, 330 (1988). https://doi.org/10.1016/0003-4916(88)90050-4
7. L. Castellani, R. D'Auria, P. Fré, *Supergravity and Superstrings: A Geometric Perspective*, vols. 1 and 2 (Published in Singapore: World Scientific, 1991)
8. L. Castellani, P. Fré, F. Giani, K. Pilch, P. van Nieuwenhuizen, Gauging of $d = 11$ supergravity? Ann. Phys. **146**, 35 (1983). https://doi.org/10.1016/0003-4916(83)90052-0
9. E. Cremmer, B. Julia, J. Scherk, Supergravity theory in eleven-dimensions. Phys. Lett. B **76**, 409 (1978). https://doi.org/10.1016/0370-2693(78)90894-8
10. R. D'Auria, P. Fré, Geometric Supergravity in d = 11 and Its Hidden Supergroup. Nucl. Phys. B **201**, 101 (1982). Erratum: [Nucl. Phys. B **206**, 496 (1982)]. https://doi.org/10.1016/0550-3213(82)90376-5

11. R. D'Auria, P. Fré, *Cartan Integrable Systems, that is Differential Free Algebras, in Supergravity*, PRINT-83-0689
12. R. D'Auria, "Geometric supergravity," Review article from the book, *Tullio Regge: An Eclectic Genius—From Quantum Gravity to Computer Play* (World Scientific Publishing Co. Pte. Ltd., 2019). [arXiv:2005.13593] [hep-th]
13. J.A. de Azcarraga, J.P. Gauntlett, J.M. Izquierdo, P.K. Townsend, Topological extensions of the supersymmetry algebra for extended objects. Phys. Rev. Lett. **63**, 2443 (1989). https://doi.org/10.1103/PhysRevLett.63.2443
14. D.E. Diaconescu, G.W. Moore, E. Witten, E(8) gauge theory, and a derivation of K theory from M theory. Adv. Theor. Math. Phys. **6**, 1031–1134 (2003). https://doi.org/10.4310/ATMP.2002.v6.n6.a2 [arXiv:hep-th/0005090] [hep-th]
15. M.J. Duff, P.S. Howe, T. Inami, K.S. Stelle, Superstrings in D=10 from Supermembranes in D=11. Phys. Lett. B **191**, 70 (1987). https://doi.org/10.1016/0370-2693(87)91323-2
16. M.J. Duff, M theory (The Theory formerly known as strings). Int. J. Mod. Phys. A **11**, 5623 (1996). [Subnucl. Ser. **34** (1997) 324] [Nucl. Phys. Proc. Suppl. **52** (1997) no.1-2, 314] https://doi.org/10.1142/S0217751X96002583 [hep-th/9608117]
17. D.T. Grasso, I.N. McArthur, p-brane superalgebras via integrability. Nucl. Phys. B **926**, 239–255 (2018). https://doi.org/10.1016/j.nuclphysb.2017.11.008 [arXiv:1709.08403] [hep-th]
18. D.T. Grasso, I.N. McArthur, Deriving all p-brane superalgebras via integrability. Nucl. Phys. B **928**, 499–519 (2018). https://doi.org/10.1016/j.nuclphysb.2018.01.026 [arXiv:1710.04497] [hep-th]
19. M. Hassaine, R. Troncoso, J. Zanelli, Poincare invariant gravity with local supersymmetry as a gauge theory for the M-algebra. Phys. Lett. B **596**, 132 (2004). https://doi.org/10.1016/j.physletb.2004.06.067 [hep-th/0306258]
20. M. Hassaine, R. Troncoso, J. Zanelli, 11D supergravity as a gauge theory for the M-algebra. PoS WC **2004**, 006 (2005). ([hep-th/0503220])
21. O. Hohm, H. Samtleben, Exceptional form of D = 11 supergravity. Phys. Rev. Lett. **111**, 231601 (2013). https://doi.org/10.1103/PhysRevLett.111.231601 [arXiv:1308.1673] [hep-th]
22. O. Hohm, H. Samtleben, Exceptional field theory. II. $E_{7(7)}$. Phys. Rev. D **89**, 066017 (2014). https://doi.org/10.1103/PhysRevD.89.066017 [arXiv:1312.4542] [hep-th]
23. O. Hohm, H. Samtleben, Consistent Kaluza-Klein truncations via exceptional field theory. JHEP **01**, 131 (2015). https://doi.org/10.1007/JHEP01(2015)131 [arXiv:1410.8145] [hep-th]
24. I.N. McArthur, Coset construction of a D-brane gauge field. Nucl. Phys. B **905**, 231–241 (2016). https://doi.org/10.1016/j.nuclphysb.2016.02.015 [arXiv:1506.02798] [hep-th]
25. H. Sati, U. Schreiber, Lie n-algebras of BPS charges. JHEP **03**, 087 (2017). https://doi.org/10.1007/JHEP03(2017)087 [arXiv:1507.08692] [math-ph]
26. J.H. Schwarz, The power of M theory. Phys. Lett. B **367**, 97 (1996). https://doi.org/10.1016/0370-2693(95)01429-2 [hep-th/9510086]
27. E. Sezgin, The M algebra. Phys. Lett. B **392**, 323 (1997). https://doi.org/10.1016/S0370-2693(96)01576-6 [hep-th/9609086]
28. D. Sullivan, Infinitesimal computations in topology. Publications Mathématiques de l'IHES **47**, 269–331 (1977)
29. P.K. Townsend, The eleven-dimensional supermembrane revisited. Phys. Lett. B **350**, 184 (1995). https://doi.org/10.1016/0370-2693(95)00397-4 [(hep-th/9501068])
30. P.K. Townsend, P-brane democracy, in *The world in eleven dimensions**, ed. by M.J. *Duff, pp. 375-389. arxiv:hep-th/9507048
31. P.K. Townsend, Four lectures on M theory, in *Trieste 1996, High energy physics and cosmology*, pp. 385–438. arxiv:hep-th/9612121
32. P.K. Townsend, M theory from its superalgebra, in *Cargese, Strings, branes and dualities*, pp. 141–177 (1997). arxiv:hep-th/9712004
33. R. Troncoso, J. Zanelli, New gauge supergravity in seven-dimensions and eleven-dimensions. Phys. Rev. D **58**, 101703 (1998). https://doi.org/10.1103/PhysRevD.58.101703 [arXiv:hep-th/9710180] [(hep-th])
34. E. Witten, On flux quantization in M theory and the effective action. J. Geom. Phys. **22**, 1–13 (1997). https://doi.org/10.1016/S0393-0440(96)00042-3 [arXiv:hep-th/9609122] [hep-th]

Defects at the Intersection: The Supergroup Side

Fabrizio Nieri

Abstract We consider two seemingly different theories in the Ω-background: one arises upon the most generic Higgsing of a 5d $\mathcal{N}=1$ U(N) gauge theory coupled to matter, yielding a 3d-1d intersecting defect; the other one arises upon simple Higgsing of a 5d $\mathcal{N}=1$ U($N|M$) supergroup gauge theory coupled to super-matter, yielding another defect. The cases $N=M=1$ are discussed in detail via equivariant localization to matrix-like models. The first theory exhibits itself a supergroup-like structure, which can be motivated via non-perturbative string dualities, and in a matter decoupling limit it is argued to be dual to a supergroup version of refined Chern-Simons theory. Furthermore, it is observed that the partition functions of the two defect theories are related by analytic continuation in one of the equivariant parameters. We find a common origin in the algebraic engineering through q-Virasoro screening currents. Another simple Higgsing of the 5d $\mathcal{N}=1$ U(1|1) yields a single component defect whose partition function is reminiscent of ordinary refined Chern-Simons on a lens space.

Keywords Defects · Supergroup Chern-Simons theory · Topological strings · Open/closed Duality

1 Intersecting Defects and Supermatrix-Like Models

Gauge invariance is one of the main principles behind our comprehension of Nature, and the dichotomy between matter particles and force mediators (fermions vs. bosons) in theories based on ordinary compact Lie groups is a fact of life. Supergroups, on the other hand, unify particles of opposite statistics: while theories exhibiting global supergroup symmetries have been long appreciated (cf. SUSY models), it is also quite interesting to study QFT (supersymmetric or not) based on gauge super-

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_16

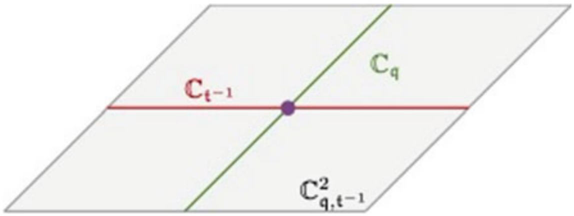

Fig. 1 Representation of the supports of the 5d theory and of the 3d-1d defects. At each point there is also a circle which is not explicitly displayed

groups. Non-unitarity is manifest due to the violation of spin-statistics, whereas the lack of a definite bilinear form on the gauge algebra requires an intrinsic non-perturbative approach. These are actually some of the features which make such theories and the closely related supermatrix models interesting to investigate. In fact, they do arise, in one way or another, in many places of theoretical physics: effective membrane dynamics [1–4], analytic continuations of unitary models [5, 6], topological strings [7–10], exotic phenomena [11, 12], instanton calculus [13] and integrability [14–16] just to mention few examples.

This note, a brief account of the results published in [17] (supplemented by original computations in Sect. 2.1), is about yet another place where supergroup gauge theories, a supergroup version of refined Chern-Simons theory in particular [18] (see also [19] for recent work in this direction), show up: SUSY theories based on ordinary gauge groups but supported on intersecting subspaces embedded in an ambient space. The intersecting gauge theories of our interest arise upon Higgsing a parent 5d SUSY gauge theory with unitary group in the Ω-background $\mathbb{C}^2_{\mathfrak{q},\mathfrak{t}^{-1}} \times \mathbb{S}^1$, while the support of the defects is given by the two orthogonal cigars $\mathbb{C}_{\mathfrak{q}} \times \mathbb{S}^1$ and $\mathbb{C}_{\mathfrak{t}^{-1}} \times \mathbb{S}^1$ intersecting at the origin along a common circle (Fig. 1).

This description of the 3d-1d coupled system allows its partition function to be computed by specialization of the instanton partition function of the parent 5d theory, which for the $\mathrm{U}(N)$ SQCD can be presented in a combinatorial form as a summation over a set of N integer partitions λ [20, 21]

$$Z_{\text{inst.}}[\text{SQCD}] \equiv \sum_{\{\lambda_A\}} \Lambda^{\sum_A |\lambda_A|} Z_{\{\lambda_A\}}[\text{SQCD}] \xrightarrow{\nu\to\nu^*} Z_{\text{vortex}}[\text{Defect}_{\mathfrak{q},\mathfrak{t}}] ,$$
$$Z_{\{\lambda_A\}}[\text{SQCD}] \equiv \prod_{A,B=1}^{N} \frac{N_{\emptyset\lambda_A}(\nu_A/\bar{\mu}_B;\mathfrak{q},\mathfrak{t}) N_{\lambda_A\emptyset}(\mu_B/\nu_A;\mathfrak{q},\mathfrak{t})}{N_{\lambda_A\lambda_B}(\nu_A/\nu_B;\mathfrak{q},\mathfrak{t})} , \tag{1}$$

where Λ denotes the instanton counting parameter, μ, $\bar{\mu}$ (anti-)fundamental flavor fugacities and ν the Coulomb branch parameters (our conventions and definitions are set in the Appendix). The truncation to the vortex part of the defect partition function is achieved by locking the Coulomb parameters to the flavour ones as follows

$$\nu_A \to \nu_A^* \equiv \mu_A \mathfrak{t}^{-r_A} \mathfrak{q}^{c_A} , \quad r, c \in \mathbb{Z}^N_{\geq 0} . \tag{2}$$

The defect theory is UV described by a pair of 3d $\mathcal{N}=2$ gauge theories with gauge groups $\mathrm{U}(r)$ and $\mathrm{U}(c)$ respectively (coupled to adjoint and fundamental/anti-fundamental chirals) and interacting through 1d chiral matter along the common $\mathbb{S}^1$ at the origin, supplemented with superpotential terms (enforcing identifications among parameters) [22–25]. The analysis of the partition function, which can be recast in a matrix-like integral

$$Z[\mathrm{Defect}_{\mathfrak{q},\mathfrak{t}}] \equiv \oint \prod_{a=1}^{r} \frac{\mathrm{d}z_a^{\mathrm{R}}}{2\pi\mathrm{i}z_a^{\mathrm{R}}} \prod_{b=1}^{c} \frac{\mathrm{d}z_b^{\mathrm{L}}}{2\pi\mathrm{i}z_b^{\mathrm{L}}} \left(\prod_a z_a^{\mathrm{R}}\right)^{-\zeta_{\mathrm{R}}} \left(\prod_b z_b^{\mathrm{L}}\right)^{\zeta_{\mathrm{L}}} \times \\ \times \Delta(z^{\mathrm{R}}, z^{\mathrm{L}}; \mathfrak{q}, \mathfrak{t}^{-1}, \mathfrak{p}) \times \prod_{A,a} \frac{(z_a^{\mathrm{R}}/\eta_{\mathrm{R}}\bar{\mu}_A; \mathfrak{q})_\infty}{(z_a^{\mathrm{R}}/\eta_{\mathrm{R}}\mu_A; \mathfrak{q})_\infty} \prod_{A,a} \frac{(z_a^{\mathrm{L}}/\eta_{\mathrm{L}}\bar{\mu}_A; \mathfrak{t}^{-1})_\infty}{(z_a^{\mathrm{L}}/\eta_{\mathrm{L}}\mu_A; \mathfrak{t}^{-1})_\infty}\,, \tag{3}$$

where $\mathfrak{p}^{1/2} \equiv \sqrt{\mathfrak{q}\mathfrak{t}^{-1}}$, $\eta_{\mathrm{R}}/\eta_{\mathrm{L}} \equiv 1/\sqrt{\mathfrak{q}\mathfrak{t}}$, $\mathfrak{q}^{\zeta_{\mathrm{R}}} \equiv \Lambda \equiv \mathfrak{t}^{\zeta_{\mathrm{L}}}$, suggests that the intersecting defects provide a deformation of a dual supergroup gauge theory, which for $N=1$ (SDEQ) can be identified as a supergroup version of refined Chern-Simons theory on $\mathbb{S}^3$ (after a matter decoupling limit). This is essentially because the integration measure features the supergroup version of the Macdonald weight function [26, 27]

$$\Delta(z^{\mathrm{R}}, z^{\mathrm{L}}; \mathfrak{q}, \mathfrak{t}^{-1}, \mathfrak{p}) \equiv \frac{\Delta_{\mathfrak{t}}(\underline{z}^{\mathrm{R}}; \mathfrak{q})\Delta_{\mathfrak{q}^{-1}}(\underline{z}^{\mathrm{L}}; \mathfrak{t}^{-1})}{\prod_{a,b}(1-\mathfrak{p}^{-1/2}z_b^{\mathrm{L}}/z_a^{\mathrm{R}})(1-\mathfrak{p}^{-1/2}z_a^{\mathrm{R}}/z_b^{\mathrm{L}})}\,, \tag{4a}$$

$$\Delta_{\mathfrak{t}}(z; \mathfrak{q}) \equiv \prod_{i\neq j} \frac{(z_i/z_j; \mathfrak{q})_\infty}{(\mathfrak{t}z_i/z_j; \mathfrak{q})_\infty}\,, \tag{4b}$$

which in scaling limit $\mathfrak{q}, \mathfrak{t} \to 1$ ($\ln\mathfrak{t}/\ln\mathfrak{q} \equiv \beta$ fixed) yields the β-deformation of the Cauchy weight function [28], simply related to the Hermitian supermatrix measure in the unrefined case $\beta = 1$.[1] The democracy between the two orthogonal planes opens up the possibility of understanding the refinement $\mathfrak{t} \neq \mathfrak{q}$ as a specific deformation away from the supergroup point: $\ln\mathfrak{q}$ and $\ln\mathfrak{t}^{-1}$ are identified with the inverse Chern-Simons couplings, strictly opposite for a supergroup theory. This can roughly be seen in the decoupling limit $\mu \to 0$, $\bar{\mu} \to \infty$, in which case the potential asymptotically contributes with $\exp[\sum_a (\ln z_a^{\mathrm{R}})^2/\ln\mathfrak{q} - \sum_b (\ln z_b^{\mathrm{L}})^2/\ln\mathfrak{t}]$. The appearance of a supergroup structure may seem surprising from the intersecting defect perspective: the parent theory is an ordinary gauge theory, while after Higgsing the degrees of freedom are even defined on different space-time components. An explanation can be offered by embedding the configuration under study into string/M-theory and then exploiting chain of non-perturbative dualities to recognise a setup engineering supergroup Chern-Simons theory [29–31] (Fig. 2).

[1] Let us note that, because of the original $\mathfrak{q} \leftrightarrow \mathfrak{t}^{-1}$ symmetry, we considered here the chamber $|\mathfrak{q}| < 1$, $|\mathfrak{t}^{-1}| < 1$, which is, however, tricky for the unrefined limit. The case $|\mathfrak{q}| < 1$, $|\mathfrak{t}| < 1$ is discussed in [17].

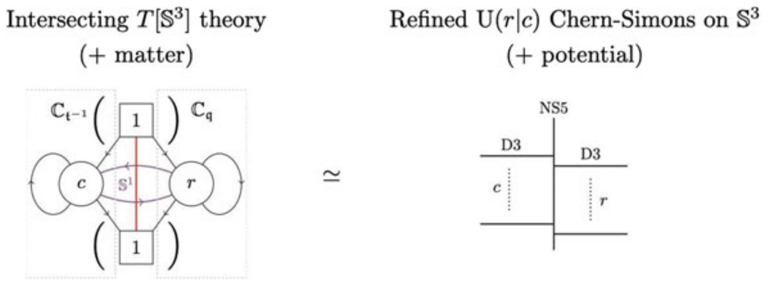

Fig. 2 The 3d[intersecting]-3d[supergroup] equivalence, here specialized to $N = 1$. It can be placed in the context of a generalized 3d-3d correspondence. The squares on the l.h.s. represent coupling to (anti-)fundamental matter (which can be decoupled)

2 Towards Non-unitary Open/closed Duality

The relation between the 5d parent theory and the 3d defect in the ordinary Higgsing process (i.e. $c = 0$), once embedded into string theory, can also be understood as a large r geometric transition (open/closed duality) [32]. Since in a generic Higgsing process the parent 5d theory (the closed side) gives rise to both $\mathfrak{q}$- and $\mathfrak{t}$-branes supporting the intersecting defect in space-time (the open side), it is natural to wonder what happens upon trying to go back by a simultaneous large (r, c) limit. One may think that the original setup must be recovered, however, this is not the only possibility: it turns out that a closed string side engineering a 5d supergroup gauge theory [13] (see [33] for a nice review) is also possible (and more natural to some extent).

A clean illustration of such dynamics can be achieved by recalling that the toric geometries/brane configurations dual to our 5d-3d-1d theories can be thought of as networks of Ding-Iohara-Miki (DIM) intertwiners [34, 35]. In the special case we are interested in (SQCD or SQED) the algebraic description can also be recast in terms of the q-Virasoro algebra [36, 37], whose screening currents $\mathbf{S}^{\pm}$ are identified with $\mathfrak{q}$- and $\mathfrak{t}$-branes. In the ordinary Higgsing only one type of screening is considered, then the open/closed duality is the observation that free field correlators involving a finite number of charges capture 3d defect partition functions [38], whereas sending their number to infinity [39] recovers the partition function of the parent 5d theory[2]

$$Z[\text{Defect}_{\mathfrak{q}}] \simeq \langle \oint \prod_{i=1,\ldots,r}^{\prec} \mathrm{d}z_i \, \mathbf{S}^+(z_i) \ \cdots \rangle \xrightarrow{r\to\infty}$$

$$\xrightarrow{r\to\infty} \langle \sum_{\{k_i\in\mathbb{Z}\}} \prod_{i\geq 1}^{\prec} x_i \mathfrak{q}^{k_i} \mathbf{S}^+(x_i \mathfrak{q}^{k_i}) \ \cdots \rangle \simeq Z_{\text{inst.}}[\mathrm{U}(1)] \,, \quad (5)$$

[2] The dependence on vortex or instanton counting parameters arises from the zero mode contributions acting on the charged Fock vacuum.

where the infinitely-many base points in the Jackson integration are at $x_i \equiv \eta_+\nu_+\mathfrak{t}^{i-1}$, for some $\eta_+ \in \mathbb{C}^\times$.[3] For simplicity, we are here limiting to the Abelian theory in 5d (dual to the resolved conifold geometry). Both the 3d integrand and the 5d instanton summands are easily recognized from the OPE relations

$$\mathbf{S}^{(\mp)}(x)\mathbf{S}^{(\pm)}(x') = \ :\mathbf{S}^{(\mp)}(x)\mathbf{S}^{(\pm)}(x'):\ \frac{(-\mathfrak{p}^{1/2}xx')^{-1}}{(1-\mathfrak{p}^{-1/2}x/x')(1-\mathfrak{p}^{-1/2}x'/x)}\ , \tag{6a}$$

$$\mathbf{S}^{(+)}(x)\mathbf{S}^{(+)}(x') = \ :\mathbf{S}^{(+)}(x)\mathbf{S}^{(+)}(x'):\ \frac{(x'/x;\mathfrak{q})_\infty(\mathfrak{p}x'/x;\mathfrak{q})_\infty}{(\mathfrak{q}x'/x;\mathfrak{q})_\infty(\mathfrak{t}x'/x;\mathfrak{q})_\infty}\, x^{2\beta}\ , \tag{6b}$$

$$\mathbf{S}^{(-)}(x)\mathbf{S}^{(-)}(x') = \ :\mathbf{S}^{(-)}(x)\mathbf{S}^{(-)}(x'):\ \frac{(x'/x;\mathfrak{t})_\infty(\mathfrak{p}^{-1}x'/x;\mathfrak{t})_\infty}{(\mathfrak{t}x'/x;\mathfrak{t})_\infty(\mathfrak{q}x'/x;\mathfrak{t})_\infty}\, x^{2\beta^{-1}}\ . \tag{6c}$$

where : : denotes the usual (free boson) normal ordering. The characteristic summation over partitions arises due to zeros in the coefficients for configurations of points outside of the set

$$\chi^+ \equiv \{\nu_+\mathfrak{t}^{i-1}\mathfrak{q}^{-\lambda_i^+},\ \lambda_i^+ \geq \lambda_{i+1}^+,\ \mathrm{i} \in [1,+\infty)\}\ . \tag{7}$$

Mimicking the same logic, we can consider free field correlators involving both types of screenings. When there are r of one type and c of the other type, the integration measure of the 3d-1d intersecting defect partition function is manifestly reproduced as in (4a).[4] When an infinite amount of both types is considered, we first generate a second dynamical set

$$\widetilde{\chi}^- \equiv \{\nu_-\mathfrak{q}^{i-1}\mathfrak{t}^{-\lambda_i^{-\vee}},\ \lambda_i^{-\vee} \geq \lambda_{i+1}^{-\vee},\ \mathrm{i} \in [1,+\infty)\}\ , \tag{8}$$

where $^\vee$ denotes transposition, and it is convenient to fix $\eta_+/\eta_- \equiv \mathfrak{p}^{1/2}$.[5] The diagonal (i.e. $++$ and $--$) OPE factors simply generate the adjoint instanton summands for the $U(1)\times U(1)$ theory

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[3] We are focusing on the adjoint sector only as the addition of fundamental matter can be implemented by inserting vertex operators, represented by the dots. Also, $\prec$ means the product runs in increasing order from left to right, and viceversa for $\succ$.

[4] Up to overall $\mathfrak{q}$- and $\mathfrak{t}$-constant which play little role for the identification.

[5] Let us note that because of the $\mathfrak{q} \leftrightarrow \mathfrak{t}$ exchange symmetry of the points in the two sets, we are naturally led to consider the chamber $|\mathfrak{q}|, |\mathfrak{t}| < 1$.

$$\prod_{x\in\chi^+}^{\prec} \mathbf{S}^+(\eta_+ x) \simeq \prod_{i,j=1}^{\ell(\lambda^+)} \frac{(\mathfrak{t}\,\mathfrak{t}^{j-i};\mathfrak{q})_\infty}{(\mathfrak{t}^{j-i};\mathfrak{q})_\infty} \frac{(\mathfrak{t}^{j-i}\mathfrak{q}^{\lambda^+_i-\lambda^+_j};\mathfrak{q})_\infty}{(\mathfrak{t}\,\mathfrak{t}^{j-i}\mathfrak{q}^{\lambda^+_i-\lambda^+_j};\mathfrak{q})_\infty} \times$$
$$\times \frac{1}{N_{\lambda^+\emptyset}(\mathfrak{t}^{\ell(\lambda^+)};\mathfrak{q},\mathfrak{t}) N_{\emptyset\lambda^+}(\mathfrak{t}^{-\ell(\lambda^+)};\mathfrak{q},\mathfrak{t})} = \frac{1}{N_{\lambda^+\lambda^+}(1;\mathfrak{q},\mathfrak{t})} \equiv Z^{\mathrm{v}}_{++}\,, \tag{9a}$$
$$\prod_{x\in\tilde{\chi}^-}^{\succ} \mathbf{S}^-(\eta_- x) \simeq \prod_{i,j=1}^{\ell(\lambda^{-\vee})} \frac{(\mathfrak{q}\,\mathfrak{q}^{j-i};\mathfrak{t})_\infty}{(\mathfrak{q}^{j-i};\mathfrak{t})_\infty} \frac{(\mathfrak{q}^{j-i}\mathfrak{t}^{\lambda^{-\vee}_i-\lambda^{-\vee}_j};\mathfrak{t})_\infty}{(\mathfrak{q}\,\mathfrak{q}^{j-i}\mathfrak{t}^{\lambda^{-\vee}_i-\lambda^{-\vee}_j};\mathfrak{t})_\infty} \times$$
$$\times \frac{1}{N_{\lambda^{-\vee}\emptyset}(\mathfrak{q}^{\ell(\lambda^{-\vee})};\mathfrak{t},\mathfrak{q}) N_{\emptyset\lambda^{-\vee}}(\mathfrak{q}^{-\ell(\lambda^{-\vee})};\mathfrak{t},\mathfrak{q})} = \frac{1}{N_{\lambda^{-\vee}\lambda^{-\vee}}(1;\mathfrak{t},\mathfrak{q})} \equiv \mathfrak{p}^{-|\lambda^-|} Z^{\mathrm{v}}_{--}\,, \tag{9b}$$

where the equalities are up to normalization (empty diagrams) and zero mode contributions. This is the first sign of a super-instanton expansion, however, the crucial information comes from the mixed (i.e. $+-$ and $-+$) terms which have to match the specific bi-fundamental-like contributions. This can checked, for instance, by using the identities

$$\frac{\prod_{(x,x')\in\tilde{\chi}^-_\emptyset\times\chi^+_\emptyset}(1-\mathfrak{p}^{-1/2}\eta_-\eta_+^{-1}x/x')}{\prod_{(x,x')\in\tilde{\chi}^-\times\chi^+}(1-\mathfrak{p}^{-1/2}\eta_-\eta_+^{-1}x/x')} =$$
$$= \frac{\prod_{i=1}^{\ell(\lambda^+)}\prod_{j=1}^{\ell(\lambda^{-\vee})}(1-\mathfrak{p}^{-1}\nu_-/\nu_+\mathfrak{t}^{-i}\mathfrak{q}^{j})}{\prod_{i=1}^{\ell(\lambda^+)}\prod_{j=1}^{\ell(\lambda^{-\vee})}(1-\mathfrak{p}^{-1}\nu_-/\nu_+\mathfrak{t}^{-\lambda^{-\vee}_j-i}\mathfrak{q}^{\lambda^+_i+j})} \times$$
$$\times N_{\lambda_+\emptyset}(\mathfrak{q}^{\ell(\lambda^{-\vee})}\nu_-/\nu_+;\mathfrak{q},\mathfrak{t}) N_{\emptyset\lambda^\vee_-}(\mathfrak{p}^{-1}\mathfrak{t}^{-\ell(\lambda^+)}\nu_-/\nu_+;\mathfrak{t},\mathfrak{q})\,, \tag{10a}$$
$$\frac{\prod_{(x,x')\in\tilde{\chi}^-_\emptyset\times\chi^+_\emptyset}(1-\mathfrak{p}^{-1/2}\eta_+\eta_-^{-1}x'/x)}{\prod_{(x,x')\in\tilde{\chi}^-\times\chi^+}(1-\mathfrak{p}^{-1/2}\eta_+\eta_-^{-1}x'/x)} =$$
$$= \frac{\prod_{i=1}^{\ell(\lambda^+)}\prod_{j=1}^{\ell(\lambda^{-\vee})}(1-\nu_+/\nu_-\mathfrak{t}^{i}\mathfrak{q}^{-j})}{\prod_{i=1}^{\ell(\lambda^+)}\prod_{j=1}^{\ell(\lambda^{-\vee})}(1-\nu_+/\nu_-\mathfrak{t}^{\lambda^{-\vee}_j+i}\mathfrak{q}^{-\lambda^+_i-j})} \times$$
$$\times N_{\emptyset\lambda_+}(\mathfrak{q}^{-\ell(\lambda^{-\vee})}\nu_+/\nu_-;\mathfrak{q},\mathfrak{t}) N_{\lambda^\vee_-\emptyset}(\mathfrak{p}^{-1}\mathfrak{t}^{\ell(\lambda^+)}\nu_+/\nu_-;\mathfrak{t},\mathfrak{q})\,. \tag{10b}$$

Once all the pieces are combined, the super-instanton partition function of the $\mathrm{U}(1|1)$ theory is reproduced. In particular, the counting parameters $\Lambda^{\pm 1}$ in the $\pm$ sectors are accounted by the zero modes of the screening currents.

2.1 Inclusion of Matter and Supergroup Higgsings

The inclusion of (anti-)fundamental matter with supergroup flavor symmetry introduces the following contributions to the super-instanton summands

$$Z^{\rm f}_{+\pm} \equiv N_{\emptyset\lambda^+}(\nu_+/\bar\mu_\pm; \mathfrak{q}, \mathfrak{t})^{\pm 1}\,, \qquad Z^{\rm fa}_{+\pm} \equiv N_{\lambda^+\emptyset}(\mu_\pm/\nu_+; \mathfrak{q}, \mathfrak{t})^{\pm 1} \tag{11a}$$

$$Z^{\rm f}_{-\pm} \equiv N_{\emptyset\lambda^{-\vee}}(\mathfrak{p}^{-1}\nu_-/\bar\mu_\pm; \mathfrak{t}, \mathfrak{q})^{\mp 1}\,, \quad Z^{\rm fa}_{-\pm} \equiv N_{\lambda^{-\vee}\emptyset}(\mathfrak{p}^{-1}\mu_\pm/\nu_-; \mathfrak{t}, \mathfrak{q})^{\mp 1}\,. \tag{11b}$$

In particular, the diagonal sectors show zeros at particular values of $\nu_\pm$ which can be used to truncate the partition function.[6] Let us consider the inclusion of one anti-fundamental, then the interesting points are at

$$\mu_+/\nu_+ = \mathfrak{t}^r\mathfrak{q}^{-c'}\,, \quad \mu_-/\nu_- = \mathfrak{p}\mathfrak{q}^c\mathfrak{t}^{-r'}\,, \quad r, c, r', c' \in \mathbb{Z}_{\geq 0}\,. \tag{12}$$

Viewing the U(1) theory as the subsector $\mathrm{U}(1|0)$, we see that the situation described in the previous Section corresponds to $c = r' = 0$, which completely freezes the negative node. With an abuse of notation, this defect can be dubbed $\mathrm{U}(r|0) \times \mathrm{U}(0|c')$, while the most general one as $\mathrm{U}(r|c) \times \mathrm{U}(r'|c')$. In the following, we consider a couple of intermediate possibilities.

Intersecting $\mathrm{U}(r|c)$ **Defect.** We are here interested in setting

$$\nu_+ \to \nu_+^* \equiv \mu_+\mathfrak{t}^{-r}\,, \quad \nu_- \to \nu_-^* \equiv \mu_-\mathfrak{p}^{-1}\mathfrak{q}^{-c}\,, \tag{13}$$

so that $\ell(\lambda^+) \leq r$, $\ell(\lambda^{-\vee}) \leq c$. At these points, the diagonal vector and fundamental contributions partially simplify

$$Z^{\rm v}_{++}Z^{\rm fa}_{++} \xrightarrow{\nu_+\to\nu_+^*} \prod_{i,j=1}^{r} \frac{(\mathfrak{t}\,\mathfrak{t}^{j-i};\mathfrak{q})_\infty}{(\mathfrak{t}^{j-i};\mathfrak{q})_\infty} \frac{(\mathfrak{t}^{j-i}\mathfrak{q}^{\lambda_i^+-\lambda_j^+};\mathfrak{q})_\infty}{(\mathfrak{t}\,\mathfrak{t}^{j-i}\mathfrak{q}^{\lambda_i^+-\lambda_j^+};\mathfrak{q})_\infty} \frac{1}{N_{\emptyset\lambda^+}(\mathfrak{t}^{-r};\mathfrak{q},\mathfrak{t})}\,, \tag{14a}$$

$$Z^{\rm v}_{--}Z^{\rm fa}_{--} \xrightarrow{\nu_-\to\nu_-^*} \prod_{i,j=1}^{c} \frac{(\mathfrak{q}\,\mathfrak{q}^{j-i};\mathfrak{t})_\infty}{(\mathfrak{q}^{j-i};\mathfrak{t})_\infty} \frac{(\mathfrak{q}^{j-i}\mathfrak{t}^{\lambda_i^{-\vee}-\lambda_j^{-\vee}};\mathfrak{t})_\infty}{(\mathfrak{q}\,\mathfrak{q}^{j-i}\mathfrak{t}^{\lambda_i^{-\vee}-\lambda_j^{-\vee}};\mathfrak{t})_\infty} \frac{\mathfrak{p}^{|\lambda_-|}}{N_{\emptyset\lambda^{-\vee}}(\mathfrak{q}^{-c};\mathfrak{t},\mathfrak{q})}\,, \tag{14b}$$

and the mixed contributions too

$$\begin{aligned} Z^{\rm v}_{+-}Z^{\rm fa}_{+-}Z^{\rm v}_{-+}Z^{\rm fa}_{-+} \xrightarrow{\nu_\pm\to\nu_\pm^*} &\prod_{i=1}^{r}\prod_{j=1}^{c} \frac{1-\mathfrak{p}\mu_+/\mu_-\mathfrak{t}^{i-r}\mathfrak{q}^{c-j}}{1-\mathfrak{p}\mu_+/\mu_-\mathfrak{t}^{\lambda_j^{-\vee}+i-r}\mathfrak{q}^{-\lambda_i^++c-j}} \times \\ &\times \prod_{i=1}^{r}\prod_{j=1}^{c} \frac{1-\mathfrak{p}^{-2}\mu_-/\mu_+\mathfrak{t}^{r-i}\mathfrak{q}^{j-c}}{1-\mathfrak{p}^{-2}\mu_-/\mu_+\mathfrak{t}^{-\lambda_j^{-\vee}+r-i}\mathfrak{q}^{\lambda_i^++j-c}} \times \\ &\times \mathfrak{p}^{-|\lambda^+|} N_{\emptyset\lambda_+}(\mathfrak{p}^2\mathfrak{t}^{-r}\mu_+/\mu_-;\mathfrak{q},\mathfrak{t}) N_{\emptyset\lambda_-^\vee}(\mathfrak{p}^{-2}\mathfrak{q}^{-c}\mu_-/\mu_+;\mathfrak{t},\mathfrak{q})\,. \end{aligned} \tag{15}$$

Eventually, the truncated super-instanton summands can be organized as the (normalized) residues at the poles

$$z_i^+ = \alpha_+\mu_+\mathfrak{t}^{i-r}\mathfrak{q}^{-\lambda_i^+}\,, \quad z_j^- = \alpha_-\mu_-\mathfrak{q}^{j-c}\mathfrak{t}^{-\lambda_j^{-\vee}} \tag{16}$$

[6] For defects induced by orbifolding see [15].

in the contour integral

$$Z_{\text{inst.}}[\mathrm{U}(1|1)+\text{fa}] \xrightarrow{\nu_\pm \to \nu_\pm^*} \oint \prod_{i=1}^{r} \frac{\mathrm{d}z_i^+}{2\pi \mathrm{i} z_i^+} \prod_{j=1}^{c} \frac{\mathrm{d}z_j^-}{2\pi \mathrm{i} z_j^-} \Big(\prod_i z_i^+\Big)^{-\zeta_+} \Big(\prod_j z_j^-\Big)^{\zeta_-} \times$$
$$\times\, \Delta(z^+, z^-; \mathfrak{q}, \mathfrak{t}, \mathfrak{p}) \prod_{i=1}^{r} \frac{(z_i^+/\alpha_+\mu_-\mathfrak{p}^{-2};\mathfrak{q})_\infty}{(z_i^+/\alpha_+\mu_+;\mathfrak{q})_\infty} \prod_{j=1}^{c} \frac{(z_j^-/\alpha_-\mu_+\mathfrak{p}^2;\mathfrak{t})_\infty}{(z_j^-/\alpha_-\mu_-;\mathfrak{t})_\infty}\,, \tag{17}$$

provided $\alpha_+/\alpha_- = \mathfrak{p}^{3/2}$, $\mathfrak{p}^{-1}\mathfrak{q}^{\zeta_+} = \Lambda = \mathfrak{p}^{-1}\mathfrak{t}^{\zeta_-}$. Comparing with the defect partition function (3) coming from the ordinary SQED, the two are related by analytic continuation $\mathfrak{t}^{-1} \to \mathfrak{t}$ and removing boundary contributions (Theta functions/$\mathfrak{t}$-constants from the measure), together with straightforward identification of parameters. In particular, the contours are different because the pole structure is different, and in the supergroup case there are no poles from the would be intersection sector.

Single Component $\mathrm{U}(r|0)\times\mathrm{U}(r'|0)$ **Defect.** We are here interested in

$$\nu_+ \to \nu_+^* \equiv \mu_+\mathfrak{t}^{-r}\,, \quad \nu_- \to \nu_-^* \equiv \mu_-\mathfrak{p}^{-1}\mathfrak{t}^{r'}\,, \tag{18}$$

so that $\ell(\lambda^+) \le r$, $\ell(\lambda^-) \le r'$. In this case, it is convenient to write the bi-fundamental contributions in the equivalent infinite product form

$$Z^{\mathrm{v}}_{+-} = \prod_{(x,x')\in\chi^+\times\chi^-} \frac{(\mathfrak{t}x/x';\mathfrak{q})_\infty}{(\mathfrak{t}^2x/x';\mathfrak{q})_\infty} \prod_{(x,x')\in\chi^+_\emptyset\times\chi^-_\emptyset} \frac{(\mathfrak{t}^2x/x';\mathfrak{q})_\infty}{(\mathfrak{t}x/x';\mathfrak{q})_\infty}\,, \tag{19a}$$

$$Z^{\mathrm{v}}_{-+} = \prod_{(x,x')\in\chi^+\times\chi^-} \frac{(\mathfrak{t}^{-1}x'/x;\mathfrak{q})_\infty}{(x'/x;\mathfrak{q})_\infty} \prod_{(x,x')\in\chi^+_\emptyset\times\chi^-_\emptyset} \frac{(x'/x;\mathfrak{q})_\infty}{(\mathfrak{t}^{-1}x'/x;\mathfrak{q})_\infty}\,, \tag{19b}$$

while for the diagonal terms we use $Z^{\mathrm{v}}_{--} = Z^{\mathrm{v}}_{++}|_{\lambda^+\to\lambda^-}$. Note that we introduced the set

$$\chi^- \equiv \{\eta_-\nu_-\mathfrak{t}^{-(i-1)}\mathfrak{q}^{\lambda_i^-}, \lambda_i^- \ge \lambda_{i+1}^-, \mathrm{i}\in[1,+\infty)\}\,, \tag{20}$$

simply related to χ^+ by $(\pm,\mathfrak{q},\mathfrak{t}) \to (\mp,\mathfrak{q}^{-1},\mathfrak{t}^{-1})$: this is why the $-$ sector is associated to a negative gauge node. At the specified points, we get the partial simplification in the $--$ sector

$$Z^{\mathrm{v}}_{--}Z^{\text{fa}}_{--} \xrightarrow{\nu_-\to\nu_-^*} \prod_{i,j=1}^{r'} \frac{(\mathfrak{t}\,\mathfrak{t}^{j-i};\mathfrak{q})_\infty}{(\mathfrak{t}^{j-i};\mathfrak{q})_\infty} \frac{(\mathfrak{t}^{j-i}\mathfrak{q}^{\lambda_i^- - \lambda_j^-};\mathfrak{q})_\infty}{(\mathfrak{t}\,\mathfrak{t}^{j-i}\mathfrak{q}^{\lambda_i^- - \lambda_j^-};\mathfrak{q})_\infty} \frac{(\mathfrak{p}^{1/2}\mathfrak{t}^{-r'})^{|\lambda^-|}}{N_{\emptyset\lambda^-}(\mathfrak{t}^{-r'};\mathfrak{q},\mathfrak{t})\, f_{\lambda^-}(\mathfrak{q},\mathfrak{t})}\,, \tag{21}$$

while in the $++$ the simplification is as before. Eventually, the truncated super-instanton summands can be organized as the (normalized) residues at the poles

$$z_i^\pm = \alpha_\pm\mu_\pm\mathfrak{t}^{\pm(i-r)}\mathfrak{q}^{\mp\lambda_i^\pm}\,, \tag{22}$$

in the contour integral

$$Z_{\text{inst.}}[\mathrm{U}(1|1)+\text{fa}] \xrightarrow{v_\pm \to v_\pm^*} \oint \prod_{i=1}^{r} \frac{\mathrm{d}z_i^+}{2\pi \mathrm{i} z_i^+} \prod_{j=1}^{r'} \frac{\mathrm{d}z_j^-}{2\pi \mathrm{i} z_j^-} \Big(\prod_i z_i^+\Big)^{-\zeta_+} \Big(\prod_j z_j^-\Big)^{-\zeta_-} \times$$
$$\times \Delta_{\mathfrak{t}}(z^+;\mathfrak{q})\Delta_{\mathfrak{t}}(z^-;\mathfrak{q}) \prod_{i=1}^{r}\prod_{j=1}^{r'} \frac{(z_i^+/z_j^- v;\mathfrak{q})_\infty}{(\mathfrak{t} z_i^+/z_j^- v;\mathfrak{q})_\infty} \frac{(v z_j^-/z_i^+;\mathfrak{q})_\infty}{(\mathfrak{t} v z_j^-/z_i^+;\mathfrak{q})_\infty} \times$$
$$\times \prod_{i=1}^{r} \frac{(\mathfrak{t}\alpha_+\mu_-/z_i^+;\mathfrak{q})_\infty}{(z_i^+/\alpha_+\mu_+;\mathfrak{q})_\infty} \prod_{i=1}^{r'} \frac{(\mathfrak{t} z_i^-/\alpha_-\mu_+;\mathfrak{q})_\infty}{(\alpha_-\mu_-/z_i^-;\mathfrak{q})_\infty} \tag{23}$$

provided $\mathfrak{q}^{\zeta_+} = \Lambda = \mathfrak{q}^{\zeta_-}\mu_-/\mu_+$, where we also set $v \equiv \mathfrak{t}\mathfrak{p}^{-1}\alpha_+/\alpha_-$ (this shift may be reabsorbed). The resulting measure looks like that of $\mathrm{U}(r+r')$ refined Chern-Simons broken to $\mathrm{U}(r)\times\mathrm{U}(r')$ by the potential. This is very reminiscent of the lens space $L(2,1)$ matrix model (with eigenvalues placed around two distinct connections), however, we do not currently have an interpretation in this direction.

3 Summary and Discussion

Motivated by a common algebraic engineering origin, we considered two a priori distinct matrix-like models: upon localization, one is associated to a coupled 3d-1d intersecting defect theory arising from Higgsing a parent 5d U(1) theory, the other one arises from Higgsing a 5d U(1|1) theory. It turns out that the two are essentially related by analytic continuation in one of the equivariant parameters. This seems to parallel what is known from other situations [5]: the $\mathbb{S}^3$ $\mathrm{U}(N_1)\times\mathrm{U}(N_2)$ ABJ(M) and the $L(2,1)$ $\mathrm{U}(N_1+N_2)$ Chern-Simons matrix models are related by analytic continuation $N_2 \to -N_2$, while from the combinatorial perspective the role of matrix sizes is played by $1/\ln\mathfrak{q}$, $-1/\ln\mathfrak{t}$ [40]. Furthermore, we considered yet another Higgsing of the 5d U(1|1) theory, and the partition function of the resulting single component defect theory resembles that of refined Chern-Simons on $L(2,1)$.

It is well-known that the large rank expansion of a matrix model and its supergroup version are equivalent up to non-perturbative effects [11] (in the unrefined case, this fits with the generic Higgsing being sensitive only to $r-c$): it would be interesting to retrace and adapt the existing analysis around supergroups, large rank dualities and non-perturbative effects in our refined setup, also in view of the vertex/anti-vertex formalism [41]. The relations between supergroup-like and ordinary theories was instrumental for understanding non-perturbative effects in topological strings [10], most notably on the local $\mathbb{P}^1\times\mathbb{P}^1$ geometry, the closed dual to Chern-Simons on $L(2,1)$: a deeper understanding of the subject reviewed in this note may help in shedding more light on seemingly different proposals [42]. On the more mathematical side, it would be interesting to study the moduli space of generalized defects and their relations to the parent instanton ones, as well as to explore the possible extension to the intersecting/supergroup setup intriguing dualities between quivers, knots and Donaldson-Thomas invariants [43].

Acknowledgements The work of FN is supported by the Simons Bridge Fellowship and the Alexander von Humboldt Foundation.

Appendix

We summarize the definitions and some property of the special functions we use throughout the main text. The (infinite) $\mathfrak{q}$-Pochhammer symbol or $\mathfrak{q}$-factorial is defined by

$$(x;\mathfrak{q})_\infty \equiv \prod_{k\geq 0}(1-\mathfrak{q}^k x)\ , \quad |\mathfrak{q}|<1\ , \tag{24}$$

and it can be extended to $|\mathfrak{q}|>1$ by means of $(\mathfrak{q}x;\mathfrak{q})_\infty \to \frac{1}{(x;\mathfrak{q}^{-1})_\infty}$. The short Jacobi Theta function is defined by

$$\Theta(x;\mathfrak{q}) \equiv (x;\mathfrak{q})_\infty(\mathfrak{q}x^{-1};\mathfrak{q})_\infty\ . \tag{25}$$

Nekrasov's function is defined according to

$$N_{\mu\nu}(x;\mathfrak{q},\mathfrak{t}) \equiv \prod_{(i,j)\in\mu}(1-x\mathfrak{q}^{\mu_i-j}\mathfrak{t}^{\nu^\vee_j-i+1})\prod_{(i,j)\in\nu}(1-x\mathfrak{q}^{-\nu_i+j-1}\mathfrak{t}^{-\mu^\vee_j+i})\ , \tag{26}$$

where μ,ν are integer partitions or Young diagrams (i.e. $\mu_i\geq\mu_{i+1}\geq 0$, $\nu_i\geq\nu_{i+1}\geq 0$) parametrized by the coordinates (i,j) of boxes running over the rows and columns respectively, with $^\vee$ the transpose operation. Its properties and diverse representations can be found in [44].

References

1. O. Aharony, O. Bergman, D.L. Jafferis, J. Maldacena, N = 6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals. JHEP **10**, 091 (2008). (arXiv:0806.1218 [hep-th])
2. O. Aharony, O. Bergman, D.L. Jafferis, Fractional M2-branes. JHEP **11**, 043 (2008). (arXiv:0807.4924 [hep-th])
3. A. Kapustin, B. Willett, I. Yaakov, Exact results for Wilson Loops in Superconformal Chern-Simons theories with matter. JHEP **03**, 089 (2010). (arXiv:0909.4559 [hep-th])
4. N. Drukker, D. Trancanelli, A supermatrix model for N = 6 super Chern-Simons-matter theory. JHEP **02**, 058 (2010). (arXiv:0912.3006 [hep-th])

5. M. Marino, P. Putrov, Exact results in ABJM theory from topological strings. JHEP **06**, 011 (2010). (arXiv:0912.3074 [hep-th])
6. H. Awata, S. Hirano, M. Shigemori, The partition function of ABJ theory. PTEP **2013**, 053B04 (2013). arXiv:1212.2966 [hep-th]
7. N. Drukker, M. Marino, P. Putrov, From weak to strong coupling in ABJM theory. Commun. Math. Phys. **306**, 511–563 (2011). (arXiv:1007.3837 [hep-th])
8. N. Drukker, M. Marino, P. Putrov, Nonperturbative aspects of ABJM theory. JHEP **11**, 141 (2011). (arXiv:1103.4844 [hep-th])
9. M. Marino, P. Putrov, ABJM theory as a Fermi gas. J. Stat. Mech. **1203**, P03001 (2012). (arXiv:1110.4066 [hep-th])
10. Y. Hatsuda, M. Marino, S. Moriyama, K. Okuyama, Non-perturbative effects and the refined topological string. JHEP **09**, 168 (2014). (arXiv:1306.1734 [hep-th])
11. C. Vafa, *Non-Unitary Holography*. arXiv:1409.1603 [hep-th]
12. R. Dijkgraaf, B. Heidenreich, P. Jefferson, C. Vafa, Negative branes, supergroups and the signature of spacetime. JHEP **02**, 050 (2018). (arXiv:1603.05665 [hep-th])
13. T. Kimura, V. Pestun, *Super Instanton Counting and Localization*. arXiv:1905.01513 [hep-th]
14. H.-Y. Chen, T. Kimura, N. Lee, Quantum elliptic calogero-moser systems from gauge origami. JHEP **02**, 108 (2020). (arXiv:1908.04928 [hep-th])
15. H.-Y. Chen, T. Kimura, N. Lee, Quantum integrable systems from supergroup gauge theories. JHEP **09**, 104 (2020). (arXiv:2003.13514 [hep-th])
16. N. Nekrasov, Superspin chains and supersymmetric gauge theories. JHEP **03**, 102 (2019). (arXiv:1811.04278 [hep-th])
17. T. Kimura, F. Nieri, Intersecting defects and supergroup gauge theory. J. Phys. A: Math. Theor. **54**, 435401 (2021). https://doi.org/10.1088/1751-8121/ac2716
18. M. Aganagic, S. Shakirov, Knot homology and refined chern-simons index. Commun. Math. Phys. **333**(1), 187–228 (2015). (arXiv:1105.5117 [hep-th])
19. L. Cassia, M. Zabzine, *On Refined Chern-Simons and Refined ABJ Matrix Models*. arXiv:2107.07525 [hep-th]
20. N.A. Nekrasov, Seiberg-Witten prepotential from instanton counting. Adv. Theor. Math. Phys. **7**(5), 831–864 (2003). (arXiv:hep-th/0206161)
21. N. Nekrasov, A. Okounkov, Seiberg-Witten theory and random partitions. Prog. Math. **244**, 525–596 (2006). (arXiv:hep-th/0306238)
22. J. Gomis, B. Le Floch, Y. Pan, W. Peelaers, Intersecting surface defects and two-dimensional CFT. Phys. Rev. D **96**(4), 045003 (2017). arXiv:1610.03501 [hep-th]
23. Y. Pan, W. Peelaers, Intersecting surface defects and instanton partition functions. JHEP **07**, 073 (2017). (arXiv:1612.04839 [hep-th])
24. F. Nieri, Y. Pan, M. Zabzine, 3d expansions of 5d instanton partition functions. JHEP **04**, 092 (2018). (arXiv:1711.06150 [hep-th])
25. F. Nieri, Y. Pan, M. Zabzine, 3d mirror symmetry from S-duality. Phys. Rev. D **98**(12), 126002 (2018). arXiv:1809.00736 [hep-th]
26. A.N. Sergeev, A.P. Veselov, Deformed macdonald-ruijsenaars operators and super macdonald polynomials. Commun. Math. Phys. **288.2**, 653–675 (2009). arXiv:0707.3129 [math.QA]
27. F. Atai, M. Hallnäs, E. Langmann, Super-macdonald polynomials: Orthogonality and Hilbert space interpretation. Commun. Math. Phys. **388**(1), 435–468. arXiv:2103.07400 [math.QA]
28. F. Atai, M. Hallnäs, E. Langmann, Orthogonality of super-Jack polynomials and a Hilbert space interpretation of deformed Calogero-Moser-Sutherland operators. Bull. London Math Soc **51.2**, 353–370 (2019). arXiv:1802.02016 [math.QA]
29. C. Vafa, *Brane/Anti-brane Systems and U(N|M) Supergroup*. arXiv:hep-th/0101218
30. T. Okuda, T. Takayanagi, Ghost D-branes. JHEP **03**, 062 (2006). (arXiv:hep-th/0601024)
31. V. Mikhaylov, E. Witten, Branes and supergroups. Commun. Math. Phys. **340**(2), 699–832 (2015). (arXiv:1410.1175 [hep-th])
32. R. Gopakumar, C. Vafa, On the gauge theory/geometry correspondence. Adv. Theor. Math. Phys. **3**, 1415–1443 (1999). (arXiv:hep-th/9811131)

33. T. Kimura, *Instanton Counting, Quantum Geometry and Algebra*, vol. 7 (Springer, 2021). arXiv:2012.11711 [hep-th]
34. H. Awata, H. Kanno, T. Matsumoto, A. Mironov, A. Morozov, A. Morozov, Y. Ohkubo, Y. Zenkevich, Explicit examples of DIM constraints for network matrix models. JHEP **07**, 103 (2016). (arXiv:1604.08366 [hep-th])
35. Y. Zenkevich, *Higgsed Network Calculus*. arXiv:1812.11961 [hep-th]
36. J. Shiraishi, H. Kubo, H. Awata, S. Odake, A Quantum deformation of the Virasoro algebra and the Macdonald symmetric functions. Lett. Math. Phys. **38**, 33–51 (1996). arXiv:q-alg/9507034 [q-alg]
37. E. Frenkel, N. Reshetikhin, Quantum affine algebras and deformations of the Virasoro and W-algebras. Commun. Math. Phys. **178**, 237–264 (1996). arXiv:q-alg/9505025 [q-alg]
38. M. Aganagic, N. Haouzi, C. Kozcaz, S. Shakirov, *Gauge/Liouville Triality*. arXiv:1309.1687 [hep-th]
39. T. Kimura, V. Pestun, Quiver W-algebras. Lett. Math. Phys. **108**(6), 1351–1381 (2018). (arXiv:1512.08533 [hep-th])
40. B. Eynard, T. Kimura, S. Ribault, *Random Matrices*. arXiv:1510.04430 [math-ph]
41. T. Kimura, Y. Sugimoto, Topological vertex/anti-vertex and supergroup gauge theory. JHEP **04**, 081 (2020). (arXiv:2001.05735 [hep-th])
42. G. Lockhart, C. Vafa, Superconformal partition functions and non-perturbative topological strings. JHEP **10**, 051 (2018). (arXiv:1210.5909 [hep-th])
43. P. Kucharski, M. Reineke, M. Stosic, P. Sułkowski, BPS states, knots and quivers. Phys. Rev. D **96**(12), 121902 (2017). arXiv:1707.02991 [hep-th]
44. H. Awata, H. Kanno, Refined BPS state counting from Nekrasov's formula and Macdonald functions. Int. J. Mod. Phys. A **24**, 2253–2306 (2009). (arXiv:0805.0191 [hep-th])

A New S-matrix Formula and Extension of the State Space in Open String Field Theory

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Abstract In this contribution, I will describe a new S-matrix formula in Witten's open string field theory. This formula is a gauge invariant combination of a classical solution Ψ, a reference tachyon vacuum solution Ψ_T, on-shell vertex operators $\{O_j\}$, and a formal object A which satisfies $Q_\Psi A = 1$ with Q_Ψ the BRST operator around the classical solution Ψ. By considering an extension of the state space, one can interpret this formula from the viewpoint of Feynman rules with what we call the unconventional propagator. I also plan to comment on the Murata-Schnabl solution, which is a hypothetical classical solution for multiple D-branes that was proposed in 2011 but has yet to be realized, and of which a topological interpretation was claimed by some authors. I argue that the Murata-Schnabl solution is realized in this extended state space, though its topological interpretation is still not clear.

Keywords Open string field theory · S-matrix · Gauge symmetry

1 Open String Field Theory

String field theory (SFT) is a general term for formulations of string theory in the style of standard (Lagrangian) formulation of quantum field theory. Witten's open SFT [1] is the simplest of the SFTs describing open strings, whose action is given by

$$S[\Phi] = -\frac{1}{g_o^2}\left(\frac{1}{2}\int \Phi * Q\Phi + \frac{1}{3}\int \Phi * \Phi * \Phi\right). \tag{1}$$

Here, Φ is an open string field, $\Phi \in \mathcal{H}$, where $\mathcal{H}$ is the state space of the open SFT; g_o is the open string coupling constant; $*$ is a binary operation $\mathcal{H} \times \mathcal{H} \to \mathcal{H}$ called the star product; $\int$ is a map from $\mathcal{H}$ to $\mathbb{C}$; Q is the worldsheet BRST operator. The action $S[\Phi]$ is invariant under the gauge transformation

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_17

$$\Phi \to \Phi' = U(Q + \Phi)U^{-1}, \tag{2}$$

where U and U^{-1} are elements of $\mathcal{H}$ at the worldsheet ghost number zero. This theory was formulated on the model of the three-dimensional Chern-Simons theory (the CS theory), so the two have a similar algebraic structure. In particular, we note that

$$\int Q\Phi = 0 \quad \text{for} \quad \forall\Phi \in \mathcal{H}. \tag{3}$$

In contrast to the CS theory, there exist elements with negative degrees in $\mathcal{H}$, which corresponds to states with the negative worldsheet ghost number. The action (1) can be defined using correlation functions of the boundary conformal field theory (BCFT) (See Appendix). For recent reviews, see [2–5].[1]

Witten's theory is one of the most studied among the SFTs. There is a good collection of studies on, for example, classical solutions or perturbative calculations under certain gauge fixing conditions. Some studies have obtained results that could not be reached by the worldsheet theory (such as description of tachyon condensation or discovery of a new boundary condition of 2 dimensional BCFT), but it seems that this is still a small number. We are looking for clues to deepen our understanding of this theory, which is comparable to the geometric description for the CS theory.

In this contribution, we introduce a recently obtained S-matrix formula [6] and the Feynman rules with an unconventional propagator [7, 8] in Sect. 2. The next section, Sect. 3, is a conceptual one, in which we discuss our prospect that the formulas obtained in Sect. 2 may help us to understand the theory better.

2 On-shell Amplitudes from Classical Solutions

For exploring new formulas, we postulated the following[2]

> Physical quantities can be read off from the classical solution which describes the systems we are interested in.

Thus, we searched for a formula which expresses the physical quantity $\mathcal{G}$ as a function of the classical solution Ψ, $\mathcal{G} = \mathcal{G}(\Psi)$. And this $\mathcal{G}(\Psi)$ should be invariant under the gauge transformation of Ψ,

$$\mathcal{G}(\Psi) = \mathcal{G}(\Psi'), \quad \Psi' = U(Q + \Psi)U^{-1}. \tag{4}$$

[1] We note that while the literature [3] are excellent lecture notes, some of the unproven perspectives contained therein may not be consistent with our discussion.

[2] The reason we believe this postulate is that the classical solution and the action include all the information about the fluctuations around the stationary point, and the physical states of our interest (particles or strings) are described by such fluctuations.

Note that we use the term gauge invariant quantity in the sense of (4), which is slightly different from the traditional terminology.

2.1 KBc Subalgebra

There is an important sub-algebra of the $*$-algebra, called the KBc-subalgebra, which is generated by the fundamental string fields K, B, and c with the ghost number 0, -1, and 1, respectively. They satisfy the following commutation relations

$$[K, B] = 0, \quad [B, c] = 1, \quad Qc = cKc, \quad QB = K. \tag{5}$$

Note that the (graded) commutator $[\phi_1, \phi_2]$ is defined with the star product by $\phi_1 * \phi_2 - (-)^{\mathrm{gh}(\phi_1)\mathrm{gh}(\phi_2)}\phi_2 * \phi_1$. Note also that we often omit the symbol $*$ when we work with $\{K, B, c\}$.

For practical calculations of physical quantities, it is necessary to formulate K, B, and c in terms of BCFT. In such a formulation, K corresponds to the line integral of the worldsheet energy-momentum tensor, and e^{xK} is the wedge state, which is a fragment of the worldsheet. B represents the line integral of the b-ghost, and c represents an insertion of the c-ghost at the boundary of the worldsheet. We also define $O \in \mathcal{H}$ corresponding to an insertion of a boundary operator $O(x)$. We present a sketchy description of these definition in Appendix. See Ref. [2] for more comprehensive description.

2.2 An S-matrix Formula

In this subsection we will present a formula which represents S-matrix (or the on-shell scattering amplitudes) around the D-brane configuration represented by a classical solution Ψ. The inputs of this S-matrix formula are the initial states and the final states of open strings, which we denote all together by the external states $\{\mathcal{O}_j\}$ ($j = 1, ..., N$). Here $\mathcal{O}_j$ is a ghost-number-one state satisfying the physical state condition around the classical solution Ψ,

$$Q_\Psi \mathcal{O}_j = 0. \tag{6}$$

In addition, we need to choose a reference tachyon vacuum solution Ψ_T.

Our S-matrix formula then reads

$$S_\Psi(\mathcal{O}_1, ..., \mathcal{O}_N) = \frac{(-1)^{N-1}}{N-2} \sum{}' \int \prod_{j=1}^{N} (A + W_\Psi)\mathcal{O}_j, \tag{7}$$

where

- A is given by $A = A_T - A_\Psi$, where A_T is a so-called homotopy state for a reference tachyon vacuum solution Ψ_T, and A_Ψ is a formal homotopy state for BRST operator around the classical solution Q_Ψ, satisfying $Q_\Psi A_\Psi = 1$;
- W_Ψ is given by
$$W_\Psi = A_T(\Psi - \Psi_T) + (\Psi - \Psi_T)A_T; \tag{8}$$
and
- $\sum'$ represents the symmetrization over $\{\mathcal{O}_j\}$.

In particular, if we consider the trivial solution $\Psi = 0$ (which represents the D-brane(s) which is used to define the theory) we obtain

$$W_{\Psi=0} = -e^K, \quad A = -e^K \frac{B}{K}. \tag{9}$$

Note that the inverse of K cannot be defined in terms of BCFT (as it cannot be expressed as a superposition of wedge states), but a prescription for calculation is given in Ref. [6].

Example

Let us consider 4-point amplitude around $\Psi = 0$. The right hand side of (7) becomes ($W \equiv W_{\Psi=0}$)

$$\begin{aligned}-\frac{1}{2}\Bigg(&\int A\mathcal{O}_1 W\mathcal{O}_2 W\mathcal{O}_3 W\mathcal{O}_4 + \int W\mathcal{O}_1 A\mathcal{O}_2 W\mathcal{O}_3 W\mathcal{O}_4 \\ &+ \int W\mathcal{O}_1 W\mathcal{O}_2 A\mathcal{O}_3 W\mathcal{O}_4 + \int W\mathcal{O}_1 W\mathcal{O}_2 W\mathcal{O}_3 A\mathcal{O}_4\Bigg) \\ &+ \text{circular permutations with respect to } \{1, 2, 3, 4\}.\end{aligned} \tag{10}$$

If Ψ_T is the Schnabl solution, then

$$A_T = B \int_0^1 e^{xK} dx. \tag{11}$$

The effect of A_0 can be regarded as a minimal subtraction of divergence caused by collisions of boundary operators (see [6] for details). The following then holds

$$\begin{aligned}&\int A\mathcal{O}_1 W\mathcal{O}_2 W\mathcal{O}_3 W\mathcal{O}_4 \\ &= \left[\int_0^1 dx\, \langle B\mathcal{O}_1(z_1)\ \mathcal{O}_2(z_2)\ \mathcal{O}_3(z_3)\ \mathcal{O}_4(z_4)\,\rangle_{C_{3+x}}\right]_{\text{minimal subtraction}},\end{aligned} \tag{12}$$

where

$$z_i = \frac{x-5}{2} + i \quad (i = 1, 2, 3, 4) \tag{13}$$

and $\langle ... \rangle_{C_r}$ denotes the correlation function on the semi-infinite cylinder C_r with circumference r,

$$C_r : \{z \mid \mathrm{Im}(z) \geq 0,\ |\mathrm{Re}(z)| \leq \tfrac{r}{2}\}/z \sim z + r. \tag{14}$$

To make (12) conform to the traditional expression for the S-matrix, we perform a conformal transformation from C_{3+x} to UHP (the upper half plane). By doing the same for all terms of (10), we can confirm that (10) is the on-shell 4-point amplitude.

2.3 Feynman Rules with an Unconventional Propagator

We also found that the on-shell amplitudes can be calculated correctly using Feynman rules with the following propagator

$$\mathcal{P}_\flat \phi = \frac{A}{2W_\Psi} * \phi + (-1)^{\mathrm{gh}(\phi)} \phi * \frac{A}{2W_\Psi}. \tag{15}$$

Here ϕ denotes a test state. In the discussion of Ref.[7], the external line φ_j for this Feynman rule is chosen as follows

$$\varphi_j = \sqrt{-W_\Psi}\, \mathcal{O}_j \sqrt{-W_\Psi}. \tag{16}$$

In particular, the propagator around the perturbative vacuum (when Ψ_T is the Schnabl solution) is given by

$$\mathcal{P}_\flat \phi = \frac{B}{2K} \frac{1}{\sqrt{e^K}} * \phi + (-1)^{\mathrm{gh}(\phi)} \phi * \frac{B}{2K} \frac{1}{\sqrt{e^K}}. \tag{17}$$

The inverse of the wedge state in (17) is canceled by the wedge state coming from (16) when calculating Feynman diagrams. Also, the inverse of K can be handled in the same way for that in (9).

Originally, the propagator (15) was found in Ref. [8] by using a general theory based on A_∞ and the homological perturbation, but the argument is formal. In particular, this argument does not give a good definition of the inverse of K, and therefore does not provide sufficient grounds for extension of this Feynman rule to off-shell or loop amplitudes.[3]

[3] As shown in Ref. [7], this propagator is related to that of the Schnabl gauge, and therefore, it is quite possible that the extension to loops requires a special treatment (regularization) as in the case of the Schnabl gauge.

3 $1/K$ May Be a Key to Understanding the Theory

Extension of the formulas (7) and (15) to off-shell or loop amplitudes is a natural question, which can be rephrased as the question whether the undefined object $1/K$ can be defined in the case where there exist off-shell boundary operators. We believe that this question is related to a better understanding of the classical solutions and to the geometrical interpretation of the theory.

3.1 Extension of the State Space

Let us assume that we can define an extended state space $\hat{\mathcal{H}} \supset \mathcal{H}$, which serves as a basis for the S-matrix formula (7) and the propagator (15). In particular, $\hat{\mathcal{H}}$ contains B/K. In this extended space $\hat{\mathcal{H}}$, we must give up the property that corresponds to the axiom (3),

$$\int Q\Phi \neq 0 \quad \text{in general}, \quad \Phi \in \hat{\mathcal{H}}. \tag{18}$$

This is because acting Q on B/K gives 1, and any Q-closed state can be written as Q-exact. The only solution to ensure that the physical state does not drop out of the S-matrix is to assume that the surface term in (18) is non-zero. This also means that considering $1/K$ necessarily implies an extension of the state space, because it can be shown from the BCFT argument that the conventional states in $\mathcal{H}$ satisfies (3).

Incidentally, we wish to comment on the sliver state $e^{\infty K}$, which is a wedge state with infinite width, $e^{\infty K} = \lim_{x\to\infty} e^{xK}$. The sliver state is often used in the past researches about $1/K$ or classical solutions. However, sliver state has some serious problems. One of the problems is that the value of the correlation function involving multiple sliver states cannot be well defined. Also, accordingly, the projector property (i.e. $e^{\infty K} * e^{\infty K} = e^{\infty K}$) is broken.

We note that our discussion of the extension of the state space is free from these problems because we do not need to consider the sliver states.

3.2 Analogy with a Universal Cover of a Manifold

In this subsection, we wish to draw an analogy between the extended state space $\hat{\mathcal{H}} \supset \mathcal{H}$ and a universal cover of a manifold. Let $\Omega^k(M)$ be the space of k-forms on a differentiable manifold M and $d_k : \Omega^k(S^1) \to \Omega^{k+1}(S^1)$ the exterior derivative. The de Rham cohomology of M is given by $H^k_{\mathrm{dR}}(M) = \mathrm{Ker}(d_k)/\mathrm{Im}(d_{k+1})$.

As a simplest example, consider S^1 and its universal covering $p : \mathbb{R} \to S^1$. We see the relationship between $\mathcal{H}$ and $\hat{\mathcal{H}}$ as analogous to $\Omega^k(S^1)$ and $\Omega^k(\mathbb{R})$.[4] Since the universal covering space $\mathbb{R}$ is contractible, from the Poincaré lemma, any closed form in $\Omega^k(\mathbb{R})$ is exact. This is similar to the story that BRST cohomology is trivial in $\hat{\mathcal{H}}$, which includes B/K.

Also, we can construct an analogue of (18) using multivalued functions on S^1. Let us parameterize S^1 with $0 \le \theta < 2\pi$ and consider a function $f(\theta) = \theta$ as an example. By extending its domain, $f(\theta)$ can be regarded as an element of $\Omega^0(\mathbb{R})$. Instead, we regard it as a branch of a multivalued function on S^1. Now, the integral of $df = d\theta$ over S^1 is not zero although df is an exact 1-form in $\Omega^1(\mathbb{R})$: $\int_0^{2\pi} d\theta = 2\pi \quad (\neq 0)$.

The story described above might be related to the problem of defining the winding number in open SFT [9], studied in connection with the Murata-Schnabl solution [10]. Murata-Schnabl solution will be briefly introduced in the next subsection.

3.3 Classical Solutions

Some simple correlation functions including $1/K$ is determined by solving the functional equations from the consistency condition. As a result, the energy of the simplest case of the Murata-Schnabl solution is uniquely determined,

$$\Psi_{\text{double}} = \frac{1}{K} c \frac{K^2 B}{K-1} c. \tag{19}$$

This solution was expected to represent two D-branes, and the energy density is consistent with this interpretation. Note that this solution appears to live in the extended state space, and its interpretation may need to be made carefully. Also, the geometrical interpretation of Witten's theory is still unclear.

Also, in Ref. [11], Ellwood presented the idea of using singular gauge transformations to understand classical solutions, and the extended state space may shed new light on this perspective.

Appendix

Here, we shall provide a sketch of the definition of the action (1) and $\{K, B, c\}$ using BCFT. Let ξ (Im $\xi \ge 0$) denote the coordinate system on the upper half plane (UHP), which is commonly used to describe open strings using BCFT. The local boundary operator is placed at the origin $\xi = 0$ and the state appears on the unit semicircle.

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[4] However, $\Omega^k(S^1)$ is not a subspace of $\Omega^k(\mathbb{R})$. In this sense, it may be more appropriate to state that $\mathcal{H}$ corresponds to the pullback of $\Omega^k(S^1)$ by p^* rather than $\Omega^k(S^1)$ itself, when we consider the relation $\hat{\mathcal{H}} \supset \mathcal{H}$.

It is convenient to map this local coordinate to that on C_2, which we call *the sliver coordinate*, by

$$z = f_s(\xi) = \frac{2}{\pi} \arctan \xi, \quad -1 \leq \mathrm{Re}\, z < 1. \tag{20}$$

Now, each term of (1) can be defined via a correlation function on C_n:

$$\int \phi_1^{\text{state}} * \phi_2^{\text{state}} = \left\langle f_s^{(-\frac{1}{2})} \circ \phi_1(0)\, f_s^{(\frac{1}{2})} \circ \phi_2(0) \right\rangle_{C_2} \tag{21}$$

$$\int \phi_1^{\text{state}} * \phi_2^{\text{state}} * \phi_3^{\text{state}} = \left\langle f_s^{(-1)} \circ \phi_1(0)\, f_s \circ \phi_2(0)\, f_s^{(1)} \circ \phi_3(0) \right\rangle_{C_3} \tag{22}$$

where $f_s^{(x)}(\xi) \equiv f_s(\xi) + x$. Note that this characterization gives definition of the star product implicitly.

Three basic building blocks $\{K, B, c\}$ are now defined by

$$K = \int_{i\infty}^{-i\infty} \frac{dz}{2\pi i} T(z)|\mathrm{id}\rangle, \quad B = \int_{i\infty}^{-i\infty} \frac{dz}{2\pi i} b(z)|\mathrm{id}\rangle, \quad c = c(\tfrac{1}{2})|\mathrm{id}\rangle, \tag{23}$$

where we have used the doubling trick to define $T(z)$ and $b(z)$ for $\mathrm{Im}(z) < 0$; $|\mathrm{id}\rangle$ is the identity of the star-product, $\lim_{x \to 0} e^{xK}$.

References

1. E. Witten, Nucl. Phys. B **268**, 253–294 (1986)
2. Y. Okawa, Prog. Theor. Phys. **128**, 1001–1060 (2012)
3. T. Erler, arXiv:1912.00521 [hep-th]
4. J. Vošmera, Doctoral thesis, Charles University, 2020 (Available via Charles University Digital Repository, 2020). http://hdl.handle.net/20.500.11956/120130
5. E. Fuchs, M. Kroyter, Phys. Rept. **502**, 89–149 (2011)
6. T. Masuda, H. Matsunaga, Prog. Theor. Exp. Phys. **2022**(1), 013B06 (2022)
7. T. Masuda, H. Matsunaga, T. Noumi, Prog. Theor. Exp. Phys. **2022**(1), 013B07 (2022)
8. T. Masuda, H. Matsunaga, arXiv:2003.05021 [hep-th]
9. H. Hata, T. Kojita, J. High Energy Phys. **01**, 088 (2012)
10. M. Murata, M. Schnabl, J. High Energy Phys. **07**, 063 (2012)
11. I. Ellwood, J. High Energy Phys. **05**, 037 (2009)

Dual Dilaton with $\mathcal{R}$ and $\mathcal{Q}$ Fluxes

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Abstract In previous works we showed that a Courant algebroid in a particular frame and the differential geometry of the sum bundle $TM \oplus T^*M$ provide a very natural geometric setting for a sector of the low energy effective limit of type II superstring theories (Supergravity theory). Given our geometric and algebraic considerations, we reproduced the NS-NS sector of the closed bosonic effective type II sting action, and an action for the inverse metric G^{-1} and the bivector Π, related to the tensors for closed strings as $(g + B)^{-1} = (G^{-1} + \Pi)$. The action depended on the stringy T-dual fluxes $\mathcal{R}$ and $\mathcal{Q}$, but the dual dilaton was missing. This short paper fills the gap.

Keywords Algebroids · Differential geometry · Supergravity models · Lagrangian theories

1 Introduction

The geometric setting for string effective actions and ultimately Supergravity is notably the generalized tangent bundle $TM \oplus T^*M$, as recognized in [8]. Geometric actions with fluxes were later constructed in the context of Generalized Geometry and Double Field Theory [1–3]. The results acquired a further meaning thanks to the Lie algebroid arguments of [4, 5]. In a recent work [7] we suggested a curvature scalar for the target space metric seen by the open strings. The T-dual fluxes $\mathcal{R}$ and $\mathcal{Q}$ were naturally encoded in the expression for the curvature scalar. To derive our result in [7] we relied not only on a Courant algebroid and on the differential geometry of the sum bundle $TM \oplus T^*M$, but also on the canonical correspondence with a dg-symplectic manifold of degree 2. Relevance was given to the graded description as it allows for a neat extrapolation of the underlying 2-cocycle structure of the Courant algebroid (the 1-gerbe $B \in \Omega^2(M)$). To keep the current paper short we will not

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_18

present this viewpoint here. Moreover, in the graded Poisson structure it is believed that the dilaton should account for an ambiguity in the quantization of the sheaf of graded functions [9].

The construction of the curvature invariant goes as follows. Requiring that the space of sections of the direct sum bundle $TM \oplus T^*M$ could have two bilinear products, namely a Courant algebroid bracket and a Lie-type of bracket, is enough to leave an affine metric connection, with totally skew-symmetric torsion, completely defined. A connection on sections of some subbundles, in particular 1-forms in $\Gamma(T^*M)$, can be defined too: One must just ask that a splitting $r : T^*M \mapsto TM \oplus T^*M$ of the short exact sequence $0 \mapsto TM \hookrightarrow TM \oplus T^*M \twoheadrightarrow T^*M \mapsto 0$ can be a morphism between vector bundles with connections, and apply it to the fully fledged connection on generalized tangent vectors. The curvature scalar of this "smaller" connection is retained in the usual fashion, i.e. by contracting the free indices of the commutator of covariant derivatives. In a similar way, with a split s of the exact Courant algebroid sequence, $s : TM \mapsto TM \oplus T^*M$ and with the associated connection on vector fields, the related curvature scalar can reproduce the Lagrangian for the NS-NS closed bosonic sector of the effective string action, if the appropriate choices are made.

In the body of the article we will apply this method starting with a particular choice of frame that depends on all the relevant fields, especially the dual dilaton, so far missing.

2 Dual Dilaton

A dual dilaton can be introduced by requiring that the volume form rescaled with the dilaton remains invariant [3], according to the formula:

$$e^{-2\phi}\sqrt{\det g}\,\mathrm{d}x = e^{-2\tilde{\phi}}\sqrt{\det G^{-1}}\,\mathrm{d}x. \tag{1}$$

An old lemma by J. Moser [10] states that this is the case if one can find a diffeomorphism of a n-dimensional compact oriented manifold with itself, for which the n-cycles are preserved. We will heavily rely on the relation (1) in our derivation.

2.1 The Courant Algebroid and Its Connection

Let us implement the observation about the dual dilaton in a vielbein $\mathcal{E}$ for $TM \oplus T^*M$, as displayed below:

$$\mathcal{E} = e^{\kappa\phi}\begin{pmatrix} \mathbb{1} & \gamma(G^{-1} - \Pi) \\ -(g+B) & \gamma\mathbb{1} \end{pmatrix}, \quad \kappa \in \mathbb{R}, \tag{2}$$

where $g(x) \in S^2(M)$ and non-degenerate, $B(x) \in \Omega^2(M)$ and $G^{-1}(x) - \Pi(x) \equiv (g(x) - B(x))^{-1}$ (open-closed string metrics relations, where G^{-1} is the symmetric part and Π a bivector). Besides, γ is

$$\gamma := \sqrt{\frac{\det g}{\det G^{-1}}}.$$

For the sake of completeness, let us just glance at how $\mathcal{E}$ transforms a basis $(e_i \oplus \tilde{e}^i)$ into another basis:

$$(e_i \oplus \tilde{e}^i) \mapsto e^{\kappa\phi}\big((e_i - (g-B)_{ij}\tilde{e}^j) \oplus \gamma((g+B)^{-1\,ij} e_j + \tilde{e}_i)\big)$$

We will exclusively work with the basis induced on $TM \oplus T^*M$ by the local chart on M. Notice moreover that if the bundle had an $O(d,d)$ structure group, then the vielbein would be reducing the structure group to $O(d) \times O(d) \times \mathbb{R}$.

Suppose that the vector bundle $TM \oplus T^*M$ encodes actually the standard exact Courant algebroid $(TM \oplus T^*M, \langle -,-\rangle, [-,-]_{\mathrm{D}}, \rho)$ (see e.g. [11]). If we take sections $U = X + \alpha$ and $V = Y + \beta$, the canonical choices for the pairing, the Dorfman bracket and the "anchor map" ρ are:

$$\langle U, V\rangle = \iota_X\beta + \iota_Y\alpha, \tag{3}$$

$$[U, V]_{\mathrm{D}} = [X, Y]_{\mathrm{Lie}} + \mathcal{L}_X\beta - \iota_Y \mathrm{d}\alpha, \tag{4}$$

$$\rho(U) = \mathrm{pr}(U) = X. \tag{5}$$

Then $\mathcal{E}$ induces a Courant algebroid homomorphism with $(TM \oplus T^*M, [-,-]_{\mathrm{D}}, \mathcal{G}, \rho)$, where $\mathcal{E}([U,V]_{\mathrm{D}}) = [\mathcal{E}U, \mathcal{E}V]_{\mathrm{D}}$,[1] and the pairing becomes

$$\mathcal{G} \equiv e^{2\kappa\phi}\begin{pmatrix} -2\,g(x) & 0 \\ 0 & 2G^{-1}(x)\gamma^2 \end{pmatrix}, \tag{6}$$

whereas

$$\rho(U) = e^{\kappa\phi}(X + \gamma(G^{-1} + \Pi)(\alpha)) \tag{7}$$

is the new anchor map.

Let us think of $\Gamma(TM \oplus T^*M)$ as a bimodule, for a Lie-like bracket $[\![-,-]\!]$ satisfying Jacobi identity, anti-symmetry and $\mathbb{R}$-linearity. Among all the possibilities, and in the basis induced by the coordinates on M, we choose it to be:

$$[\![U, V]\!] = \big(\rho(U)Y^i - \rho(V)X^i\big)\,\partial_i \oplus (\rho(U)\beta_i - \rho(V)\alpha_i)\,\mathrm{d}x^i. \tag{8}$$

The map $\rho : \Gamma(TM \oplus T^*M) \mapsto \Gamma(TM)$ is the aforementioned anchor. Then an affine connection on the sections of the Courant algebroid, with completely skew torsion, and metric with respect to the Courant algebroid pairing, can be extracted in the following way, from the difference of the two brackets:

[1] More explicitly, the Dorfman bracket on the LHS is just written in the new basis.

$$\langle [U, V]_{\mathrm{D}} - [\![U, V]\!], W\rangle = \langle \nabla_W U, V\rangle, \quad W \in \Gamma(TM \oplus T^*M). \tag{9}$$

Conventionally Courant algebroid connections are represented by ∇_- rather than $\nabla_{\rho(-)}$, i.e. it is usual to not display the anchor map.

The interesting non-flat connection arises when working in the $\mathcal{E}$ basis for $[-, -]_{\mathrm{D}}$:

$$\mathcal{G}\left([U, V]_{\mathrm{D}} - [\![U, V]\!], W\right) = \mathcal{G}\left(\nabla_W U, V\right).$$

However we will not need a fully-fledged connection, as we want to look at the dual vector spaces TM and T^*M, dual with respect to the canonical pairing of vector fields with 1-forms. For a specific instance of κ and dimension of the base manifold M, the TM case was already studied in [6]. We will comment on this later. Let us hence focus on T^*M. For our current purpose, we will inspect the short exact sequence:

$$0 \mapsto TM \overset{\Delta}{\hookrightarrow} TM \oplus T^*M \overset{\Delta^*}{\twoheadrightarrow} T^*M \mapsto 0.$$

Here, Δ embeds TM into $TM \oplus T^*M$ in the following way:

$$\Delta(X) = \frac{1}{2} e^{-\kappa\phi}\left(g^{-1}(g + B)(X) + \gamma^{-1} G(X)\right). \tag{10}$$

Using the Courant algebroid metric $\mathcal{G}$ to identify $TM \oplus T^*M$ with its dual, we get the surjective map $\Delta^* : \Gamma(TM \oplus T^*M) \twoheadrightarrow \Gamma(T^*M)$, which sends sections of the generalized tangent bundle into sections of the cotangent bundle in this way:

$$\Delta^*(X + \alpha) = e^{\kappa\phi}\left(-(g - B)(X) + \gamma\alpha\right).$$

A closer inspection should suffice to convince oneself that $\ker(\Delta^*) = \mathrm{im}(\Delta)$, as it should be.

The short exact sequence can be split with the help of $r : \Gamma(T^*M) \mapsto \Gamma(TM \oplus T^*M)$,

$$r(\alpha) = e^{-\kappa\phi}\gamma^{-1}\alpha. \tag{11}$$

Now one can demand to work with a connection on covectors by implementing only r-generalized vectors in the formula for the connection (9):

$$\mathcal{G}\left([r(\alpha), r(\beta)] - [\![r(\alpha), r(\beta)]\!], r(\nu)\right) \equiv \mathcal{G}(\nabla_r(\nu) r(\alpha), r(\beta)) =: 2G^{-1}(\widetilde{\nabla}_{\rho(\nu)}\alpha, \beta). \tag{12}$$

Here we defined $\widetilde{\nabla} : \Gamma(T^*M) \mapsto \Gamma(TM \otimes T^*M)$ as

$$\nabla_{r(\nu)} r(\alpha) =: r(\widetilde{\nabla}_{\rho(\nu)}\alpha)$$

and used the induced metric $\mathcal{G}(r(-), r(-)) = 2G^{-1}(-, -)$.

Now the splitting (11) combined with the Courant algebroid bracket will exactly forget all the scalar factors so that, upon a clever extraction of some terms resulting from $[\![r(\alpha), r(\beta)]\!]$, one obtains, combining all the expressions together, the connection for the case $\phi = 0, \gamma = 1$ in the coordinate basis:

$$2G^{kl}\boldsymbol{\Gamma}^{ij}{}_k = \left(G^{-1}+\Pi\right)^{im}\partial_m\left(G^{-1}+\Pi\right)^{jl} + 2\left(G^{-1}+\Pi\right)^{[j|m}\partial_m\left(G^{-1}+\Pi\right)^{|i]l}. \tag{13}$$

Of course this is not the end of the story, as some additional terms from the Lie-like bracket due to the derivative $\rho(r(-))$ hitting the scalar factors have not been unveiled yet. Let us denote

$$-\partial D_\kappa = e^{\kappa\phi}\gamma\partial(e^{-\kappa\phi}\gamma^{-1}) = -\kappa\partial\phi - \frac{1}{2}g^{ln}\partial g_{ln} + \frac{1}{2}G_{ln}\partial G^{ln}. \tag{14}$$

Eventually, in the holonomic coordinate basis, the connection coefficients can be checked to be:

$$\boldsymbol{\Gamma}^{ij}{}_k + \left(\delta^i_k\left(G^{-1}+\Pi\right)^{jm} - G_{pk}G^{ij}\left(G^{-1}+\Pi\right)^{pm}\right)\partial_m D_\kappa \equiv \boldsymbol{\Gamma}^{ij}{}_k + T^{ij}{}_k. \tag{15}$$

The partial derivatives are the result of differentiating $e^{\kappa\phi}$ and γ. For the sake of convenience the last two summands are called $T^{ij}{}_k \in C^\infty(\vee^2 TM \wedge T^*M)$. $T^{ij}{}_k$ is antisymmetric in j, k and symmetric in i, k and j, i respectively.

2.2 The Curvature Scalar

A curvature scalar built from the Riemann curvature tensor Riem:

$$\mathrm{Riem}(U, V, W) = [\nabla_U, \nabla_V]\, W - \nabla_{[\![U,V]\!]} W, \quad U, V, W \in \Gamma(TM \oplus T^*M)$$

yields an invariant Lagrangian. On the r-sections of course the Riemann curvature tensors for the connections ∇ and $\widetilde{\nabla}$ are related by $r(\mathrm{Riem}(U, V, W)) = \mathrm{Riem}(r(U), r(V), r(W))$. We want to focus especially on the Ricci curvature tensor of the connection (15) contracted with the non-symmetric combination $G^{-1} - \Pi$:

$$\mathrm{Riem}_m{}^{lij}\delta^m_i G_{lp}\left(G^{-1}-\Pi\right)^{pq} G_{qj}. \tag{16}$$

For the formula of the Riemann curvature due to $\boldsymbol{\Gamma}^{ij}{}_k$ we refer to our previous paper [7]. Now let us instead establish how $T^{ij}{}_k$ contributes. Its contribution can be written as:

$$-T^{mk}{}_i G_{mp} G_{qk} \widetilde{\nabla}^i_{\phi_0,\gamma_1} \left(G^{-1} - \Pi\right)^{pq}$$
$$+\left(\mathrm{Tor}^{im}{}_l T^{lk}{}_i + 2T^{[i|l}{}_i T^{|m]k}{}_l\right) \left(G^{-1} - \Pi\right)^{pq} G_{mp} G_{qk} \tag{17}$$

In the above expression, $\widetilde{\nabla}^i_{\phi_0,\gamma_1}$ is the connection when $\phi = 0$, $\gamma = 1$, which has the totally antisymmetric torsion Tor^{ijk}:

$$\mathrm{Tor}^{ijk} = 2\mathcal{R}^{ijk} + 2\mathcal{Q}^{ij}{}_l G^{lk} + 4G^{r[i|}\mathcal{Q}^{|j]k}{}_r. \tag{18}$$

Here we have introduced the fluxes in the Supergravity frame (which is the frame with metric g, 2-form B and $\phi \in C^\infty(M)$ solving the bosonic part of the Supergravity action). Their local expressions are:

$$\mathcal{R}^{ijk} := 3\Pi^{[i|m}\partial_m \Pi^{|jk]}, \tag{19}$$
$$\mathcal{Q}^{ij}{}_k := \partial_k \Pi^{ij}. \tag{20}$$

For the first term in (17) we have integrated by part against the volume form and used that $\widetilde{\nabla}^i_{\phi_0,\gamma_1}$ is a metric connection for G. Then $T^{mki} = -T^{mik}$ dictates that the covariant derivative of Π must be completely antisymmetrized.[2] However, since T is symmetric in the first two slots, this term drops out of the expression.

The second term in (17) is contracted just with the bivector. Disclosing the torsion in its components, the final result is:

$$2\left(G^{-1} + \Pi\right)^{st} \partial_t D_\kappa \left[G_{si}(\mathcal{R}^{imk} - \mathcal{Q}^{imk} + \mathcal{Q}^{ikm})\Pi_{mk} + \mathcal{Q}^{mk}{}_s \Pi_{mk}\right]. \tag{21}$$

As of the remaining bit, it yields the following:

$$(1-d)(d-2)\left((G^{-1} + \Pi)^{ki}\partial_i D_\kappa\right)^2 \tag{22}$$

We can now combine all the pieces together and couple the curvature scalar to $e^{-2\tilde{\phi}}\sqrt{\det G^{-1}}$. In doing so, one must not forget about integration by parts with $\widetilde{\nabla}_{\phi_0,\gamma_1}$, as this hits $e^{-2\tilde{\phi}}$. Before going on with the result, let us follow the strategy of our work [6]: there, the kinetic term for the dilaton in 10 dimensions turned out to be accounted by a conformal factor of $e^{-2\phi/3}$, which rescaled the Riemannian metric g. Thus replacing $G \mapsto e^{2\tilde{\phi}/3}G$ twice in (16) and taking care of the square root of its determinant, but still displaying d dimensions and κ factors though these are actually fixed:

$$e^{-2\tilde{\phi}}\sqrt{\det G^{-1}} \equiv e^{(-4\kappa + d\kappa)\tilde{\phi}}\sqrt{\det G^{-1}}.$$

[2] With a very little effort one can notice that

$$\widetilde{\nabla}^{[i}_{\phi_0,\gamma_1}\Pi^{pq]} = \partial^{[i}\Pi^{pq]} + \mathrm{Tor}^{ip}{}_l \Pi^{lq} - \mathrm{Tor}^{iq}{}_l \Pi^{lp} + \mathrm{Tor}^{pq}{}_l \Pi^{li}.$$

.

Furthermore, $\partial D_\kappa \equiv e^{\kappa\tilde{\phi}}\partial e^{-\kappa\tilde{\phi}} = -\kappa\partial\tilde{\phi}$, so we get:

$$-\kappa^2(1-d)(d-4)\left((G^{-1}+\Pi)^{ki}\partial_i\tilde{\phi}\right)^2 . \tag{23}$$

Putting (21), (22) and (23) together in the action S, we get

$$\begin{aligned} S = \int \mathrm{d}x\, \sqrt{\det G^{-1}} e^{(d-4)\kappa\tilde{\phi}} \big[& 2\kappa^2(1-d)\left((G^{-1}+\Pi)^{ki}\partial_i\tilde{\phi}\right)^2 + \\ & -2\kappa\left(G^{-1}+\Pi\right)^{st}\partial_t\tilde{\phi}\left[G_{si}(\mathcal{R}^{imk}-2\mathcal{Q}^{imk})\Pi_{mk}+\mathcal{Q}^{mk}{}_s\Pi_{mk}\right] \\ & + \mathrm{R}_G - \frac{1}{12}\mathcal{R}^2 - \frac{1}{2}\mathcal{R}^{lmi}\mathcal{Q}^{jk}{}_i G_{lj}G_{mk} - \frac{1}{4}\mathcal{Q}^{jl}{}_m\mathcal{Q}^{kn}{}_i G_{jk}G_{ln}G^{mi} \\ & - \frac{1}{2}\mathcal{Q}^{lj}{}_k\mathcal{Q}^{km}{}_l G_{jm}\big]. \end{aligned} \tag{24}$$

In this formula, R_G is the Ricci curvature scalar of the symmetric part of the connection. Remarkably, compared to literature on the topic, it does not simply come from a tensor constructed with the G metric only, but it entails the bivector Π, because of the anchor map (7).

In particular, in 10 dimensions and when $\kappa = -1/3$, the kinetic term for the dual dilaton is:

$$-2\left((G^{-1}+\Pi)^{ki}\partial_i\tilde{\phi}\right)^2 .$$

3 Conclusion

In this short note we suggested a way to embed the dual dilaton in the construction of an invariant action for the open strings metric and the T-dual fluxes $\mathcal{R}$ and $\mathcal{Q}$. Our findings complement our previous works on the same topic [6, 7]. More importantly, one of the features of our particular Ansatz is that it encompasses the effective closed string action for the NS-NS fields: This is obtained when one deploys a splitting $s : \Gamma(TM) \mapsto \Gamma(TM \oplus T^*M)$, $\rho \circ s = \mathrm{id}_{TM}$ to focus on vector fields in (9), and then builds the curvature invariant according to the same technique.

Thus with the present work we think to have provided a new clear algebraic and geometric picture for the background fields for strings and the local fluxes.

Acknowledgements Thanks to prof. Peter Schupp from Jacobs University Bremen for bringing the dual dilaton in gravity with fluxes to my attention. Thanks to dr. Afentoulidis Almpanis for discussion. A warm thanks to the organizers of "Lie Theory and its Applications in Physics" 2021 (LT-14) for this very interesting workshop oriented at strengthening the bonds between representation theory, algebra and geometry with mathematical physics. I acknowledge GAČR EXPRO 19-28628X for financial support.

References

1. D. Andriot, A. Betz, JHEP **12**, 083 (2013)
2. D. Andriot, O. Hohm, M. Larfors, D. Lust, P. Patalong, Fortsch. Phys. **60**, 1150–1186 (2012)
3. D. Andriot, O. Hohm, M. Larfors, D. Lust, P. Patalong, Phys. Rev. Lett. **108** (2012)
4. R. Blumenhagen, A. Deser, E. Plauschinn, F. Rennecke, Phys. Lett. B **720**, 215–218 (2013)
5. R. Blumenhagen, A. Deser, E. Plauschinn, F. Rennecke, C. Schmid, Fortsch. Phys. **61**, 893–925 (2013)
6. E. Boffo, P. Schupp, JHEP **01**, 007 (2020)
7. E. Boffo, P. Schupp, JHEP **12**, 143 (2021)
8. A. Coimbra, C. Strickland-Constable, D. Waldram, JHEP **11**, 091 (2011)
9. M. Gualtieri, Ann. Math. **174(1)**, 75–123 (2011). http://www.jstor.org/stable/23030560
10. J. Moser, https://doi.org/10.1090/s0002-9947-1965-0182927-5
11. D. Roytenberg, A. Weinstein, Lett. Math. Phys. **46**, 81–93 (1998)

Representation Theory

On 1-Dimensional Modules over the Super-Yangian of the Superalgebra $Q(1)$

Elena Poletaeva

Abstract Let $Q(n)$ be the queer Lie superalgebra. We determine conditions under which two 1-dimensional modules over the super-Yangian of $Q(1)$ can be extended nontrivially, and thus belong to the same block of the category of finite-dimensional $YQ(1)$-modules. We use these results to determine conditions under which two 1-dimensional modules over the finite W-algebra for $Q(n)$ can be extended nontrivially.

Keywords Queer Lie superalgebra · Yangian · Finite W-algebra

1 Introduction

In [8] we classified irreducible representations of the finite W-algebra W^n for the *queer* Lie superalgebra $Q(n)$ associated with the principal nilpotent coadjoint orbits (they are all finite-dimensional), as well as irreducible finite-dimensional representations of the super-Yangian $YQ(1)$ of $Q(1)$ (Theorem 4.7 and Theorem 5.13). An interesting problem arising from these classifications is to describe blocks of the subcategories of finite-dimensional $YQ(1)$-modules and finite-dimensional W^n-modules admitting a given generalized central character χ. If $\chi = 0$, then the simple modules in these subcategories are 1-dimensional. In this paper we determine when two 1-dimensional $YQ(1)$-modules can be extended nontrivially, and thus belong to the same block (Theorem 1). We use these results and results of [8] to determine when two 1-dimensional W^n-modules can be extended nontrivially (Theorem 2).

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_19

2 Preliminaries

Recall that the Lie superalgebra $\mathfrak{g} = Q(n)$ is defined as follows (see [2]). Equip $\mathbb{C}^{n|n}$ with the odd operator ζ such that $\zeta^2 = -\mathrm{Id}$. Then $Q(n)$ is the centralizer of ζ in the Lie superalgebra $\mathfrak{gl}(n|n)$. It is easy to see that $Q(n)$ consists of matrices of the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where A, B are $n \times n$-matrices. Let

$$\{e_{i,j},\, f_{i,j} \,|\, i, j = 1, \ldots, n\}$$

denote the basis in $Q(n)$ consisting of elementary even and odd matrices.

3 The Super Yangian of $Q(1)$

The Yangians $YQ(n)$ associated with the Lie superalgebras $Q(n)$ were defined by M. L. Nazarov ([3, 4]). Recall that $YQ(1)$ is the associative unital superalgebra over $\mathbb{C}$ with the countable set of generators $T_{i,j}^{(m)}$, where $m = 1, 2, \ldots$ and $i, j = \pm 1$. The $\mathbb{Z}_2$-grading of $YQ(1)$ is defined as follows:

$$p(T_{i,j}^{(m)}) = p(i) + p(j), \text{ where } p(1) = 0 \text{ and } p(-1) = 1.$$

To write the defining relations for these generators, we employ the formal series in $YQ(1)[[u^{-1}]]$:

$$T_{i,j}(u) = \delta_{ij} \cdot 1 + T_{i,j}^{(1)} u^{-1} + T_{i,j}^{(2)} u^{-2} + \ldots.$$

Then for all possible indices i, j, k, l we have the relations

$$\begin{aligned}
&(u^2 - v^2)[T_{i,j}(u), T_{k,l}(v)] \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} \\
&= (u+v)(T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u)) \\
&-(u-v)(T_{-k,j}(u)T_{-i,l}(v) - T_{k,-j}(v)T_{i,-l}(u)) \cdot (-1)^{p(k)+p(l)}.
\end{aligned} \tag{1}$$

Here v is a formal parameter independent of u, so that (1) is an equality in the algebra of formal Laurent series in u^{-1}, v^{-1} with coefficients in $YQ(1)$. For all indices i, j we also have the relations

$$T_{i,j}(-u) = T_{-i,-j}(u). \tag{2}$$

The relations (1) and (2) are equivalent to the following defining relations:

$$([T_{i,j}^{(m+1)}, T_{k,l}^{(r-1)}] - [T_{i,j}^{(m-1)}, T_{k,l}^{(r+1)}]) \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} =$$

$$T_{k,j}^{(m)} T_{i,l}^{(r-1)} + T_{k,j}^{(m-1)} T_{i,l}^{(r)} - T_{k,j}^{(r-1)} T_{i,l}^{(m)} - T_{k,j}^{(r)} T_{i,l}^{(m-1)} \tag{3}$$

$$+(-1)^{p(k)+p(l)}(-T_{-k,j}^{(m)} T_{-i,l}^{(r-1)} + T_{-k,j}^{(m-1)} T_{-i,l}^{(r)} + T_{k,-j}^{(r-1)} T_{i,-l}^{(m)} - T_{k,-j}^{(r)} T_{i,-l}^{(m-1)}),$$

$$T_{-i,-j}^{(m)} = (-1)^m T_{i,j}^{(m)}, \tag{4}$$

where $m, r = 1, \ldots$ and $T_{i,j}^{(0)} = \delta_{ij}$. Recall that $YQ(1)$ is a *Hopf superalgebra* (see [4]) with comultiplication given by the formula

$$\Delta(T_{i,j}^{(r)}) = \sum_{s=0}^{r} \sum_{k} (-1)^{(p(i)+p(k))(p(j)+p(k))} T_{i,k}^{(s)} \otimes T_{k,j}^{(r-s)}.$$

The *evaluation homomorphism* $ev : YQ(1) \to U(Q(1))$ is defined as follows:

$$T_{1,1}^{(1)} \mapsto -e_{1,1}, \quad T_{1,-1}^{(1)} \mapsto f_{1,1}, \quad T_{i,j}^{(0)} \mapsto \delta_{i,j}, \quad T_{i,j}^{(r)} \mapsto 0 \text{ for } r > 1, i, j = \pm 1.$$

4 1-Dimensional $YQ(1)$-Modules

We classified simple finite-dimensional $YQ(1)$-modules in [8]. Here we recall the description of 1-dimensional $YQ(1)$-modules.

Remark 1 Note that $[T_{1,1}^{(k)}, T_{1,1}^{(m)}] = 0$ if $k + m$ is even (see [6, Proposition 6.4]).

Definition 1 Let **A** be the commutative subalgebra in $YQ(1)$ generated by $T_{1,1}^{(2k)}$ for $k \geq 0$. Let

$$f(u) = 1 + \sum_{k>0} f_{2k} u^{-2k}.$$

Let Γ_f be the corresponding 1-dimensional **A**-module, where the action of

$$T_{1,1}(u^{-2}) = \sum_{k \geq 0} T_{1,1}^{(2k)} u^{-2k}$$

is given by the generating function $f(u)$.

Recall that for any Hopf superalgebra R, the ideal (R_1) generated by all odd elements is a Hopf ideal and the quotient $R/(R_1)$ is a Hopf algebra.

Proposition 1 ([8, Lemma 5.11]) *The quotient* $YQ(1)/(YQ(1)_1)$ *is isomorphic to* $\mathbf{A} \simeq \mathbb{C}[T_{1,1}^{(2k)}]_{k>0}$, *with comultiplication*

$$\Delta T_{1,1}(u^{-2}) = T_{1,1}(u^{-2}) \otimes T_{1,1}(u^{-2}).$$

Thus we can lift an $\mathbf{A}$-module Γ_f to a $YQ(1)$-module.

Proposition 2 ([8, Lemma 5.12]) *The isomorphism classes of* 1*-dimensional* $YQ(1)$*-modules are in bijection with the set* $\{\Gamma_f\}$. *Furthermore, we have the identity* $\Gamma_f \otimes \Gamma_g \simeq \Gamma_{fg}$.

5 The Category $YQ(1)$–Mod

We described the center Z of $YQ(1)$ in [8]. Let

$$\eta_i = \left(-\frac{1}{2}\right)^i \operatorname{ad}^i T_{1,1}^{(2)}(T_{1,-1}^{(1)}), \quad Z_{2i} = \frac{1}{2}[\eta_0, \eta_{2i}],$$

where $\operatorname{ad}^i T_{1,1}^{(2)}$ is the i-power of the adjoint endomorphism $\operatorname{ad} T_{1,1}^{(2)}$. The elements $\{Z_{2i} \mid i \in \mathbb{N}\}$ are algebraically independent generators of the center of $YQ(1)$.

Let $YQ(1)$–mod be the category of finite-dimensional $YQ(1)$-modules. A $YQ(1)$-module M admits *generalized central character* χ if for any $z \in Z$ and $m \in M$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that $(z - \chi(z))^n \cdot m = 0$. Let $(YQ(1))^\chi$–mod be the full subcategory of modules admitting generalized central character χ. The category $YQ(1)$–mod is the direct sum of the subcategories $(YQ(1))^\chi$–mod, as χ ranges over the central characters for which $(YQ(1))^\chi$–mod is nonempty.

Recall that simple modules are partitioned into *blocks*. If two simple modules M_1 and M_2 can be extended nontrivially, i.e., if there is a non-split short exact sequence $0 \longrightarrow M_i \longrightarrow M \longrightarrow M_j \longrightarrow 0$ with $\{i, j\} = \{1, 2\}$, then M_1 and M_2 belong to the same block, and we will say that they are *linked*. Here we agree that M_i is linked to itself. More generally, if there is a finite sequence of simple modules $M = M_1, M_2, \ldots, M_n = N$ such that adjacent pairs belong to the same block, then modules M and N belong to this block. A module M belongs to a block if all its composition factors do. Each block lies in a single $(YQ(1))^\chi$–mod. However, different blocks can belong to the same $(YQ(1))^\chi$–mod: see [1].

5.1 The Subcategory $(YQ(1))^{\chi=0}$–Mod

The simple even modules in the subcategory $(YQ(1))^{\chi=0}$–mod are exactly the 1-dimensional modules Γ_f (see [8]). Let Γ_f and Γ_g be two $YQ(1)$-modules, where

$$f(u) = \sum_{k\geq 0} a_{2k}u^{-2k}, \quad g(u) = \sum_{k\geq 0} b_{2k}u^{-2k}, \quad a_0 = b_0 = 1.$$

Recall that Γ_f is linked to itself. If $f \neq g$, then one can easily check that the short exact sequence

$$0 \longrightarrow \Gamma_f \longrightarrow M \longrightarrow \Gamma_g \longrightarrow 0$$

splits. Indeed, we have the following relations in $YQ(1)$:

$$[T^{(2k)}_{1,1}, T^{(1)}_{1,-1}] = 2T^{(2k)}_{1,-1}. \tag{5}$$

$$[T^{(2)}_{1,1}, T^{(2k)}_{1,-1}] = 2T^{(2k+1)}_{1,-1} + 2T^{(2k)}_{1,-1} - 2T^{(2k)}_{1,1}T^{(1)}_{1,-1}. \tag{6}$$

$$[T^{(1)}_{1,-1}, T^{(2k+1)}_{1,-1}] = -2T^{(2k+1)}_{1,1}. \tag{7}$$

All odd generators $T^{(r)}_{1,-1}$ act on M by zero, since M is a purely even module. Then $T^{(2k+1)}_{1,1}$ also acts on M by zero by (7). Note that $T^{(2k)}_{1,1}$ acts on M as $\begin{pmatrix} a_{2k} & c_{2k} \\ 0 & b_{2k} \end{pmatrix}$, and there exists m such that $a_{2m} \neq b_{2m}$, since $f \neq g$. We can choose a basis in M so that $c_{2m} = 0$. Then $c_{2k} = 0$ for all k, since $T^{(2k)}_{1,1}$ commute. Hence $M \simeq \Gamma_f \oplus \Gamma_g$.

Let Π be the parity functor $\Pi(X) = X \otimes \mathbb{C}^{0|1}$. We will determine when Γ_f is linked with $\Pi(\Gamma_g)$.

Theorem 1 *$Ext^1(\Pi(\Gamma_g), \Gamma_f) \neq 0$ if and only if $b_{2k} = a_{2k} - x_{2k}$, where x_2 is an arbitrary complex number and x_{2k} for $k > 1$ is defined by the recurrence relation*

$$x_{2k+2} = \left(\frac{(x_2 - 2)x_{2k}}{4} + a_{2k}\right)x_2. \tag{8}$$

Proof Note that the short exact sequence

$$0 \longrightarrow \Gamma_f \longrightarrow M \longrightarrow \Pi(\Gamma_g) \longrightarrow 0$$

is non-split if and only if $T^{(1)}_{1,-1}$ does not act by zero. Indeed, if $T^{(1)}_{1,-1}$ acts by zero, then $T^{(2k)}_{1,-1}$ and $T^{(2k+1)}_{1,-1}$ also act by zero for all k by (5) and (6), but then $M \simeq \Gamma_f \oplus \Pi(\Gamma_g)$. Clearly, if $M \simeq \Gamma_f \oplus \Pi(\Gamma_g)$, then all odd generators act by zero.

Hence $\mathrm{Ext}^1(\Pi(\Gamma_g), \Gamma_f) \neq 0$ if and only if one can define a representation $\rho : YQ(1) \longrightarrow \mathrm{End}(\mathbb{C}^{1|1})$ such that (up to equivalence)

$$\rho(T_{1,1}^{(2k)}) = \begin{pmatrix} a_{2k} & 0 \\ 0 & b_{2k} \end{pmatrix}, \quad \rho(T_{1,-1}^{(1)}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

Then

$$\rho(T_{1,-1}^{(2k)}) = \begin{pmatrix} 0 & \frac{a_{2k}-b_{2k}}{2} \\ 0 & 0 \end{pmatrix}, \tag{10}$$

$$\rho(T_{1,-1}^{(2k+1)}) = \begin{pmatrix} 0 & \frac{(a_2-b_2)(a_{2k}-b_{2k})}{4} + \frac{a_{2k}+b_{2k}}{2} \\ 0 & 0 \end{pmatrix}, \tag{11}$$

$$\rho(T_{1,1}^{(2k+1)}) = 0. \tag{12}$$

Here (10) follows from (9) and the relation (5), (11) follows from (9), (10), and (6), and (12) follows from (11) and (7).

Let $x_{2k} = a_{2k} - b_{2k}$. The recurrence relation (3) with $m = 2k - 1$ and $r = 2p + 2$ gives the relation

$$\begin{aligned} &([T_{1,1}^{(2k)}, T_{1,-1}^{(2p+1)}] - [T_{1,1}^{(2k-2)}, T_{1,-1}^{(2p+3)}]) = \\ &T_{1,1}^{(2k-1)} T_{1,-1}^{(2p+1)} + T_{1,1}^{(2k-2)} T_{1,-1}^{(2p+2)} - T_{1,1}^{(2p+1)} T_{1,-1}^{(2k-1)} - T_{1,1}^{(2p+2)} T_{1,-1}^{(2k-2)} \\ &+T_{-1,1}^{(2k-1)} T_{-1,-1}^{(2p+1)} - T_{-1,1}^{(2k-2)} T_{-1,-1}^{(2p+2)} - T_{1,-1}^{(2p+1)} T_{1,1}^{(2k-1)} + T_{1,-1}^{(2p+2)} T_{1,1}^{(2k-2)}. \end{aligned} \tag{13}$$

From (13) and (9)–(12) we obtain the relation

$$\frac{x_2 x_{2p} x_{2k}}{4} + \frac{2a_{2p} - x_{2p}}{2} x_{2k} - \frac{x_2 x_{2p+2} x_{2k-2}}{4} = \frac{x_{2p+2}}{2}(2a_{2k-2} - x_{2k-2}).$$

If $p = 0$ (and $a_0 = 1$, $x_0 = 0$) we have

$$x_{2k} = (\frac{x_2 x_{2k-2}}{4} + a_{2k-2} - \frac{x_{2k-2}}{2}) x_2,$$

which is equivalent to (8). One can check that ρ preserves the relations (3). □

Corollary 1 *$Ext^1(\Gamma_f, \Pi(\Gamma_g)) \neq 0$ if and only if $b_{2k} = a_{2k} + x_{2k}$, where x_2 is an arbitrary complex number and x_{2k} for $k > 1$ is defined by the recurrence relation*

$$x_{2k+2} = \Big(\frac{(x_2 + 2) x_{2k}}{4} + a_{2k}\Big) x_2. \tag{14}$$

6 The Finite W-Algebra for $Q(n)$

Let W^n be the *finite W-algebra* associated with a principal even nilpotent element φ in the coadjoint representation of $\mathfrak{g} = Q(n)$. Let us recall its definition (see [10]). We fix the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ to be the set of matrices with diagonal A and B. By $\mathfrak{n}^+$ (respectively, $\mathfrak{n}^-$) we denote the nilpotent subalgebras consisting of matrices with strictly upper triangular (respectively, low triangular) A and B.

The Lie superalgebra $\mathfrak{g}$ has the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, and we set $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$. Choose $\varphi \in \mathfrak{g}^*$ such that

$$\varphi(f_{i,j}) = 0, \quad \varphi(e_{i,j}) = \delta_{i,j+1}.$$

Let I_φ be the left ideal in $U(\mathfrak{g})$ generated by $x - \varphi(x)$ for all $x \in \mathfrak{n}^-$. Let $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_\varphi$ be the natural projection. Then

$$W^n = \{\pi(y) \in U(\mathfrak{g})/I_\varphi \mid \mathrm{ad}(x)y \in I_\varphi \text{ for all } x \in \mathfrak{n}^-\}.$$

Using the identification of $U(\mathfrak{g})/I_\varphi$ with the Whittaker module $U(\mathfrak{g}) \otimes_{U(\mathfrak{n}^-)} \mathbb{C}_\varphi \simeq U(\mathfrak{b}) \otimes \mathbb{C}$, we can consider W^n as a subalgebra of $U(\mathfrak{b})$. The natural projection $\vartheta : U(\mathfrak{b}) \to U(\mathfrak{h})$ with the kernel $\mathfrak{n}^+ U(\mathfrak{b})$ is called the *Harish-Chandra homomorphism*. It is proven in [6] that the restriction of ϑ to W^n is injective. We will identify W^n with $\vartheta(W^n) \subset U(\mathfrak{h})$.

Example 1 $n = 2$, $\mathfrak{h} = \mathrm{span}\{e_{1,1}, e_{2,2} \mid f_{1,1}, f_{2,2}\}$, where $f_{1,1}^2 = e_{1,1}$ and $f_{2,2}^2 = e_{2,2}$. Then W^2 realized as a subalgebra of $U(\mathfrak{h})$ has the following generators:

$$z_0 = e_{1,1} + e_{2,2}, \; z_1 = e_{1,1}e_{2,2} + f_{1,1}f_{2,2} \text{ (even)},$$

$$\phi_0 = f_{1,1} - f_{2,2}, \; \phi_1 = e_{1,1}f_{2,2} + e_{2,2}f_{1,1} \text{ (odd)}.$$

7 W^n Is a Quotient of $YQ(1)$

Definition 2 (a) Define $\Delta_l : YQ(1) \longrightarrow YQ(1)^{\otimes l}$ by

$$\Delta_l := \Delta_{l-1,l} \circ \cdots \circ \Delta_{2,3} \circ \Delta.$$

(b) Let $\varphi_n : YQ(1) \to U(Q(1))^{\otimes n} \simeq U(\mathfrak{h})$ be $\varphi_n := ev^{\otimes n} \circ \Delta_n$.

Proposition 3 ([7, Corollary 5.16]) *The map φ_n is a surjective homomorphism from $YQ(1)$ onto W^n, realized as a subalgebra of $U(\mathfrak{h})$:*

$$\varphi_n(YQ(1)) = \vartheta(W^n) \simeq W^n.$$

Note that W^{m+n} is a subalgebra of $W^m \otimes W^n$ ([8, Lemma 3.3]). The following diagram commutes:

$$\begin{array}{ccc} YQ(1) & \xrightarrow{\Delta} & YQ(1) \otimes YQ(1) \\ {\scriptstyle \varphi_{m+n}}\downarrow & & {\scriptstyle \varphi_m \otimes \varphi_n}\downarrow \\ W^{m+n} & \longrightarrow & W^m \otimes W^n \end{array} \tag{15}$$

8 1-Dimensional W^n-Modules

We classified simple W^n-modules in [8, Theorem 4.7]. Here we recall the construction of 1-dimensional W^n-modules. Let $r, p \in \mathbb{N}$ and $r + 2p = n$, $\mathbf{t} = (t_1, \ldots, t_p) \in \mathbb{C}^p$, $t_1, \ldots, t_p \neq 0$. Recall that there is an embedding $W^n \hookrightarrow W^r \otimes (W^2)^{\otimes p}$ ([8], Corollary 3.4). Let Γ_t be the simple W^2-module of dimension $(1|0)$ on which ϕ_0, ϕ_1 and z_0 act by zero and z_1 acts by the scalar t. Set $S(\mathbf{t}) := \mathbb{C} \otimes \Gamma_{t_1} \otimes \cdots \otimes \Gamma_{t_p}$, where the first term $\mathbb{C}$ in the tensor product denotes the trivial W^r-module.

Proposition 4 (see [8, Theorem 4.7]) *(a) Every* 1*-dimensional W^n-module is isomorphic to $S(\mathbf{t})$ up to change of parity.*
(b) Two W^n-modules $S(\mathbf{t})$ and $S(\mathbf{t}')$ are isomorphic if and only if $\mathbf{t}' = \sigma(\mathbf{t})$ for some $\sigma \in S_p$.

Proposition 5 (see [8, Proposition 5.19]) *The* 1*-dimensional $YQ(1)$-module Γ_f is lifted from some W^m-module if and only if $f \in \mathbb{C}[u^{-2}]$. Moreover, the smallest such m is equal to the degree of the polynomial f.*

Remark 2 Note that $m = 2p$ is even. Proposition 4 and the diagram (15) imply that $S(t_1, \ldots, t_p) \simeq \Gamma_f$, where $f = \prod_{i=1}^{p}(1 + t_i u^{-2})$.

9 The Category W^n–Mod

Recall that the image of the center of $U(\mathfrak{g})$ under the Harish-Chandra homomorphism is generated by the Q-symmetric polynomials (see [5]). In [6] we proved that the center of W^n coincides with the image of the center of $U(\mathfrak{g})$ under the Harish-Chandra homomorphism, and hence can also be identified with the ring of *Q-symmetric polynomials*. The center of W^n coincides with $\varphi_n(Z)$, where Z is the center of $YQ(1)$ (see [8]).

Let W^n–mod be the category of finite-dimensional W^n-modules. Let $(W^n)^\chi$–mod be the full subcategory of modules admitting generalized central character χ. The category W^n–mod is the direct sum of subcategories $(W^n)^\chi$–mod, as χ ranges over the central characters for which $(W^n)^\chi$–mod is nonempty. In [9] we described blocks of the category W^2–mod.

9.1 The Subcategory $(W^n)^{\chi=0}$–Mod

Note that simple even modules in the subcategory $(W^n)^{\chi=0}$–mod are exactly the 1-dimensional modules $S(\mathbf{t})$ (see [8]). Let σ_k denote the kth elementary symmetric polynomial.

Theorem 2 *Fix $\mathbf{t} = (t_1, \ldots, t_p)$ and $\mathbf{t}' = (t'_1, \ldots, t'_q)$, where p and q are less than or equal to $\frac{n}{2}$. Consider the W^n-modules $S(\mathbf{t})$ and $S(\mathbf{t}')$. Define $a_{2k} = \sigma_k(t_1, \ldots, t_p)$ for $k = 1, \ldots, p$, $a_{2k} = 0$ for $k > p$. Similarly, define $b_{2k} = \sigma_k(t'_1, \ldots, t'_q)$ for $k = 1, \ldots, q$, $b_{2k} = 0$ for $k > q$. Let $x_{2k} = a_{2k} - b_{2k}$.*
(a) If $S(\mathbf{t})$ is a nontrivial W^n-module, then $Ext^1(\Pi(S(\mathbf{t}')), S(\mathbf{t})) \neq 0$ if and only if $x_2 \neq 0$ and x_{2k} satisfy the recurrence relation (8).
(b) If $S(\mathbf{t}) = \mathbb{C}^{1|0}$ is the trivial W^n-module, then $Ext^1(\Pi(S(\mathbf{t}')), S(\mathbf{t})) \neq 0$ if and only if $S(\mathbf{t}') = \mathbb{C}^{1|0}$ or $\mathbf{t}' = (t'_1)$ with $t'_1 = -2$.

Proof Suppose that $\mathrm{Ext}^1(\Pi(S(\mathbf{t}')), S(\mathbf{t})) \neq 0$. Lift $S(\mathbf{t})$ and $S(\mathbf{t}')$ to $YQ(1)$-modules Γ_f and Γ_g, respectively, where

$$f(u) = 1 + \sum_{k>0} a_{2k} u^{-2k} \text{ and } g(u) = 1 + \sum_{k>0} b_{2k} u^{-2k}.$$

Then $\mathrm{Ext}^1(\Pi(\Gamma_g), \Gamma_f)) \neq 0$. Hence by Theorem 1, $x_{2k} = a_{2k} - b_{2k}$ satisfy (8). Note that $x_2 \neq 0$, since otherwise all $x_{2k} = 0$ and so $S(\mathbf{t}')$ is isomorphic to $S(\mathbf{t})$. However, $S(\mathbf{t})$ is linked with $\Pi(S(\mathbf{t}))$ only if $S(\mathbf{t})$ is the trivial module (see [9]).

Conversely, if $x_{2k} = a_{2k} - b_{2k}$ satisfy (8), then the lifted modules Γ_f and $\Pi(\Gamma_g)$ are linked. Because $x_2 \neq 0$, (8) implies that

$$\frac{(x_2 - 2)x_{2k}}{4} + a_{2k} = 0$$

for $2k \geq n$ if n is even and for $2k \geq n - 1$ if n is odd. Then $\rho(T^{(r)}_{1,-1}) = 0$ if $r > n$ by (10) and (11), and $\rho(T^{(r)}_{1,1}) = 0$ if $r > n$ by (9) and (12).

The kernel of the surjective homomorphism $\varphi_n : YQ(1) \longrightarrow W^n$ is generated by $T^{(r)}_{1,1}$ and $T^{(r)}_{1,-1}$, where $r > n$. This allows one to define a representation $\mu : W^n \longrightarrow \mathrm{End}(\mathbb{C}^{1|1})$ such that $\rho = \mu \circ \varphi_n$. Thus $S(\mathbf{t})$ is linked with $\Pi(S(\mathbf{t}'))$.

If $S(\mathbf{t}) = \mathbb{C}^{1|0}$, then $a_{2k} = 0$ for $k \geq 1$. From (8), $x_2 = 0$ or $x_2 = 2$. In the first case $x_{2k} = 0$ and $b_{2k} = 0$ for $k \geq 1$. Hence $S(\mathbf{t}')$ is the trivial module. In the second case, $x_2 = 2$ and $x_{2k} = 0$ for $k \geq 2$, $b_2 = -2$, and $b_{2k} = 0$ for $k \geq 2$. Hence $\mathbf{t}' = (t'_1)$ with $t'_1 = -2$. □

Remark 3 Suppose that Γ_f is lifted from a nontrivial module $S(\mathbf{t})$, and assume that $\mathrm{Ext}^1(\Pi(\Gamma_g), \Gamma_f) \neq 0$. Note that Γ_g is a lift from some W^n-module $S(\mathbf{t}')$ if and only if in (8) we have $x_{n+2} = 0$ if n is even and $x_{n+1} = 0$ if n is odd. This means that x_2 is a (nonzero) root of the polynomial of degree n (respectively, $n - 1$) defined by the recurrence relation (8) if n is even (respectively, odd). Then we set $b_{2k} = a_{2k} - x_{2k}$

and find $\mathbf{t}' = (t'_1, \ldots, t'_q)$ such that $b_{2k} = \sigma_k(\mathbf{t}')$. Here $\mathbf{t}'$ is defined up to permutation of $t'_1, \ldots, t'_q$, and we delete all zero entries. Then $\mathrm{Ext}^1(\Pi(S(\mathbf{t}')), S(\mathbf{t})) \neq 0$, and moreover all modules $S(\mathbf{t}')$ satisfying the above formula are obtained in this way.

Example 2 (*see* [9]) Let $n = 2$, so that by (8), $x_4 = (\frac{x_2^2}{4} - \frac{x_2}{2} + a_2)x_2$. Then x_2 must satisfy $\frac{x_2^2}{4} - \frac{x_2}{2} + a_2 = 0$.

Corollary 2 *(a) If $S(\mathbf{t})$ is a nontrivial W^n-module, then $\mathrm{Ext}^1(S(\mathbf{t}), \Pi(S(\mathbf{t}'))) \neq 0$ if and only if $x_{2k} := b_{2k} - a_{2k}$ satisfies the recurrence relation (14) for all k and $x_2 \neq 0$.*
(b) If $S(\mathbf{t})$ is a trivial W^n-module, then $\mathrm{Ext}^1(S(\mathbf{t}), \Pi(S(\mathbf{t}'))) \neq 0$ if and only if $S(\mathbf{t}') = \mathbb{C}^{1|0}$ or $\mathbf{t}' = (t'_1)$ with $t'_1 = -2$.

Acknowledgements This research was supported by Simons Foundation Collaboration Grant 354874. I thank the organizers of the 14th International Workshop "Lie Theory and Its Applications in Physics" (LT-14), 21-25 June 2021, Sofia, Bulgaria, for a very interesting conference, and I am grateful to V. Serganova for very helpful discussions. I also thank the referee for carefully reading the manuscript and for many detailed comments and suggestions.

References

1. J. Humphreys, *Representations of Semisimple Lie algebras in the BGG Category $\mathcal{O}$*, *Graduate Studies in Mathematics*, vol. 94 (American Mathematical Society, Providence, RI, 2008), pp. 30–32
2. V. G. Kac, Adv. Math. **26**, 8–96 (1977)
3. M. Nazarov, in P. P. Kulish (ed.) *Quantum Groups, Lecture Notes in Mathematics*, vol. 1510 (Springer, Berlin-Heidelberg, 1992), pp. 90–97
4. M. Nazarov, Commun. Math. Phys. **208**, 195–223 (1999)
5. A. Sergeev, Lett. Math. Phys. **7**, 177–179 (1983)
6. E. Poletaeva, V. Serganova, Adv. Math. **300**, 320–359 (2016). arXiv:1403.3866v1 [math RT]
7. E. Poletaeva, V. Serganova, J. Math. Phys. **58**(11), 111701 (2017). arXiv:1705.10200v1 [math RT]
8. E. Poletaeva, V. Serganova, J. Algebra **570**, 140–163 (2021). arXiv:1903.05272v3 [math RT]
9. E. Poletaeva, J. Phys.: Conf. Ser. **1416**, 012028 (2019)
10. A. Premet, Adv. Math. **170**, 1–55 (2002)

A Klein Operator for Paraparticles

Nedialka I. Stoilova and Joris Van der Jeugt

Abstract It has been known for a long time that there are two non-trivial possibilities for the relative commutation relations between a set of m parafermions and a set of n parabosons. These two choices are known as "relative parafermion type" and "relative paraboson type", and correspond to quite different underlying algebraic structures. In this short note we show how the two possibilities are related by a so-called Klein transformation.

Keywords Parafermions · Parabosons · Klein transformation

The standard creation and annihilation operators of identical particles satisfy canonical commutation (boson) or anticommutation (fermion) relations, expressed by means of commutators or anticommutators. In 1953 Green [1] generalized bosons to so-called parabosons and fermions to parafermions, by postulating certain triple relations for the creation and annihilation operators, rather than just (anti)commutators. A system of m parafermion creation and annihilation operators $f_j^\pm$ $(j = 1, \ldots, m)$ is determined by

$$[[f_j^\xi, f_k^\eta], f_l^\epsilon] = |\epsilon - \eta|\delta_{kl} f_j^\xi - |\epsilon - \xi|\delta_{jl} f_k^\eta, \tag{1}$$

where $j, k, l \in \{1, 2, \ldots, m\}$ and $\eta, \epsilon, \xi \in \{+, -\}$ (to be interpreted as $+1$ and -1 in the algebraic expressions $\epsilon - \xi$ and $\epsilon - \eta$). Similarly, a system of n pairs of parabosons $b_j^\pm$ satisfies

$$[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon] = (\epsilon - \xi)\delta_{jl} b_k^\eta + (\epsilon - \eta)\delta_{kl} b_j^\xi. \tag{2}$$

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*, Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_20

These triple relations involve nested (anti)commutators, just like the Jacobi identity of Lie (super)algebras. And indeed, later it was shown [2, 3] that the parafermionic algebra determined by (1) is the orthogonal Lie algebra $\mathfrak{so}(2m+1)$, and that the parabosonic algebra determined by (2) is the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ [4].

Greenberg and Messiah [5] considered combined systems of parafermions and parabosons. Apart from two trivial combinations (where the parafermions and parabosons mutually commute or anticommute), they found two non-trivial relative commutation relations between parafermions and parabosons, also expressed by means of triple relations. The first of these are the relative parafermion relations, determined by:

$$\begin{aligned} &[[f_j^\xi, f_k^\eta], b_l^\epsilon] = 0, \qquad [\{b_j^\xi, b_k^\eta\}, f_l^\epsilon] = 0, \\ &[[f_j^\xi, b_k^\eta], f_l^\epsilon] = -|\epsilon - \xi|\delta_{jl} b_k^\eta, \qquad \{[f_j^\xi, b_k^\eta], b_l^\epsilon\} = (\epsilon - \eta)\delta_{kl} f_j^\xi. \end{aligned} \tag{3}$$

The second are the so-called relative paraboson relations, and will appear later in this paper.

The parastatistics algebra with relative parafermion relations, determined by (1)–(3), was identified by Palev [6] and is the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$.

When dealing with parastatistics, a major object of study is the Fock space. By definition the parastatistics Fock space of order p is the Hilbert space with vacuum vector $|0\rangle$, defined by means of

$$\begin{aligned} &\langle 0|0\rangle = 1, \qquad f_j^-|0\rangle = 0, \qquad b_j^-|0\rangle = 0, \qquad (f_j^\pm)^\dagger = f_j^\mp, \qquad (b_j^\pm)^\dagger = b_j^\mp, \\ &[f_j^-, f_k^+]|0\rangle = p\delta_{jk}|0\rangle, \qquad \{b_j^-, b_k^+\}|0\rangle = p\delta_{jk}|0\rangle, \end{aligned} \tag{4}$$

and by irreducibility under the action of the elements $f_j^\pm, b_j^\pm$.

The purpose of this short contribution is to show that a Klein operator [7–9] can be constructed, and that new operators $\tilde{b}_k^\pm$ and $\tilde{f}_j^\pm$ can be defined in terms of the operators $b_k^\pm$ and $f_j^\pm$ in such a way that these new operators satisfy the parastatistics algebra with relative paraboson relations.

First of all, let us define the following elements in terms of the paraoperators $b_j^\pm$ and $f_i^\pm$

$$\begin{aligned} h_i &= -\frac{1}{2}[f_i^-, f_i^+] \qquad (i = 1, \ldots, m) \\ h_{m+j} &= \frac{1}{2}\{b_j^-, b_j^+\} \qquad (j = 1, \ldots, n). \end{aligned} \tag{5}$$

From the triple relations (1)–(3), it is easy to deduce that

$$\begin{aligned}
&h_i f_j^{\pm} = f_j^{\pm}(h_i \pm \delta_{ij}) \qquad (i = 1, \ldots, m;\ j = 1, \ldots, m)\\
&h_i b_k^{\pm} = b_k^{\pm} h_i \qquad (i = 1, \ldots, m;\ k = 1, \ldots, n)\\
&h_{m+i} f_j^{\pm} = f_j^{\pm} h_{m+i} \qquad (i = 1, \ldots, n;\ j = 1, \ldots, m)\\
&h_{m+i} b_k^{\pm} = b_k^{\pm}(h_{m+i} \pm \delta_{ik}) \qquad (i = 1, \ldots, n;\ k = 1, \ldots, n).
\end{aligned} \tag{6}$$

So if we put

$$H = h_1 + h_2 + \cdots + h_{m+n}, \tag{7}$$

then

$$\begin{aligned}
&Hf_j^{\pm} = f_j^{\pm}(H \pm 1) \qquad (j = 1, \ldots, m)\\
&Hb_k^{\pm} = b_k^{\pm}(H \pm 1) \qquad (k = 1, \ldots, n).
\end{aligned} \tag{8}$$

All these relations are purely algebraic, i.e. they follow from the triple relations (1)–(3), and hold in the algebra generated by the $2(m+n)$ elements $b_k^{\pm}$ and $f_j^{\pm}$.

Following (8), we would like to define an operator of the form $(-1)^H$. In order to have a proper meaning for this, it is useful to work in the Fock space of order p, characterized by (4), so that the abstract algebraic elements become operators acting in this Fock space. Following (5), one finds that

$$H|0\rangle = -\frac{p}{2}(m-n)|0\rangle. \tag{9}$$

So it is convenient to define

$$N = H + \frac{p}{2}(m-n) \tag{10}$$

and the Klein operator K by

$$K = (-1)^N. \tag{11}$$

Then $K|0\rangle = |0\rangle$ and $K^2|0\rangle = |0\rangle$. Since the commutator of N with $f_j^{\pm}$ and $b_k^{\pm}$ is the same as that of H (given by (8)), this implies that K^2 acts as the identity operator in the Fock space. Moreover, from (8) it follows that

$$Kf_j^{\pm} + f_j^{\pm}K = 0, \qquad Kb_k^{\pm} + b_k^{\pm}K = 0 \qquad (j = 1, \ldots, m;\ k = 1, \ldots, n). \tag{12}$$

In fact, one could also continue to work purely algebraically, and extend the algebra generated by the generators $b_k^{\pm}$ and $f_j^{\pm}$, subject to the relations (1)–(3), by an abstract element K satisfying

$$K^2 = 1, \qquad \{K, f_j^{\pm}\} = 0, \qquad \{K, b_k^{\pm}\} = 0 \qquad (j = 1, \ldots, m;\ k = 1, \ldots, n). \tag{13}$$

The previous analysis just shows that such an operator K exists in the Fock space of the paraoperators.

Let us now proceed to the main construction. Define new operators

$$\tilde{f}_j^{\pm} = \pm f_j^{\pm} K = \mp K f_j^{\pm} \qquad (j = 1, \ldots, m)$$
$$\tilde{b}_k^{\pm} = b_k^{\pm} \qquad (k = 1, \ldots, n). \tag{14}$$

The purpose is now to examine the triple relations for the new set of operators $\tilde{b}_k^{\pm}$ and $\tilde{f}_j^{\pm}$. Since the $\tilde{b}_k^{\pm}$ are the same as the $b_k^{\pm}$, one has

$$[\{\tilde{b}_j^{\xi}, \tilde{b}_k^{\eta}\}, \tilde{b}_l^{\epsilon}] = (\epsilon - \xi)\delta_{jl}\tilde{b}_k^{\eta} + (\epsilon - \eta)\delta_{kl}\tilde{b}_j^{\xi}. \tag{15}$$

Next, using $\tilde{f}_j^{\xi} = \xi f_j^{\xi} K$, $\tilde{f}_k^{\eta} = \eta f_k^{\eta} K$, $K^2 = 1$ and (13), one has

$$[\tilde{f}_j^{\xi}, \tilde{f}_k^{\eta}] = -\xi\eta[f_j^{\xi}, f_k^{\eta}].$$

This implies that

$$\begin{aligned}[][[\tilde{f}_j^{\xi}, \tilde{f}_k^{\eta}], \tilde{f}_l^{\epsilon}] &= -\xi\eta\epsilon(|\epsilon - \eta|\delta_{kl} f_j^{\xi} - |\epsilon - \xi|\delta_{jl} f_k^{\eta})K \\ &= -\eta\epsilon|\epsilon - \eta|\delta_{kl}\tilde{f}_j^{\xi} + \xi\epsilon|\epsilon - \xi|\delta_{jl}\tilde{f}_k^{\eta}.\end{aligned}$$

But for the allowed values of $\xi, \eta, \epsilon \in \{-1, +1\}$, one has $-\eta\epsilon|\epsilon - \eta| = |\epsilon - \eta|$, and similar for the second factor above. Therefore, the elements $\tilde{f}_j^{\pm}$ satisfy the usual parafermion triple relations

$$[[\tilde{f}_j^{\xi}, \tilde{f}_k^{\eta}], \tilde{f}_l^{\epsilon}] = |\epsilon - \eta|\delta_{kl}\tilde{f}_j^{\xi} - |\epsilon - \xi|\delta_{jl}\tilde{f}_k^{\eta}. \tag{16}$$

Next, let us turn to the “relative relations.” From the earlier observations, it follows already that

$$[[\tilde{f}_j^{\xi}, \tilde{f}_k^{\eta}], \tilde{b}_l^{\epsilon}] = 0, \qquad [\{\tilde{b}_j^{\xi}, \tilde{b}_k^{\eta}\}, \tilde{f}_l^{\epsilon}] = 0.$$

Furthermore,

$$\{\tilde{f}_j^{\xi}, \tilde{b}_k^{\eta}\} = \xi f_j^{\xi} K b_k^{\eta} + \xi b_k^{\eta} f_j^{\xi} K = -\xi[f_j^{\xi}, b_k^{\eta}]K,$$

and then

$$\begin{aligned}\{\{\tilde{f}_j^{\xi}, \tilde{b}_k^{\eta}\}, \tilde{f}_l^{\epsilon}\} &= -\xi[f_j^{\xi}, b_k^{\eta}]K\epsilon f_l^{\epsilon} K - \xi\epsilon f_l^{\epsilon} K[f_j^{\xi}, b_k^{\eta}]K \\ &= \epsilon\xi[f_j^{\xi}, b_k^{\eta}]f_l^{\epsilon} - \epsilon\xi f_l^{\epsilon}[f_j^{\xi}, b_k^{\eta}] \\ &= \epsilon\xi[[f_j^{\xi}, b_k^{\eta}], f_l^{\epsilon}] = -\epsilon\xi|\epsilon - \xi|\delta_{jl} b_k^{\eta} = |\epsilon - \xi|\delta_{jl}\tilde{b}_k^{\eta}.\end{aligned}$$

In a similar way, one finds

$$
\begin{aligned}
[\{\tilde{f}_j^\xi, \tilde{b}_k^\eta\}, \tilde{b}_l^\epsilon] &= -\xi [f_j^\xi, b_k^\eta] K b_l^\epsilon + \xi b_l^\epsilon [f_j^\xi, b_k^\eta] K \\
&= \xi [f_j^\xi, b_k^\eta] b_l^\epsilon K + \xi b_l^\epsilon [f_j^\xi, b_k^\eta] K \\
&= \xi \{[f_j^\xi, b_k^\eta], b_l^\epsilon\} K = \xi(\epsilon - \eta)\delta_{kl} f_j^\xi K = (\epsilon - \eta)\delta_{kl} \tilde{f}_j^\xi.
\end{aligned}
$$

In other words, the new operators $\tilde{b}_k^\pm$ and $\tilde{f}_j^\pm$ satisfy (15), (16) and

$$
\begin{aligned}
&[[\tilde{f}_j^\xi, \tilde{f}_k^\eta], \tilde{b}_l^\epsilon] = 0, \qquad [\{\tilde{b}_j^\xi, \tilde{b}_k^\eta\}, \tilde{f}_l^\epsilon] = 0, \\
&\{\{\tilde{f}_j^\xi, \tilde{b}_k^\eta\}, \tilde{f}_l^\epsilon\} = |\epsilon - \xi| \delta_{jl} \tilde{b}_k^\eta, \qquad [\{\tilde{f}_j^\xi, \tilde{b}_k^\eta\}, \tilde{b}_l^\epsilon] = (\epsilon - \eta)\delta_{kl} \tilde{f}_j^\xi.
\end{aligned} \tag{17}
$$

But (15)–(17) are exactly the relations for a mixed set of paraparticles satisfying the relative paraboson relations [5, 12].

In short, we have shown that the simple Klein transformation (14) maps the paraoperators with relative parafermion relations to the paraoperators with relative paraboson relations.

Observe that the algebra generated by the paraoperators $\tilde{b}_k^\pm$ and $\tilde{f}_j^\pm$, subject to the relations (15)–(17), is no longer (the enveloping algebra of) a Lie algebra or a Lie superalgebra. It was identified as a certain $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra in [10–13]. In the notation of Tolstoy [13], the parastatistics algebra with relative paraboson relations would be $\mathfrak{osp}(1, 2m|2n, 0)$. In [14], this $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra was denoted as $\mathfrak{pso}(2m+1|2n)$.

The Fock spaces of order p for the parastatistics algebra with relative parafermion relations were studied in [15]. They correspond to a class of lowest weight representations of the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$. In a similar way, the Fock spaces of order p for the parastatistics algebra with relative paraboson relations were studied in [14], corresponding to a class of lowest weight representations of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{pso}(2m+1|2n)$. Although one is dealing with different algebraic structures (in terms of gradings, commutators and anticommutators), the similarity between these representations was striking. Now that we have identified the Klein transformation relating these structures, the similarity becomes completely clear.

Acknowledgements N. Stoilova was supported by the Bulgarian National Science Fund, grant KP-06-N28/6, and J. Van der Jeugt was supported by the EOS Research Project 30889451.

References

1. H.S. Green, Phys. Rev. **90**, 270–273 (1953)
2. S. Kamefuchi, Y. Takahashi, Nucl. Phys. **36**, 177–206 (1962)
3. C. Ryan, E.C.G. Sudarshan, Nucl. Phys. **47**, 207–211 (1963)
4. A.Ch. Ganchev, T.D. Palev, J. Math. Phys. **21**, 797–799 (1980)
5. O.W. Greenberg, A.M. Messiah, Phys. Rev. **138**(5B), 1155–1167 (1965)
6. T.D. Palev, J. Math. Phys. **23**, 1100–1102 (1982)

7. O. Klein, J. de Phys. **9**, 1–12 (1938)
8. G. Lüders, Z. Naturforsch. **13a**, 254–260 (1958)
9. M.A. Vasil'ev, JETP Lett. **40**, 1261–1264 (1984)
10. W. Yang, S. Jing, Sci China (Ser A) **44**, 1167–1173 (2001)
11. W. Yang, S. Jing, Mod. Phys. Lett. A **16**, 963–971 (2001)
12. K. Kanakoglou, A. Herrera-Aguilar, J. Phys.: Conf. Ser. **287**, 011237 (2011)
13. V.N. Tolstoy, Phys. Part. Nucl. Lett. **11**, 933–937 (2014)
14. N.I. Stoilova, J. Van der Jeugt, J. Phys. A: Math. Theor. **51**, 135201 (2018). (17pp)
15. N.I. Stoilova, J. Van der Jeugt, J. Phys. A: Math. Theor. **48**, 155202 (2015). (16pp)

Principal and Complementary Series Representations at the Late-Time Boundary of de Sitter

Gizem Şengör and Constantinos Skordis

Abstract We demonstrate how free massive scalar fields in the set up that usually appears in early universe inflationary studies, correspond to the principal series and complementary series representations of the group SO(d+1,1) by introducing late-time operators and computing their two-point functions.

Keywords Quantum field theory on de Sitter · Realization of de sitter representations · Principal series representations · Complementary series representations

1 Introduction

Representations of Lie groups play an important role in quantum field theory since Wigner pointed out that elementary particles correspond to unitary irreducible representations of the isometry group of Minkowski spacetime, the Poincare group [9]. Of relevance to cosmology, is the de Sitter spacetime, a maximally symmetric spacetime with a positive cosmological constant, whose isometry group is also the conformal group of Euclidean space in one less dimension. The presence of conformal symmetries in de Sitter suggest the possibility to approach de Sitter physics in the framework of holography. Holography has been a valuable tool for studies on Anti de Sitter, a maximally symmetric spacetime with a negative cosmological constant and its applicability to the other maximally symmetric spacetimes is an active area of investigation. Following [6] and [7], we summarize how to recognize the unitary irreducible representations of the de Sitter isometry group $SO(d+1, 1)$

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_21

at the late-time boundary so as to gather clues on the inner workings of quantum field theory and the holographic nature of de Sitter spacetime.

2 The de Sitter Geometry, The de Sitter Group and the Late-Time Boundary

The aim of this section is to introduce notation and some background information so as to establish connection between mathematics and physics literatures with focus on quantum field theory.

The de Sitter geometry is the vacuum solution to Einstein equations with a positive cosmological constant [2]. In global coordinates, with the metric convention $diag(-,+,+,\dots)$, the metric for $d+1$ dimensional de Sitter is

$$ds^2_{global} = g^{\mu\nu}_{global} dX^\mu dX^\nu = -dT^2 + \frac{1}{H^2} cosh^2(HT) d\Omega^2_d, \tag{1}$$

where $d\Omega^2_d$ denotes the metric on d dimensional sphere and H is the Hubble constant associated with the de Sitter scale ($l_{dS} = H^{-1}_{dS}$). The time coordinate runs in the range $T \in (-\infty, \infty)$. Due to the behaviour of the cosine hyperbolic function, this spacetime undergoes accelerated expansion in the interval $T \in [0, \infty)$. Within the cosmic history of our universe we have evidence of two epochs of accelerated expansion. One of these is the primordial epoch of inflation and the second is the current epoch of dark energy domination. Inflationary and dark energy epochs correspond to de Sitter like epochs and a lot of information about these epochs can be obtained by studying quantized fields on de Sitter.

The de Sitter geometry has two conformal boundaries which lie along the time direction. These are referred to as the *early-time boundary*, denoted by $\mathcal{I}^-$, and the *late-time boundary*, $\mathcal{I}^+$. Figure 1(left) shows these boundaries in light blue, for the conformal diagram of de Sitter in global coordinates. We will carry out our calculations in the so called planar patch or Poincare patch coordinates where the metric is

$$ds^2_{planar} = -dt^2 + e^{2Ht} d\mathbf{x}^2 = \frac{-d\eta^2 + d\mathbf{x}^2}{H^2|\eta|^2}. \tag{2}$$

As shown in Fig. 1(right) these coordinates have access to the entire late-time boundary that we are interested in while they give access to only a single point from the early-time boundary. We will work in terms of conformal time η which runs in the range $\eta \in (-\infty, 0]$, and the late-time limit corresponds to the limit $\eta \to 0$. This coordinate system is the one used in inflationary studies, where late-time correlation functions [4], mainly two-point and three-point functions, can be used to put observational constraints on inflationary interactions with comparison to cosmic microwave background radiation observations [5].

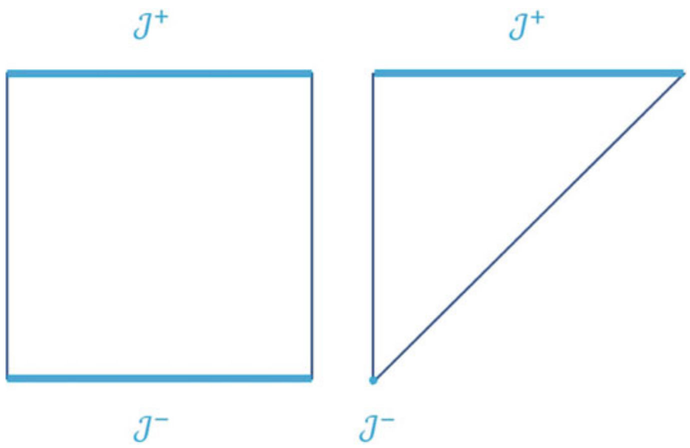

Fig. 1 Conformal diagrams for de Sitter: in global coordinates (left), in planar coordinates (right)

The isometries of $d+1$ dimensional de Sitter geometry correspond to the group $SO(d+1,1)$, also referred to as the de Sitter group. The representation theory of $SO(d+1,1)$ is well established in the mathematics literature initiating from the works of Harish-Chandra. Here we will follow the monograph [3]. A recent short review can also be found in [8].

The de Sitter group is composed of linear transformations on the real $d+2$ dimensional vector space that leave the following quadratic form invariant

$$\xi^2 = \xi\gamma\xi = \xi_1^2 + \cdots + \xi_{d+1}^2 - \xi_0^2 \tag{3}$$

where γ denotes the metric on flat $d+2$ dimensional spacetime with the nonzero components being $\gamma_{ii} = \cdots = \gamma_{d+1d+1} = -\gamma_{00} = 1$. The de Sitter transformations include dilatations with the corresponding subgroup denoted by $A = SO(1,1)$, spatial rotations $M = SO(d)$, spatial translations $\tilde{N}$ and special conformal transformations N. Moreover there is the maximally compact subgroup $K = SO(d+1)$. An important feature of de Sitter spacetime that sets it apart from the other two maximally symmetric spacetimes, Minkowski and Anti de Sitter, is that the de Sitter group does not involve global time translations.

The representations of the de Sitter group are induced by the parabolic subgroup $P = NAM$. Under a dilatation where $\mathbf{x} \to \lambda\mathbf{x}$ an operator among the representations of the de Sitter group transform as

$$\mathcal{O}(\lambda\mathbf{x}) = \lambda^{-\Delta}\mathcal{O}(\mathbf{x}). \tag{4}$$

The exponent Δ is called the scaling dimension and for the group $SO(d+1,1)$ it has the following form

$$\Delta = \frac{d}{2} + c \tag{5}$$

where c is called the *scaling weight* and it determines which category a given representation belongs to.

The eigenvalues of the quadratic Casimir operator are $\mathcal{C} = l(l + d - 2) + c^2 - \frac{d^2}{4}$ and therefore the representations are labeled by spin l the label for representations of $M = SO(d)$ and the scaling weight c, denoted in a compact way as $\chi = [l, c]$. The unitary representations of the de Sitter group fall under four different categories where for each category the range of c and the well defined inner product is different. Representations with purely imaginary scaling weight belong to the *principal series* representations, these are irreducible and have a straightforward inner product. Three different categories span unitary representations with real scaling weight, the irreducible *complementary series* and *discrete series* representations and the reducible *exceptional series* representations. The well defined inner product involves intertwining operators for these categories. The range of c and the accompanying intertwining operator in the inner product differs for each of these categories. We refer the reader to [3] for the definitions of appropriate intertwining operators and to [6] for a summary and a practical use of the intertwining operator in the case of complementary series representations.

3 The Late-Time Operators

We will introduce late-time operators that correspond to free quantized scalar fields on de Sitter following [6]. Our assumption is that the scalar field does not effect the geometry, the metric is fixed to be the de Sitter metric. A free, massive scalar field on de Sitter has the following action

$$S = \int d^d x d\eta \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right] \tag{6}$$

where g denotes the determinant of the metric.

For convenience we will consider the fourier modes of the field, $\phi_k(\eta)$, in momentum space $\mathbf{k}$. For a quantized field, the mode decomposition involves *annihilation*, $a_{\mathbf{k}}$, and *creation*, $a^\dagger_{\mathbf{k}}$, operators. These operators obey the following nontrivial commutation relation

$$\left[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'} \right] = (2\pi)^d \delta^{(d)}(\mathbf{k} - \mathbf{k}'). \tag{7}$$

The annihilation operator annihilates the vacuum state $|0\rangle$, while the creation operator with a given momentum acting on the vacuum state creates a state with the specified momentum

$$a_{\mathbf{k}}|0\rangle = 0, \quad a^\dagger_{\mathbf{k}}|0\rangle = |\mathbf{k}\rangle. \tag{8}$$

The states created in this way are normalized with respect to

$$\langle \mathbf{k}|\mathbf{k}'\rangle = (2\pi)^d \delta^{(d)}(\mathbf{k}-\mathbf{k}'). \tag{9}$$

With these operators the scalar field is decomposed into its fourier modes as follows

$$\phi(\mathbf{x},\eta) = \int \frac{d^dk}{(2\pi)^d}\left[\phi_k(\eta)a_{\mathbf{k}} + \phi_k^*(\eta)a^{\dagger}_{-\mathbf{k}}\right]e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{10}$$

where $*$ denotes complex conjugation and $\phi(\mathbf{x},\eta)$ is a real field while the modes $\phi_k(\eta)$ are complex valued.

We demand Bunch-Davies initial conditions [1], which require the field to behave as if it was on Minkowski at early times. The mode functions that satisfy the equations of motion with these initial conditions are given in terms of Hankel functions. The solutions split into two branches depending on how heavy the mass of the field is with respect to the Hubble scale as follows

$$for\ m < \frac{d}{2}H:\ \phi_k^L(\eta) = |\eta|^{\frac{d}{2}}H_\nu^{(1)}(k|\eta|),\ where\ \nu^2 = \frac{d^2}{4} - \frac{m^2}{H^2} \tag{11}$$

we call these as light fields, and

$$for\ m > \frac{d}{2}H:\ \phi_k^H(\eta) = |\eta|^{\frac{d}{2}}e^{-\rho\pi/2}H_{i\rho}^{(1)}(k|\eta|),\ where\ \rho^2 = \frac{m^2}{H^2} - \frac{d^2}{4} \tag{12}$$

we call these as heavy fields. Our notation is such that ν and ρ are real and positive.

In the late-time limit, as $\eta \to 0$, the field goes to

$$\lim_{\eta\to 0}\phi(\mathbf{x},\eta) = \int \frac{d^dk}{(2\pi)^d}\left[|\eta|^{\frac{d}{2}-\mu_p}\alpha^p(\mathbf{k}) + |\eta|^{\frac{d}{2}+\mu_p}\beta^p(\mathbf{k})\right]e^{i\mathbf{k}\cdot\mathbf{x}} \tag{13}$$

where the label p indicates light and heavy $p = L, H$ and $\mu_L = \nu$, $\mu_H = i\rho$. The operators $\alpha^p(\mathbf{k})$ and $\beta^p(\mathbf{k})$ are the *late-time operators*. They are build out of annihilation and creation operators. Moreover the late-time operator $\alpha^p(\mathbf{k})$ has momentum dependence $k^{-\mu_p}$ while $\beta^p(\mathbf{k})$ has k^{μ_p}. Looking at the scaling dimensions of these operators we recognize that their scaling dimensions match the format of scaling dimensions for the unitary irreducible representations of the de Sitter group with scaling weights $c = \pm\mu_p$ [6], with plus sign for the β operator, minus for the α operator.

We can define states by acting on the vacuum state with the late-time operators, such as

$$|\alpha^p(\mathbf{k})\rangle \equiv \mathcal{N}^p\alpha^p(\mathbf{k})|0\rangle \tag{14}$$

where $\mathcal{N}^p$ is the normalization to be determined and the same argument follows for $\beta^p(\mathbf{k})$. Such states build from light late-time operators with $c = \pm\nu$ are normalizable with respect to the complementary series inner product in the range $0 < m < \frac{d}{2}H$ while such states build from any of the heavy late-time operators with $c = \pm i\rho$ are normalizable with respect to the principal series inner product. From these observations we can identify our normalized heavy late-time operators

$$\alpha_{\rm N}^{\rm H}(\mathbf{k}) = \sqrt{\rho\pi\sinh(\rho\pi)}\left[-i\,\tfrac{\Gamma(i\rho)}{\pi}e^{-\rho\pi}a_{\mathbf{k}} + \tfrac{1}{\sinh(\rho\pi)\Gamma(1-i\rho)}a^{\dagger}_{-\mathbf{k}}\right]\left(\tfrac{k}{2}\right)^{-i\rho} \quad (15)$$

$$\beta_{\rm N}^{\rm H}(\mathbf{k}) = \sqrt{\rho\pi\sinh(\rho\pi)}\left[\tfrac{e^{\rho\pi}}{\sinh(\rho\pi)\Gamma(1+i\rho)}a_{\mathbf{k}} + i\,\tfrac{\Gamma(-i\rho)}{\pi}a^{\dagger}_{-\mathbf{k}}\right]\left(\tfrac{k}{2}\right)^{i\rho} \quad (16)$$

as the principal series representations of the de Sitter group, and the normalized light late-time operators

$$\alpha_{\rm N}^{\rm L}(\mathbf{k}) = -i2^{\nu/2}\left[a_{\mathbf{k}} - a^{\dagger}_{-\mathbf{k}}\right]k^{-\nu}, \quad (17)$$

$$\beta_{\rm N}^{\rm L}(\mathbf{k}) = 2^{-\nu/2}\left[\frac{1+i\cot(\pi\nu)}{1-i\cot(\pi\nu)}a_{\mathbf{k}} + a^{\dagger}_{-\mathbf{k}}\right]k^{\nu} \quad (18)$$

as the complementary series representations of the de Sitter group [6, 7]. The α_N^p and β_N^p have nontrivial commutation relations inherited from the commutation relation of the annihilation and creation operators.

4 The Late-Time Two-Point Functions

At this point it is straight forward to calculate the two-point functions defined as $\langle\mathcal{O}_1(\mathbf{k})\mathcal{O}_2(\mathbf{k})\rangle \equiv \langle 0|\mathcal{O}_1(\mathbf{k})\mathcal{O}_2(\mathbf{k})|0\rangle$. We obtain the following list [7]

$$\begin{aligned}
\langle\alpha_{\rm N}^{\rm L}(\mathbf{k})\alpha_{\rm N}^{\rm L}(\mathbf{k}')\rangle &= 2^{\nu}k^{-2\nu}(2\pi)^d\delta^{(d)}(\mathbf{k}+\mathbf{k}'),\\
\langle\beta_{\rm N}^{\rm L}(\mathbf{k})\beta_{\rm N}^{\rm L}(\mathbf{k}')\rangle &= \frac{k^{2\nu}}{2^{\nu}}\frac{1+i\cot(\pi\nu)}{1-i\cot(\pi\nu)}(2\pi)^d\delta^{(d)}(\mathbf{k}+\mathbf{k}'),\\
\langle\alpha_{\rm N}^{\rm L}(\mathbf{k})\beta_{\rm N}^{\rm L}(\mathbf{k}')\rangle &= -i(2\pi)^d\delta^{(d)}(\mathbf{k}+\mathbf{k}'),\ \langle\beta_{\rm N}^{\rm L}(\mathbf{k})\alpha_{\rm N}^{\rm L}(\mathbf{k}')\rangle =\\
&= i\frac{1+i\cot(\pi\nu)}{1-i\cot(\pi\nu)}(2\pi)^d\delta^{(d)}(\mathbf{k}+\mathbf{k}'),
\end{aligned} \quad (19)$$

for the complementary series two-point functions, while for the principal series

$$\begin{aligned}
\langle\alpha_{\rm N}^{\rm H}(\mathbf{k})\alpha_{\rm N}^{\rm H}(\mathbf{k}')\rangle &= -\frac{\Gamma(1+i\rho)}{\Gamma(1-i\rho)}e^{-\rho\pi}(2\pi)^d\delta^{(d)}(\mathbf{k}+\mathbf{k}')\left(\frac{k}{2}\right)^{-2i\rho},\\
\langle\beta_{\rm N}^{\rm H}(\mathbf{k})\beta_{\rm N}^{\rm H}(\mathbf{k}')\rangle &= i\rho\frac{\Gamma(-i\rho)}{\Gamma(1+i\rho)}e^{\rho\pi}(2\pi)^d\delta^{(d)}(\mathbf{k}+\mathbf{k}')\left(\frac{k}{2}\right)^{2i\rho}\\
\langle\alpha_{\rm N}^{\rm H}(\mathbf{k})\beta_{\rm N}^{\rm H}(\mathbf{k}')\rangle &= e^{-\rho\pi}(2\pi)^d\delta^{(d)}(\mathbf{k}+\mathbf{k}'),\\
\langle\beta_{\rm N}^{\rm H}(\mathbf{k})\alpha_{\rm N}^{\rm H}(\mathbf{k}')\rangle &= e^{\rho\pi}(2\pi)^d\delta^{(d)}(\mathbf{k}+\mathbf{k}').
\end{aligned} \quad (20)$$

In [7] how these two-point functions contribute to field and conjugate momentum two-point functions at late-times which are related to observable quantities are discussed, in canonical and wavefunction quantization.

With the list of two-point functions in (19) and (20) we hope to provide further insight on the structure of two-point functions in momentum space in general dimensions, in the presence of $SO(d+1, 1)$ invariance, in a way that make the cases of principal and complementary series representations comparable to each other starting from the case of scalar operators. To do so we have grounded our analysis on how the unitary irreducible representations present themselves at the late-time boundary of de Sitter spacetime.

Our work has some overlap with other studies which we now discuss. In [10], the authors consider a different construction to obtain principal series operators at the early time boundary of dS_3 and analyse their two-point function at large separation in terms of cluster decomposition axiom of Euclidean quantum field theory based on [11]. Another example is [12] that focusses on two dimensions, where the structure of principal series two-point functions in the presence of $SL(2, R)$ symmetries, which include representations of $SO(2, 1)$, is analysed in position space. Also working in two dimensions, [13] provides a holographic toy model, a quantum mechanical model that captures the principal series representations. While we considered operators at the late-time boundary in momentum space that correspond to unitary irreducible representations of $SO(d+1, 1)$, [14] considers the construction of local bulk fields based on the representation theory by incorporating the properties of the representations into the definition of annihilation and creation operators.

Acknowledgements GŞ is supported by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 840709-SymAcc. CS acknowledges support from the European Structural and Investment Funds and the Czech Ministry of Education, Youth and Sports (MSMT) (Project CoGraDS -CZ.02.1.01/0.0/0.0/15003/0000437).

References

1. T.S. Bunch, P.C.W. Davies, Proc. Roy. Soc. Lond. **A360**, 117 (1978)
2. W. de Sitter, Mon. Not. Roy. Astron. Soc. **78**, 3 (1917)
3. V.K. Dobrev, G. Mack, V.B. Petkova, S.G. Petrova, I.T. Todorov, Harmonic analysis on the n-dimensional Lorentz group and its application to conformal quantum field theory. Lect. Notes Phys. **63**, 1 (1977)
4. J.M. Maldacena, JHEP **05**, 013 (2003). ([astro-ph/0210603])
5. Planck collaboration, Astron. Astrophys. **641**, A10 (2020). [arXiv: 1807.06211]
6. S. Sengor, C. Skordis, JHEP **06**, 041 (2020). (arXiv:1912.09885 [hep-th])
7. S. Sengor, C. Skordis, arXiv:2110.01635 [hep-th]
8. Z. Sun, arXiv:2111.04591 [hep-th]
9. E. Wigner, Ann. Math. **40**, 149 (1939)
10. A. Chatterjee, D. Lowe, Phys. Rev. D **92**, 084038 (2015)
11. K. Ostrawalder, R. Schrader, Commun. Math. Phys. **31**, 83 (1973); K. Ostrawalder, R. Schrader, Commun. Math. Phys. **42**, 281 (1975)
12. D. Anninos, D.M. Hofman, J. Kruthoff, SciPost Phys. **7**, 054 (2019)

13. T. Anous, J. Skulte, SciPost Phys. **9**, 028 (2020)
14. E. Joung, J. Mourad, R. Parentani, JHEP **0709**, 030 (2007); E. Joung, J. Mourad, R. Parentani, JHEP **0608**, 082 (2006)
15. S. Preuss, A. Demchuk Jr., M. Stuke, Appl. Phys. A
16. M.K. Slifka, J.L. Whitton, J. Mol. Med., https://doi.org/10.1007/s001090000086
17. S.E. Smith, in *Neuromuscular Junction*, ed. by E. Zaimis. Handbook of Experimental Pharmacology, vol. 42 (Springer, Heidelberg, 1976), p. 593

Bulk Reconstruction from a Scalar CFT at the Boundary by the Smearing with the Flow Equation

Sinya Aoki, Janos Balog, Tetsuya Onogi, and Shuichi Yokoyama

Abstract We explain our proposal for an alternative bulk reconstruction of AdS/CFT correspondences from a scalar field by the flow method. By smearing and then normalizing a primary field in a d dimensional CFT, we construct a bulk field, through which a $d+1$ dimensional AdS space emerges. The content of this proceeding is based on a talk given by S. Aoki at the 14th International Workshop "Lie Theory and its Applications in Physics" (LT-14), 21-25 June 2021, Sofia, Bulgaria (on-line).

Keywords AdS/CFT correspondence · Bulk reconstruction · Flow equation

1 Introduction

The AdS/CFT correspondence [1] is a key to understand a holographic nature of gravity and may give a hint for quantum gravity. Although plenty of circumstance evidences exist, an essential mechanism why AdS/CFT correspondence realizes has not been fully established yet. While the correspondence may be explained by the string theory, an alternative but more universal mechanism might exist.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_22

One of key questions is how an additional dimension of the AdS emerges from the CFT at the boundary. One approach, called the HKLL bulk reconstruction [2, 3], gives a relation between a bulk local operator in the AdS and CFT operators at the boundary with Lorentzian signature.

Recently we have proposed an alternative bulk reconstruction by the so-called flow method by the path-integral with Euclidean signature [4–9], which is explained in this talk.

2 Bulk Reconstruction by the Flow

2.1 From Conformal Symmetry to Bulk Symmetry

Let us consider a non-singlet primary field of a conformal field theory (CFT) on a d dimensional Euclidean space, whose 2-pt function behaves as

$$\langle \varphi^a(x)\varphi^b(y)\rangle = \delta^{ab}\frac{1}{|x-y|^{2\Delta}}, \tag{1}$$

where Δ is a conformal dimension of φ^a and $a, b \ldots$ represent indices of a global symmetry such as an $O(N)$ symmetry.

We first make one parameter extension of the field φ^a via the generalized flow equation as

$$(-\alpha\eta\partial_\eta^2 + \beta\partial_\eta)\phi^a(x,\eta) = \Box\phi^a(x,\eta), \quad \phi^a(x,0) = \varphi^a(x), \tag{2}$$

where α, β are parameters to control this extension. We regard this process as the smearing of the field φ^a, since the flow equation becomes the heat equation at $\alpha = 0$. We then normalize ϕ^a as

$$\sigma^a(X) = \frac{\phi^a(x,\eta)}{\sqrt{\langle\phi(x,\eta)\cdot\phi(x,\eta)\rangle}} \Rightarrow \langle\sigma(X)\cdot\sigma(X)\rangle = 1, \tag{3}$$

where the average is taken for the CFT in d dimensions as in (1), $X := (z, x)$ is a $d+1$ dimensional coordinate with $\eta := \alpha z^2/4$, and $F \cdot G := \sum_a F^a G^a$. The smearing followed by normalization may be interpreted as a kind of the renormalization group transformation. The field σ^a is expressed in the smeared form by the integral as

$$\sigma^a(X) = \int d^d y\, h(z, x-y)\varphi^a(y). \tag{4}$$

The smearing kernel $h(z, x)$ is determined by conditions that the conformal transformation U to φ^a, given by

$$U\varphi^a(y)U^\dagger := \tilde{\varphi}^a(y) = J^\Delta(y)\varphi^a(\tilde{y}), \tag{5}$$

generates the coordinate transformation to σ^a in $d+1$ dimensions as

$$U\sigma^a(X)U^\dagger := \tilde{\sigma}^a(X) = \sigma^a(\tilde{X}), \tag{6}$$

where

1. translation $\tilde{y}^\mu = y^\mu + a^\mu$, $J(y) = 1$, $\tilde{x}^\mu = x^\mu + a$, $\tilde{z} = z$,
2. rotation $\tilde{y}^\mu = \Omega^\mu{}_\nu y^\nu$, $J(y) = 1$, $\tilde{x}^\mu = \Omega^\mu{}_\nu x^\nu$, $\tilde{z} = z$,
3. dilatation $\tilde{y}^\mu = \lambda y^\mu$, $J(y) = \lambda$, $\tilde{X}^A = \lambda X^A$,
4. inversion $\tilde{y}^\mu = y^\mu / y^2$, $J(y) = 1/y^2$, $\tilde{X}^A = X^A / X^2$,

which generate SO($d+1$, 1) transformation. Transformations 1–3 imply

$$\sigma^a(X) = z^{\Delta - d} \int d^d y\, \Sigma\left(1 + \frac{(x-y)^2}{z^2}\right)\varphi^a(y), \tag{7}$$

where Σ is an arbitrary function. The transformation 4 to the CFT operator in the above equation generates

$$\begin{aligned} z^{\Delta-d}\int d^d y\, \Sigma\left(1+\frac{(x-y)^2}{z^2}\right)\frac{\varphi(q=\tilde{y})}{(y^2)^\Delta} &= \\ = \int d^d q\, \Sigma\left(1+\frac{(x-\tilde{q})^2}{z^2}\right)(zq^2)^{\Delta-d}\varphi(q) &= \\ = \left(\frac{\tilde{z}}{\tilde{X}^2}\right)^{\Delta-d}\int d^d q\, \Sigma\left(\frac{\tilde{X}^2}{q^2}\left[1+\frac{(\tilde{x}-q)^2}{\tilde{z}^2}\right]\right)(q^2)^{\Delta-d}\varphi(q), & \end{aligned} \tag{8}$$

while the bulk operator transforms as

$$\sigma^a(\tilde{X}) = \tilde{z}^{\Delta-d}\int d^d q\, \Sigma\left(1+\frac{(\tilde{x}-q)^2}{\tilde{z}^2}\right)\varphi(q). \tag{9}$$

Thus the symmetry implies $\Sigma(u) = \Sigma_0 u^{\Delta-d}$, which finally gives

$$\sigma^a(X) = \int d^d y\, h(z, x-y)\varphi^a(y), \quad h(z,x) = \Sigma_0\left(\frac{z}{z^2+x^2}\right)^{d-\Delta}, \tag{10}$$

where $\Delta < d$ is necessary for this expression to be regarded as a smearing. After the Wick rotation, this smearing kernel is identical to the one in the HKLL [3], though $\Delta > d-1$ is required for the HKLL.

2.2 Some properties

It is easy to see the kernel h satisfies $(\Box_{\text{AdS}} - m^2)h(X) = 0$, where $\Box_{\text{AdS}} := z^2(\partial_z^2 + \Box) - (d-1)z\partial_z$ and $m^2R^2 := (\Delta - d)\Delta$ with some length scale R, which turns out to be the AdS radius. Furthermore, $h(X)$ corresponds to a solution to the flow Eq. (2) with $\beta/\alpha = (d-2)/2 - \Delta$. As a result, $\sigma^a(X)$ satisfies equations of motion for a scalar field in the Euclidean AdS space as

$$(\Box_{\text{AdS}} - m^2)\sigma^a(X) = 0. \tag{11}$$

Since, for $\Delta < d/2$,

$$\lim_{z\to 0} \frac{z^{d-2\Delta}}{(x^2+z^2)^{d-\Delta}} \sim \begin{cases} z^{d-2\Delta} \to 0, \; x \neq 0, \\ z^{-d} \to \infty, \;\; x = 0, \end{cases} \tag{12}$$

$$\int d^dx \, \frac{z^{d-2\Delta}}{(x^2+z^2)^{d-\Delta}} = \frac{\pi^{d/2}\Gamma(d/2-\Delta)}{\Gamma(d-\Delta)} := \frac{1}{\Lambda}, \tag{13}$$

we can write

$$\lim_{z\to 0} \frac{z^{d-2\Delta}}{(x^2+z^2)^{d-\Delta}} = \frac{\delta^{(d)}(x)}{\Lambda}. \tag{14}$$

Thus the smearing function satisfies

$$\lim_{z\to 0} h(z,x) = \frac{\Sigma_0}{\Lambda} z^{\Delta}\delta^{(d)}(x), \tag{15}$$

which leads to a BDHM relation [10] as

$$\lim_{z\to 0} z^{-\Delta}\sigma^a(X) = \frac{\Sigma_0}{\Lambda}\varphi^a(x). \tag{16}$$

Using a singlet bulk composite scalar field $S(x) := \sum_a \sigma^a(X)\sigma^a(X)$, we can define a bulk to boundary correlation function as

$$\mathcal{F}_O(X,y) := \langle S(X)O(y)\rangle \tag{17}$$

where $O(y)$ is an arbitrary singlet scalar primary field with a conformal dimension Δ_O at the boundary. A combination of the conformal symmetry and the bulk symmetry in the previous subsection leads to

$$\mathcal{F}_O(X,y) = C_O\left(\frac{z}{(x-y)^2+z^2}\right)^{\Delta_O}, \tag{18}$$

which is exact in the sense that only unknown constant C_O depends on the detail of the boundary CFT such as coupling constants, and reproduces the standard prediction by the AdS/CFT correspondence. Furthermore, $\mathcal{F}$ satisfies

$$(\Box^X_{\text{ADS}} - m_O^2)\mathcal{F}(X, y) = 0, \quad m_O^2 := \frac{\Delta_O(\Delta_O - d)}{R^2}. \tag{19}$$

3 Symmetries and Bulk Geometry

In this section we investigate what kind of space the $d+1$ dimensional coordinate X describes.

3.1 Constraints to a Generic Correlation Function

Let us consider a generic correlation function, which contains m bulk fields and s boundary fields, given by

$$\left\langle \prod_{i=1}^m G^i_{A_1^i A_2^i \cdots A_{n_i}^i}(X_i) \prod_{j=1}^s O^j_{\mu_1^j \mu_2^j \cdots \mu_{l_j}^j}(y_j) \right\rangle, \tag{20}$$

where $A_1^i A_2^i \cdots A_{n_i}^i$ is a tensor index of a bulk operator G^i, while $\mu_1^j \mu_2^j \cdots \mu_{l_j}^j$ is that for a boundary operator O^j. The conformal symmetry at the boundary with the associated coordinate transformation in the bulk gives an exact quantum relation as

$$\left\langle \prod_{i=1}^m G^i_{A_1^i A_2^i \cdots A_{n_i}^i}(\tilde{X}_i) \prod_{j=1}^s O^j_{\mu_1^j \mu_2^j \cdots \mu_{l_j}^j}(\tilde{y}_j) \right\rangle = \prod_{i=1}^m \frac{\partial X_I^{B_1^i}}{\partial \tilde{X}_I^{A_1^i}} \frac{\partial X_I^{B_2^i}}{\partial \tilde{X}_I^{A_2^i}} \cdots \frac{\partial X_I^{B_{n_i}^i}}{\partial \tilde{X}_I^{A_{n_i}^i}}$$
$$\times \prod_{j=1}^s J(y_j)^{-\Delta_j} \frac{\partial y_j^{\nu_1^j}}{\partial \tilde{y}_j^{\mu_1^j}} \frac{\partial y_j^{\nu_2^j}}{\partial \tilde{y}_j^{\mu_2^j}} \cdots \frac{\partial y_j^{\nu_{l_j}^j}}{\partial \tilde{y}_j^{\mu_{l_j}^j}} \left\langle \prod_{i=1}^m G^i_{B_1^i B_2^i \cdots B_{n_i}^i}(X_i) \prod_{j=1}^s O^j_{\nu_1^j \nu_2^j \cdots \nu_{l_j}^j}(y_j) \right\rangle. \tag{21}$$

3.2 Metric Field

We define the bulk metric field as [11]

$$g_{AB}(X) := \ell^2 \sum_{a=1}^N \partial_A \sigma^a(X) \partial_B \sigma^a(X), \tag{22}$$

which is the simplest among singlet and symmetric 2nd rank tensors in the bulk, where ℓ is some length scale. It is worth noting that this metric field is finite without ultra-violate (UV) divergence thank to the smearing.

The metric becomes classical in the large N limit due to the large N factorization as [11]

$$\langle g_{AB}(X_1) g_{CD}(X_2)\rangle = \langle g_{AB}(X_1)\rangle\langle g_{CD}(X_2)\rangle + O(1/N), \tag{23}$$

which make a vacuum expectation value (VEV) of the quantum Einstein tensor G_{AB} classical:

$$\langle G_{AB}(g_{CD})\rangle = G_{AB}(\langle g_{CD}\rangle) + O(1/N). \tag{24}$$

A classical geometry appears after quantum averages.

The VEV of the metric can be interpreted as the (Bures) information metric [4], which defines a distance between (mixed state) density matrices ρ and $\rho + d\rho$ as

$$d^2(\rho, \rho + d\rho) := \frac{1}{2}\mathrm{tr}[d\rho\, G], \tag{25}$$

where G satisfies $\rho G + G\rho = d\rho$. In our case, ρ is given by a mixed state as

$$\rho(X) := \sum_{a=1}^{N} |\sigma^a(X)\rangle\langle\sigma^a(X)|, \tag{26}$$

which represents N entangled pairs, where the inner product between states is defined by $\langle\sigma^a(X)|\sigma^b(Y)\rangle = \delta^{ab}\langle\sigma(X)\cdot\sigma(Y)\rangle/N$. Using this definition, we obtain

$$\ell^2 d^2(\rho, \rho + d\rho) = \langle g_{AB}(X)\rangle dX^A dX^B. \tag{27}$$

Thus the VEV of the metric field $g_{AB}(X)$ defines a distance in the bulk space through $d^2(\rho, \rho + d\rho)$ in unit of ℓ^2.

3.3 VEV of the Metric Field

Let us calculate the VEV of the metric field. The constraint (21) applied to g_{AB} leads to

$$\langle 0|g_{AB}(X)|0\rangle = R^2\frac{\delta_{AB}}{z^2}, \tag{28}$$

which describes the AdS space in the Poincare coordinate, where an unknown constant R is the radius of the AdS space. The explicit form of the 2-pt function of CFT in (1) gives $R^2 = \ell^2\Delta(d-\Delta)/(d+1)$, which is positive since $\Delta < d/2$. The boundary CFT generates the bulk AdS, and the bulk symmetry in the previous section is equal to the AdS isometry on $\langle 0|g_{AB}(X)|0\rangle$. Thus the AdS/CFT correspondence is realized by the flow construction.

3.4 Scalar Excited State Contribution

We finally consider how excited states modify the AdS structure described by the VEV of the metric field. As the simplest example, we calculate the VEV of the metric in the presence of the source J coupled to the singlet CFT scalar field at the origin in the radial quantization as

$$\bar{g}_{AB}(X) = \langle 0|g_{AB}(X)e^{JO(0)}|0\rangle = \langle 0|g_{AB}(X)|0\rangle + J\langle 0|g_{AB}(X)|S\rangle + O(J^2), \tag{29}$$

where we need to calculate

$$\langle 0|g_{AB}(X)|S\rangle = \lim_{y^2\to 0} G_{AB}(X, y), \quad G_{AB}(X, y) := \langle 0|g_{AB}(X)O(y)|0\rangle. \tag{30}$$

The constraint (21) to G_{AB} reads

$$J(y)^{\Delta_O} G_{AB}(X, y) = \frac{\partial \tilde{X}^C}{\partial X^A}\frac{\partial \tilde{X}^D}{\partial X^B} G_{CD}(\tilde{X}, \tilde{y}), \tag{31}$$

which leads to

$$G_{AB}(X, y) = T^{\Delta_O}(X, y)\left[a_1\frac{\delta_{AB}}{z^2} + a_2\frac{T_A(X, y)T_B(X, y)}{T^2(X, y)}\right], \tag{32}$$

where a_1 and a_2 are unknown constants, and

$$T(X, y) := \frac{z}{(x-y)^2 + z^2}, \quad T_A(X, y) := \partial_A T(X, y). \tag{33}$$

We thus obtain

$$\bar{g}_{AB}(X) = \frac{\delta_{AB}}{z^2}\left[R^2 + a_1 J\left(\frac{z}{x^2+z^2}\right)^{\Delta_O}\right] + a_2 J T_A T_B\left(\frac{z}{x^2+z^2}\right)^{\Delta_O - 2} \tag{34}$$

at the 1st order in J, where

$$T_z := \frac{x^2 - z^2}{(x^2+z^2)^2}, \quad T_\mu := -\frac{2x_\mu z}{(x^2+z^2)^2}. \tag{35}$$

This metric describes the asymptotically AdS space since

$$\bar{g}_{AB}(X) = R^2 \frac{\delta_{AB}}{z^2} \left[1 + O\left(z^{\Delta_O}\right)\right], \tag{36}$$

as $z \to 0$.

4 Summary and Discussion

Smearing and normalizing the non-singlet primary field φ^a with $\Delta < d/2$ at the boundary give the bulk field σ^a. Through this process, the conformal symmetry turns into the bulk coordinate transformation, which leads to the expected behavior for the bulk to boundary correlation function. As the VEV of the metric field describes the AdS space, the AdS/CFT correspondence is naturally realized by our method.

As one of future problems, let us consider a bulk to bulk correlation function for the scalar, which is evaluated as

$$\langle S(X_1)S(X_2)\rangle = 1 + \frac{2}{N} G^2(X_1, X_2) + \langle S(X_1)S(X_2)\rangle_c, \tag{37}$$

where

$$G(X_1, X_2) = {}_2F_1\left(\frac{\Delta}{2}, \frac{d-\Delta}{2}; \frac{d+1}{2}; 1 - \frac{R_{12}^2}{4}\right), \; R_{12} := \frac{(x_1 - x_2)^2 + z_1^2 + z_2^2}{z_1 z_2}, \tag{38}$$

which is non-singular at $X_1 = X_2$. Since the connected part given by the last term is absent for the free CFT, the corresponding bulk theory is non-local (stringy). Thus the interaction in the CFT is required to recover the locality of the bulk theory.

Acknowledgements This work is supported in part by the Grant-in-Aid of the Japanese Ministry of Education, Sciences and Technology, Sports and Culture (MEXT) for Scientific Research (Nos. JP16H03978, JP18K03620, JP18H05236, JP19K03847), and by the NKFIH grant K134946.

References

1. J.M. Maldacena, Adv. Theor. Math. Phys. **2**, 231–252 (1998)
2. A. Hamilton, D.N. Kabat, G. Lifschytz, D.A. Lowe, Phys. Rev. D **73**, 086003 (2006)
3. A. Hamilton, D.N. Kabat, G. Lifschytz, D.A. Lowe, Phys. Rev. D **74**, 066009 (2006)
4. S. Aoki, S. Yokoyama, PTEP **2018**(3), 031B01 (2018)
5. S. Aoki, S. Yokoyama, Nucl. Phys. B **933**, 262–274 (2018)
6. S. Aoki, J. Balog, S. Yokoyama, PTEP **2019**(4), 043B06 (2019)
7. S. Aoki, S. Yokoyama, K. Yoshida, Phys. Rev. D **99**(12), 126002 (2019)
8. S. Aoki, J. Balog, S. Yokoyama, K. Yoshida, Phys. Rev. Res. **2**(1), 013169 (2020)
9. S. Aoki, T. Onogi, S. Yokoyama, Phys. Lett. B **814**, 136104 (2021)

10. T. Banks, M.R. Douglas, G.T. Horowitz, E.J. Martinec, [arXiv:hep-th/9808016] [hep-th]
11. S. Aoki, K. Kikuchi, T. Onogi, PTEP **2015**(10), 101B01 (2015).https://doi.org/10.1093/ptep/ptv131 [arXiv:1505.00131] [hep-th]

Building Momentum Kernel from Shapovalov Form

Yihong Wang and Chih-Hao Fu

Abstract In this text, we consider structures and physical objects from scattering amplitudes in the framework of a Lie algebra with space-time momentum as its simple roots. We identify the momentum kernel for scattering amplitudes with the Shapovalov form on basis vectors of the Verma module, at both the classical and quantum level. We then take a step forward and show how the Feynman diagrams emerges from the Shapovalov dual of the Verma module basis vectors.

Keywords Scattering amplitudes · Colour-kinematics duality · Shapovalov form · Planar binary tree · Kawai-Lewellen-tye relations · Feynman propagator

1 Introduction

Although seeking an alternative theoretical construct that can at the same time reproduces field theory amplitudes was one of the initial motive and achievement for string theory, the study of respective amplitudes of string and field theory later picked up different emphasis. While the field theory amplitude calculations are nowadays largely driven by the demand for theoretical predictions directly comparable with data from particle colliders, the study of string amplitudes, besides its original scope to explore possible consistent theories, later picked up more emphasis in the CFT and quantum algebra context [4–6], viewing string amplitudes as conformal blocks. The two dis-

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_23

ciplines reunited in recent years [19] and it has become clear to physicists that some of the observable features of field theory amplitudes are actually linked to properties which were previously known only for string amplitudes such as the KLT relation between open and closed string amplitudes [12, 19] and Plahte identity [18]. In the point particle limit KLT relation reduces to the double copy relation connecting Yang-Mills and gravity amplitudes [2, 3]

$$A_{GR} = (-1)^n \sum_{\gamma,\, \beta \in S_{n-2}} \tilde{A}_{YM}\,(n, \gamma(n-1), ..\gamma(2), 1)\; S[\gamma|\beta] \qquad (1)$$
$$A_{YM}\,(1, \beta(2), .., \beta(n-1), n) \,/ \sum_{1 \le i < j \le n} 2k_i \cdot k_j,$$

whereas the Plahte identity for open string amplitudes reduces to Bern-Carrasco-Johansson (BCJ) relation for Yang-Mills amplitudes [1], which in turn was understood to be equivalent to the fact that the amplitudes are expressible as sum of terms indexed by binary trees. In this paper we shall denote the set of full binary trees with leaves $\{1, 2, ..., n-1\}$ as $BT\,(\{1, 2, ..., n-1\})$, so that according to the assumption of BCJ, the Yang-Mills amplitude is expressible as the following sum.

$$A_{YM}\,(1, 2, ..., n) = \sum_{\Gamma \in BT(\{1,2,...,n-1\})} \frac{N\,(\Gamma)}{p\,(\Gamma)}, \qquad (2)$$

where $p\,(\Gamma)$ is the Feynman propagator determined by the corresponding tree diagram Γ, namely the associative structure of its leaves. The numerator factor $N\,(\Gamma)$, known to amplitude theorists as the BCJ numerator, is assumed to carry an additional algebraic structure: By mapping all the leaves in binary tree Γ to generators of a Lie algebra and non-leaf nodes to Lie-brackets, one can construct a Lie algebra element that naturally associates with Γ. The numerator N is then a map that takes the corresponding Lie algebra element to a real number, linearly homomorphic in the sense that if three binary trees $\Gamma_i, i = 1, 2, 3$ are related by cyclic permutations of sub-trees (so that they are respectively associated to algebra elements of the form $[...\,[[O_1, O_2]\,, O_3]\,...]\,, [...\,[[O_2, O_3]\,, O_1]\,...]\,, [...\,[[O_3, O_1]\,, O_2]\,...]$), then their corresponding BCJ numerators are subject to the constraint imposed by Jacobi identity.

$$N\,(\Gamma_1) + N\,(\Gamma_2) + N\,(\Gamma_3) = 0 \qquad (3)$$

We would like to point out that although it is sometimes more efficient to exploit the above Jacobi identity to translate amplitude as a sum of basis numerators $N(\Gamma)$, the original expression (2), as it was demonstrated in [7, 8, 16], is especially convenient for algebraic discussion, as it manifest the correspondence between numerators and binary trees, which naturally characterises the recursive and combinatoric property.

From string theory perspective, the string lift of the BCJ numerators can be understood as conformal blocks, while it was shown in previous work [9] that the screening operators within naturally define a quantum algebra with the external particle momenta identified as its roots. We feel that it is then natural to conjecture that objects in the settings of BCJ duality such as propagator matrix and momentum kernel can be equally well translated to algebraic terms, in particular that when taking point particle limit, the momentum kernel should reduce to purely a Lie algebra object. In this text we confirm this thinking by showing that the momentum kernel $S[\gamma|\beta]$ indeed derives from Shapovalov form of the corresponding Lie algebra. Furthermore, we note that even though field theory scattering amplitudes arise from seemingly independent definition as sum of Feynman diagrams, the fact that propagator matrix is known to amplitude theorists as the inverse of momentum kernel [13] suggests a similar algebra interpretation. In this text we show that the same algebraic setting prescribes dual vectors as trivalent Feynman diagrams dressed with propagators.

2 Conventions

The aim of this text is to introduce algebraic concept and method into the study of field theory and string amplitudes. To avoid confusion and repetition, in this section we assemble a prior summary of notations for the subjects in Lie algebra, quantum algebra and graph theory we will encounter in the line of our discussion.

First of all, the positive and negative simple root generators in the Chevalley basis for Lie algebra are denoted by E_i and F_i, and their Lie bracket, the elements in Cartan sub-algebra are denoted by H_i. For simplicity we normalisation the roots to $\alpha_i \cdot \alpha_i = 2$. Under this normalisation for the roots, the generators are subject to the following relations.

$$\left[F_i, E_j\right] = \delta_{ij} H_i, \qquad \left[H_i, E_j\right] = \alpha_i \cdot \alpha_j E_j, \qquad \left[H_i, F_j\right] = -\alpha_i \cdot \alpha_j F_j. \tag{4}$$

The Shapovalov form $\langle \quad , \quad \rangle$ is defined recursively on a Verma module M_λ (generated by the vector v_λ with weight equals λ).

$$\langle V_\lambda, V_\lambda \rangle = 1, \qquad \langle F_i V_1, V_2 \rangle = \langle V_1, E_i V_2 \rangle. \tag{5}$$

By definition the Shapovalov form is only nonzero between two vector of the same weight. In this text we usually discuss Shapovalov in the subspace with a fixed weight $\sum_{i=1}^n \alpha_i + \lambda$ in M_λ, which is spanned by the basis $\left\{ V\left(\sum_{i=1}^n \alpha_i, \sigma\right) \middle| \sigma \in S_n \right\}$, where $V\left(\sum_{i=1}^n \alpha_i, \sigma\right)$ is our shorthand notation for the following vector.

$$V\left(\sum_{i=1}^n \alpha_i, \sigma\right) = \left(\prod F_{\sigma(i)}\right) V_\lambda = F_{\sigma(1)} F_{\sigma(2)} \ldots F_{\sigma(n)} V_\lambda. \tag{6}$$

For quantum algebra, we use lower case letters for the generators $\{e_i, f_i\}$ and the module vectors. And we use the subscript q to emphasize objects defined in the context of a quantum algebra, for example the Verma module generated by the vector v_λ is denoted by $M_{\lambda,q}$ and the Shapovalov form is denoted by $\langle\ ,\ \rangle_q$.

We also follow the frequently used abbreviation expression in quantum group literature

$$[z]_q = \frac{q^z - q^{-z}}{q - q^{-1}} \tag{7}$$

to write the defining relations between the generators $\{e_i, f_i\}$

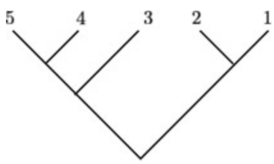
$$\begin{aligned} &q^{h_{\alpha_i}} e_{\alpha_j} q^{-h_{\alpha_i}} = q^{\alpha_i \cdot \alpha_j} e_{\alpha_i}, \qquad q^{h_{\alpha_i}} f_{\alpha_j} q^{-h_{\alpha_i}} = q^{\alpha_i \cdot \alpha_j} f_{\alpha_i}, \\ &\left[e_{\alpha_i}, e_{-\alpha_j}\right] = \delta_{\alpha_i \alpha_j} \left[h_{\alpha_i}\right]_q . \end{aligned} \tag{8}$$

The Shapovalov for quantum algebra $\langle\ ,\ \rangle_q$ is defined on the Verma module $M_{\lambda,q}$ by

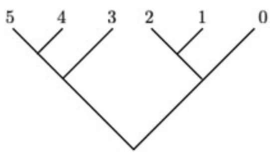
$$\langle v_\lambda, v_\lambda \rangle_q = 1, \quad \langle f_i v_1, v_2 \rangle_q = \langle v_1, e_i v_2 \rangle_q . \tag{9}$$

The weight $\sum_{i=1}^n \alpha_i + \lambda$ subspace of $M_{\lambda,q}$ is spanned by the set of vectors $\left\{ f_{\sigma(1)} f_{\sigma(2)} \dots f_{\sigma(n)} \middle| \sigma \in S_n \right\}$. In later discussion we will use the notation $v\left(\sum_{i=1}^n \alpha_i + \lambda, \sigma\right)$ for the vectors of the form $\left(\prod_{i=1}^n f_{\sigma(i)}\right) v_\lambda$.

Note that we have not include the non-simple root vector in the definition for Lie algebra, as in this text we will write them as iterated Lie brackets of the generators. For an ordered set $\{E_1, E_2, ..., E_n\}$ there are C_{n-1} (Catalan number) different ways to parenthesise and construct a Lie algebra element with root $\sum_{i=1}^n \alpha_n$. This set of algebra elements is in bijection with the set of full binary trees with leaves $\{1, 2, ..., n\}$, which we will denote by $BT(\{1, 2, ..., n\})$ throughout this text. One can translate an algebraic element in the form of an iteration of $n-1$ Lie brackets on n generators to a full binary tree by listing n leaves in the order of the generators and connecting each pair of leaves or sub-trees inside one Lie bracket by a node. By the same translation, one can read off an algebra element from a full binary tree. For example, the binary tree corresponds to $[[[E_5, E_4], E_3], [E_2, E_1]]$ is :

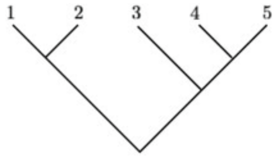


Additionally, by identifying V_λ as the last leaf l_0, and omitting all bracket nesting to it, one can map binary trees to vectors in M_λ, for example, the following binary tree corresponds to $[[E_5, E_4], E_3][E_2, E_1] V_\lambda$.

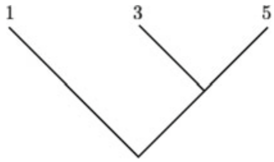


In the discussion that follows for dual vectors we will need the map from binary trees to algebra elements and the module vectors $\mathcal{O} : BT \mapsto L$ and $\mathcal{V} : BT \mapsto M_\lambda$, where BT is the set of all binary trees.

Moreover for simplicity let us introduce some operations on the binary tree. First of all, we will denote the ordered set of all leaves of Γ as $\ell(\Gamma)$, and we will define deletion $\backslash$ the insertion $\wedge$ and the joint action $\vee$ on binary trees: We define $\Gamma \backslash A$ as a map from a binary tree Γ and a subset of its leaves to a smaller tree by removing leaves in the set A, for example for $A = \{2, 4\}$ and Γ the following binary tree



$\Gamma \backslash A$ is the following tree



We define the insertion $n \wedge \Gamma$ as a map from a binary tree and new leaf n to the set of binary trees with n as their first leaf and will reduce to Γ after deleting the leaf $\{n\}$, that is $n \wedge \Gamma = \left\{ \Gamma' \middle| \Gamma' \backslash \{n\} = \Gamma, \Gamma' \in BT(\{n\} \cup \ell(\Gamma)) \right\}$. The joint action $\Gamma_1 \vee \Gamma_2 \in BT(\ell(\Gamma_1) \cup \ell(\Gamma_2))$ is defined as the binary tree constructed by joining Γ_1 to the left and Γ_2 to the right of the root node.

Finally there are two frequently used physical variables associated with the binary trees that will emerge in later discussions, the Mandelstam variables and the Feynman propagators. The Mandelstam variables $s(A)$ are defined for a set of root vectors, $s(A) = \sum_{E_i, E_j \in A} \alpha_i \cdot \alpha_j$. For an edge e of a binary tree, the Mandelstam variable naturally associate to it is $s(\ell(e))$ where $\ell(e)$ is the set of leaves connecting to the upper node of the edge e. Following this notation the Feynman propagator for a binary tree Γ is understood as the product of all Mandelstam variables for its edges

$$p(\Gamma) = \prod_{e \in \Gamma} s(\ell(e)) \tag{10}$$

3 The Relation Between Momentum Kernel and Shapovalov Form

The momentum kernel $S[\sigma|\tau]$ [2, 3] appears as the inner product when pairing two sets of Yang-Mills amplitudes into a gravity amplitude [12], as well as the transition matrix in the linear transformation from amplitudes to the BCJ numerators [13]. It has the following explicit formula as a function of a set of space-time momentum $\{k_i\}$, two permutations acting on the set σ and τ, and a reference momentum p:

$$\begin{aligned} &S[p, \sigma(1), \ldots, \sigma(n)|\, p, \tau(1), \ldots, \tau(n)] \\ &= \prod_{t=1}^{n} \left(p \cdot k_{\sigma(t)} + \sum_{q>t}^{k} \theta(\sigma(t), \sigma(q))\, k_{\sigma(t)} \cdot k_{\sigma(q)} \right), \end{aligned} \tag{11}$$

where $\theta(\sigma(t), \sigma(q))$ is defined as:

$$\theta(\sigma(t), \sigma(q)) = \begin{cases} 1, & (\sigma(t) - \sigma(q)) \left(\sigma\tau^{-1}(t) - \sigma\tau^{-1}(q)\right) < 0 \\ 0, & (\sigma(t) - \sigma(q)) \left(\sigma\tau^{-1}(t) - \sigma\tau^{-1}(q)\right) > 0 \end{cases}. \tag{12}$$

On the other hand, according to our previous work [9], the string lift of momentum kernel is the transition coefficient matrix between two basis of state vectors generated by screening operators with different choices of integration contour. The fact that the momentum kernel is expressible as a transition matrix implies the that it might be constructible as a bilinear form between two vectors in a module over a free Lie algebra with simple roots equal to the space-time momentum, and one of the two vectors should be of the form $F_{\tau(1)} F_{\tau(2)} \ldots F_{\tau(n-1)} V_{k_1}$. Indeed it was shown by Frost, Mafra and Mason [7, 8] that $S[\sigma|\tau]$ can be written as a straightforward Kronecker delta bilinear form imposing on $F_{\tau(1)} F_{\tau(2)} \ldots F_{\tau(n-1)} V_{k_1}$ and another vector constructed recursively by the S-map defined in [14]. We in the following discussion seek an alternative approach by replacing the Kronecker delta by Shapovalov form and evaluating it on $F_{\tau(1)} F_{\tau(2)} \ldots F_{\tau(n-1)} V_{\lambda}$ with another vector of the same weight, which is straightforward for vectors that are images of a binary tree under the map $\mathcal{V} : BT \mapsto M_{\lambda}$. For example, for any binary tree Γ with leaves $\{n, n-1, \ldots 1, 0\}$, its Shapovalov with the basis vector $V\left(\sum_{i=1}^{n} \alpha_i + \lambda, I\right)$ can be calculated recursively by flipping F_n in $V\left(\sum_{i=1}^{n} \alpha_i + \lambda, I\right)$ to contract with $\mathcal{V}(\Gamma)$. The result would be the product of the Shapovalov between two lower weight vectors and the factor $\alpha_n \cdot \left(\alpha_{Adjacent}(\Gamma, n)\right)$, where $\left(\alpha_{Adjacent}(\Gamma, n)\right)$ is the sum of roots of leaves on the subtree on Γ that is directly connecting to the same node with the leaf n (with a negative sign if its on the left of n). Therefore by iteration, the Shapovalov form is equal to the product of factors from the flipping of each F_i in $V\left(\sum_{i=1}^{n} \alpha_i + \lambda, I\right)$.

$$\begin{aligned}
&\left\langle \mathcal{V}(\Gamma), V\left(\sum_{i=1}^{n}\alpha_i+\lambda, I\right)\right\rangle \\
&= \langle \mathcal{V}(\Gamma), F_n \ldots F_2 F_1 V_\lambda \rangle \\
&= \alpha_n \cdot \left(\alpha_{Adjacent}(\Gamma, n)\right)\left\langle \mathcal{V}(\Gamma\backslash\{n\}), V\left(\sum_{i=1}^{n-2}\alpha_i, I\right)\right\rangle \\
&= \prod_{i=0}^{n} \alpha_{n-i} \cdot \left(\alpha_{Adjacent}(\Gamma\backslash\{n, n-1, ..., n-i+1\}, n-i)\right).
\end{aligned} \tag{13}$$

With this pattern in (13) one can compare it to the formula of the momentum kernel and construct a tree by matching sub-tree for each factor, which turns out to be a nested tree corresponding to the vector $V\left(\sum_{i=1}^{n}\alpha_i+\lambda, \sigma\right)$. We now have the pair of vectors which will reproduce $S[\sigma|\tau]$ in (11) up to the identification $\lambda = p,\ \alpha_l = k_l$,

$$\langle V\left(\sum_{i=1}^{n}\alpha_i, \sigma\right), V\left(\sum_{i=1}^{n}\alpha_i, \tau\right)\rangle = \prod_{i=0}^{n-2}\alpha_{\tau(n-i)} \cdot \left(\lambda + \sum_{\substack{\tau^{-1}\sigma(j)<n-i,\\ \tau^{-1}(j)<n-i}} \alpha_j\right). \tag{14}$$

Moreover the q-deformed version of (14) can be immediately identified as an algebraic construction for the string KLT kernel defined in [2, 3]

$$\begin{aligned}
&\mathcal{S}_{\alpha'}[\sigma(1), \ldots, \sigma(n)|\,\tau(1), \ldots, \tau(n)] \\
&= \left(\pi\alpha'/2\right)^{-n}\prod_{t=1}^{n}\sin\left(\pi\alpha'\left(p\cdot k_{\sigma(t)} + \sum_{q>t}^{k}\theta(\sigma(t), \sigma(q))\, k_{\sigma(t)}\cdot k_{\sigma(q)}\right)\right),
\end{aligned} \tag{15}$$

where $\theta(\sigma(t), \sigma(q))$ is the same function as in the momentum kernel formula defined in (12). It is straightforward to verify that the Shapovalov form $\langle v\left(\sum_{i=1}^{n}\alpha_i, \sigma\right), v\left(\sum_{i=1}^{n}\alpha_i, I\right)\rangle_q$ is exactly the q-deformation of (14), which can be explicitly written as $\prod_{i=0}^{n-2}\left[\alpha_{n-i}\cdot\left(\lambda+\sum_{\sigma(j)<n-i, j<n-i}\alpha_j\right)\right]_q$. Upon identifying the weights and roots as space-time momentum $\lambda = p,\ \alpha_l = k_l,\ q = e^{\alpha'\pi i/2}$, it matches (15) up to an overall factor $\left(\frac{2i}{e^{\alpha'\pi i/2}-e^{-\alpha'\pi i/2}}\right)^n$

4 Explaining Feynman Propagators as Shapovalov Dual Vectors

As we mentioned earlier, the momentum kernel $S[\sigma|\tau]$ appears as the transition matrix from amplitudes to BCJ numerators. Now that we have shown that $S[\sigma|\tau]$ is in fact the Shapovalov form $\langle V\left(\sum_{i=1}^{n}\alpha_i+\lambda, \sigma\right), V\left(\sum_{i=1}^{n}\alpha_i+\ \lambda, \tau\right)\rangle$, it is reasonable to assume there is a way to identify the natural transition matrix from BCJ numerators to amplitudes, namely the Feynman propagators, with the Shapovalov dual of the vectors $V\left(\sum_{i=1}^{n}\alpha_i, \sigma\right)$. As it turns out, the ϕ^3-theory Feynman diagrams, or the binary trees with ordered leaves dressed with Feynman propagators, naturally emerges as coefficients in the dual vector $V^*\left(\sum_{i=1}^{n}\alpha_i, \sigma\right)$:

$$V^*\left(\sum_{i=1}^{n}\alpha_i + \lambda, \sigma\right) = \left(F_{\sigma(n)}\dots F_{\sigma(2)}F_{\sigma(1)}V_\lambda\right)^* = \sum_{\Gamma\in BT(\{\sigma(n),\dots,\sigma(2),\sigma(1),0\})}\frac{\mathcal{V}(\Gamma)}{p(\Gamma)}. \tag{16}$$

This formula for the Shapovalov dual vectors can be proved inductively. As for a generic configuration for the roots $\{\alpha_i\}$ and weight λ , the defining property for Shapovalov form (5) combined with the definition of the dual vectors induces a recursion relation

$$E_n V^*\left(\sum_{i=1}^{n-1}\alpha_i + \lambda, \sigma\right) = \delta_{\sigma(n),n} V\left(\sum_{i=1}^{n-1}\alpha_i + \lambda,\ \sigma|_{\{1,\dots,n-1\}}\right)^* \tag{17}$$

that uniquely determines the dual vectors. Therefore, showing (17) holds for (16) is sufficient to prove (16) the correct explicit expression for the dual vector. In other words, we need to show the result of acting E_l on the binary tree summation formula (16) is either 0 or a summation of binary trees with one leaf fewer, depending on if l coincide with the first leaf of the trees:

$$E_l \sum_{\Gamma\in BT(\{n,n-1,\dots,2,1,0\})}\frac{\mathcal{V}(\Gamma)}{p(\Gamma)} = \delta_{l,n}\sum_{\Gamma\in BT(\{n-1,\dots,2,1,0\})}\frac{\mathcal{V}(\Gamma)}{p(\Gamma)} \tag{18}$$

In fact this recursion relation (18) holds for individual binary trees, that is, for any binary tree $\Gamma \in BT(\{n-1, \dots, 2, 1, 0\})$, the following identity holds.

$$E_l \sum_{\Gamma'\in n\wedge\Gamma} p\left(\Gamma'\right)\mathcal{V}\left(\Gamma'\right) = \delta_{l,n}\, p(\Gamma)\,\mathcal{V}(\Gamma) \tag{19}$$

which can be proved recursively as one can check if (19) and its algebra analog for $\mathcal{O}(\Gamma)$ are true for any binary tree Γ_1 , then they hold for $\Gamma_1 \vee \Gamma_2$ for any Γ_2.

On the other hand, we observe $BT(\{n, \dots, 2, 1, 0\})$ is the disjoint union of the sets $n \wedge \Gamma$ with $\Gamma \in BT(\{n-1, \dots, 2, 1, 0\})$. As obviously the intersection of two such sets is the empty set for any two different binary trees Γ_1 and Γ_2, and the union of all $n \wedge \Gamma$ for all trees in $BT(\{n-1, \dots, 2, 1, 0\})$ is the set of all binary trees with leaves $\{n, \dots, 2, 1, 0\}$. Combining this observation for the set $BT(\{n, \dots, 2, 1, 0\})$ with the recursion for individual tree vectors (19), we have arrived at a proof for (18), and furthermore for the claimed explicit expression for the dual vectors (16).

By imposing Jacobi identities and writing Lie brackets as commutators in (16), we can expand the dual vectors on the basis vectors $V\left(\sum_{i=1}^{n}\alpha_i,\sigma\right)$.

$$V^{*}\left(\sum_{i=1}^{n}\alpha_i,\sigma\right) = \sum_{\tau\in S_n}\sum_{\substack{\Gamma\in BT(\{\tau(n),\ldots,\tau(2),\tau(1),0\})\\ \cap BT(\{\sigma(n),\ldots,\sigma(2),\sigma(1),0\})}} p(\Gamma)\,V\left(\sum_{i=1}^{n}\alpha_i,\tau\right). \tag{20}$$

The sum $b(\sigma|\tau)=\sum_{\substack{\Gamma\in BT(\{\tau(n),\ldots,\tau(2),\tau(1),0\})\\ \cap BT(\{\sigma(n),\ldots,\sigma(2),\sigma(1),0\})}} p(\Gamma)$ is known to amplitude theorists as the ϕ^3 Berends-Giele current, which was defined and proved recursively to be the inverse of momentum kernel $S[\sigma|\tau]$ in [15].

Finally we want to remark on the Shapovalov dual vectors for the quantum algebra. As we showed at the end of last section, the way we constructed momentum kernel by Shapovalov form extends naturally to the string KLT kernel. However, the quantum version of the dual vector expression (16) involves some new structure and therefore is not as straightforward. In (16) the sum is over all binary trees with ordered leaves, which are the 0-faces of an associahedron. In principle, as suggested by [17], to get the quantum analog for (16), one need to sum over all faces of the associahedron. Each of the face, when evaluating its Feynman propagator, can be treated as a (contracted) binary tree, and the Mandelstam variable for each edge $s(\ell(e))$ needs to be replaced by its quantum counterpart $q^{s(\ell(e))}-q^{-s(\ell(e))}$. We leave this part of the discussion to future work.

5 Conclusion and Discussion

In this text we presented an explicit algebraic construction for the momentum kernel as well as its quantum parallel for sting KLT kernel. Although the momentum kernel is structurally much simpler comparing to the BCJ numerators for Yang-Mills theory, its algebraic construction captures some of the key features of the algebraic construction for it. For example, the numerators should be considered as a bilinear form evaluating on module vectors implied by the string numerator formula. In fact, Yang-Mills BCJ numerators can also be derived by evaluating Shapovalov form on the Verma module of a Lie algebra with roots $\{\epsilon_i, k_i-\epsilon_i\}$, where k_i is the space time momentum and ϵ_i is the polarisation vector for the ith particle [10, 11]. It could be interesting to look into the detailed structure and physical interpretation of this algebraic construction of Yang-Mills BCJ numerator.

Acknowledgements We thank the organizers of LT-14 for the opportunity to present our work. YW is supported by China National Natural Science Funds for Distinguished Young Scholar (Grant No. 12105062).

References

1. Z. Bern, J.J.M. Carrasco, H. Johansson, Phys. Rev. D **78**, 085011 (2008). https://doi.org/10.1103/PhysRevD.78.085011
2. N.E.J. Bjerrum-Bohr, P.H. Damgaard, B. Feng, T. Sondergaard, Phys. Rev. D **82**, 107702 (2010). https://doi.org/10.1103/PhysRevD.82.107702
3. N.E.J. Bjerrum-Bohr, P.H. Damgaard, B. Feng, T. Sondergaard, JHEP **09**, 067 (2010). https://doi.org/10.1007/JHEP09(2010)067
4. Vl. S. Dotsenko, Vladimir A. Fateev, Conformal algebra and multipoint correlation functions in 2D statistical models. Nucl. Phys. B **240**(3), 312–348 (1984)
5. Vl. S. Dotsenko, Vladimir A. Fateev, Four-point correlation functions and the operator algebra in the 2D conformal invariant theories with the central charge $C<1$. Nucl. Phys. B **251**, 691–734 (1985)
6. B.L. Feigin, D.B. Fuks, Invariant skew-symmetric differential operators on the line and Verma modules over the Virasoro algebra. Funct. Anal. Appl. **16**(2), 114–126 (1982)
7. H. Frost, C.R. Mafra, L. Mason, arXiv:2012.00519 [hep-th]
8. H. Frost, L. Mason, arXiv:1912.04198 [hep-th]
9. C.H. Fu, P. Vanhove, Y. Wang, JHEP **09**, 141 (2018). https://doi.org/10.1007/JHEP09(2018)141
10. C.H. Fu, Y. Wang, Work in progress
11. P. Goddard, D.I. Olive, DAMTP-83/22
12. H. Kawai, D.C. Lewellen, S.H.H. Tye, Nucl. Phys. B **269**, 1–23 (1986). https://doi.org/10.1016/0550-3213(86)90362-7
13. M. Kiermaier, *Gravity as the Square of Gauge Theory, Talk at Amplitudes, 2010* (University of London, Queen Mary, 2010)
14. C.R. Mafra, O. Schlotterer, JHEP **07**, 153 (2014). https://doi.org/10.1007/JHEP07(2014)153 [arXiv:1404.4986 [hep-th]]
15. C.R. Mafra, JHEP **07**, 080 (2016). https://doi.org/10.1007/JHEP07(2016)080 [arXiv:1603.09731 [hep-th]]
16. C.R. Mafra, *Algebraic Combinatorics, Resurgence, Moulds and Applications (CARMA)*. arXiv:2011.14413 [math.CO]
17. S.J. Matsubara-Heo, arXiv:2010.14142 [math.CO]
18. E. Plahte, Symmetry properties of dual tree-graph n-point amplitudes. Il Nuovo Cimento A (1971–1996) **66**(4), 713–733 (1970)
19. E. Witten, Commun. Math. Phys. **252**, 189–258 (2004). https://doi.org/10.1007/s00220-004-1187-3 [arXiv:hep-th/0312171 [hep-th]]

Action of w_0 on V^L for Orthogonal and Exceptional Groups

Ilia Smilga

Abstract In this note, we present some results that partially answer the following question. Let G be a simple real Lie group; what is the set of representations V of G in which the longest element w_0 of the restricted Weyl group W acts nontrivially on the subspace V^L of V formed by vectors that are invariant by L, the centralizer of a maximal split torus of G? We give a conjectural answer to that question, as well as the experimental results that back this conjecture, when G is either an orthogonal group (real form of $\mathrm{SO}_n(\mathbb{C})$ for some n) or an exceptional group.

Keywords Weyl group · Levi subgroup · Branching rules · Highest Weight · Representation theory

1 Basic Notations and Statement of Problem

Let G be a semisimple real Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$. We start by establishing the notations for some well-known objects related to $\mathfrak{g}$.

- We choose in $\mathfrak{g}$ a *Cartan subspace* $\mathfrak{a}$ (an abelian subalgebra of $\mathfrak{g}$ whose elements are diagonalizable over $\mathbb{R}$ and which is maximal for these properties).
- We choose in $\mathfrak{g}^{\mathbb{C}}$ a *Cartan subalgebra* $\mathfrak{h}^{\mathbb{C}}$ (an abelian subalgebra of $\mathfrak{g}^{\mathbb{C}}$ whose elements are diagonalizable and which is maximal for these properties) that contains $\mathfrak{a}$.
- We denote $L := Z_G(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ in G, $\mathfrak{l}$ its Lie algebra.
- Let Δ be the set of roots of $\mathfrak{g}^{\mathbb{C}}$ in $(\mathfrak{h}^{\mathbb{C}})^*$. We shall identify $(\mathfrak{h}^{\mathbb{C}})^*$ with $\mathfrak{h}^{\mathbb{C}}$ via the Killing form. We call $\mathfrak{h}_{(\mathbb{R})}$ the $\mathbb{R}$-linear span of Δ; it is given by the formula $\mathfrak{h}_{(\mathbb{R})} = \mathfrak{a} \oplus i\mathfrak{a}^{\perp}$.
- We choose on $\mathfrak{h}_{(\mathbb{R})}$ a lexicographical ordering that "puts $\mathfrak{a}$ first", i.e. such that every vector whose orthogonal projection onto $\mathfrak{a}$ is positive is itself positive. We

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_24

call Δ^+ the set of roots in Δ that are positive with respect to this ordering, and we let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots in Δ^+ (where r is the rank of $\mathfrak{g}^{\mathbb{C}}$). Let $\varpi_1, \dots, \varpi_r$ be the corresponding fundamental weights.

- We introduce the *dominant Weyl chamber*

$$\mathfrak{h}^+ := \{X \in \mathfrak{h} \mid \forall i = 1, \dots, r, \quad \alpha_i(X) \geq 0\},$$

and the *dominant restricted Weyl chamber*

$$\mathfrak{a}^+ := \mathfrak{h}^+ \cap \mathfrak{a}.$$

- We introduce the *restricted Weyl group* $W := N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ of G. Then $\mathfrak{a}^+$ is a fundamental domain for the action of W on $\mathfrak{a}$. We define the *longest element* of the restricted Weyl group as the unique element $w_0 \in W$ such that $w_0(\mathfrak{a}^+) = -\mathfrak{a}^+$.
- For each *dominant integral weight* λ of $\mathfrak{g}^{\mathbb{C}}$ (i.e. linear combination of the fundamental weights ϖ_i with nonnegative integer coefficients), we denote by V_λ the irreducible representation of $\mathfrak{g}$ with highest weight λ.

Our goal is to study the action of W, and more specifically of w_0, on various representations V of G. Note however that this action is ill-defined: indeed if we want to see the abstract element $w_0 \in W = N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ as the projection of some concrete map $\tilde{w_0} \in N_G(\mathfrak{a}) \subset G$, then $\tilde{w_0}$ is defined only up to multiplication by an element of $Z_G(\mathfrak{a}) = L$, whose action on V can of course be nontrivial.

This naturally suggests the idea of restricting to L-invariant vectors. Given a representation V of $\mathfrak{g}$, we denote

$$V^L := \{v \in V \mid \forall l \in L, \quad l \cdot v = v\}$$

the L-invariant subspace of V: then W, and in particular w_0, has a well-defined action on V^L.

Our goal is to characterize, for a given semisimple real Lie group G, the representations V of G for which the action of w_0 on V^L is nontrivial. This problem naturally splits into two subproblems (see [11] for a more extended discussion):

Problem 1 Given a semisimple Lie algebra $\mathfrak{g}$ and a dominant integral weight λ, give a simple necessary and sufficient condition for having $V_\lambda^{\mathfrak{l}} \neq 0$.

Problem 2 Given a simple Lie algebra $\mathfrak{g}$ and a dominant integral weight λ, assuming that $V_\lambda^{\mathfrak{l}} \neq 0$, give: (i) a simple necessary and sufficient condition for having $w_0|_{V_\lambda^{\mathfrak{l}}} = \pm\,\mathrm{Id}$; (ii) a criterion to determine the actual sign.

In [14], we have already completely solved Problem 1. In [8], we have solved Problem 2 in the case where $\mathfrak{g}$ is split. In this note, we shall present our recent work on this latter problem.

2 Background and Motivation

These two problems arose from the author's work in geometry. The interest of this particular algebraic property is that it furnishes a sufficient, and presumably necessary, condition for another, geometric property of V. Namely, the author obtained the following result:

Theorem 1 ([13]) *Let G be a semisimple real Lie group, V a representation of G. Suppose that the action of w_0 on V^L is nontrivial. Then there exists, in the affine group $G \ltimes V$, a subgroup Γ whose linear part is Zariski-dense in G, which is free of rank at least 2, and acts properly discontinuously on the affine space corresponding to V.*

He, and other people, also proved the converse statement in some special cases:

Theorem 2 *The converse holds, for irreducible V:*

- *[12] if G is split, but not of type A_n ($n \geq 2$), D_{2n+1} or E_6;*
- *[12] if G is split, has one of these types, and V satisfies a very restrictive additional assumption (see [12] for the precise statement);*
- *[2] if $G = \mathrm{SO}(p, q)$ for arbitrary p and q, and $V = \mathbb{R}^{p+q}$ is the standard representation.*

Moreover, it seems plausible that, by combining the approaches of [2, 12], we might prove the converse in all generality. This geometric property is related to the so-called Auslander conjecture [3], which is an important conjecture that has stood for more than fifty years and generated an enormous amount of work: see e.g. [1, 6, 7, 9, 10] and many others. For the statement of the conjecture as well as a more comprehensive survey of past work on it, we refer to [5].

3 Statement of Main Results

We have run some numerical experiments that allow us to conjecture the answer to Problem 2(i) in the case where $\mathfrak{g}^{\mathbb{C}}$ is of type B_r, D_r or exceptional. Indeed, our numerical experiments prove this conjecture in some particular cases (see Theorem 1 below).

In the case where $\mathfrak{g}^{\mathbb{C}}$ is of type A_r or C_r, we have also made some computations in low rank. Unfortunately, the data we have is not sufficient to be able to predict the general pattern (and we do not have enough computational power to generate more); see also the final remark in [11] for more details.

Conjecture 1 Assume that $\mathfrak{g}^{\mathbb{C}}$ is of type B_r (for some $r \geq 1$), D_r (for some $r \geq 3$), E_6, E_7, E_8, F_4 or G_2. Let λ be a dominant integral weight of $\mathfrak{g}^{\mathbb{C}}$, V_λ the irreducible (complex) representation of $\mathfrak{g}$ with highest weight λ. Then:
(i) If λ is one of the weights listed in Table 1, then $w_0|_{V_\lambda^L} = \pm\,\mathrm{Id}$.
(ii) If $V_\lambda^L \neq 0$ (this can be looked up in [14, Table 2]) but λ does not occur in Table 3, then $w_0|_{V_\lambda^L} \neq \pm\,\mathrm{Id}$.

Table 1 Values of λ for which $w_0|_{V_\lambda^L} = \pm\,\mathrm{Id}$, for various algebras $\mathfrak{g}$. The fundamental weights ϖ_i are numbered using the Bourbaki ordering [4]. The coefficients k, l and m range in the nonnegative integers. Note that the lists may contain duplicates.
Real forms of $B_r = \mathfrak{so}_{2r+1}(\mathbb{C})$ ($r \geq 1$). In $\mathfrak{so}(p, q)$, we assume $p \leq q$

The algebra $\mathfrak{g}$	Weights λ	Conditions on indices	Conditions on coefficients
$\mathfrak{so}(p, q)$ $p \leq \frac{p+q}{4}$ $p+q$ odd	$\lambda = k\varpi_i + l\varpi_{2p}$	$i = 1$ or $2p-1$	any k, any l
		$2 = i = 2p-2$	any k, any l
		$i = 2$ or $2p-2$	$k \leq 2$, any l
		$2 < i < 2p-2$ $\wedge\ 2 \mid i$	$k \leq 1$, any l
$\mathfrak{so}(p, q)$ $p = \frac{p+q+1}{4}$	$\lambda = k\varpi_i + l\varpi_{q-p}$	$i = 1$	any k; $l \leq 2$
		$2 = i = q-p-1$	any k; $l \leq 2$
		$2 = i < q-p-1$	$k \leq 2$, $l \leq 2$
		$2 < i < q-p-1$ $\wedge\ 2 \mid i$	$k \leq 1$, $l \leq 2$
		$2 < i = q-p-1$	$k \leq 2$, $l \leq 2$
		$i = q-p$	any k, any l
$\mathfrak{so}(p, q)$ $p > \frac{p+q+1}{4}$ $p+q$ odd	$\lambda = k\varpi_i + l\varpi_{q-p}$	$i = 1$	any k; $l \leq 1$
		$2 = i < q-p$	$k \leq 2$, $l \leq 1$
		$2 < i < q-p$ $\wedge\ 2 \mid i$	$k \leq 1$, $l \leq 1$
	$\lambda = k\varpi_i$	$2 = i = \frac{p+q-1}{2}$	any k
		$q-p < i = 2$	$k \leq 2$
		$q-p < i$	$k \leq 1$
		$q-p+1 = i = \frac{p+q-1}{2}$	$k \leq 4$
		$q-p < i = \frac{p+q-1}{2}$	$k \leq 2$

Table 2 Values of λ for which $w_0|_{V_\lambda^L} = \pm\,\mathrm{Id}$, for various algebras $\mathfrak{g}$. The fundamental weights ϖ_i are numbered using the Bourbaki ordering [4]. The coefficients k, l and m range in the nonnegative integers. Note that the lists may contain duplicates. Real forms of $D_r = \mathfrak{so}_{2r}(\mathbb{C})$ $(r \geq 3)$. In $\mathfrak{so}(p,q)$, we always assume $p \leq q$; and we denote by $r := \frac{p+q}{2}$ the (complex) rank

The algebra $\mathfrak{g}$	Weights λ	Conditions on indices	Conditions on coefficients
$\mathfrak{so}(p,q)$ $\quad p \leq \frac{p+q}{4} - 1$, $p+q$ even	Same as for $p \leq \frac{p+q}{4}$ in the B_r case		
$\mathfrak{so}(p,q)$ $\quad p = \frac{p+q-2}{4}$	Same as for $p \leq \frac{p+q}{4}$ in the B_r case, but with ϖ_{2p} replaced by $(\varpi_{2p} + \varpi_{2p+1})$		
$\mathfrak{so}(p,q)$ $\quad p = \frac{p+q}{4}$	Same as for $p \leq \frac{p+q}{4}$ in the B_r case		
	Same as for $p \leq \frac{p+q}{4}$ in the B_r case, but with ϖ_{2p} replaced by ϖ_{2p-1}		
$\mathfrak{so}(p,q)$ $\quad p > \frac{p+q}{4}$, $p+q \equiv 0 \pmod 4$	$\lambda = k\varpi_i$	$i = 1$	any k
		$i = 2$	$k \leq 2$
		$2 < i < r-1 \;\wedge\; 2 \mid i$	$k \leq 1$
		$i \in \{r-1, r\} \;\wedge\; r = 4$	any k
		$i \in \{r-1, r\} \;\wedge\; p = \frac{p+q}{4} + 1$	$k \leq 4$
		$i \in \{r-1, r\}$	$k \leq 2$
$\mathfrak{so}(p,q)$ $\quad p > \frac{p+q}{4}$, $p+q \equiv 2 \pmod 4$	$\lambda = k\varpi_i$	$i = 1$	any k
		$2 = i < q-p-1$	$k \leq 2$
		$2 < i < q-p-1 \;\wedge\; 2 \mid i$	$k \leq 1$
		$i \in \{r-1, r\} \;\wedge\; i \leq q-p+1$	any k
$\mathfrak{so}^*(6)$	$\lambda = k\varpi_1 + l\varpi_i$	$i \in \{2, 3\}$	any k, any l
$\mathfrak{so}^*(8)$: see $\mathfrak{so}(6,2)$, to which it is isomorphic			
$\mathfrak{so}^*(10)$	$\lambda = k\varpi_i$	$i \in \{1, 4, 5\}$	any k
$\mathfrak{so}^*(12)$	$\lambda = k\varpi_i$	$i \in \{1, 2, 6\}$	any k
		$i = 4$	$k \leq 1$
		$i = 5$	$k \leq 2$
$\mathfrak{so}^*(2r)$ $\quad r > 5$, r odd	$\lambda = k\varpi_i$	$i = 1$	any k
$\mathfrak{so}^*(2r)$ $\quad r > 6$, r even	$\lambda = k\varpi_i$	$i = 1$	any k
		$i = 2$	$k \leq 2$
		$2 < i < r-1 \;\wedge\; 2 \mid i$	$k \leq 1$
		$i \in \{r-1, r\}$	$k \leq 4$

Table 3 Values of λ for which $w_0|_{V^L_\lambda} = \pm \mathrm{Id}$, for various algebras $\mathfrak{g}$. The fundamental weights ϖ_i are numbered using the Bourbaki ordering [4]. The coefficients k, l and m range in the nonnegative integers. Note that the lists may contain duplicates. Real forms of exceptional algebras

The algebra $\mathfrak{g}$	Weights λ	Conditions on indices	Conditions on coefficients
E I, E II	$\lambda = 0$		
E III	$\lambda = k\varpi_i$	$i \in \{1, 6\}$	any k
E IV	$\lambda = k\varpi_2 + l\varpi_i$	$i \in \{1, 3, 5, 6\}$	any k, any l
E V, E VI	$\lambda = k\varpi_i$	$i = 1$	$k \leq 2$
	$\lambda = k\varpi_i$	$i \in \{6, 7\}$	$k \leq 1$
E VII	$\lambda = k\varpi_i$	$i = 1$	any k
	$\lambda = k\varpi_i$	$i \in \{6, 7\}$	$k \leq 1$
E VIII, E IX	$\lambda = k\varpi_i$	$i = 1$	$k \leq 1$
	$\lambda = k\varpi_i$	$i = 8$	$k \leq 2$
F I	$\lambda = k\varpi_i$	$i \in \{1, 4\}$	$k \leq 2$
F II	$\lambda = k\varpi_1 + l\varpi_2 + m\varpi_i$	$i \in \{3, 4\}$	any k, any l, any m
G	$\lambda = k\varpi_i$	$i \in \{1, 2\}$	$k \leq 2$

Here are the cases in which we have checked this conjecture. (In a few special cases, when useful and feasible, we have actually gone a bit beyond the cutoff figures listed below; these details would be too tedious to list.)

Proposition 1 • *Conjecture 1(ii) holds for all real forms of B_r with $r \leq 7$, of D_r with $r \leq 9$, and of all exceptional algebras.*

• *Conjecture 1(i) holds for all the algebras in the same list, for weights $\lambda = \sum_{i=1}^r c_i \varpi_i$ satisfying $c_i \leq 3p_i$ for all coefficients c_i, where p_i is the least positive integer such that $V^L_{p_i \varpi_i} \neq 0$ (except in the case $i \in \{2p, 2p+1\}$ for $\mathfrak{g} = \mathfrak{so}(p, q)$ with $p = \frac{p+q-2}{4}$, where we convene that $p_i = 1$.)*

The proof of Proposition 1 relies on the additivity property [11, Proposition 1(iii)], which reduces it to a finite number of computations; and on an algorithm to compute the restriction of w_0 to V^L that the author has recently developed and implemented in the LiE software [15]. The details of that algorithm will be published in a subsequent paper.

Acknowledgements The author is supported by the European Research Council (ERC) under the European Union Horizon 2020 research and innovation programme (ERC starting grant DiGGeS, grant agreement No. 715982).

References

1. H. Abels, G.A. Margulis, G.A. Soifer, On the Zariski closure of the linear part of a properly discontinuous group of affine transformations. J. Diff. Geom. **60**, 315–344 (2002)
2. H. Abels, G.A. Margulis, G.A. Soifer, The linear part of an affine group acting properly discontinuously and leaving a quadratic form invariant. Geom. Dedic. **153**, 1–46 (2011)
3. L. Auslander, The structure of complete locally affine manifolds. Topology **3**, 131–139 (1964)
4. N. Bourbaki. *Éléments de Mathématique, Groupes et Algèbres de Lie : chapitres 4, 5 et 6* (Hermann, 1968)
5. J. Danciger, T.A. Drumm, W.M. Goldman, I. Smilga, Proper actions of discrete groups of affine transformations, in *Dynamics, Geometry, Number Theory: the Impact of Margulis on Modern Mathematics* (University of Chicago Press, 2022). arXiv:2002.09520
6. J. Danciger, F. Guéritaud, F. Kassel, Margulis spacetimes via the arc complex. Invent. Math. **204**(1), 133–193 (2016)
7. D. Fried, W.M. Goldman, Three-dimensional affine crystallographic groups. Adv. Math. **47**, 1–49 (1983)
8. B. Le Floch, I. Smilga, Action of Weyl group on zero-weight space. C. R. Math. Acad. Sci. Paris **356**(8), 852–858 (2018)
9. G.A. Margulis, Free properly discontinuous groups of affine transformations. Dokl. Akad. Nauk SSSR **272**, 785–788 (1983)
10. J. Milnor, On fundamental groups of complete affinely flat manifolds. Adv. Math. **25**, 178–187 (1977)
11. I. Smilga, Action of the restricted Weyl group on the L-invariant vectors of a representation, in *Proceedings of the XIII International Workshop "Lie Theory and Its Applications in Physics" (Varna, Bulgaria, June 2019)*, volume 335 of *Springer Proceedings in Mathematics and Statistics*, ed. by V. Dobrev (Springer, Singapore, 2020), pp. 365–372
12. I. Smilga, Construction of Milnorian representations. Geom. Dedic. **206**, 55–73 (2020)
13. I. Smilga, Proper affine actions: a sufficient criterion. Math. Ann. (2022). arXiv:1612.08942
14. I. Smilga, Representations having vectors fixed by a Levi subgroup. J. of Algebra (2022). arXiv:2002.10928
15. M.A.A. van Leeuwen, A.M. Cohen, B. Lisser, *LiE, A Package for Lie Group Computations* (2000). http://wwwmathlabo.univ-poitiers.fr/~maavl/LiE/

Pairs of Spectral Projections of Spin Operators

Ood Shabtai

Abstract We discuss the semiclassical behavior of an arbitrary bivariate polynomial, evaluated on certain spectral projections of spin operators, and contrast it with the behavior of the polynomial when evaluated on random pairs of projections. The discrepancy is closely related to a type of Slepian concentration problem, which is also addressed. This is a survey article.

Keywords Spectral projections · Quantization · Slepian concentration problem

1 Introduction

This paper is a survey of recent findings about pairs of spectral projections of spin operators. The main results (Theorems 1, 2 and 3) can be found in [10], where they are proven using earlier results from [9]. The only previously unpublished result included here is Theorem 4.

The real Lie algebra $\mathfrak{su}(2)$ is often specified by a basis $\{u_1, u_2, u_3\}$, such that the commutation relations between the basis elements are

$$[u_1, u_2] = u_3,\ [u_2, u_3] = u_1,\ [u_3, u_1] = u_2.$$

For every $n \in \mathbb{N}$ there exists a unique (up to equivalence) n-dimensional irreducible representation ρ_n of $\mathfrak{su}(2)$. Moreover, we can assume that $i\rho_n(u)$ is self-adjoint for every $u \in \mathfrak{su}(2)$. The operators

$$J_1 = i\rho_n(u_1),\ J_2 = i\rho_n(u_2),\ J_3 = i\rho_n(u_3)$$

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_25

are used in quantum mechanics to model spin angular momentum ([1]). Their spectrum is identical, and it equals the set $\sigma_n = \{j, j-1, ..., -j\}$, where $2j+1=n$.

Let $\mathcal{A}$ denote the complex unital algebra generated by two non-commuting variables x, y which satisfy $x^2 = x$, $y^2 = y$. Let $\left(\mathrm{G}_k(n), \mu_{k,n}\right)$ denote the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$, equipped with the uniform probability measure[1]. In what follows, a subspace $V \in \mathrm{G}_k(n)$ is identified with the orthogonal projection $P : \mathbb{C}^n \to V$.

The following two questions were raised by D. Kazhdan, who has also essentially predicted the phenomena described in Theorems 1 and 2.

Questions. Fix $0 < \alpha \leq \frac{1}{2}$, and define intervals $(0, \alpha_n) \subset \mathbb{R}$ containing exactly $\lfloor \alpha n \rfloor$ elements of σ_n. Let $P_{1,\alpha,n} = \mathbb{1}_{(0,\alpha_n)}(J_1)$, $P_{3,\alpha,n} = \mathbb{1}_{(0,\alpha_n)}(J_3)$. Fix $0 \neq f \in \mathcal{A}$.

1. What is the behavior of $f\left(P_{1,\alpha,n}, P_{3,\alpha,n}\right)$ in the semiclassical limit $n \to \infty$?
2. How does it compare with $f(P, Q)$, where $P, Q \in \mathrm{G}_{\lfloor \alpha n \rfloor}(n)$ are random?

We note that polynomials in two orthogonal projections can appear quite naturally in quantum theory ([5, 6, 10]), hence some of our motivation to consider the full algebra $\mathcal{A}$ rather than a few specific elements. The structure of the algebra generated by two projections is well-understood ([2]), and so are the asymptotic properties of random pairs of projections ([3]).

As it turns out, the asymptotic behavior of $P_{1,\alpha,n}, P_{3,\alpha,n}$ is quite unlike that of random pairs $P, Q \in \mathrm{G}_{\lfloor \alpha n \rfloor}(n)$ (see Figs. 1 and 2). The discrepancy is closely related to the Slepian spectral concentration problem ([4, 11–13]) associated with $P_{1,\alpha,n}, P_{3,\alpha,n}$ (Theorem 3 addresses the case $\alpha = \frac{1}{2}$). Roughly speaking, the Slepian spectral concentration problem associated with a pair of non-commuting orthogonal projections is to find vectors which are optimally "localized" with respect to the ranges of both projections.

2 Main Results

Question 1 is addressed in Theorem 1. When $\alpha < \frac{1}{2}$, Theorems 1 and 2 together shed some light on Question 2. In the case $\alpha = \frac{1}{2}$, Question 2 is addressed in Theorem 3. We let $\sigma(A)$ denote the spectrum of a linear operator A.

Theorem 1 ([10]) *There exists $M_f > 0$, depending only on f, such that $\lim_{n\to\infty} \|f(P_{1,\alpha,n}, P_{3,\alpha,n})\|_{\mathrm{op}} = M_f$. M_f is a universal, tight upper bound for $\|f(P, Q)\|_{\mathrm{op}}$, where P, Q are completely arbitrary orthogonal projections (on some separable complex Hilbert space).*[2]

[1] $\mu_{k,n}$ is the unique probability measure invariant under the action of the unitary group on $\mathrm{G}_k(n)$.

[2] Throughout, all Hilbert spaces are assumed to be separable and complex.

Let $\mathbb{C}[z, w]$ be the algebra of complex polynomials in commuting variables, and let $\mathcal{I}$ denote the ideal generated by $z^2 - z, w^2 - w$. Let $\Phi : \mathcal{A} \to \mathbb{C}[z, w]/\mathcal{I}$ map $f(x, y)$ to $f(z, w)$. For conciseness, we focus on $f \in \ker(\Phi)$.

Theorem 2 ([10]) *Let* $\Omega_{n,\alpha} = \mathrm{G}_{\lfloor \alpha n \rfloor}(n) \times \mathrm{G}_{\lfloor \alpha n \rfloor}(n)$, $\nu_n = \mu_{\lfloor \alpha n \rfloor, n} \times \mu_{\lfloor \alpha n \rfloor, n}$. *Fix* $0 \neq f \in \ker(\Phi)$. *There exists a continuous, piecewise smooth function* $\psi_f : [0, 1] \to [0, \infty)$*, such that* $\psi_f(0) = \psi_f(1) = 0$, $\max_{[0,1]} \psi_f = M_f$*, and*

$$\max_{[0,4\alpha(1-\alpha)]} \psi_f = I_\alpha^f = \lim_{n\to\infty} \int_{\Omega_{n,\alpha}} \|f(P, Q)\|_{\mathrm{op}} d\nu_{n,\alpha}.$$

ψ_f essentially appears in the literature on the subject [2]: for any orthogonal projections P, Q we have that $\|f(P, Q)\|_{\mathrm{op}} = \max_{\sigma(PQP)} \psi_f$. Since $\psi_f(0) = 0$, if α is small enough then $0 \approx I_\alpha^f < M_f$. Moreover,

Corollary 1 *For every* $0 < \alpha < \frac{1}{2}$ *there exists* $f \in \ker(\Phi)$ *with* $I_\alpha^f < M_f$. *Thus,* $f(P_{1,\alpha,n}, P_{3,\alpha,n})$ *behave unlike random* $f(P, Q)$ *as* $n \to \infty$.

Example 1 If $f \in \mathcal{A}$ is of the form $f(x, y) = f_1(xy)xy - f_1(yx)yx$, where f_1 is a univariate polynomial, then $\psi_f(t) = |f_1(t)|\sqrt{t(1-t)}$. Thus, for instance, if $f(x, y) = (xy)^{k+1} - (yx)^{k+1}$, we have that

$$\psi_f(t) = t^k\sqrt{t(1-t)},\ M_f = \max_{[0,1]} \psi_f = \psi_f\left(\frac{2k+1}{2k+2}\right),$$

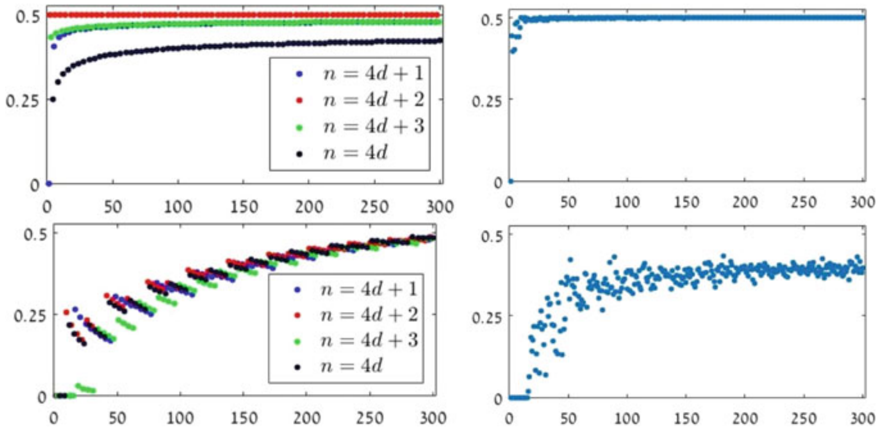

Fig. 1 $\|[P_{3,\alpha,n}, P_{1,\alpha,n}]\|_{\mathrm{op}}$ as a function of n (left), $\|[P, Q]\|_{\mathrm{op}}$ as a function of n for random $P, Q \in \mathrm{G}_{\lfloor \alpha n \rfloor}(n)$ (right), when $\alpha = \frac{1}{2}$ (top), $\frac{1}{16}$ (bottom). For $f(x, y) = [x, y]$, $I_{\frac{1}{2}}^f = M_f = \frac{1}{2}$ and $I_{\frac{1}{16}}^f = \frac{7\sqrt{15}}{64} \approx 0.423$

and $I_\alpha^f = \begin{cases} (4\alpha(1-\alpha))^{k+\frac{1}{2}}(1-2\alpha) & 0 < \alpha < \frac{1}{2}\left(1-\frac{1}{\sqrt{2k+2}}\right) \\ M_f & \frac{1}{2}\left(1-\frac{1}{\sqrt{2k+2}}\right) \le \alpha \le \frac{1}{2} \end{cases}$.

The example is illustrated in Fig. 1 for $k = 0$.

2.1 Slepian Spectral Concentration

Theorems 1 and 2 fail to distinguish between $P_{1,\frac{1}{2},n}$, $P_{3,\frac{1}{2},n}$ and random pairs of projections (since $I^f_{\frac{1}{2}} = M_f$ for every $f \in \mathcal{A}$). The following result (which is of independent interest) implies that $P_{1,\frac{1}{2},n}$, $P_{3,\alpha,n}$ are non-generic as $n \to \infty$ for every $0 < \alpha \le \frac{1}{2}$. Note that $\dim\left(\mathrm{Im}(P_{3,\alpha,n})\right) = \lfloor \alpha n \rfloor$.

Theorem 3 ([10]) *Let $N_n(s,t)$ be the number of eigenvalues of*

$$R_{\alpha,n} = P_{3,\alpha,n} P_{1,\frac{1}{2},n} P_{3,\alpha,n} \in \mathrm{End}\left(\mathrm{Im}(P_{3,\alpha,n})\right)$$

lying in the interval $[s,t]$, where $0 \le s < t \le 1$. Fix $0 < t < \frac{1}{2}$. Then

$$\lim_{n\to\infty} \frac{1}{\alpha n} N_n(0,t) = \lim_{n\to\infty} \frac{1}{\alpha n} N_n(1-t,1) = \frac{1}{2},$$

and $N_n(t, 1-t) = \mathcal{O}(\log n)$.

The proof of Theorem 3 is modelled on that of an analogous result [4] on pairs of spectral projections corresponding to generators of finite Heisenberg groups. The clustering of the eigenvalues of $R_{\alpha,n}$ near 0 and 1 (see Fig. 2) is typical in the context of Slepian spectral concentration problems, which normally involve pairs of (spectral) projections analogous to $P_{3,\alpha,n}$, $P_{1,\alpha,n}$. The eigenvectors of $R_{\alpha,n}$ corresponding to eigenvalues $\lambda \approx 1$ can be viewed as highly "localized" with respect to both $\mathrm{Im}\left(P_{3,\alpha,n}\right)$ and $\mathrm{Im}\left(P_{1,\frac{1}{2},n}\right)$.

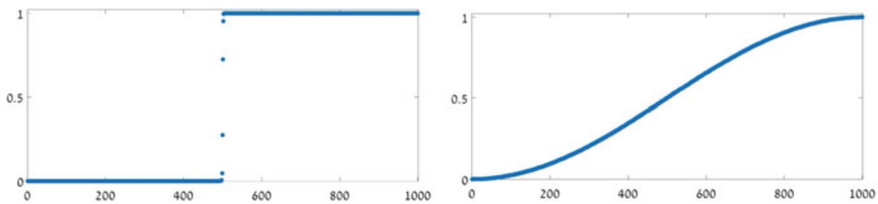

Fig. 2 The sorted eigenvalues of $R_{\frac{1}{2},n} \in \mathrm{End}\left(\mathrm{Im}(P_{3,\frac{1}{2},n})\right)$ (left) and of $PQP \in \mathrm{End}\left(\mathrm{Im}(P)\right)$ (right), where $P, Q \in \mathrm{G}_{\lfloor \frac{1}{2} n \rfloor}(n)$ are random and $n = 2000$

The relatively large gaps between the eigenvalues of $R_{\alpha,n}$ lying away from 0, 1 cause the convergence rate of the norm of "most" polynomials in $P_{1,\frac{1}{2},n}$, $P_{3,\alpha,n}$ to be relatively slow. In particular, using simple measure-theoretic arguments, it can be shown that

Theorem 4 *For almost all* $t_0 \in (0, 1)$*, if* $f \in \ker(\Phi)$ *satisfies that* $M_f = \max_{[0,1]} \psi_f = \psi_f(t_0)$*, that the maximum is unique, and that* $\psi''_f(t_0)$ *is well defined and non-zero, then*

$$\limsup_n \left((\log n)^2 \left| \| f(P_{1,\frac{1}{2},n}, P_{3,\alpha,n}) \|_{\mathrm{op}} - M_f \right| \right) > 0.$$

It is not difficult to produce quite explicit examples of such $f \in \ker(\Phi)$; at the same time, in the case of commutators ($f(x, y) = xy - yx$), the convergence rate is not known to us.

3 Concluding Remarks

3.1 More Cases

In [9], we studied several examples of pairs of spectral projections corresponding to non-commuting quantum observables, with an emphasis on the case of spin operators and (essentially) on the pairs $P_{1,\alpha,n}$, $P_{3,\alpha,n}$. Notably, we also considered pairs of spectral projections coming from position and momentum operators on $L^2(\mathbb{R})$, and from generators of finite Heisenberg groups (acting on $l^2(\mathbb{Z}_n)$, where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$). The pairs of projections studied in [9] were all found to be unitarily equivalent in the semi-classical limit. Consequently, Theorem 1 holds for them as well.

3.2 Numerical Phenomena of Spectral Projections of Spin Operators

The numerical simulations of the spectral projections of J_1, J_3 feature various curious properties, and it would be interesting to study them further.

A notable feature of both images on the left of Fig. 1 is the apparent dependence of $c_n = \left\| \left[P_{1,\alpha,n}, P_{3,\alpha,n} \right] \right\|_{\mathrm{op}}$ on the dimension of the representation n modulo 4. When $\alpha = \frac{1}{2}$ (top-left image), the sub-sequences $\{c_{4d+k}\}_{d\in\mathbb{N}}$ ($k = 0, 1, 2, 3$) appear essentially as graphs of four distinct “nice” functions, such that $c_{4d+2} = \frac{1}{2}$ for every $d \in \mathbb{N}$. In fact, the latter was established by L. Polterovich, but otherwise the modulo 4 dependence on n is unproven. Surprisingly, the same type of behavior is exhibited (numerically) by pairs of projections corresponding to the generators of finite Heisenberg groups [9]. The case $\alpha = \frac{1}{16}$ (bottom-left image) provides an

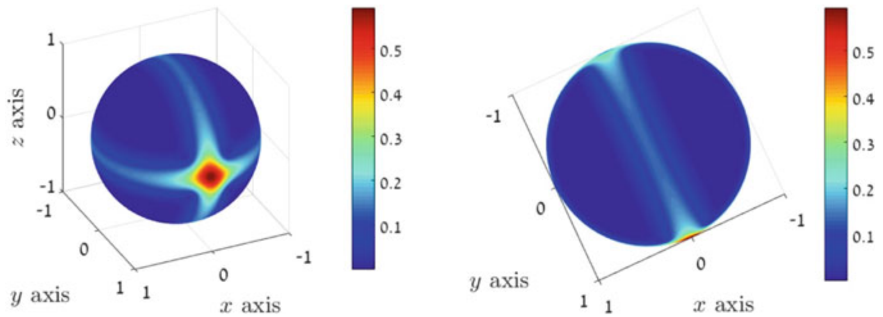

Fig. 3 (originally by Y. Le Floch) The modulus of an (unit) eigenvector of $\left[P_{1,\frac{1}{2},n}, P_{3,\frac{1}{2},n}\right]$, $n = 101$, corresponding to the extremal eigenvalue $\approx \frac{1}{2}$, realized as a polynomial on $\mathbb{C}$, then projected to S^2 using the stereographic projection

example of another curious, unproven phenomenon: when $\alpha = \frac{1}{4p}$, the sub-sequences $\{c_{4d+k}\}_{d\in\mathbb{N}}$ appear essentially as graphs of four distinct "piecewise nice" functions, such that every "piece" is of length p.

Remark 1 The dependence on n modulo 4 appears to be a special case of a general pattern: for pairs of spectral projections associated with J_3 and $\cos\frac{\pi}{p}J_3 + \sin\frac{\pi}{p}J_1$, the sequence c_n seems to depend on $n \mod 2p$.

Finally, we note that J_1, J_2 and J_3 can be obtained, up to normalization, through geometric quantization [8] of the Cartesian coordinate functions x_1, x_2, x_3 on the two-dimensional sphere [7]. In this framework, the properties of the commutator $[P_{1,\frac{1}{2},n}, P_{3,\frac{1}{2},n}]$ seem to be related to the boundaries of the hemispheres $\{x_1 > 0\}$, $\{x_3 > 0\}$ (see Fig. 3).

References

1. L.C. Biedenharn, J.D. Louck, *Angular Momentum in Quantum Physics: Theory and Application, Encyclopedia of Mathematics and its Applications*, vol. 8 (Addison-Wesley Publishing Company, Reading, MA, 1981)
2. A. Böttcher, I.M. Spitkovsky, A gentle guide to the basics of two projections theory. Linear. Algebra. Appl. **432**, 1412–1459 (2010)
3. B. Collins, Product of random projections, Jacobi ensembles and universality problems arising from free probability. Probab. Theory. Related. Fields **133**, 315–344 (2005)
4. A. Edelman, P. McCorquodale, S. Toledo, The future fast fourier transform? SIAM J. Sci. Comput. **20**(3), 1094–1114 (1998)
5. T. Heinosaari, J. Kiukas, D. Reitzner, Coexistence of effects from an algebra of two projections. J. Phys. A Math. Theor. **47**, 225301 (2014)
6. J. Kiukas, R.F. Werner, Maximal violation of Bell inequalities by position measurements. J. Math. Phys. **51**, 072105 (2010)
7. Y. Le Floch, *A Brief Introduction to Berezin-Toeplitz Operators on Compact Kähler Manifolds* (Springer International Publishing, CRM Short Courses, 2014)

8. X. Ma, Geometric quantization on Kähler and symplectic manifolds. Proc. Internat. Congr. Math. Hyderabad, India **II**, 785–810 (2010)
9. O. Shabtai, Commutators of spectral projections of spin operators. J. Lie. Theory. **31**(3), 599–624 (2020)
10. O. Shabtai, On polynomials in spectral projections of spin operators. Lett. Math. Phys. **111** (2021)
11. D. Slepian, H.O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty V: the discrete case. Bell. Syst. Tech. J. **57**, 1371–1430 (1978)
12. J.M. Varah, The prolate matrix. Linear. Algebra. Appl. **187**, 269–278 (1993)
13. L.L. Wang, A review of prolate spheroidal wave functions from the perspective of spectral methods. J. Math. Study. **50**(2), 101–143 (2017)

Integrable Systems

Algebraic Engineering and Integrable Hierarchies

Jean-Emile Bourgine

Abstract Algebraic engineering consists in constructing observables of supersymmetric gauge theories within the representation theory of a quantum group. It is based on its realization as a branes system in string theory which is mapped to a network of modules. This algebraic construction brings new perspectives on many important properties of gauge theories, including AGT correspondence, dualities and integrability. In this proceeding, recent advances on this topic are briefly reviewed. Then, the underlying quantum group is used to revisit the relation between topological strings and integrable hierarchies.

Keywords String theory · Brane systems · Topological strings · Quantum groups · Integrable hierarchies

1 Introduction

Despite the lack of observational support in high energy physics experiments, the study of supersymmetric Quantum Field Theories (QFT) remains an active area of research. One of the main reasons is that supersymmetry provides a class of theories for which exact non-perturbative calculations are possible, thus giving the opportunity to study phenomena beyond the reach of the standard approach in high energy physics. These theories form a testing ground for string theory predictions, including descriptions of black hole, AdS/CFT correspondence, brane constructions of gauge theories, dualities,... In this way, they serve a similar purpose as the integrable systems in statistical physics or quantum mechanics. As it turns out, they also share some of their underlying mathematical structures.

Over the last few years, the algebraic engineering technique has been developed with the aim of deriving the BPS observables of supersymmetric QFT from the repre-

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_26

sentation theory of a quantum group. In practice, these observables are obtained from an operator acting on a network of modules which is closely related to the brane system engineering the QFT in string theory. Its name derives from this proximity with the geometric engineering program [1]. The algebraic engineering is non-perturbative by design, and can also be applied to non-Lagrangian theories. The underlying quantum group structure enters in several of the QFT key properties, like the description of Coulomb branches (cohomological Hall algebras), and certain dualities and correspondences (BPS/CFT correspondence or S-duality). Besides, these algebras exhibit many interesting connections with several important topics of mathematical physics like symmetric polynomials, W-algebras, integrable systems and, as we shall see, integrable hierarchies.

2 Algebraic Engineering of Supersymmetric Gauge Theories

To explain how the algebraic construction works, we focus here on the example of the pure $U(2)$ 5D $\mathcal{N}=1$ gauge theory. It is a five dimensional supersymmetric version of the Yang-Mills theory with gauge group $U(2)$ that preserves eight supercharges. This theory is realized in type IIB string theory as a system of 5-branes, called (p,q)-branes, that are bound states of p D5 and q NS5 branes. They are arranged in the ten-dimensional spacetime as shown on Fig. 1, and directions 01234 corresponding to the 5D spacetime of the gauge theory.

The choice of quantum group is mainly determined by the spacetime, here $\mathbb{C}_{\varepsilon_1}\times\mathbb{C}_{\varepsilon_2}\times S^1_R$. The non-compact directions $\mathbb{C}$ are regularized by turning-on a B-field in string theory, which provides the supersymmetry preserving IR regulators $\varepsilon_1,\varepsilon_2$ (Ω-deformation). These directions are associated to the affinization of the quantum group, and in this case the underlying Lie algebra is toroidal, i.e. twice affine. Compact directions S^1 correspond to trigonometric/elliptic deformations. Here, the quantum group is the quantum toroidal $\mathfrak{gl}(1)$ algebra (or Ding-Iohara-Miki algebra) with parameters $(q_1,q_2)=(e^{R\varepsilon_1},e^{R\varepsilon_2})$, we will denote it $\mathcal{E}(q_1,q_2)$ for short.

To each 5-brane is associated a bosonic Fock space with states labeled by partitions λ. This space is a module for the Fock representation $\rho_u^{(q,p)}$ of the algebra $\mathcal{E}(q_1,q_2)$.

	0	1	2	3	4	5	6	7	8	9
NS5	x	x	x	x	x	x				
D5	x	x	x	x	x		x			
(p,q)	x	x	x	x	x	$\varphi_{p,q}$	$\varphi_{p,q}$			

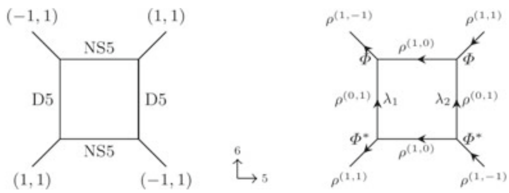

Fig. 1 Brane system engineering the $U(2)$ 5D $\mathcal{N}=1$ gauge theory (left), and its corresponding network of representations (right)

The levels correspond to the brane charges (p, q) and the weight u encodes their position in the (56)-plane. In this way, we obtain the network of representations shown on Fig. 1 (right). An intertwining operator between the representations $\rho^{(1,n+1)}$ and $\rho^{(0,1)} \otimes \rho^{(1,n)}$ has been attached to each vertex, Φ or Φ^* depending on the orientation. Gluing these operators along the internal edges, we can construct an operator $\mathfrak{T}$ that intertwines between the representations associated to incoming and outgoing edges.

$$\mathfrak{T} = \sum_{\lambda_1,\lambda_2} \Phi^*_{\lambda_1} \Phi^*_{\lambda_2} \otimes \Phi_{\lambda_1} \Phi_{\lambda_2}. \tag{1}$$

This operator encodes some of the BPS observables of the QFT. For instance, the partition function is obtained as the vacuum expectation value $\mathcal{Z}[U(2)] = (\langle\emptyset| \otimes \langle\emptyset|)\, \mathfrak{T}\, (|\emptyset\rangle \otimes |\emptyset\rangle)$.

The algebraic engineering technique has been applied successfully to many supersymmetry QFT with a known brane realization. Elliptic or degenerate versions of the original formalism [2] have been applied to 6D and 4D gauge theories respectively in [3, 4]. Algebras of higher rank were used in [5] to discuss gauge theories on ALE spaces, a setup that also led to the introduction of a new family of quantum toroidal algebras [6]. While most applications require the use of a toroidal quantum group, 3D $\mathcal{N} = 2$ gauge theories on $\mathbb{C}_\varepsilon \times S^1$ were recently addressed using a shifted quantum affine $\mathfrak{sl}(2)$ algebra [7], a simpler algebraic framework than the one employed previously in [8]. Other important results include the study of D-type quiver gauge theories, the derivation of qq-characters (i.e. generating function of Wilson loops) [9], and the description of brane crossings using R-matrices [10].

3 Integrable Hierarchies and Topological Strings

3.1 Integrable Hierarchies

An integrable hierarchy is an infinite set of commuting Hamiltonian flows that form compatible partial differential equations. A typical example is the Kadomtsev-Petviashvili (KP) hierarchy that starts with the equation $3u_{yy} = \partial_x (4u_t - 6uu_x - u_{xxx})$ modeling water waves with long wavelengths [11]. Solutions of the hierarchy can be written using a τ-function $\tau(\boldsymbol{t})$ depending on infinitely many 'times' parameters $\boldsymbol{t} = (t_1, t_2, \cdots)$. It satisfies the Hirota bilinear equation

$$\oint_\infty e^{\sum_{k>0}(t_k - t'_k) z^k} \tau(\boldsymbol{t} - [z^{-1}])\tau(\boldsymbol{t}' + [z^{-1}])dz = 0. \tag{2}$$

with $\boldsymbol{t} \pm [z] = (t_1 \pm z, t_2 \pm \frac{1}{2}z^2, \cdots)$ which implies that the function $u(\boldsymbol{t}) = 2\partial_x^2 \log \tau(\boldsymbol{t})$ solves the KP hierarchy with $t_1 = x$, $t_2 = y$ and $t_3 = t$.

In the 1980's, the Kyoto School of integrability developed a formalism to construct τ-functions based on the representation of the algebra $\widehat{\mathfrak{gl}}(\infty)$ on the 2D Dirac fermionic Fock space. This Fock space is built upon the vacuum state $|\emptyset\rangle$ annihilated by positive modes ψ_r, $\bar{\psi}_r$ with $\{\psi_r, \bar{\psi}_s\} = \delta_{r+s}$ [12]. In this construction, the τ-function takes the canonical form

$$\tau_G(\boldsymbol{t}) = \langle\emptyset|e^{\sum_{k>0} t_k J_k} G|\emptyset\rangle \tag{3}$$

where $J_k = \sum_r :\bar{\psi}_{k-r}\psi_r$ are the Heisenberg algebra generators obtained by bosonization. Solutions are indexed by the operator G that must satisfy the basic bilinear condition $[G \otimes G, \Psi] = 0$ with $\Psi = \sum_r \bar{\psi}_{-r} \otimes \psi_r$. This condition is satisfied by all the elements of the group

$$\mathrm{GL}(\infty) = \{e^{\sum_{r,s} a_{r,s} :\bar{\psi}_r \psi_s:}, \quad a_{r,s} \in \mathbb{C}\}, \tag{4}$$

where $E_{r,s} =: \bar{\psi}_r \psi_s :$ are the generators of the Lie algebra $\widehat{\mathfrak{gl}}(\infty)$.

The relation between topological strings and integrable hierarchies follows from the presence of a $W_{1+\infty}$ algebra of symmetries [13]. We will use here its q-deformed version, with the quantum parameter related to the string coupling as $q = e^{g_{\mathrm{str}}}$. This algebra is equivalent to $\widehat{\mathfrak{gl}}(\infty)$ under the transformation

$$W_{m,n} = \sum_{r\in\mathbb{Z}+1/2} q^{-(r+1/2)n} E_{m-r,r}. \tag{5}$$

In a series of papers starting from [14], Nakatsu and Takasaki (NT) have developed a method to associate τ-functions of integrable hierarchies to time-deformations of certain topological strings amplitudes. In particular, starting from the topological vertex in the melting crystal picture where it is seen as a counting function for plane partitions [15], they defined the deformation

$$Z(q) = \sum_\lambda (s_\lambda(q^{-\rho}))^2 \quad\rightarrow\quad Z(q, \boldsymbol{t}) = \sum_\lambda (s_\lambda(q^{-\rho}))^2 e^{\sum_{k>0} t_k \Omega_k(\lambda)}. \tag{6}$$

Here $Z(q)$ is the MacMahon counting function, $s_\lambda(q^{-\rho})$ are Schur polynomials specialized at $q^{-\rho} = (q^{1/2}, q^{3/2}, \cdots)$ and $\Omega_k(\lambda)$ are the eigenvalues of the operators $W_{0,-k}$ in a certain basis $|\lambda\rangle$ of the fermionic Fock space. They have shown that $Z(q, \boldsymbol{t})$ is a tau function of a trigonometric version of the KP hierarchy called 1D Toda hierarchy. It takes the canonical form (3) with the operator $G = F^{-1}\Phi_\emptyset\Phi_\emptyset F^{-1}$. We will come back to this expression shortly.

3.2 Refined Topological Strings

Topological strings amplitudes can be refined by the introduction of an extra parameter t which originates from the correspondence with partition functions of 5D $\mathcal{N}=1$ gauge theories [16]. Starting from the previous observations, our paper [17] addresses the following questions:

- Can we extend the Nakatsu-Takasaki construction to the refined case?
- Is there any connection with the algebraic engineering?

As we have seen, the construction relies heavily on the presence of the quantum $W_{1+\infty}$ symmetries. This algebra is deformed into the quantum toroidal $\mathfrak{gl}(1)$ algebra $\mathcal{E}(q_1, q_2)$, with $(q_1, q_2) = (q, t^{-1})$, which has no longer a fermionic representation, but still possesses bosonic ones: the Fock representations $\rho^{(q,p)}$ mentioned earlier. To answer the two previous questions, the NT construction was revisited with an emphasis on the role of the intertwiner Φ and the group SL$(2,\mathbb{Z})$ of automorphisms for $\mathcal{E}(q_1, q_2)$.

The refinement deforms Schur polynomials into Macdonald polynomials P_λ, and the natural deformation of the τ-function is

$$Z(t,q) = \sum_\lambda \frac{(\iota P_\lambda(t^{-\rho}))^2}{\langle P_\lambda, P_\lambda\rangle_{q,t}} \quad \to \quad Z(t,q,\boldsymbol{t}) = \sum_\lambda \frac{(\iota P_\lambda(t^{-\rho}))^2}{\langle P_\lambda, P_\lambda\rangle_{q,t}} e^{\sum_{k>0} t_k \Omega_k(\lambda)}, \quad (7)$$

where $\left\langle P_\lambda, P_\mu\right\rangle_{q,t}$ is Macdonald's scalar product [18]. The times t_k are coupled to the eigenvalues $\Omega_k(\lambda)$ of the dual Cartan modes b_{-k} which indeed act diagonally on the Macdonald basis $|P_\lambda\rangle$ in the Fock representation $\rho^{(1,0)}$.[1] These modes reduce to $W_{0,-k}$ in the limit $q = t$ where $\mathcal{E}(q, t^{-1}) \to q - W_{1+\infty}$.

Before going any further, we need to derive two important properties that extend the *shift symmetries* observed by Nakatsu and Takasaki. In this process, we will find a new algebraic understanding for them. To do so, we introduce the generators $\mathcal{S}$ and $\mathcal{T}$ of the SL$(2,\mathbb{Z})$ group of automorphisms of the algebra $\mathcal{E}(q_1, q_2)$. The automorphism $\mathcal{S}$ has been introduced by Miki in [19], it realizes the $\mathcal{S}$-duality of type IIB string theory [20, 21], and is used here to define the dual Cartan modes $b_{-k} = \mathcal{S}\cdot a_{-k}$. On the other hand, $\mathcal{T}$ is related to the framing factors of topological strings which also correspond to the Chern-Simons term in the Lagrangian of 5D $\mathcal{N}=1$ gauge theories [9]. These automorphisms generate isomorphisms between Fock representations of different levels: $\rho^{(0,1)} \approx \rho^{(1,0)} \circ \mathcal{S}$, $\rho^{(1,1)} \approx \rho^{(1,0)} \circ \mathcal{T}$. Recall that Φ is an intertwiner between the representations $\rho^{(1,1)}$ and $\rho^{(0,1)} \otimes \rho^{(1,0)}$. Using the automorphisms, the intertwining relation takes the form

$$\rho^{(1,0)}(\mathcal{T}\cdot e)\Phi = \Phi\left(\rho^{(1,0)} \circ \mathcal{S} \otimes \rho^{(1,0)}\, \Delta(e)\right), \quad e \in \mathcal{E}(q_1, q_2). \quad (8)$$

[1] This basis built by exploiting the isomorphism with the ring of symmetric polynomials $J_{-k} \equiv p_k(x)$.

where $\varDelta$ is the coproduct of the quantum group. Projecting $\varPhi$ on the vacuum state in the representation $\rho^{(0,1)}$, we find

$$\boxed{\rho^{(1,0)}(\mathcal{T}\cdot b_{-k})\varPhi_\emptyset = \varPhi_\emptyset\rho^{(1,0)}(b_{-k}).} \tag{9}$$

It generalizes the first shift symmetry, with $W_{0,k}\to b_{-k}$ and $W_{k,-k}\to\mathcal{T}\cdot b_{-k}$. The second shift symmetry has a completely different origin: it is the realization of the automorphism $\mathcal{T}\mathcal{S}^{-1}$ on the modes a_k in the Fock representation $\rho^{(1,0)}$ as the adjoint action of an operator F,

$$\boxed{\rho^{(1,0)}(\mathcal{T}\mathcal{S}^{-1}\cdot a_k) = F\rho^{(1,0)}(a_k)F^{-1},\quad F|P_\lambda\rangle = \prod_{(i,j)\in\lambda} q_1^{i-1}q_2^{j-1}|P_\lambda\rangle.} \tag{10}$$

The eigenvalues of F coincide with the framing factors of topological strings and we call this operator the framing operator. Armed with these two properties, we can now proceed to study the time-deformed amplitude $Z(t,q,\boldsymbol{t})$.

From the known matrix elements $\langle\emptyset|\varPhi_\emptyset|P_\lambda\rangle = \iota P_\lambda(t^{-\rho})$ and $\langle P_\lambda|\varPhi_\emptyset|\emptyset\rangle = \iota P_\lambda(t^{-\rho})$, the deformed amplitude 7 can be rewritten in the form of a Fock space correlator

$$Z(t,q,\boldsymbol{t}) = \sum_\lambda \frac{\langle\emptyset|\varPhi_\emptyset\rho^{(1,0)}(e^{\sum_k t_k b_{-k}})|P_\lambda\rangle\langle P_\lambda|\varPhi_\emptyset|\emptyset\rangle}{\langle P_\lambda, P_\lambda\rangle_{q,t}}. \tag{11}$$

The summation is performed using the Macdonald basis closure relation,

$$Z(t,q,\boldsymbol{t}) = \langle\emptyset|\varPhi_\emptyset\rho^{(1,0)}(e^{\sum_k t_k b_{-k}})\varPhi_\emptyset|\emptyset\rangle. \tag{12}$$

At this stage, we need to move the exponential of $\sum_k t_k b_{-k}$ to the left, which is a non-trivial operation. For this purpose, we use the first shift symmetry 9 to write

$$Z(t,q,\boldsymbol{t}) = \langle\emptyset|\rho^{(1,0)}(\mathcal{T}\cdot e^{\sum_k t_k b_{-k}})\varPhi_\emptyset\varPhi_\emptyset|\emptyset\rangle. \tag{13}$$

Then, the second shift symmetry 10 together with $F|\emptyset\rangle = |\emptyset\rangle$ gives

$$Z(t,q,\boldsymbol{t}) = \langle\emptyset|\rho^{(1,0)}(e^{\sum_k t_k a_k})F^{-1}\varPhi_\emptyset\varPhi_\emptyset F^{-1}|\emptyset\rangle. \tag{14}$$

To conclude, we observe that, in the Fock representation $\rho^{(1,0)}$, the Cartan modes are identified with the Heisenberg modes: $\rho^{(1,0)}(a_k) = \gamma_k J_k$, up to a known factor $\gamma_k(q_1,q_2)$. Thus, we recover the canonical expression 3 with the rescaled times $t_k\to\gamma_k t_k$ and the operator $G = F^{-1}\varPhi_\emptyset\varPhi_\emptyset F^{-1}$.

This derivation is the main result in [17], it provides a direct link with the algebraic engineering technique. Indeed, the operator $\Phi_\emptyset$ entering in G can be seen as a trivial version of the operator $\mathfrak{T}$ discussed in the previous section. This link could be exploited to generalize the construction to various amplitudes, including the conifold discussed in [22].

4 Discussion

Can we conclude that the refined quantity $Z(t, q, \boldsymbol{t})$ is the tau function of an integrable hierarchy? It would be so if the deformed operator G still obeyed the canonical bilinear relation $[G \otimes G, \Psi] = 0$. While this is a well-defined question since Ψ can be bosonized, it is surprisingly difficult to answer directly. Instead, it is possible to examine the Hirota equation perturbatively and check whether $Z(t, q, \boldsymbol{t})$ is a solution. And it is not. To understand why, we must introduce the more general framework of Kac and Wakimoto [23] in which Ψ is replaced by the Casimir operator of different algebras. Indeed, writing the group elements $G \in \mathrm{GL}(\infty)$ in the exponential form $G = e^g$, we observe that the basic bilinear condition becomes $[\Delta(g), \Psi] = 0$ where $\Delta(g) = g \otimes 1 + 1 \otimes g$ is the co-commutative coproduct. After refinement, this coproduct is replaced by the non-trivial Drinfeld coproduct of $\mathcal{E}(q_1, q_2)$, and the operator Ψ becomes the Casimir satisfying $[\left(\rho^{(1,0)} \otimes \rho^{(1,0)}\right) \Delta(e), \Psi] = 0$ for all $e \in \mathcal{E}(q_1, q_2)$. In fact, Ψ can be identified with the screening charges of the q-Virasoro algebra [24] using the decomposition of the representation $\left(\rho^{(1,0)} \otimes \rho^{(1,0)}\right) \Delta$ into q-Virasoro and Heisenberg factors observed in [25]. Work is in progress to show that $\Delta(G)$ does commute with these operators, and derive the refined version of the Hirota bilinear equation.

Meanwhile, our observation brings the possibility to generalize the correspondence between topological strings amplitudes and integrable hierarchies. For instance, the trinion amplitudes T_N studied in [26] are associated to the intertwining operators $\Phi^{(N)}$ between representations of higher levels $\rho^{(N,N)}$ and $\rho^{(0,N)} \otimes \rho^{(N,0)}$ which could be used to define τ-functions of more involved hierarchies. Conversely, it could also be used to extend the algebraic engineering using the quantum algebras associated to reduced hierarchies [27].

Finally, topological strings have another equivalent description called the *B-model formulation*. In this formulation, amplitudes are written as matrix model integrals that are also known τ-functions. The refinement replaces the matrix model with a β-ensemble, its integrals are no-longer τ-functions but still exhibit a rich mathematical structure: quantum curves, topological recursion,... This work could bring a new perspective on the mirror symmetry between A and B models.

References

1. S. H. Katz, A. Klemm, C. Vafa, Geometric engineering of quantum field theories. Nucl. Phys. B **497**, 173–195 (1997). https://doi.org/10.1016/S0550-3213(97)00282-4, arXiv:hep-th/9609239
2. H. Awata, B. Feigin, J. Shiraishi, Quantum algebraic approach to refined topological vertex. JHEP **03**, 041 (2012). https://doi.org/10.1007/JHEP03(2012)041, http://arxiv.org/abs/1112.6074arXiv:1112.6074 [hep-th]
3. O. Foda, R.-D. Zhu, An elliptic topological vertex. Nucl. Phys. **B936**, 448–471 (2018). https://doi.org/10.1016/j.nuclphysb.2018.10.001, http://arxiv.org/abs/1801.04943arXiv:1801.04943 [hep-th]
4. J.E. Bourgine, K. Zhang, A note on the algebraic engineering of 4D $\mathcal{N} = 2$ super Yang-Mills theories. Phys. Lett. **B789**, 610–619 (2019). https://doi.org/10.1016/j.physletb.2018.11.066, http://arxiv.org/abs/1809.08861arXiv:1809.08861 [hep-th]
5. H. Awata, H. Kanno, A. Mironov, A. Morozov, K. Suetake, Y. Zenkevich, (q, t)-KZ equations for quantum toroidal algebra and Nekrasov partition functions on ALE spaces. JHEP **03**, 192 (2018). https://doi.org/10.1007/JHEP03(2018)192, arXiv:1712.08016 [hep-th]
6. J.-E. Bourgine, S. Jeong, New quantum toroidal algebras from 5D $\mathcal{N} = 1$ instantons on orbifolds. JHEP **05**, 127 (2020). arXiv:1906.01625 [hep-th]
7. J.-E. Bourgine, Engineering 3D $\mathcal{N} = 2$ theories using the quantum affine $sl(2)$ algebra. arXiv:2107.10063 [hep-th]
8. Y. Zenkevich, Higgsed network calculus. JHEP **08**(149) (2021). arXiv:1812.11961 [hep-th]
9. J.-E. Bourgine, M. Fukuda, K. Harada, Y. Matsuo, R.-D. Zhu, (p, q)-webs of DIM representations, 5d $\mathcal{N} = 1$ instanton partition functions and qq-characters. JHEP **11**, 034 (2017). https://doi.org/10.1007/JHEP11(2017)034, arXiv:1703.10759 [hep-th]
10. Y. Zenkevich, *Mixed Network Calculus*. arXiv:2012.15563 [hep-th]
11. B.B. Kadomtsev, V.I. Petviashvili, On the stability of solitary waves in weakly dispersing media. Dokl. Akad. Nauk. SSSR. **192**(4), 753–756 (1970)
12. M. Jimbo, T. Miwa, Solitons and infinite dimensional lie algebras. Publ. Res. Inst. Math. Sci. Kyoto **19**, 943 (1983). https://doi.org/10.2977/prims/1195182017
13. M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino, C. Vafa, Topological strings and integrable hierarchies. Commun. Math. Phys. **261**, 451–516 (2006). https://doi.org/10.1007/s00220-005-1448-9. arXiv:hep-th/0312085 [hep-th]
14. T. Nakatsu, K. Takasaki, Melting crystal, quantum torus and toda hierarchy. Commun. Math. Phys. **285**, 445–468 (2009). https://doi.org/10.1007/s00220-008-0583-5, arXiv:0710.5339 [hep-th]
15. A. Okounkov, N. Reshetikhin, C. Vafa, Quantum Calabi-Yau and classical crystals. Prog. Math. **244**, 597 (2006). https://doi.org/10.1007/0-8176-4467-9_16, arXiv:hep-th/0309208 [hep-th]
16. A. Iqbal, C. Kozcaz, C. Vafa, The refined topological vertex. JHEP **2009**(0910), 069 (2009). http://arxiv.org/abs/hep-th/0701156, arXiv:hep-th/0701156 [hep-th]
17. J.-E. Bourgine, Intertwining operator and integrable hierarchies from topological strings. JHEP **05**, 216 (2021). https://doi.org/10.1007/JHEP05(2021)216, arXiv:2101.09925 [hep-th]
18. I. Macdonald, *Symmetric Functions and Hall Polynomials*
19. K. Miki, A (q, γ) analog of the $W_{1+\infty}$ algebra. J. Math. Phys. **48**(12), 3520 (2007). https://doi.org/10.1063/1.2823979
20. J.-E. Bourgine, Fiber-base duality from the algebraic perspective. JHEP **03**, 003 (2019). https://doi.org/10.1007/JHEP03(2019)003, arXiv:1810.00301 [hep-th]
21. M. Fukuda, Y. Ohkubo, J. Shiraishi, Non-stationary Ruijsenaars functions for $\kappa = t^{-1/N}$ and intertwining operators of Ding-Iohara-Miki algebra, SIGMA **16**, 116 (2020). arXiv:2002.00243 [math.QA]
22. K. Takasaki, A modified melting crystal model and the Ablowitz–Ladik hierarchy. J. Phys. A **46**, 245202 (2013). https://doi.org/10.1088/1751-8113/46/24/245202, arXiv:1302.6129 [math-ph]

23. V. Kac, M. Wakimoto, Exceptional hierarchies of soliton equations. Proc. Symp. Pure. Math. **49** (1989)
24. J. Shiraishi, H. Kubo, H. Awata, S. Odake, A Quantum deformation of the Virasoro algebra and the Macdonald symmetric functions. Lett. Math. Phys. **38** 33–51 (1996). https://doi.org/10.1007/BF00398297, arXiv:q-alg/9507034
25. B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi, S. Yanagida, A commutative algebra on degenerate CP^1 and Macdonald polynomials. J. Math. Phys. **50**(9), 095215 (2009). https://doi.org/10.1063/1.3192773, arXiv:0904.2291 [math.CO]
26. I. Coman, E. Pomoni, J. Teschner, Trinion conformal blocks from topological strings. JHEP **09**, 078 (2020). https://doi.org/10.1007/JHEP09(2020)078, arXiv:1906.06351 [hep-th]
27. J.-E. Bourgine, Quantum $\mathrm{W}1+\infty$ subalgebras of BCD type and symmetric polynomials. J. Math. Phys. **62**(6), 063505 (2021). https://doi.org/10.1063/5.0049355, arXiv:2101.03877 [hep-th]

Nested Bethe Ansatz for RTT–Algebra $\mathcal{A}_n$

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Abstract This contribution continues our recent studies on the algebraic Bethe Ansatz for the RTT–algebras of $\mathrm{sp}(2n)$ and $\mathrm{o}(2n)$ types. In these studies, we faced RTT–algebras which we called $\mathcal{A}_n$. The next step in our construction of the Bethe vectors for the RTT-algebras of type $\mathrm{sp}(2n)$ and $\mathrm{o}(2n)$ is to find the Bethe vectors for the RTT–algebras $\mathcal{A}_n$. This paper deals with the construction of the Bethe vectors of the RTT–algebra $\mathcal{A}_n$ using the Bethe vectors of the RTT–algebra $\mathcal{A}_{n-1}$.

Keywords RTT algebras · Nested algebraic Bethe ansatz · Bethe vectors

1 Introduction

In studying the algebraic Bethe Ansatz for the RTT–algebras of type $\mathrm{sp}(2n)$ and $\mathrm{o}(2n)$ [1, 2], we discovered some RTT–algebras which we called $\mathcal{A}_n$. The main result of this contribution is the assertion that for the construction of eigenvalues and eigenvectors of the transfer–matrix of the RTT–algebras of type $\mathrm{sp}(2n)$ and $\mathrm{o}(2n)$ it is enough to find eigenvalues and eigenvectors for the RTT–algebra $\mathcal{A}_n$.

In this work, we deal with the nested Bethe Ansatz for the RTT–algebra $\mathcal{A}_n$. We show how to construct eigenvectors for the RTT–algebra $\mathcal{A}_n$ by using eigenvectors of the RTT–algebra $\mathcal{A}_{n-1}$.

Note that the RTT–algebra $\mathcal{A}_{n-1}$ is not the RTT–subalgebra $\mathcal{A}_n$. However, $\mathcal{A}_n$ contains two RTT–subalgebras $\mathcal{A}_n^{(+)}$ and $\mathcal{A}_n^{(-)}$, which are of type $\mathrm{gl}(n)$. The RTT–algebras $\mathcal{A}_{n-1}^{(\pm)}$ are already the RTT–subalgebras of $\mathcal{A}_n^{(\pm)}$. As we will see later, we can construct some eigenvectors for the RTT–algebras $\mathcal{A}_n$ as Bethe vectors of the

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_27

RTT–algebras $\mathcal{A}_n^{(\pm)}$, i.e. as the Bethe vectors for the RTT–algebras of type gl(n). Our result for such eigenvectors is the same as for the nested Bethe Ansatz for the RTT–algebras of gl(n), which can be found in [4]. In this sense, our construction is a certain generalization of the nested Bethe Ansatz for the RTT–algebras of type gl(n).

The proofs of many claims are only a suitable but long adjustment of the Yang–Baxter and the RTT–equations. Due to the limited length of contributions in the Proceedings, we did not include them in this paper.

2 The RTT–Algebra $\mathcal{A}_n$

We denote $\mathbf{E}_k^i$ and $\mathbf{E}_{-k}^{-i}$, where $i, k = 1, \ldots, n$, the matrices $\left(\mathbf{E}_k^i\right)_s^r = \left(\mathbf{E}_{-k}^{-i}\right)_{-s}^{-r} = \delta_k^r \delta_s^i$. Then the relations $\mathbf{E}_k^i \mathbf{E}_s^r = \delta_s^i \mathbf{E}_k^r$, $\sum_{i=1}^n \mathbf{E}_i^i = \mathbf{I}_+$ and $\sum_{i=1}^n \mathbf{E}_{-i}^{-i} = \mathbf{I}_-$ hold.

The RTT–algebra $\mathcal{A}_n$ is an associative algebra with a unit that is generated by the elements $T_k^i(x)$ and $T_{-k}^{-i}(x)$, where $i, k = 1, \ldots, n$. If we introduce the monodromy matrix $\mathbf{T}(x) = \mathbf{T}^{(+)}(x) + \mathbf{T}^{(-)}(x)$, where

$$\mathbf{T}^{(+)}(x) = \sum_{i,k=1}^n \mathbf{E}_i^k \otimes T_k^i(x)\,, \qquad \mathbf{T}^{(-)}(x) = \sum_{i,k=1}^n \mathbf{E}_{-i}^{-k} \otimes T_{-k}^{-i}(x)\,,$$

the commutation relations between the generators are defined by the RTT–equation

$$\mathbf{R}_{1,2}(x, y)\mathbf{T}_1(x)\mathbf{T}_2(y) = \mathbf{T}_2(y)\mathbf{T}_1(x)\mathbf{R}_{1,2}(x, y), \tag{1}$$

where the R–matrix is $\mathbf{R}(x, y) = \mathbf{R}^{(+,+)}(x, y) + \mathbf{R}^{(+,-)}(x, y) + \mathbf{R}^{(-,+)}(x, y) + \mathbf{R}^{(-,-)}(x, y)$,

$$\begin{aligned}
\mathbf{R}^{(+,+)}(x, y) &= \frac{1}{f(x, y)} \Big(\mathbf{I}_+ \otimes \mathbf{I}_+ + g(x, y) \sum_{i,k=1}^n \mathbf{E}_k^i \otimes \mathbf{E}_i^k\Big),\\
\mathbf{R}^{(+,-)}(x, y) &= \mathbf{I}_+ \otimes \mathbf{I}_- - k(x, y) \sum_{i,k=1}^n \mathbf{E}_k^i \otimes \mathbf{E}_{-k}^{-i}\,,\\
\mathbf{R}^{(-,+)}(x, y) &= \mathbf{I}_- \otimes \mathbf{I}_+ - h(x, y) \sum_{i,k=1}^n \mathbf{E}_{-k}^{-i} \otimes \mathbf{E}_k^i\,,\\
\mathbf{R}^{(-,-)}(x, y) &= \frac{1}{f(x, y)} \Big(\mathbf{I}_- \otimes \mathbf{I}_- + g(x, y) \sum_{i,k=1}^n \mathbf{E}_{-k}^{-i} \otimes \mathbf{E}_{-i}^{-k}\Big),\\
g(x, y) &= \frac{1}{x-y}\,, \qquad\qquad f(x, y) = \frac{x-y+1}{x-y}\,,\\
h(x, y) &= \frac{1}{x-y+n-\eta}\,, \qquad k(x, y) = \frac{1}{x-y+\eta}
\end{aligned}$$

and η is any number. For $\eta = -1$ we obtain the RTT–algebra connected with the RTT–algebra of sp$(2n)$ type and for $\eta = 1$ the RTT–algebra connected with the RTT-algebra of o$(2n)$ type.

By direct calculation, it can be verified that this R–matrix satisfies the Yang–Baxter equation

$$\mathbf{R}_{1,2}(x, y)\mathbf{R}_{1,3}(x, z)\mathbf{R}_{2,3}(y, z) = \mathbf{R}_{2,3}(y, z)\mathbf{R}_{1,3}(x, z)\mathbf{R}_{1,2}(x, y) \tag{2}$$

and has the inverse R–matrix. Therefore, it defines the RTT–algebra that we denote by $\mathcal{A}_n$.

It is easily seen that the RTT–equation (1) can be written as

$$\mathbf{R}_{1,2}^{(\epsilon_1,\epsilon_2)}(x,y)\mathbf{T}_1^{(\epsilon_1)}(x)\mathbf{T}_2^{(\epsilon_2)}(y) = \mathbf{T}_2^{(\epsilon_2)}(y)\mathbf{T}_1^{(\epsilon_1)}(x)\mathbf{R}_{1,2}^{(\epsilon_1,\epsilon_2)}(x,y)\,, \tag{3}$$

where $\epsilon_1,\ \epsilon_2 = \pm$. From this form of the RTT–equation it is clear that in the RTT–algebra $\mathcal{A}_n$ there are two RTT-subalgebras $\mathcal{A}_n^{(+)}$ and $\mathcal{A}_n^{(-)}$, which are generated by the elements $T_k^i(x)$ and $T_{-k}^{-i}(x)$, where $i,\ k = 1,\ \ldots,\ n$.

Using the RTT–equation (3), it is possible to show that in the RTT–algebra $\mathcal{A}_n$ the operators

$$H^{(+)}(x) = \mathrm{Tr}\mathbf{T}^{(+)}(x) = \sum_{i=1}^{n} T_i^i(x)\,, \qquad H^{(-)}(x) = \mathrm{Tr}\mathbf{T}^{(-)}(x) = \sum_{i=1}^{n} T_{-i}^{-i}(x)$$

mutually commute.

We deal with the representations of the RTT–algebra $\mathcal{A}_n$ on the vector space $\mathcal{W} = \mathcal{A}_n\omega$, where ω is a vacuum vector for which the relations

$$\begin{array}{ll} T_k^i(x)\omega = 0 & \text{for } 1 \le i < k \le n\,,\ T_i^i(x)\omega = \lambda_i(x)\,,\omega \\ T_{-i}^{-k}(x)\omega = 0 & \text{for } 1 \le i < k \le n\,,\ T_{-i}^{-i}(x)\omega = \lambda_{-i}(x)\omega \end{array}$$

hold. Our goal is to find in the vector space $\mathcal{W}$ common eigenvectors of the operators $H^{(\pm)}(x)$.

In the RTT–algebra $\mathcal{A}_n$ there are two RTT–subalgebras $\tilde{\mathcal{A}}^{(+)} = \mathcal{A}_{n-1}^{(+)}$ and $\tilde{\mathcal{A}}^{(-)} = \mathcal{A}_{n-1}^{(-)}$ of gl$(n-1)$ type, which are generated by the elements $T_k^i(x)$ and $T_{-k}^{-i}(x)$, where $i,\ k = 1,\ \ldots,\ n-1$.

First, we will consider the subspace $\tilde{\mathcal{W}}$ generated by the elements $\tilde{\mathcal{A}}^{(+)}\tilde{\mathcal{A}}^{(-)}\omega$.

Proposition 1 *The relations*

$$T_n^i(x)w = T_{-i}^{-n}(x)w = 0\,, \quad T_n^n(x)w = \lambda_n(x)w\,, \quad T_{-n}^{-n}(x)w = \lambda_{-n}(x)w \tag{4}$$

hold for any $w \in \tilde{\mathcal{W}}$ and $i = 1,\ 2,\ \ldots,\ n-1$.

Proposition 2 *The space $\tilde{\mathcal{W}}$ is invariant with respect to $\tilde{\mathcal{A}}^{(+)}$ and $\tilde{\mathcal{A}}^{(-)}$.*

Proposition 3 *If we define*

$$\tilde{\mathbf{T}}^{(+)}(x) = \sum_{i,k=1}^{n-1} \mathbf{E}_i^k \otimes T_k^i(x)\,, \qquad \tilde{\mathbf{T}}^{(-)}(x) = \sum_{i,k=1}^{n-1} \mathbf{E}_{-i}^{-k} \otimes T_{-k}^{-i}(x)$$

the commutation relations for $T_k^i(x)$ and $T_{-k}^{-i}(x)$, where $i,\ k = 1,\ \ldots,\ n-1$, reduced to the space $\tilde{\mathcal{W}}$ can be written in the form of the RTT–equation

$$\tilde{\mathbf{R}}_{1,2}^{(\epsilon_1,\epsilon_2)}(x,y)\tilde{\mathbf{T}}_1^{(\epsilon_1)}(x)\tilde{\mathbf{T}}_2^{(\epsilon_2)}(y) = \tilde{\mathbf{T}}_2^{(\epsilon_2)}(y)\tilde{\mathbf{T}}_1^{(\epsilon_1)}(x)\tilde{\mathbf{R}}_{1,2}^{(\epsilon_1,\epsilon_2)}(x,y)$$

where $\epsilon_1,\ \epsilon_2 = \pm$ and

$$\tilde{\mathbf{R}}^{(+,+)}_{1,2}(x,y) = \frac{1}{f(x,y)}\Big(\tilde{\mathbf{I}}_+ \otimes \tilde{\mathbf{I}}_+ + g(x,y)\sum_{i,k=1}^{n-1} \mathbf{E}^i_k \otimes \mathbf{E}^k_i\Big),$$
$$\tilde{\mathbf{R}}^{(-,-)}_{1,2}(x,y) = \frac{1}{f(x,y)}\Big(\tilde{\mathbf{I}}_- \otimes \tilde{\mathbf{I}}_- + g(x,y)\sum_{i,k=1}^{n-1} \mathbf{E}^{-i}_{-k} \otimes \mathbf{E}^{-k}_{-i}\Big),$$
$$\tilde{\mathbf{R}}^{(+,-)}_{1,2}(x,y) = \tilde{\mathbf{I}}_+ \otimes \tilde{\mathbf{I}}_- - k(x,y)\sum_{i,k=1}^{n-1} \mathbf{E}^i_k \otimes \mathbf{E}^{-i}_{-k},$$
$$\tilde{\mathbf{R}}^{(-,+)}_{1,2}(x,y) = \tilde{\mathbf{I}}_- \otimes \tilde{\mathbf{I}}_+ - \tilde{h}(x,y)\sum_{i,k=1}^{n-1} \mathbf{E}^{-i}_{-k} \otimes \mathbf{E}^i_k$$
$$\tilde{\mathbf{I}}_+ = \sum_{k=1}^{n-1}\mathbf{E}^k_k\,, \qquad \tilde{\mathbf{I}}_- = \sum_{k=1}^{n-1}\mathbf{E}^{-k}_{-k}\,, \qquad \tilde{h}(x,y) = \frac{1}{x-y+n-1-\eta}\,.$$

The following theorem immediately follows from Proposition 3.

Theorem 1 *The action of the operators $T^i_k(x)$ and $T^{-i}_{-k}(x)$, where $i,\ k = 1,\ \ldots,\ n-1$, in the space $\tilde{\mathcal{W}}$ forms the RTT–algebra $\mathcal{A}_{n-1}$.*

3 General Form of Common Eigenvectors of $H^{(+)}(x)$ and $H^{(-)}(x)$

Let $\mathbf{v} = (v_1, v_2, \ldots, v_P)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_Q)$ be ordered sets of mutually different numbers. We will search for a general shape of the common eigenvectors $H^{(+)}(x)$ and $H^{(-)}(x)$ in the form

$$\mathfrak{B}(\mathbf{v},\mathbf{w}) = \sum_{k_1,\ldots,k_P=1}^{n-1}\ \sum_{r_1,\ldots,r_Q=1}^{n-1} T^n_{k_1}(v_1)\ldots T^n_{k_P}(v_P)$$
$$T^{-r_1}_{-n}(w_1)\ldots T^{-r_Q}_{-n}(w_Q)\Phi^{k_1,\ldots,k_P}_{-r_1,\ldots,-r_Q}\,,$$

where $\Phi^{k_1,\ldots,k_P}_{-r_1,\ldots,-r_Q} \in \tilde{\mathcal{W}}$.

We will consider $(n-1)$–dimensional spaces $\mathcal{V}_+$ and $\mathcal{V}_-$ with the base $\mathbf{e}_k$ and $\mathbf{e}_{-r}$ and denote $\mathbf{f}^k$ and $\mathbf{f}^{-r}$ their dual base in dual spaces $\mathcal{V}^*_+$ and $\mathcal{V}^*_-$.

Let us define

$$\mathbf{b}^{(+)}(v) = \sum_{k=1}^{n-1}\mathbf{f}^k \otimes T^n_k(v) \in \mathcal{V}^*_+ \otimes \mathcal{A}_n$$
$$\mathbf{b}^{(-)}(w) = \sum_{r=1}^{n-1}\mathbf{e}_{-r} \otimes T^{-r}_{-n}(w) \in \mathcal{V}_- \otimes \mathcal{A}_n$$

and denote

$$\mathbf{b}^{(+)}_{1^*,\ldots,P^*}(\mathbf{v}) = \mathbf{b}^{(+)}_{1^*}(v_1)\mathbf{b}^{(+)}_{2^*}(v_2)\ldots\mathbf{b}^{(+)}_{P^*}(v_P) \in \mathcal{V}^*_1 \otimes \mathcal{V}^*_2 \otimes \ldots \otimes \mathcal{V}^*_P \otimes \mathcal{A}_n$$
$$\mathbf{b}^{(-)}_{1,\ldots,Q}(\mathbf{w}) = \mathbf{b}^{(-)}_1(w_1)\mathbf{b}^{(-)}_2(w_2)\ldots\mathbf{b}^{(-)}_Q(w_Q) \in \mathcal{V}_{-1} \otimes \mathcal{V}_{-2} \otimes \ldots \otimes \mathcal{V}_{-Q} \otimes \mathcal{A}_n.$$

Explicitly, we have

$$\mathbf{b}^{(+)}_{1^*,\dots,P^*}(\mathbf{v}) = \sum_{k_1,\dots,k_P=1}^{n-1} \mathbf{f}^{k_1} \otimes \dots \otimes \mathbf{f}^{k_P} \otimes T^n_{k_1}(v_1)\dots T^n_{k_P}(v_P)$$
$$\mathbf{b}^{(-)}_{1,\dots,Q}(\mathbf{w}) = \sum_{r_1,\dots,r_Q=1}^{n-1} \mathbf{e}_{-r_1} \otimes \dots \otimes \mathbf{e}_{-r_Q} \otimes T^{-r_1}_{-n}(w_1)\dots T^{-r_Q}_{-n}(w_Q)\,.$$

If we introduce $\mathbf{\Phi} \in \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_P \otimes \mathcal{V}^*_{-1} \otimes \cdots \otimes \mathcal{V}^*_{-Q} \otimes \tilde{W}$

$$\mathbf{\Phi} = \sum_{k_1,\dots,k_P=1}^{n-1} \sum_{r_1,\dots,r_Q=1}^{n-1} \mathbf{e}_{k_1} \otimes \dots \otimes \mathbf{e}_{k_P} \otimes \mathbf{f}^{-r_1} \otimes \dots \otimes \mathbf{f}^{-r_Q} \otimes \Phi^{k_1,\dots,k_P}_{-r_1,\dots,-r_Q} =$$
$$= \sum_{\mathbf{k},\mathbf{r}} \mathbf{e}_{\mathbf{k}} \otimes \mathbf{f}^{-\mathbf{r}} \otimes \Phi^{\mathbf{k}}_{-\mathbf{r}}\,,$$

where

$$\Phi^{k_1,k_2,\dots,k_P}_{-r_1,-r_2,\dots,-r_Q} = \Phi^{\mathbf{k}}_{-\mathbf{r}} \in \tilde{\mathcal{W}}\,,$$
$$\mathbf{e}_{\mathbf{k}} = \mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2} \otimes \dots \otimes \mathbf{e}_{k_P} \in \left(\mathcal{V}_+\right)^{\otimes P},$$
$$\mathbf{f}^{-\mathbf{r}} = \mathbf{f}^{-r_1} \otimes \mathbf{f}^{-r_2} \otimes \dots \otimes \mathbf{f}^{-r_Q} \in \left(\mathcal{V}^*_-\right)^{\otimes Q},$$

the assumed shape of the eigenvectors can be written as

$$\mathfrak{B}(\mathbf{v},\mathbf{w}) = \left\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\mathbf{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\mathbf{w}),\, \mathbf{\Phi} \right\rangle.$$

4 Bethe Vectors and Bethe Condition

Our goal is to write the action of the operators $T^n_n(x)$, $T^{-n}_{-n}(x)$, $\tilde{\mathbf{T}}^{(+)}$ and $\tilde{\mathbf{T}}^{(-)}$ on the assumed form of the Bethe vectors using the operators that act only on $\mathbf{\Phi}$. We do not explicitly mention these relations in this paper, even though the Theorems of this part are their consequence.

For $\mathbf{v} = (v_1, v_2, \dots, v_P)$ we introduce a set $\overline{v} = \{v_1, v_2, \dots, v_P\}$, denote

$$\mathbf{v}_k = (v_1,\dots,v_{k-1},v_{k+1},\dots,v_P)\,,\ \ \overline{v}_k = \overline{v} \setminus \{v_k\}\,,$$
$$F(x;\overline{v}) = \prod_{v_k\in\overline{v}} f(x,v_k)\,, \qquad F(\overline{v};x) = \prod_{v_k\in\overline{v}} f(v_k,x)\,.$$

and define

$$\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\mathbf{v};\mathbf{w}) = \widehat{\mathbf{R}}^{(+,-)}_{0;1^*,\dots,Q^*}(x;\mathbf{w})\tilde{\mathbf{T}}^{(+)}_0(x)\widehat{\mathbf{R}}^{(+,+)}_{0;1,\dots,P}(x;\mathbf{v}) =$$
$$= \sum_{i,k=1}^{n-1} \mathbf{E}^k_i \otimes \widehat{T}^i_k(x;\mathbf{v};\mathbf{w})$$
$$\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\mathbf{v};\mathbf{w}) = \widehat{\mathbf{R}}^{(-,-)}_{0;1^*,\dots,Q^*}(x;\mathbf{w})\tilde{\mathbf{T}}^{(-)}_0(x)\widehat{\mathbf{R}}^{(-,+)}_{0;1,\dots,P}(x;\mathbf{v}) =$$
$$= \sum_{i,k=1}^{n-1} \mathbf{E}^{-k}_{-i} \otimes \widehat{T}^{-i}_{-k}(x;\mathbf{v};\mathbf{w})\,,$$

where

$$
\begin{aligned}
&\widehat{\mathbf{R}}^{(+,+)}_{0;1,\dots,P}(x;\mathbf{v}) = \widehat{\mathbf{R}}^{(+,+)}_{0_+,P_+}(x,v_P)\dots\widehat{\mathbf{R}}^{(+,+)}_{0_+,2_+}(x,v_2)\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(x,v_1)\\
&\widehat{\mathbf{R}}^{(+,-)}_{0;1^*,\dots,Q^*}(x;\mathbf{w}) = \widehat{\mathbf{R}}^{(+,-)}_{0_+,1^*_-}(x,w_1)\widehat{\mathbf{R}}^{(+,-)}_{0_+,2^*_-}(x,w_2)\dots\widehat{\mathbf{R}}^{(+,-)}_{0_+,Q^*_-}(x,w_Q)\\
&\widehat{\mathbf{R}}^{(-,-)}_{0;1^*,\dots,Q^*}(x;\mathbf{w}) = \widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{0_-,2^*_-}(x,w_2)\dots\widehat{\mathbf{R}}^{(-,-)}_{0_-,Q^*_-}(x,w_Q)\\
&\widehat{\mathbf{R}}^{(-,+)}_{0;1,\dots,P}(x;\mathbf{v}) = \widehat{\mathbf{R}}^{(-,+)}_{0_-,P_+}(x,v_P)\dots\widehat{\mathbf{R}}^{(-,+)}_{0_-,2_+}(x,v_2)\widehat{\mathbf{R}}^{(-,+)}_{0_-,1_+}(x,v_1)\\
&\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(x,v) = \frac{1}{f(x,v)}\left(\tilde{\mathbf{I}}_+\otimes\tilde{\mathbf{I}}_+ + g(x,v)\sum_{i,k=1}^{n-1}\mathbf{E}^i_k\otimes\mathbf{E}^k_i\right)\\
&\widehat{\mathbf{R}}^{(+,-)}_{0_+,1^*_-}(x,w) = \tilde{\mathbf{I}}_+\otimes\tilde{\mathbf{I}}^*_- - \tilde{h}(w,x)\sum_{r,s=1}^{n-1}\mathbf{E}^r_s\otimes\mathbf{F}^{-r}_{-s}\\
&\tilde{\mathbf{R}}^{(-,+)}_{0_-,1_+}(x,v) = \tilde{\mathbf{I}}_-\otimes\tilde{\mathbf{I}}_+ - \tilde{h}(x,v)\sum_{i,k=1}^{n-1}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^i_k\\
&\widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(x,w) = \frac{1}{f(w,x)}\left(\tilde{\mathbf{I}}_{0_-}\otimes\tilde{\mathbf{I}}^*_{1_-} + g(w,x)\sum_{r,s=1}^{n-1}\mathbf{E}^{-r}_{-s}\otimes\mathbf{F}^{-s}_{-r}\right).
\end{aligned}
$$

The main results of this paper are the following three Theorems.

Theorem 2 *Let* $\boldsymbol{\Phi} = \sum_{\mathbf{k},\mathbf{r}}\mathbf{e}_{\mathbf{k}}\otimes\mathbf{f}^{-\mathbf{r}}\otimes\Phi^{\mathbf{k}}_{-\mathbf{r}}$, *where* $\Phi^{\mathbf{k}}_{-\mathbf{r}}\in\tilde{\mathcal{W}}$ *is a common eigenvector of the operators*

$$
\begin{aligned}
\widehat{H}^{(+)}(x;\mathbf{v};\mathbf{w}) &= \mathrm{Tr}_0\left(\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\mathbf{v};\mathbf{w})\right) = \sum_{i=1}^{n-1}\widehat{T}^i_i(x;\mathbf{v};\mathbf{w})\\
\widehat{H}^{(-)}(x;\mathbf{v};\mathbf{w}) &= \mathrm{Tr}_0\left(\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\mathbf{v};\mathbf{w})\right) = \sum_{i=1}^{n-1}\widehat{T}^{-i}_{-i}(x;\mathbf{v};\mathbf{w})
\end{aligned}
$$

with eigenvalues $\mu^{(+)}(x;\mathbf{v};\mathbf{w})$ *and* $\mu^{(-)}(x;\mathbf{v};\mathbf{w})$. *If the Bethe conditions*

$$
\begin{aligned}
&\lambda_n(v_\ell)F(\overline{v}_\ell;v_\ell)F(\overline{w};v_\ell-n+1+\eta) = \mu^{(+)}(v_\ell;\mathbf{v};\mathbf{w})F(v_\ell;\overline{v}_\ell)\\
&\lambda_{-n}(w_s)F(w_s;\overline{w}_s)F(w_s+n-1-\eta;\overline{v}) = \mu^{(-)}(w_s;\mathbf{v};\mathbf{w})F(\overline{w}_s;w_s)
\end{aligned}
\tag{5}
$$

are fulfilled for any $v_\ell\in\overline{v}$ *and* $w_s\in\overline{w}$, *the vector*

$$\mathfrak{B}(\mathbf{v};\mathbf{w}) = \left\langle\mathbf{b}^{(+)}_{1^*,\dots,P^*}(\mathbf{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\mathbf{w}),\boldsymbol{\Phi}\right\rangle$$

is a common eigenvector of the operators $H^{(+)}(x)$ *and* $H^{(-)}(x)$ *with eigenvalues*

$$
\begin{aligned}
E^{(+)}(x;\mathbf{v};\mathbf{w}) &= \lambda_n(x)F(\overline{v};x)F(\overline{w};x-n+1+\eta) + \mu^{(+)}(x;\mathbf{v},\mathbf{w})F(x;\overline{v})\\
E^{(-)}(x;\mathbf{v};\mathbf{w}) &= \lambda_{-n}(x)F(x;\overline{w})F(x+n-1-\eta;\overline{v}) + \mu^{(-)}(x;\mathbf{v};\mathbf{w})F(\overline{w};x)\,.
\end{aligned}
$$

Theorem 3 *The operators* $\widehat{T}^i_k(x;\mathbf{v};\mathbf{w})$ *and* $\widehat{T}^{-i}_{-k}(x;\mathbf{v};\mathbf{w})$ *are for any* $\mathbf{v}$ *and* $\mathbf{w}$ *generators of the RTT–algebra of* $\mathcal{A}_{n-1}$ *type.*

The following Theorem shows that

$$\widehat{\boldsymbol{\Omega}} = \underbrace{\mathbf{e}_{n-1}\otimes\dots\otimes\mathbf{e}_{n-1}}_{P\times}\otimes\underbrace{\mathbf{f}^{-n+1}\otimes\dots\otimes\mathbf{f}^{-n+1}}_{Q\times}\otimes\omega$$

is a vacuum vector for the representation of the RTT–algebra $\mathcal{A}_{n-1}$, which is generated by $\widehat{T}^i_k(x;\mathbf{v};\mathbf{w})$ and $\widehat{T}^{-i}_{-k}(x;\mathbf{v};\mathbf{w})$.

Theorem 4 *For the vector* $\widehat{\boldsymbol{\Omega}}$ *and* $i,\ k=1,\ \ldots,\ n-1$

$$\begin{array}{ll} \widehat{T}^i_k(x;\mathbf{v},\mathbf{w})\widehat{\boldsymbol{\Omega}}=0 \quad \text{for} \quad i<k\,, & \widehat{T}^{-i}_{-k}(x;\mathbf{v};\mathbf{w})\widehat{\boldsymbol{\Omega}}=0 \quad \text{for} \quad k<i \\ \widehat{T}^i_i(x;\mathbf{v};\mathbf{w})\widehat{\boldsymbol{\Omega}}=\nu_i(x;\mathbf{v};\mathbf{w})\widehat{\boldsymbol{\Omega}} & \widehat{T}^{-i}_{-i}(x;\mathbf{v};\mathbf{w})\widehat{\boldsymbol{\Omega}}=\nu_{-i}(x;\mathbf{v};\mathbf{w})\widehat{\boldsymbol{\Omega}} \end{array}$$

where

$$\begin{array}{ll} \nu_i(x;\mathbf{v};\mathbf{w})=\lambda_i(x)F(\overline{v};x+1) & \text{for } 1\le i<n-1 \\ \nu_{n-1}(x;\mathbf{v};\mathbf{w})=\lambda_{n-1}(x)F(x-n+1+\eta;\overline{w}) & \\ \nu_{-i}(x;\mathbf{v};\mathbf{w})=\lambda_{-i}(x)F(x-1;\overline{w}) & \text{for } 1\le i<n-1 \\ \nu_{-n+1}(x;\mathbf{v};\mathbf{w})=\lambda_{-n+1}(x)F(\overline{v};x+n-1-\eta) & \end{array}$$

are valid.

These three theorems show that to find the Bethe vectors $\mathfrak{B}(\mathbf{v};\mathbf{w})$ for the RTT–algebra $\mathcal{A}_n$, it is sufficient to find the Bethe vectors for the RTT–algebra $\mathcal{A}_{n-1}$ that is generated by the operators $\widehat{T}^i_k(x;\mathbf{v};\mathbf{w})$, $\widehat{T}^{-i}_{-k}(x;\mathbf{v};\mathbf{w})$, where $i,k=1,\ldots,n-1$, and that has a vacuum vector $\widehat{\boldsymbol{\Omega}}$.

5 Conclusion

The paper describes the construction of eigenvectors for the representations of the RTT–algebra $\mathcal{A}_n$ by using the highest weight vectors for the representation of the RTT–algebra $\mathcal{A}_{n-1}$. We meet these RTT–algebras [1, 2] while studying the algebraic Bethe Ansatz for the RTT–algebras of sp$(2n)$ and o$(2n)$ types.

In the special cases, when $\overline{v}$ or $\overline{w}$ is an empty set, our construction is known as the algebraic nested Bethe Ansatz, which was formulated in [4]. So our construction of the Bethe vectors is a generalization of the algebraic nested Bethe Ansatz to the RTT–algebra of $\mathcal{A}_n$ type.

For the RTT–algebra of $\mathcal{A}_2$ type we get from Theorems 2–4 the Bethe vectors

$$\mathfrak{B}_2(\mathbf{v};\mathbf{w})=T^2_1(\mathbf{v})T^{-1}_{-2}(\mathbf{w})\omega$$

and the Bethe conditions

$$\begin{array}{l} \lambda_2(v_\ell)F(\overline{v}_\ell;v_\ell)F(\overline{w};v_\ell-1+\eta)=\lambda_1(v_\ell)F(v_\ell-1+\eta;\overline{w})F(v_\ell;\overline{v}_\ell) \\ \lambda_{-2}(w_s)F(w_s;\overline{w}_s)F(w_s+1-\eta;\overline{v})=\lambda_{-1}(w_s)F(\overline{v};w_s+1-\eta)\,F(\overline{w}_s;w_s), \end{array}$$

which we found for this algebra and $\nu=-1$ in [3].

For higher n it is possible by means of Theorems 2–4 to step-by-step decrease the value of n and thus obtain an explicit form of the Bethe vectors. For the RTT–algebra of gl(n) type this procedure leads to trace-formula [5]. We intend to publish a similar explicit form of the Bethe vectors for the RTT–algebras $\mathcal{A}_n$, of sp$(2n)$ and o$(2n)$ types in the near future.

References

1. Č Burdík, O. Navrátil, Phys. Part. Nucl. **69**, 936–942 (2018)
2. Č Burdík, O. Navrátil, Phys. Atom. Nucl. **81**, 810–814 (2018)
3. Č Burdík, O. Navrátil, Theor. Math. Phys. **198**, 1–16 (2019)
4. P.P. Kulish, N. Yu. Reshetikhin, J. Phys. A. **16**, L591–L596 (1983)
5. E. Mukhin, V. Tarasov, A. Varchenko, J. Stat. Mech. **8**, P08002 (2006); preprint: math.QA/0605015

Lie Reductions and Exact Solutions of Generalized Kawahara Equations

Olena Vaneeva, Olena Magda, and Alexander Zhalij

Abstract We complete the classical Lie symmetry analysis of a class of generalized Kawahara equations with time dependent coefficients by classification of Lie reductions of equations from this class. Some exact Lie-invariant solutions are also constructed.

Keywords Kawahara equation · Lie symmetry · Lie reduction · Invariant solution · Exact solution · Group analysis

1 Introduction

The classical Kawahara equation appeared in the literature as early as in 1972 as model in solitary wave theory of the form

$$u_t + \alpha u u_x + \beta u_{xxx} + \sigma u_{xxxxx} = 0,$$

where α was fixed as $3/2$, while β and σ are nonzero constants representing effect of dispersion [3]. Later a number of generalizations of Kawahara equations were proposed (see introduction in [4] for literature overview). To the best of our knowledge Kawahara equations with three time dependent coefficients firstly were considered

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_28

in [2] though the complete Lie symmetry classification for such equations was not achieved therein.

The most general class of generalized Kawahara equations with time dependent coefficients has the form

$$u_t + \alpha(t)f(u)u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \quad f_u\alpha\beta\sigma \neq 0, \tag{1}$$

where f, α, β and σ are smooth nonvanishing functions of their variables. Some Lie symmetries and conservation laws of such equations were found in [1]. Transformation properties of the class (1) were exhaustively investigated in our work [10]. It was proved that this class is not normalized but can be presented as the union of two disjoint normalized subclasses which are singled out by the conditions $f_{uu} \neq 0$ and $f_{uu} = 0$ (the rigorous theory on the transformation properties of classes of DEs and the related notions can be found in [9]). The respective extended generalized equivalence groups were constructed. It was shown that the optimal gauging of arbitrary elements is the gauging $\alpha = 1$ and it is realized by the family of point transformations $\tilde{t} = \int_{t_0}^{t}\alpha(y)\,\mathrm{d}y$, $\tilde{x} = x$, $\tilde{u} = u$, from the equivalence group of the class. Without loss of generality we can restrict ourselves by the investigation of the class

$$u_t + f(u)u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \quad f_u\beta\sigma \neq 0, \tag{2}$$

instead of its superclass (1). The complete Lie symmetry classification of equations (2) (resp. (1)) was carried out in [10, Sect. 4]. In order to finalize the classical group analysis of the generalized Kawahara equations with time dependent coefficients in this paper we present the classification of their Lie reductions and construct some exact solutions.

2 Lie Reductions and Exact Solutions

The reduction method with respect to subalgebras of Lie invariance algebras is algorithmic and well-known [5]. As equations from class (2) are $(1 + 1)$-dimensional nonlinear partial differential equations, Lie reductions of them with respect to one-dimensional subalgebras of their maximal Lie invariance algebras will lead to ordinary differential equations (ODEs). In order to get inequivalent reductions one should use subalgebras from the so called *optimal system* (see Sect. 3.3 in [5]).

Consider the general form of Lie symmetry operator $Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$, which forms a basis of the respective one-dimensional subalgebra from the constructed optimal system, then the Ansatz is found as a solution of the invariant surface condition $Q[u] := \tau u_t + \xi u_x - \eta = 0$. In practice, the corresponding characteristic system $\frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}u}{\eta}$ should be solved.

The kernel of the maximal Lie invariance algebras of equations from the class (2) with $f_{uu} \neq 0$ coincides with the one-dimensional algebra $\langle\partial_x\rangle$. Consider one by one all the inequivalent equations from the class (2) which admit Lie symmetry extensions (numbers of cases correspond to those presented in [10, Table 1]. Together with each equation L_i from the class (2) we list its maximal Lie invariance algebra $A^{\max}$, the type of $A^{\max}$ according to the classification of low-dimensional algebras presented in [6], the optimal system (OS) of one-dimensional subalgebras, Ansatze and the respective invariant variable ω, and finally the reduced ordinary differential equation RL_i. In each case we skip reductions with respect to the subalgebra $\langle \mathfrak{g}_0 = \partial_x \rangle$ as they lead to trivial constant solutions only. Below a, ρ and C are arbitrary constants, λ and δ are nonzero constants and $\varepsilon = \{-1, 0, 1\}$.

Case 1. $L_1\colon u_t + f(u)u_x + \lambda t^2 u_{xxx} + \delta t^4 u_{xxxxx} = 0$,

$A^{\max} = \langle \partial_x,\ t\partial_t + x\partial_x \rangle$ (non-Abelian algebra A_2),

OS: $\{\langle \mathfrak{g}_0 = \partial_x \rangle, \langle \mathfrak{g}_1 = t\partial_t + x\partial_x \rangle\}$,

$\mathfrak{g}_1\colon$ Ansatz is $u = \varphi(\omega)$, where $\omega = \dfrac{x}{t}$,

$RL_1\colon \delta\varphi''''' + \lambda\varphi''' + f(\varphi)\varphi' - \omega\varphi' = 0.$

Case 2. $L_2\colon u_t + f(u)u_x + \lambda u_{xxx} + \delta u_{xxxxx} = 0$,

$A^{\max} = \langle \partial_x,\ \partial_t \rangle$ (Abelian algebra $2A_1$),

OS: $\{\langle \mathfrak{g}_0 = \partial_x \rangle, \langle \mathfrak{g}_2 = \partial_t + a\partial_x \rangle\}$,

$\mathfrak{g}_2\colon$ Ansatz is $u = \varphi(\omega)$, where $\omega = x - at$,

$RL_2\colon \delta\varphi''''' + \lambda\varphi''' + f(\varphi)\varphi' - a\varphi' = 0.$

Case 3. $L_3\colon u_t + \ln(u)u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0$,

$A^{\max} = \langle \partial_x,\ t\partial_x + u\partial_u \rangle$ (Abelian algebra $2A_1$),

OS: $\{\langle \mathfrak{g}_0 = \partial_x \rangle, \langle \mathfrak{g}_3 = (t + a)\partial_x + u\partial_u \rangle\}$,

$\mathfrak{g}_3\colon$ Ansatz is $u = e^{\frac{x}{t+a}}\varphi(\omega)$, where $\omega = t$,

$RL_3\colon \varphi' + \dfrac{\varphi \ln\varphi}{\omega + a} + \dfrac{\beta(\omega)\varphi}{(\omega + a)^3} + \dfrac{\sigma(\omega)\varphi}{(\omega + a)^5} = 0.$

This ODE can be integrated, $\varphi = \exp\left(\dfrac{C - \int \frac{\beta(\omega)}{(\omega+a)^2}\mathrm{d}\omega - \int \frac{\sigma(\omega)}{(\omega+a)^4}\mathrm{d}\omega}{\omega + a}\right)$. The respective exact solution of the equation L_3 is

$$u = \exp\left(\frac{x + C - \int \frac{\beta(t)}{(t+a)^2}\mathrm{d}t - \int \frac{\sigma(t)}{(t+a)^4}\mathrm{d}t}{t + a}\right).$$

Case 4. $L_4\colon u_t + \ln(u)u_x + \lambda t^2 u_{xxx} + \delta t^4 u_{xxxxx} = 0$,

$A^{\max} = \langle \partial_x,\ t\partial_x + u\partial_u,\ t\partial_t + x\partial_x \rangle$ (algebra of the type $A_1 \oplus A_2$),

OS: $\{\langle \mathfrak{g}_0 = \partial_x \rangle, \langle \mathfrak{g}_4^a = t\partial_t + (x + at)\partial_x + au\partial_u \rangle, \langle \mathfrak{g}_4^\varepsilon = (t + \varepsilon)\partial_x + u\partial_u \rangle\}$,

$\mathfrak{g}_4^a$: Ansatz is $u = t^a\varphi(\omega)$, where $\omega = \dfrac{x}{t} - a\ln t$,

$RL_{4,a}$: $\delta\varphi''''' + \lambda\varphi''' + (\ln\varphi - \omega - a)\varphi' + a\varphi = 0$;

$\mathfrak{g}_4^\varepsilon$: Ansatz is $u = e^{\frac{x}{t+\varepsilon}}\varphi(\omega)$, where $\omega = t$,

$RL_{4,\varepsilon}$: $\varphi' + \dfrac{\varphi\ln\varphi}{\omega+\varepsilon} + \dfrac{\lambda\omega^2\varphi}{(\omega+\varepsilon)^3} + \dfrac{\delta\omega^4\varphi}{(\omega+\varepsilon)^5} = 0.$

The equation $RL_{4,\varepsilon}$ is a particular case of the equation RL_3 and can be integrated. The respective exact solution of the equation L_4 is

$$u = \exp\left(\frac{x + C + \lambda\left(\frac{\varepsilon^2}{t+\varepsilon} + 2\varepsilon\ln(t+\varepsilon) - t\right) + \delta\left(\frac{\varepsilon^2}{3}\,\frac{18t^2+30\varepsilon t+13\varepsilon^2}{(t+\varepsilon)^3} + 4\varepsilon\ln(t+\varepsilon) - t\right)}{t+\varepsilon}\right).$$

Case 5. L_5: $u_t + \ln(u)u_x + \lambda u_{xxx} + \delta u_{xxxxx} = 0$,

$A^{\max} = \langle\partial_x,\ t\partial_x + u\partial_u,\ \partial_t\rangle$ (the Weyl algebra $A_{3.1}$)

OS: $\{\langle\mathfrak{g}_0 = \partial_x\rangle, \langle\mathfrak{g}_5 = \partial_t\rangle, \langle\mathfrak{g}_5^a = a\partial_t + t\partial_x + u\partial_u\rangle\}$,

$\mathfrak{g}_5$: Ansatz is $u = \varphi(\omega)$, where $\omega = x$,

RL_5: $\delta\varphi''''' + \lambda\varphi''' + \ln\varphi\varphi' = 0$;

$\mathfrak{g}_5^a$: Ansatz is $u = e^{\frac{t}{a}}\varphi(\omega)$, where $\omega = x - \frac{t^2}{2a}$, $a \neq 0$,

$RL_{5,a}$: $\delta\varphi''''' + \lambda\varphi''' + \ln\varphi\varphi' + \frac{1}{a}\varphi = 0$;

$\mathfrak{g}_5^0$: Ansatz is $u = e^{\frac{x}{t}}\varphi(\omega)$, where $\omega = t$,

$RL_{5,0}$: $\varphi' + \dfrac{\varphi\ln\varphi}{\omega} + \dfrac{\lambda\varphi}{\omega^3} + \dfrac{\delta\varphi}{\omega^5} = 0.$

The solution $\varphi = \exp\left(\frac{C\omega^3+3\lambda\omega^2+\delta}{3\omega^4}\right)$ of the latter equation leads to the following exact solution of the equation L_5

$$u = \exp\left(\frac{(3x+C)t^3 + 3\lambda t^2 + \delta}{3t^4}\right).$$

It should be noted that the reductions performed with respect to the subalgebras g_4^ε and g_5^0 are particular cases of the reduction with respect to g_3.

Case 6.1. $L_{6.1}$: $u_t + u^n u_x + \lambda t^\rho u_{xxx} + \delta t^{\frac{5\rho+2}{3}} u_{xxxxx} = 0$, $\rho \neq -1$, $n \neq 0$,

$A^{\max} = \langle\partial_x,\ 3nt\partial_t + (\rho+1)nx\partial_x + (\rho-2)u\partial_u\rangle$ (non-Abelian algebra A_2),

OS: $\{\langle\mathfrak{g}_0 = \partial_x\rangle, \langle\mathfrak{g}_{6.1} = 3nt\partial_t + (\rho+1)nx\partial_x + (\rho-2)u\partial_u\rangle\}$,

$\mathfrak{g}_{6.1}$: Ansatz is $u = t^{\frac{\rho-2}{3n}}\varphi(\omega)$, where $\omega = xt^{-\frac{\rho+1}{3}}$,

$RL_{6.1}\colon \delta\varphi''''' + \lambda\varphi''' + \varphi^n\varphi' - \frac{\rho+1}{3}\omega\varphi' + \frac{\rho-2}{3n}\varphi = 0.$

Case 6.2. $L_{6.2}\colon u_t + u^n u_x + \lambda t^{-1}u_{xxx} + \delta t^{-1}u_{xxxxx} = 0,\ n \neq 0,$

$A^{\max} = \langle \partial_x,\ nt\partial_t - u\partial_u\rangle$ (Abelian algebra $2A_1$),

OS: $\{\langle \mathfrak{g}_0 = \partial_x\rangle, \langle \mathfrak{g}_{6.2} = nt\partial_t + a\partial_x - u\partial_u\rangle\}$,

$\mathfrak{g}_{6.2}\colon$ Ansatz is $u = t^{-\frac{1}{n}}\varphi(\omega)$, where $\omega = x - \frac{a}{n}\ln t$,

$RL_{6.2}\colon \delta\varphi''''' + \lambda\varphi''' + \varphi^n\varphi' - \frac{a}{n}\varphi' - \frac{1}{n}\varphi = 0.$

Case 7. $L_7\colon u_t + u^n u_x + \lambda e^t u_{xxx} + \delta e^{\frac{5}{3}t}u_{xxxxx} = 0,\ n \neq 0,$

$A^{\max} = \langle \partial_x,\ 3n\partial_t + nx\partial_x + u\partial_u\rangle$ (non-Abelian algebra A_2),

OS: $\{\langle \mathfrak{g}_0 = \partial_x\rangle, \langle \mathfrak{g}_7 = 3n\partial_t + nx\partial_x + u\partial_u\rangle\}$,

$\mathfrak{g}_7\colon$ Ansatz is $u = e^{\frac{t}{3n}}\varphi(\omega)$, where $\omega = xe^{-\frac{t}{3}}$,

$RL_7\colon \delta\varphi''''' + \lambda\varphi''' + \varphi^n\varphi' - \frac{1}{3}\omega\varphi' + \frac{1}{3n}\varphi = 0.$

Case 8.1. $L_{8.1}\colon u_t + e^u u_x + \lambda t^\rho u_{xxx} + \delta t^{\frac{5\rho+2}{3}}u_{xxxxx} = 0,\ \rho \neq -1,$

$A^{\max} = \langle \partial_x,\ 3t\partial_t + (\rho+1)x\partial_x + (\rho-2)\partial_u\rangle$ (non-Abelian algebra A_2),

OS: $\{\langle \mathfrak{g}_0 = \partial_x\rangle, \langle \mathfrak{g}_{8.1} = 3t\partial_t + (\rho+1)x\partial_x + (\rho-2)\partial_u\rangle\}$,

$\mathfrak{g}_{8.1}\colon$ Ansatz is $u = \varphi(\omega) + \frac{\rho-2}{3}\ln t$, where $\omega = xt^{-\frac{\rho+1}{3}}$,

$RL_{8.1}\colon \delta\varphi''''' + \lambda\varphi''' + e^\varphi\varphi' - \frac{\rho+1}{3}\omega\varphi' + \frac{\rho-2}{3} = 0.$

Case 8.2. $L_{8.2}\colon u_t + e^u u_x + \lambda t^{-1}u_{xxx} + \delta t^{-1}u_{xxxxx} = 0,$

$A^{\max} = \langle \partial_x,\ t\partial_t - \partial_u\rangle$ (Abelian algebra $2A_1$),

OS: $\{\langle \mathfrak{g}_0 = \partial_x\rangle, \langle \mathfrak{g}_{8.2} = t\partial_t + a\partial_x - \partial_u\rangle\}$,

$\mathfrak{g}_{8.2}\colon$ Ansatz is $u = \varphi(\omega) - \ln t$, where $\omega = x - a\ln t$,

$RL_{8.2}\colon \delta\varphi''''' + \lambda\varphi''' + e^\varphi\varphi' - a\varphi' - 1 = 0.$

Case 9. $L_9\colon u_t + e^u u_x + \lambda e^t u_{xxx} + \delta e^{\frac{5}{3}t}u_{xxxxx} = 0,$

$A^{\max} = \langle \partial_x,\ 3\partial_t + x\partial_x + \partial_u\rangle$ (non-Abelian algebra A_2),

OS: $\{\langle \mathfrak{g}_0 = \partial_x\rangle, \langle \mathfrak{g}_9 = 3\partial_t + x\partial_x + \partial_u\rangle\}$,

$\mathfrak{g}_9\colon$ Ansatz is $u = \varphi(\omega) + \frac{t}{3}$, where $\omega = xe^{-\frac{t}{3}}$,

$RL_9\colon \delta\varphi''''' + \lambda\varphi''' + e^\varphi\varphi' - \frac{1}{3}\omega\varphi' + \frac{1}{3} = 0.$

The reductions presented in Cases 2, 6.1, 6.2 and 7 are valid also for the case $f(u) = u$ (or $n = 1$). Following [4], where this case was investigated in details, we list three additional inequivalent reductions of the equations

$$u_t + uu_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \quad \beta\sigma \neq 0, \tag{3}$$

for the completeness of the presented results.

The Ansatz $u = \varphi(\omega) + \frac{x}{t+a}$, where $\omega = t$, constructed using the subalgebra $\langle (t + a)\partial_x + \partial_u \rangle$ reduces any equation from the class (3)

to the first order ODE $(\omega + a)\varphi' + \varphi = 0$. Its solution $\varphi = \frac{C}{\omega+a}$ leads to the so-called 'degenerate' solution $u = \frac{x+C}{t+a}$ of Eq. (3).

The Ansatz $u = 2t/a + \varphi(\omega)$, where $\omega = x - t^2/a$, $a \neq 0$, resulted from the subalgebra $\langle a\partial_t + 2t\partial_x + 2\partial_u \rangle$, reduces the equation

$$u_t + uu_x + \lambda u_{xxx} + \delta u_{xxxxx} = 0,$$

to the ODE $\delta\varphi''''' + \lambda\varphi''' + \varphi\varphi' + 2/a = 0$.

The last case concerns the equation

$$u_t + uu_x + \lambda(t^2+1)^{\frac{1}{2}} e^{3\nu \arctan t} u_{xxx} + \delta(t^2+1)^{\frac{3}{2}} e^{5\nu \arctan t} u_{xxxxx} = 0, \tag{4}$$

where ν is an arbitrary constant. The Ansatz $u = \frac{e^{\nu \arctan t}}{\sqrt{t^2+1}}\varphi(\omega) + \frac{xt}{t^2+1}$, where $\omega = \frac{xe^{-\nu \arctan t}}{\sqrt{t^2+1}}$, constructed using the subalgebra $\langle (t^2 + 1)\partial_t + (t + \nu)x\partial_x + (x + (\nu - t)u)\partial_u \rangle$ reduces Eq. (4) to the ODE $\delta\varphi''''' + \lambda\varphi''' + (\varphi - \nu\omega)\varphi' + \nu\varphi + \omega = 0$.

The classification of inequivalent Lie reductions of Eq. (2) (resp. (1)) is completed. The obtained reductions may be used also for solving the related invariant boundary value problems (see [7, 8] for the details).

Acknowledgements OV would like to thank the Organizing Committee of LT-14 and especially Prof. Vladimir Dobrev for the support of her participation. She also acknowledges the financial support provided by NAS of Ukraine within the project 0121U110543.

References

1. M.L. Gandarias, M. Rosa, E. Recio, S. Anco, AIP Conference Proceedings **1836**, 020072, 6 pp (2017)
2. L. Kaur, R.K. Gupta, Math. Meth. Appl. Sci. **36**(5), 584–600 (2013)
3. T. Kawahara, J. Phys. Soc. Jpn. **33**, 260–271 (1972)
4. O. Kuriksha, S. Pošta, O. Vaneeva, J. Phys. A: Math. Theor. **47**, 045201, 19 pp (2014)
5. P.J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd edn. (Springer, New York, 2000)
6. J. Patera, P. Winternitz, J. Math. Phys. **18**, 1449–1455 (1977)
7. O.O. Vaneeva, N.C. Papanicolaou, M.A. Christou, C. Sophocleous, Commun. Nonlinear Sci. Numer. Simulat. **19**(9), 3074–3085 (2014)
8. O.O. Vaneeva, C. Sophocleous, P.G.L. Leach, J. Engrg. Math. **91**(1), 165–176 (2015)
9. O.O. Vaneeva, A. Bihlo, R.O. Popovych, Commun. Nonlinear Sci. Numer. Simulat. **91**, 105419 (2020)
10. O. Vaneeva, O. Magda, A. Zhalij, Equivalence groupoid and enhanced group classification of a class of generalized Kawahara equations, in *Lie Theory and Its Application in Physics*, ed. by V. Dobrev. Springer Proceedings in Mathematics & Statistics, vol. 335 (Springer, 2020), pp. 329–340

Several Exactly Solvable Quantum Mechanical Systems and the SWKB Quantization Condition

Yuta Nasuda

Abstract We study the SWKB quantization conditions for two classes of exactly solvable quantum mechanical systems: one is the system whose eigenfunctions contain the multi-indexed polynomials as the main parts and the other is obtained through the Krein–Adler transformation. We show that the condition equation is always independent of $\hbar$ and the condition is not exactly satisfied but holds with a certain degree of accuracy. These results are based on our previous work [12]. A brief review on exactly solvable quantum mechanics is also presented.

Keywords Supersymmetric quantum mechanics · SWKB quantization condition · Exactly solvable Schrödinger equation · Shape invariance · Multi-indexed Laguerre and Jacobi polynomials · Krein-Adler transformation

1 Introduction

Supersymmetric quantum mechanics (SUSY QM) [4, 19] has been giving insights into the solvability in one-dimensional (1-d) quantum mechanics. It answers why Schrödinger equations with certain class of 1-d potentials are analytically solvable. Also, several exactly solvable (ES) potentials have been constructed within the context of SUSY QM.

In 1985, Comtet et al. proposed a WKB-like quantization condition in the context of SUSY QM [3]. This condition, which is referred to as the SWKB quantization condition, seems to have deep physical insights, for it successfully reproduces the exact bound-state spectra for a certain class of ES systems [6]. Studies on the condition and other classes of ES potentials have been carried out so far [2, 5, 10, 12, 13]. However, the condition does not give exact energy eigenvalues for other classes of ES systems. It was once believed that the condition equation is somehow equiva-

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_29

lent to a sufficient condition for the exact solvability known as the shape invariance (SI) [7]. However, Bougie et al. claimed that a system with SI, whose eigenfunctions are expressed using the exceptional orthogonal polynomials, does not satisfy the condition recently [2].

In literature, one often comes across careless treatment on $\hbar$-dependency. In dealing with $\hbar$-dependency properly, we have come to realize $\hbar$ is removed completely from the SWKB quantization condition [12]. Also, considering the fact that the superpotential depends on $\hbar$ in general, the SWKB quantization condition is not derived from the WKB quantization condition. These mean that the SWKB condition should not be discussed in the context of the semi-classical regime of the quantum systems like WKB formalism. In the WKB approximation, one expands a wave function in powers of $\hbar$. Then, what is the counterpart of $\hbar$ in the SWKB scheme? Our investigation of the SWKB condition will be completed when this "unknown parameter" is identified. As an initial step, we verify the claim by Bougie et al. [2] and examine the SWKB condition for further classes of ES systems to show the breaking of the condition equation might be depicted by the "unknown parameter".

2 Exactly Solvable Quantum Mechanics

We discuss time-independent, 1-d ES Schrödinger equation, or the eigenvalue problem of the Hamiltonian $\mathcal{H}$:

$$\mathcal{H}\psi(x) = \mathcal{E}\psi(x)\ , \quad \mathcal{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\ , \quad x \in (x_1, x_2)\ . \tag{1}$$

Hereafter, we fix $2m = 1$ but retain $\hbar$. The eigenvalue problem is *exactly solvable* (ES), when all the discrete eigenvalues $\{\mathcal{E}_n\}$ and the corresponding eigenfunctions $\{\psi_n(x)\}$ are obtained explicitly: $\mathcal{H}\psi_n(x) = \mathcal{E}_n\psi_n(x)$. We assume that $\mathcal{H}$ is bounded from below so that $\mathcal{H}$ has infinitely or finitely many discrete eigenvalues, and also we set the ground-state eigenvalue to be zero; $0 = \mathcal{E}_0 < \mathcal{E}_1 < \mathcal{E}_2 < \cdots$. Also, the eigenfunctions $\{\psi_n(x)\}$ are orthogonal

$$\int_{x_1}^{x_2} \psi_n^*(x)\psi_m(x)\,\mathrm{d}x = h_n\delta_{nm}\ , \tag{2}$$

where h_n is some constant.

2.1 *SUSY QM and Shape Invariance*

The Hamiltonian we are dealing with, which is denoted by $\mathcal{H}^{[0]}$ in this subsection, can be factorized in the following form:

$$\mathcal{H}^{[0]} \equiv \mathcal{A}^{\dagger}\mathcal{A}\ , \quad \mathcal{A} := \hbar\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\ , \quad \mathcal{A}^{\dagger} = -\hbar\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\ , \tag{3}$$

where $W(x)$ is often referred to as the *superpotential*. The superpotential $W(x)$ can be expressed using the ground-state wave function $\psi_0(x)$;

$$\mathcal{A}\psi_0(x) = 0\ , \qquad \therefore\ W(x) = -\hbar\frac{\mathrm{d}}{\mathrm{d}x}\ln|\psi_0(x)|\ . \tag{4}$$

In the context of SUSY QM, the potential $V(x)$ is formally given by

$$V(x) = W(x)^2 - \hbar\frac{\mathrm{d}W}{\mathrm{d}x} = \hbar^2\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}\ln|\psi_0(x)|\right)^2 + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln|\psi_0(x)|\right]\ . \tag{5}$$

Here, we define a Hamiltonian by changing the order of $\mathcal{A}^{\dagger}$ and $\mathcal{A}$ in $\mathcal{H}^{[0]}$;

$$\mathcal{H}^{[1]} := \mathcal{A}\mathcal{A}^{\dagger}\ . \tag{6}$$

The partner Hamiltonians: $\mathcal{H}^{[0]}$ and $\mathcal{H}^{[1]}$, are shown to be iso-spectral, except for the ground state of $\mathcal{H}^{[0]}$. Moreover, eigenstates of the partner Hamiltonians are related via $\mathcal{A}$ and $\mathcal{A}^{\dagger}$ each other. These are verified through the following *intertwining relations*: $\mathcal{A}\mathcal{H}^{[0]} = \mathcal{H}^{[1]}\mathcal{A}$, $\mathcal{A}^{\dagger}\mathcal{H}^{[1]} = \mathcal{H}^{[0]}\mathcal{A}^{\dagger}$.

Those Hamiltonians usually contain several parameters $\mathbf{a} \equiv (a_1, a_2, \ldots)$, and we write the parameter dependency of the Hamiltonians explicitly as $\mathcal{H}^{[i]} = \mathcal{H}^{[i]}(\mathbf{a})$. When $\mathcal{H}^{[1]}$ are related to $\mathcal{H}^{[0]}$ by

$$\mathcal{H}^{[1]}(\mathbf{a}) = \mathcal{H}^{[0]}(f(\mathbf{a})) + \epsilon(\mathbf{a})\ , \tag{7}$$

with f being some function, e.g., $f(a) = a + 1$, and $\epsilon(\mathbf{a})$ being a constant, the Hamiltonians are said to be *shape invariant* (SI) [7]. The shape invariance (SI) is a sufficient condition of the exact solvability of the Schrödinger equation. Sometimes, Eq. (7) is called the SI transformation. We will give three typical examples and a novel class of potentials with SI in the subsequent subsection.

2.2 *Exactly Solvable Quantum Mechanical Systems*

ES quantum mechanical potentials have been studied since the early days of quantum mechanics. Among them is the factorization method [8], which is now understood in connection with SI. This class of ES potentials are referred to as the *conventional SI systems*. The term "conventional" reflects the fact that these systems were already known in the 1950's.

After that, a number of classes of ES systems have been constructed in relation to the conventional SI systems. A part of those ES quantum mechanical systems

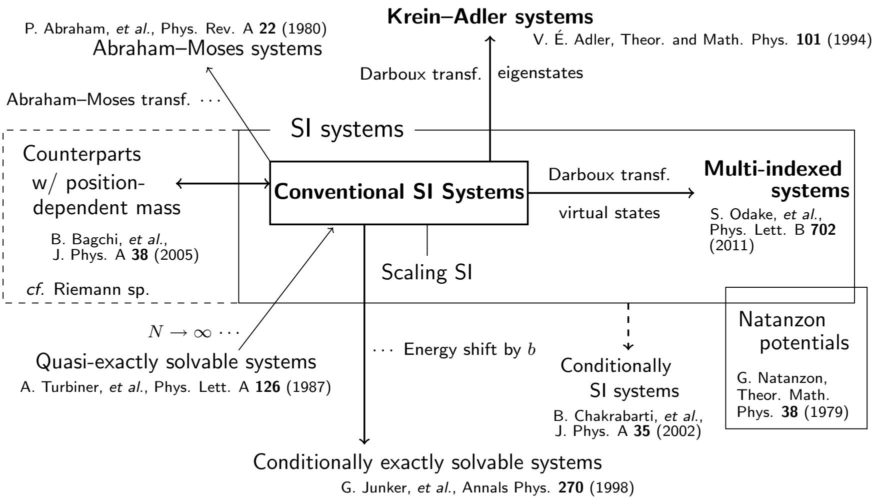

Fig. 1 Several classes of ES quantum mechanical systems in connection with the conventional SI systems. We cover the following three systems in this article: the conventional SI, the multi-indexed and the Krein–Adler systems

(references are given in the figure) and how they are related to the conventional SI systems are shown in Fig. 1. Among them, we discuss the multi-indexed and the Krein–Adler systems with the SWKB quantization condition. As we shall see later in the subsection, these systems are obtained by modifying the conventional SI systems.

In Ref. [2], Bougie et al. dealt with a special case of the multi-indexed system. We carry out further investigation on a general case of this class of ES systems. The Krein–Adler system is another class of ES systems which is constructed through the same transformation as the multi-indexed system. A difference between the two classes are the seed solutions of the transformation. This difference causes a different property on SI of the resulting systems; the multi-indexed system is SI, while the Krein–Adler system is no longer SI. Then, it is quite natural to ask how about the exactness of the SWKB condition, which we shall discuss in Sect. 3.

We first introduce three typical examples of the conventional SI systems, and then by transforming the three, we give the formulae for both the multi-indexed and Krein–Adler systems corresponding to the three conventional SI systems.

Conventional SI systems: The 1-d harmonic oscillator (H), the radial oscillator (L) and the Pöschl–Teller potential (J) are significant examples of the conventional SI systems. The potentials are given by

$$V^{(*)}(x) = \begin{cases} \omega^2 x^2 - \hbar\omega\,, \quad x \in (-\infty, \infty) & * = \mathrm{H}\,, \\ \omega^2 x^2 + \dfrac{\hbar^2 g(g-1)}{x^2} - \hbar\omega(2g+1)\,, \quad x \in (0, \infty) & * = \mathrm{L}\,, \\ \dfrac{\hbar^2 g(g-1)}{\sin^2 x} + \dfrac{\hbar^2 h(h-1)}{\cos^2 x} - \hbar^2(g+h)^2\,, \quad x \in \left(0, \dfrac{\pi}{2}\right) & * = \mathrm{J}\,, \end{cases} \tag{8}$$

where $\omega > 0$ and $g, h > 1/2$. Parameters change during the SI transformations as (H): $\emptyset$, (L): $g \to g+1$ and (J): $(g, h) \to (g+1, h+1)$. The energy eigenvalues and the corresponding wave functions are given as follows:

$$\mathcal{E}_n^{(*)} = \begin{cases} 2n\hbar\omega & * = \mathrm{H}\,, \\ 4n\hbar\omega & * = \mathrm{L}\,, \\ 4\hbar^2 n(n+g+h) & * = \mathrm{J}\,, \end{cases} \tag{9}$$

and

$$\psi_n^{(*)}(x) = \begin{cases} \mathrm{e}^{-\frac{\xi^2}{2}} H_n(\xi) & * = \mathrm{H}\,, \\ \mathrm{e}^{-\frac{z}{2}} z^{\frac{g}{2}} L_n^{(g-\frac{1}{2})}(z) & * = \mathrm{L}\,, \\ (1-y)^{\frac{g}{2}}(1+y)^{\frac{h}{2}} P_n^{(g-\frac{1}{2}, h-\frac{1}{2})}(y) & * = \mathrm{J}\,. \end{cases} \tag{10}$$

For all of them, $n \in \mathbb{Z}_{\geq 0}$. In Eq. (10), H_n, $L_n^{(\alpha)}$, $P_n^{(\alpha,\beta)}$ denote the Hermite, Laguerre, Jacobi polynomials respectively, and $\xi \equiv \sqrt{\omega/\hbar}\, x$, $z \equiv \xi^2$ and $y \equiv \cos 2x$.

Multi-indexed systems: One can obtain another class of SI systems whose eigenfunctions are expressed with the multi-indexed Laguerre and Jacobi polynomials through the multiple Darboux transformation of the conventional SI systems with $* =$ L, J [16]. We employ the virtual-state wave functions $\left\{\{\varphi_n^{(*),\mathrm{I}}(x)\}, \{\varphi_n^{(*),\mathrm{II}}(x)\}\right\}$ as the seed solutions of the transformation, with $n \in \mathcal{D} = \mathcal{D}^{\mathrm{I}} \cup \mathcal{D}^{\mathrm{II}} = \{d_1^{\mathrm{I}}, \ldots, d_M^{\mathrm{I}}\} \cup \{d_1^{\mathrm{II}}, \ldots, d_N^{\mathrm{II}}\}$ where $d_1^{\mathrm{I}} < \cdots < d_M^{\mathrm{I}}, d_1^{\mathrm{II}} < \cdots < d_N^{\mathrm{II}} \in \mathbb{Z}_{>0}$. Note that the special cases $[\mathcal{D}^{\mathrm{I}} = \{\ell\}$ and $\mathcal{D}^{\mathrm{II}} = \emptyset]$ and $[\mathcal{D}^{\mathrm{I}} = \emptyset$ and $\mathcal{D}^{\mathrm{II}} = \{\ell\}]$ are called the type I and the type II X_ℓ-Laguerre/Jacobi system, respectively [14, 15, 17, 18].

The resulting systems are

$$\begin{aligned} \mathcal{H}_{\mathcal{D}}^{(\mathrm{MI},*)} := {} & -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^{(*)}(x) \\ & -2\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \left| \mathrm{W}\left[\varphi_{d_1^{\mathrm{I}}}^{(*),\mathrm{I}}, \ldots, \varphi_{d_M^{\mathrm{I}}}^{(*),\mathrm{I}}, \varphi_{d_1^{\mathrm{II}}}^{(*),\mathrm{II}}, \ldots, \varphi_{d_N^{\mathrm{II}}}^{(*),\mathrm{II}}\right](x) \right|\,, \end{aligned} \tag{11}$$

in which $* =$ L, J and $\mathrm{W}[f_1, \cdots, f_n](x)$ is the Wronskian. The energy eigenvalues and the corresponding wave functions are given as follows:

$$\mathcal{E}_{\mathcal{D};n}^{(\mathrm{MI},*)} = \mathcal{E}_n^{(*)}\,, \tag{12}$$

$$\psi_{\mathcal{D};n}^{(\mathrm{MI},*)}(x) = \frac{\mathrm{W}\left[\varphi_{d_1^{\mathrm{I}}}^{(*),\mathrm{I}}, \ldots, \varphi_{d_M^{\mathrm{I}}}^{(*),\mathrm{I}}, \varphi_{d_1^{\mathrm{II}}}^{(*),\mathrm{II}}, \ldots, \varphi_{d_N^{\mathrm{II}}}^{(*),\mathrm{II}}, \psi_n^{(*)}\right](x)}{\mathrm{W}\left[\varphi_{d_1^{\mathrm{I}}}^{(*),\mathrm{I}}, \ldots, \varphi_{d_M^{\mathrm{I}}}^{(*),\mathrm{I}}, \varphi_{d_1^{\mathrm{II}}}^{(*),\mathrm{II}}, \ldots, \varphi_{d_N^{\mathrm{II}}}^{(*),\mathrm{II}}\right](x)}\,. \tag{13}$$

Krein–Adler systems: The Krein–Adler transformation [1, 11] is a multiple Darboux transformation by choosing several eigenfunctions of the original systems as the seed solutions. The systems obtained through the Krein–Adler transformation of the conventional SI systems (8) are called the Krein–Adler systems, which is not SI any more. To obtain this class of system, one can choose the seed solutions as $\{\psi_n^{(*)}(x)\}$ with $n \in \mathcal{D} = \{d_1, d_1+1 < d_2, d_2+1 < \cdots < d_N, d_N+1\}$ where $d_1, \ldots, d_N \in \mathbb{Z}_{\geq 0}$ and $N \in \mathbb{Z}_{>0}$ in general. This transformation is interpreted as the deletion of the eigenstates of the original systems labeled by $\mathcal{D}$. Here, we restrict ourselves to the case where the number of the seed solutions are two, i.e., $\psi_d^{(*)}(x)$ and $\psi_{d+1}^{(*)}(x)$.

The resulting deformed systems read

$$\mathcal{H}_{\mathcal{D}}^{(\mathrm{KA},*)} := -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^{(*)}(x) - 2\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \left| \mathrm{W}\left[\psi_d^{(*)}, \psi_{d+1}^{(*)}\right](x) \right| , \tag{14}$$

in which $* = \mathrm{H, L, J}$. The energy eigenvalues and the corresponding wave functions are given as follows:

$$\mathcal{E}_{\mathcal{D};n}^{(\mathrm{KA},*)} = \mathcal{E}_{\breve{n}}^{(*)} , \quad \psi_{\mathcal{D};n}^{(\mathrm{KA},*)}(x) = \frac{\mathrm{W}\left[\psi_d^{(*)}, \psi_{d+1}^{(*)}, \psi_{\breve{n}}^{(*)}\right](x)}{\mathrm{W}\left[\psi_d^{(*)}, \psi_{d+1}^{(*)}\right](x)} , \tag{15}$$

where

$$\breve{n} := \begin{cases} n & (0 \leq n \leq d-1) \\ n+2 & (n \geq d) \end{cases} \tag{16}$$

with the number of nodes n.

3 SWKB Quantization Condition

The well-known WKB quantization condition is given by

$$\int_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \sqrt{\mathcal{E}_n - V(x)}\,\mathrm{d}x = \left(n + \frac{1}{2}\right)\pi\hbar , \quad n \in \mathbb{Z}_{\geq 0} , \tag{17}$$

in which x_{L} and x_{R} are the classical turning points; $V(x_{\mathrm{L}}) = V(x_{\mathrm{R}}) = \mathcal{E}_n$. On the other hand, in the context of SUSY QM, a WKB-like quantization condition was proposed by Comtet et al. [3], which reads

$$\int_a^b \sqrt{\mathcal{E}_n - W(x)^2}\,\mathrm{d}x = n\pi\hbar , \quad n \in \mathbb{Z}_{\geq 0} , \tag{18}$$

where a, b are the roots of $W(x)^2 = \mathcal{E}_n$. This condition is called the *SWKB quantization condition* in the literature. In some cases, the equation $W(x)^2 = \mathcal{E}_n$ has more than two roots: $\{(a_i, b_i);\ i = 1, 2, \ldots\}$. We employ a prescription of replacing the l.h.s. of Eq. (18) as follows [12]:

$$\int_a^b \sqrt{\mathcal{E}_n - W(x)^2}\,\mathrm{d}x \quad\to\quad \sum_i \int_{a_i}^{b_i} \sqrt{\mathcal{E}_n - W(x)^2}\,\mathrm{d}x\ . \tag{19}$$

The SWKB condition is exact condition for the ground state of any system by construction. Also, it has been demonstrated that for all conventional SI systems, the SWKB condition reproduces exact bound-state spectra [6]. For other ES potentials, this is not an exact quantization condition [2, 5, 10, 12, 13].

We emphasize here that, the superpotential $W(x)$ depends on $\hbar$ in general, and the SWKB condition is not derived from the WKB quantization condition. It is now almost certain that the SWKB formalism means nothing about semi-classical approximation. Then, what the condition means? Unfortunately, there is still no answer for the question. In order to get closer to the answer, it is still worth examining how the SWKB condition works for several ES systems.

3.1 Multi-indexed Systems and SWKB Condition

For the multi-indexed systems, the condition equation (18) reduces to

$$\mathrm{L}:\quad \int_{a'}^{b'} \sqrt{n - z\left(\frac{\mathrm{d}}{\mathrm{d}z}\ln\left|\psi_{\mathcal{D};0}^{(\mathrm{MI,L})}(x)\right|\right)^2}\,\frac{\mathrm{d}z}{\sqrt{z}} = n\pi\ , \tag{20}$$

$$\mathrm{J}:\quad \int_{a'}^{b'} \sqrt{n(n+g+h) - (1-y^2)\left(\frac{\mathrm{d}}{\mathrm{d}y}\ln\left|\psi_{\mathcal{D};0}^{(\mathrm{MI,J})}(x)\right|\right)^2}\,\frac{\mathrm{d}y}{\sqrt{1-y^2}} = n\pi\ , \tag{21}$$

with a' and b' being the roots of the equations where inside the square roots are put to zero. Note that these formulae depend on g, h, but *are independent of* $\hbar, \omega$.

We calculate the l.h.s. of Eqs. (20) and (21), which is denoted by I, numerically to see the accuracy of the SWKB conditions for the multi-indexed systems. Here, we choose the case of the multi-indexed Laguerre system with $g = 5$ and $\mathcal{D} = \{1\} \cup \{2\}$ as an illustrative example. For more examples, see Ref. [12]. We plot in Fig. 2 the integrals I and the relative errors Err defined by

$$\mathrm{Err} := \frac{I - n\pi}{I}\ , \quad n \in \mathbb{Z}_{>0}\ . \tag{22}$$

For the special case of $n = 0$, where the condition equations (20), (21) are exact, we define $\mathrm{Err} = 0$.

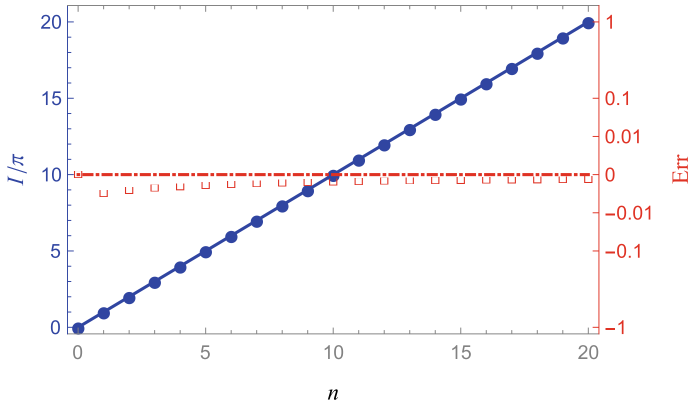

Fig. 2 The accuracy of the SWKB condition for a multi-indexed Laguerre system. The blue dots are the values of the l.h.s. of Eq. (20) and the red squares are the corresponding errors Err, while the blue line and the red chain line mean that the SWKB condition is exact on these lines. We adopt the following rescaling for the plot of the errors: $\mathrm{Err} \to \mathrm{sgn}(\mathrm{Err})2^{\log_{10}|\mathrm{Err}|}$

From Fig. 2, one can immediately see that, except for the ground state, the condition equation (20) is not exactly satisfied. Also, we have replicated the calculation in Ref. [2], which contains problematical treatments of $\hbar$-dependency. Moreover, we have confirmed that the same thing as the case of Laguerre can be said for the case of the multi-indexed Jacobi system [12]. In conclusion, the claim by Bougie et al. still holds after the explicit $\hbar$-dependency is properly taken into account, and it can be applied to the wider class of the multi-indexed systems.

Although the SWKB condition is not an exact quantization condition for this class of potentials, it is notable that the errors are always less than 10^{-2}. As n grows, $|\mathrm{Err}|$ gradually reduces, and in the limit $n \to \infty$, the SWKB condition will be restored. For larger g, the errors are even smaller. Therefore, one can say that the SWKB condition for the multi-indexed systems is not an exact but an *approximate* quantization condition. For further and detailed discussions, see our previous work [12].

3.2 Krein–Adler Systems and SWKB Condition

For the Krein–Adler systems, the condition equation (18) becomes

$$\mathrm{H}: \quad \int_{a'}^{b'} \sqrt{2\breve{n} - \left(\frac{\mathrm{d}}{\mathrm{d}\xi} \ln \left|\psi_{\mathcal{D};0}^{(\mathrm{KA,H})}(x)\right|\right)^2}\, \mathrm{d}\xi = n\pi \ , \tag{23}$$

$$\mathrm{L}: \quad \int_{a'}^{b'} \sqrt{\breve{n} - z\left(\frac{\mathrm{d}}{\mathrm{d}z} \ln \left|\psi_{\mathcal{D};0}^{(\mathrm{KA,L})}(x)\right|\right)^2}\, \frac{\mathrm{d}z}{\sqrt{z}} = n\pi \ , \tag{24}$$

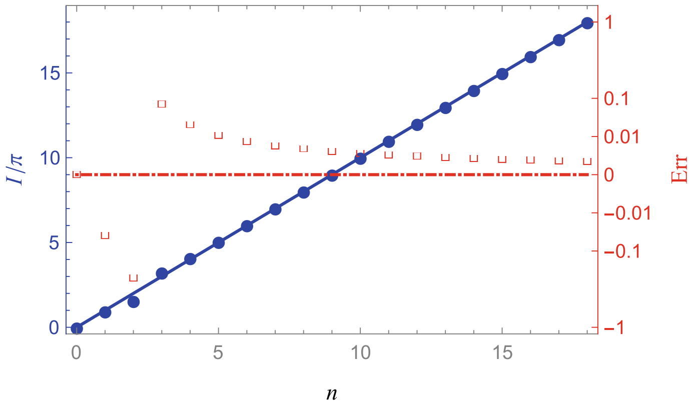

Fig. 3 The accuracy of the SWKB condition for a Krein–Adler Hermite system. The blue dots are the values of the l.h.s. of Eq. (23) and the red squares are the corresponding errors Err, while the blue line and the red chain line mean that the SWKB condition is exact on these lines. We adopt the following rescaling for the plot of the errors: Err $\to$ sgn(Err)$2^{\log_{10}|\mathrm{Err}|}$

$$\mathrm{J}:\quad \int_{a'}^{b'} \sqrt{\breve{n}(\breve{n}+g+h)-(1-y^2)\left(\frac{\mathrm{d}}{\mathrm{d}y}\ln\left|\psi_{\mathcal{D};0}^{(\mathrm{KA,J})}(x)\right|\right)^2}\frac{\mathrm{d}y}{\sqrt{1-y^2}} = n\pi\ , \tag{25}$$

in which a' and b' are the roots of the equations where inside the square roots equal zero. Note that these formulae depend on g, h, but *are totally independent of* $\hbar$, ω, which is just the same as the case of the multi-indexed systems.

We numerically examine the integral of the condition equations (23)–(25), which is denoted by I, to show how accurate they are for the Krein–Adler systems. Our example here is Krein–Adler Hermite system with $\mathcal{D} = \{3, 4\}$, for which we plot the integrals I and the relative error Err (22) in Fig. 3. For more examples, see Ref. [12].

The numerical analysis shows that, except for $n = 0$, the condition equation (23) is not satisfied again. We have also confirmed that the same can be said for the cases of Krein–Adler Laguerre and Jacobi. Moreover, the results indicate similar behaviors. First, the integrals deviate around the deleted levels, i.e., around $n = 2, 3$ or $\breve{n} = 2, 5$ for our current example. The maximal errors are $|\mathrm{Err}| \gtrsim 10^{-1}$. The errors tend to be of opposite sign between the below and the above of the deleted levels. Second, as n goes to infinity, |Err| gets closer to zero and the SWKB condition tends to be exact, which is again the same behavior as the case in the previous subsection. The SWKB condition for the Krein–Adler systems is also an *approximate* condition, except for n's around the deleted levels.

We would like to discuss from our numerical results what guarantees the approximate satisfaction of the SWKB condition. Considering that the Krein–Adler transformation is a deformation of the distribution of energy spectra, and the fact that the maximal errors are seen around the deleted levels $\mathcal{D}$ and |Err| becomes smaller as n steps away from $\mathcal{D}$, it may be the whole distribution of the energy eigenvalues that is responsible for the approximate satisfaction of the SWKB condition. The modifications of the conventional SI systems change the level structures of the systems, and so do the values of the integral of the SWKB condition. This is how the exactness of the condition equation breaks.

We note at the end of this subsection that interesting things happen for larger d. For the details, see our previous work [12].

4 Conclusion

We studied the SWKB conditions for the multi-indexed systems and the Krein–Adler systems. We showed that $\hbar$ is always factored out of the l.h.s. of Eq. (18), hence the condition equation is totally independent of $\hbar$. Then, we numerically computed the integral in the SWKB conditions. The results clearly show that the condition equations are not exactly satisfied, but hold with some degree of accuracy for the systems we studied. As Bougie et al. suggested, the translational SI is not responsible for the exactness of the SWKB condition.

It seems from our analyses that the level structures of the system play an important role in the SWKB formalism. We conjecture that the approximate satisfaction of the SWKB condition is guaranteed by the level structure of the systems. In order to confirm the conjecture, the analyses on ES potentials where the level structures are modified with a continuous parameter [9] might help. We report the results in Ref. [13].

As was mentioned in Sect. 1, we believe that identifying the "unknown parameter" is necessary for a full understanding of the condition. Our works constitute an earlier stage of this resolution. Once we fully understand the SWKB condition, not only we can obtain approximate energy spectra for unsolvable systems, which are often found in context of physical applications, but also we might be able to reach further mathematical implications of the ES quantum mechanical systems. One also may be able to construct novel solvable systems through the SWKB formalism.

Acknowledgements The author would like to thank Ryu Sasaki and Nobuyuki Sawado for their careful and useful advice and comments. We also appreciate Naruhiko Aizawa and Atsushi Nakamula for valuable discussions. Discussions during the YITP workshop YITP-W-20-03 on "Strings and Fields 2020" , and YITP workshop YITP-W-21-04 on "Strings and Fields 2021" have been useful to complete this work.

References

1. V.É. Adler, Theor. Math. Phys. **101**, 1381–1386 (1994)
2. J. Bougie, A. Gangopadhyaya, C. Rasinariu, J. Phys. A: Math. Theor. **51**, 375202 (2018)
3. A. Comtet, A.D. Bandrauk, D.K. Campbell, Phys. Lett. B **150**, 159–162 (1985)
4. F. Cooper, A. Khare, U.P. Sukhatme, Phys. Rept. **251**, 267–385 (1995)
5. D. DeLaney, M.M. Nieto, Phys. Lett. B **247**, 301–308 (1990)
6. R. Dutt, A. Khare, U.P. Sukhatme, Phys. Lett. B **181**, 295–298 (1986)

7. L.E. Gendenshtein, JETP Lett. **38**, 356–358 (1983)
8. L. Infeld, T.E. Hull, Rev. Mod. Phys. **23**, 21–68 (1951)
9. G. Junker, P. Roy, Ann. Phys. **270**, 155–177 (1998)
10. A. Khare, Y. Varshni, Phys. Lett. A **142**, 1–4 (1989)
11. M.G. Krein, Proc. USSR Acad. Sci. **113**, 970–973 (1957)
12. Y. Nasuda, N. Sawado, Mod. Phys. Lett. A **36**, 2150025 (2021)
13. Y. Nasuda, N. Sawado, arXiv:2108.12567
14. S. Odake, R. Sasaki, Phys. Lett. B **679**, 414–417 (2009)
15. S. Odake, R. Sasaki, Phys. Lett. B **684**, 173–176 (2010)
16. S. Odake, R. Sasaki, Phys. Lett. B **702**, 164–170 (2011)
17. C. Quesne, J. Phys. A: Math. Theor. **41**, 392001 (2008)
18. R. Sasaki, S. Tsujimoto, A. Zhedanov, J. Phys. A: Math. Theor. **43**, 315204 (2010)
19. E. Witten, Nucl. Phys. B **188**, 513–554 (1981)

Automorphic Symmetries and AdS_n Integrable Deformations

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Abstract We develop technique based on boost automorphism for finding new lattice integrable models with various dimensions of local Hilbert spaces. We initiate the method by implementing it for two-dimensional models and resolve classification problem, which not only confirms known vertex model solution space, but also extends to the new $\mathfrak{sl}_2$ deformed sector. The generalisation of the approach for string integrable models is provided and allows to find new integrable deformations and associated R-matrices. Hence our new integrable solutions appear to be of non-difference form that admit AdS_2 and AdS_3 S-matrices as special cases, we also obtain embedding of double deformed sigma model R-matrix into our solution. The braiding and crossing for the novel models as well as their emergence with respect to the deformation parameter k are shown. The present contribution is based on series of works [21, 23, 25, 27] and relevant to the questions discussed below.

Keywords Boost automorphism · AdS/CFT integrability · Deformations · Sigma models · R-matrix · Integrable transformations

1 The Method: Automorphic Symmetry

Introduction. An integrable spin chain is characterised by the hierarchy of mutually commuting conserved quantities, *e.g.* charge operators $\mathbb{Q}_2$, $\mathbb{Q}_3$, $\mathbb{Q}_4$, ..., $\mathbb{Q}_r$ where r denotes interaction range. In this respect, one considers local vector space $\mathfrak{H} \simeq \mathbb{C}^n$, $n = 2s + 1$. Then spin-s chain configuration space is given by the L-fold tensor product

$$\text{Complete Fock space:} \quad H = \mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_L \equiv \bigotimes_{i=1}^{L} \mathfrak{H}_i \qquad \mathfrak{H}_i \simeq \mathfrak{H} \tag{1}$$

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_30

We will consider homogeneous periodic integrable spin chains (lattice integrability) with the **N**earest-**N**eighbour interaction (NN). In magnon propagating systems by definition one can obtain an analogue of momentum operator (shifts) $\mathbb{Q}_1 \equiv P$ and 2-site NN charge – Hamiltonian $\mathbb{Q}_2 \equiv H$. Quantum integrability of such a system is now characterised by *quantum* R-matrix that satisfies *quantum* ***Y**ang-**B**axter **E**quation* [1–3, 5]

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v) \tag{2}$$

where R-matrix properties and structure imply underlying integrability of the model. R-matrix is an operator on spectral parameters associated to the local spaces, specifically for a number of integrable classes such dependence is of difference (or additive) form:

$$R_{ab}(u,v) \to R_{ab}(u-v) \tag{3}$$

$\mathcal{B}$**oost operator**. In general, in order to obtain conserved charges, one needs to define R-monodromy and apply log-derivative

$$\mathbb{Q} = \sum_{n=1}^{L} R^{-1}_{n,n+1}(0)\frac{d}{du}R_{n,n+1} \equiv \sum_{n} \mathcal{Q}_{n,n+1} \tag{4}$$

with $\mathcal{Q}$ to denote local densities. Range r local density can be defined as

$$\mathbb{Q}_r \equiv \sum_n \mathcal{Q}_{n,n+1,\ldots,n+r-1} \tag{5}$$

however a method is required to construct all higher charges in the commuting hierarchy. By Master Symmetries it is possible to show that the tower of conserved charges $\mathbb{Q}_r$, $r = 2, 3, 4, \ldots n$ can be recursively generated by the *boost operator* $\mathcal{B}[\mathbb{Q}_2]$ [4, 20]

$$\mathcal{B}[\mathbb{Q}_2] \equiv \sum_{k=-\infty}^{\infty} k\mathcal{H}_{k,k+1} \quad \to \quad \mathbb{Q}_{r+1} \simeq [\mathcal{B}[\mathbb{Q}_2], \mathbb{Q}_r] \tag{6}$$

with Hamiltonian density $\mathcal{H}_{k,k+1}$. Strictly, the boost operator $\mathcal{B}[\cdot]$ is well-defined on *infinite length spin chains*, which contrasts our closed spin chain system. However the $\mathcal{B}$-generated conserved charges and their commutators represent finite range operators, so that such construction can be shown to be consistently implemented on spin chains with periodic conditions.

To address the new models, one can demand a generic ansatz for $\mathcal{H}$ based on an appropriate representation of the underlying symmetry algebra and consistent generator basis [21]. Specifically, starting from $\mathcal{Q}_{ij} = \mathcal{H}_{ij} \equiv A_{\mu\nu}\, \sigma^{\mu} \otimes \sigma^{\nu}$ ansatz and commutation of $\mathbb{Q}_r$ charges with finite range tensor embedding $\sigma^{\mu}_k =$

$\mathbb{1} \otimes \ldots \underbrace{\otimes \sigma^{\mu} \otimes}_{\mathrm{k}} \cdots \otimes \mathbb{1}$, as well as use of symmetry algebra, we can obtain an algebraic system on the $A_{\mu\nu}$ coefficients. In general, $[\mathbb{Q}_r, \mathbb{Q}_s]$ commutator provides $\frac{1}{2}(3^{r+s-1} - 1)$ polynomial equations of degree $r + s - 2$ [13]. To note, that the existence of the first commutator for the studied systems was sufficient to completely resolve for the Hamiltonian. However analytic (recursive) proof of the sufficiency of $[\mathbb{Q}_2, \mathbb{Q}_3] = 0$ is still not present for all integrable sectors [7].

Moreover a set of integrable transformations [21] is needed to reduce the complete solution space to the characteristic generators, which allow to show all distinct integrable classes. The necessary and sufficient transformation symmetries include: Norms and shifts of R, Reparametrisation $R(\mathfrak{f}(u), \mathfrak{f}(v))$, Local Basis Transform $R^{\mathcal{V}}(u, v) = [\mathcal{V}(u) \otimes \mathcal{V}(v)]\, R(u, v)\, [\mathcal{V}(u) \otimes \mathcal{V}(v)]^{-1}$, Discrete Transform of PRP-type, Twisting $[\mathfrak{T}_1(u) \otimes \mathfrak{T}_2(v)]\, R\, [\mathfrak{T}_2(u) \otimes \mathfrak{T}_1(v)]^{-1}$ as a part of the R-matrix symmetry $\left[\mathfrak{T}_{1,2} \otimes \mathfrak{T}_{1,2}, R\right] = 0$.

Bottom-up construction: R-matrix. The next goal is to construct R-matrix to each associated generating Hamiltonian $\mathcal{H}$ (class), for that one needs to consider expansion of the R matrix

$$R = P + P\mathcal{H}u + \sum_{n\geq 2} R^{(n)} u^n \tag{7}$$

where P is permutation operator. If one substitutes R-matrix Ansatz to YBE, one could potentially solve it recursively for the coefficients in the expansion, but in fact, in a number of cases it becomes impossible to identify the right expansion sequence. Instead the *Hamilton-Cayley* theorem argument can be imposed on the R-expansion 7 with the specific set of functional constraints, which led to $R \equiv R\left(\mathbb{1}, \mathcal{H}, \mathcal{H}^2, \mathcal{H}^3\right)$ resolution. With this construction one can find all $\mathbb{C}^2$ integrable R-matrices, however in higher dimensions $\mathbb{C}^n$ and arbitrary spectral dependence a stronger generalised approach for finding the R-matrix is proven.

$\mathfrak{sl}_2$ **sector.** As a quick test of the technique provided above, one can consider models with two-dimensional local space $\mathbb{C}^2$ and generic ansatz for $\mathcal{H}$. It turned not only to show full agreement with the set of integrable models that are found from the YBE resolution (*i.e.* Heisenberg class, *-magnets, multivertex etc), but also find new higher parametric integrable models in the $\mathfrak{sl}_2$ sector. Some of these new classes exhibit *non-diagonalisability* and *nilpotency* of the $\mathcal{H}$, but others develop conserved charges with *non-trivial Jordan blocks*, which leads to important results and corollaries. Some R_X matrices from characteristic X classes include

$$R_1(u) = \begin{pmatrix} 1 & \frac{a_1(e^{a_5 u}-1)}{a_5} & \frac{a_2(1-e^{-a_5 u})}{a_5} & \frac{a_1 a_3 + a_2 a_4}{a_5^2}(\cosh(a_5 u) - 1) \\ 0 & 0 & e^{-a_5 u} & \frac{a_4(1-e^{-a_5 u})}{a_5} \\ 0 & e^{a_5 u} & 0 & \frac{a_3(e^{a_5 u}-1)}{a_5} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

or another extended model, which in the parametric reduction limit admits the *Kulish-Stolin* model[1]

$$R_6(u) = (1 - a_1 u)(1 + 2a_1 u)\begin{pmatrix} 1 & a_2 u & a_2 u & -a_2^2 u^2(2a_1 u + 1) \\ 0 & \frac{2a_1 u}{2a_1 u+1} & \frac{1}{2a_1 u+1} & -a_2 u \\ 0 & \frac{1}{2a_1 u+1} & \frac{2a_1 u}{2a_1 u+1} & -a_2 u \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

It is important to find all underlying Yangian deformations $\mathcal{Y}^*[\mathfrak{sl}_2]$ and associated quantum groups. Also known that Belavin-Drinfeld cohomological classification of quantum symmetries for the novel models is an important question on its own.

2 $\mathcal{B}$ in AdS Integrability

AdS/CFT integrability [8–10, 12] implies agreement of global symmetries on both sides of the correspondence, e.g. $\mathcal{N} = 4$ superconformal symmetry and $AdS_5 \times S^5$ superspace isometries are described by covering supergroup $\widetilde{PSU}(2,2|4)$. The corresponding worldsheet model (σ-models) integrability is based on $\mathfrak{psu}(2,2|4)$ Lie superalgebra and its broken versions. In this setting, the scattering process is described by the S- or R-matrix with arbitrary dependence on the spectral parameter.

$\mathcal{B}$ **generalisation.** To address novel results in AdS string background sector we would need to develop boost automorphism method for operators with generic spectral dependence [24], which intermediately will result in non-additive form of YBE. One will be able to obtain nontrivial constraint system from the commuting tower of the new $\mathbb{Q}_r$ charges

$$\mathcal{B}[\mathbb{Q}_2] = \sum_{k=-\infty}^{+\infty} k\mathcal{H}_{k,k+1}(\theta) + \partial_\theta \qquad \mathbb{Q}_{r+1} = [\mathcal{B}[\mathbb{Q}_2], \mathbb{Q}_r] \quad r > 1 \tag{10}$$

$$\left[\mathbb{Q}_{r+1}, \mathbb{Q}_2\right] \Rightarrow [[\mathcal{B}[\mathbb{Q}_2], \mathbb{Q}_r], \mathbb{Q}_2] + [d_\theta \mathbb{Q}_r, \mathbb{Q}_2] = 0 \tag{11}$$

from here follows the first order nonlinear ODE coupled system. For the trigonometric and elliptic sectors in AdS spin chain picture generic ansatz would be

$$\begin{aligned} \mathcal{H} = & h_1 \mathbb{1} + h_2(\sigma_z \otimes \mathbb{1} - \mathbb{1} \otimes \sigma_z) + h_3 \sigma_+ \otimes \sigma_- + h_4 \sigma_- \otimes \sigma_+ + \\ & h_5(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) + h_6 \sigma_z \otimes \sigma_z + h_7 \sigma_- \otimes \sigma_- + h_8 \sigma_+ \otimes \sigma_+ \end{aligned} \tag{12}$$

with h_k to be a function on the spectral parameter and $\sigma^\pm = \dfrac{1}{2}(\sigma_x \pm i\sigma_y)$.

[1] Known one-parameter family of $\mathfrak{sl}_2$ models, which also could be related to [6]. In the past it was conjectured, that higher-parametric generalisations might exist, however it appeared non-resolvable by RTT-approach, and in our computation we prove their existence and relations.

Sutherland type couple. To obtain R-matrix in such setting, one can expand YBE to the first order, associate spectral parameters along with R-matrix identifications, which will result in coupled differential R-system

$$\begin{cases} [R_{13}R_{23}, \mathcal{H}_{12}(u)] = (\partial_u R_{13})R_{23} - R_{13}(\partial_u R_{23}) & u_1 = u_2 \equiv u \\ [R_{13}R_{12}, \mathcal{H}_{23}(v)] = (\partial_v R_{13})R_{12} - R_{13}(\partial_v R_{12}) & u_2 = u_3 \equiv v \end{cases} \tag{13}$$

which appears sufficient to fix $R_{ij} = R_{ij}(u, v)$ and equations of the system constitute a reduction from the *Sutherland* equation.

6v: Trigonometric. We find two classes out of four, whose R-matrices exhibit completely arbitrary spectral dependence and provide deformations relevant for AdS_3 models (incl. specific worldsheet model deformations).

8v: Elliptic. Also in elliptic sector we find two novel 8-vertex classes

8-vertex A class

$$\begin{aligned} &r_1(u, v) = r_4(u, v) = \mathrm{sn}(\eta + z) && r_3(u, v) = r_2(u, v) = \mathrm{sn}(z) \\ &r_5(u, v) = r_6(u, v) = \mathrm{sn}(\eta) && r_7(u, v) = r_8(u, v) = k\mathrm{sn}(\eta)\mathrm{sn}(z)\mathrm{sn}(\eta + z) \end{aligned}$$

8-vertex B class

$$\begin{aligned} r_1(u, v) &= \frac{1}{\sqrt{\sin\eta(u)\sin\eta(v)}}\left(\sin\eta_+\frac{\mathrm{cn}}{\mathrm{dn}} - \cos\eta_+\mathrm{sn}\right) \\ r_2(u, v) &= \frac{\pm 1}{\sqrt{\sin\eta(u)\sin\eta(v)}}\left(\cos\eta_-\mathrm{sn} + \sin\eta_-\frac{\mathrm{cn}}{\mathrm{dn}}\right) \\ r_3(u, v) &= \frac{\pm 1}{\sqrt{\sin\eta(u)\sin\eta(v)}}\left(\cos\eta_-\mathrm{sn} - \sin\eta_-\frac{\mathrm{cn}}{\mathrm{dn}}\right) \\ r_4(u, v) &= \frac{1}{\sqrt{\sin\eta(u)\sin\eta(v)}}\left(\sin\eta_+\frac{\mathrm{cn}}{\mathrm{dn}} + \cos\eta_+\mathrm{sn}\right) \\ r_5(u, v) &= r_6(u, v) = 1 \\ r_7(u, v) &= r_8(u, v) = k\frac{\mathrm{sn}\,\mathrm{cn}}{\mathrm{dn}} \end{aligned}$$

where r_i agree with 8v positions of (12), the deformation parameter k, arbitrary $\eta(u)$ in $\eta_\pm \equiv \frac{\eta(u)-\eta(v)}{2}$ and Jacobi elliptic functions $\mathrm{xn} = \mathrm{xn}(u - v, k^2)$ to be $\{u, v\}$ dependent. Current class appears in the AdS_2 integrable background.

Free Fermions. Important to note that these classes also satisfy algebraic integrable constraint – *Free fermion* condition [25]. The corresponding characteristic constraint $[r_1r_4 + r_2r_3 - (r_5r_6 + r_7r_8)]^2 \cdot [r_1r_2r_3r_4]^{-1} = \mathcal{K}$ implies Baxter condition for $\mathcal{K} \neq 0$ or Free fermion when $\mathcal{K} = 0$.

3 AdS_2 and AdS_3 Integrable Backgrounds

Completeness of AdS/CFT correspondence requires integrable string backgrounds of distinct dimensionality, which leads to different amounts of preserved supersymmetry and distinct properties of integrable model.

$AdS_3 \times S^3 \times \mathcal{M}_4$. In our case AdS_3/CFT_2 [17] defines $AdS_3 \times S^3 \times \mathcal{M}^4$ background under two geometries that preserve 16 supercharges

$$\begin{cases} \mathcal{M}^4 = T^4, \text{ with } \mathfrak{psu}(1,1|2)^2 \\ \mathcal{M}^4 = S^3 \times S^1, \text{ with } \mathfrak{d}(2,1;\alpha)^2 \sim \mathfrak{d}(2,1;\alpha)_L \oplus \mathfrak{d}(2,1;\alpha)_R \oplus \mathfrak{u}(1) \end{cases}$$

α is related to the relative radii of the spheres. As it was stated, the underlying R-matrix is of trigonometric type and we find novel 6-vertex B type model to constitute same chirality AdS_3 Hamiltonian. It can admit either continuous family of deformations (*spectral functional shifts*) when mapped to *6-vB* or *single-parameter elliptic deformation* when mapped to *8-vB*. In the present setting, we also confirm that $AdS_3 \times S^3 \times T^4$ R-/S-matrix [15, 16] can be obtained from $AdS_3 \times S^3 \times S^3 \times S^1$ [18] by appropriate limits. Importantly, we also show that our *6-vB* model allows to embed the two-parameter q-deformed R-matrix [14] that underlies double deformed Metsaev-Tseytlin model [11].

$AdS_2 \times S^2 \times T^6$. The $AdS_2 \times S^2 \times T^6$ [19] model contains $\frac{PSU(1,1|2)}{SO(1,1)\times SO(2)}$ supercoset, with $\mathbb{Z}_4 \in \mathfrak{psu}(1,1|2)$, which implies classical integrability, but the construction lacks gauge fixing of κ-symmetry. Scattering process on this background can be captured by elliptic deformation of the R-matrix. In this case, the new *8-vB* type, which admits *single-parameter* deformation, represents deformation of the (massive) $AdS_2 \times S^2 \times T^6$.

Crossing symmetry. Generically individual blocks 4×4 obey YBE, Crossing symmetry and Braiding unitarity, however it is necessary to show that the full scattering operator respects them as well.[2] Considering the full boso-fermion R-matrix, the crossing symmetry is satisfied for arbitrary k in both AdS_3 and AdS_2 deformations.

4 Conclusions and Remarks

We have developed a method that allows to find new integrable models and their deformations without direct resolution of the YBE. To achieve that, one is required to use automorphic symmetry to build conserved charges and apply property of the integrable commuting hierarchy. Application of invariant transformations to the solution space provides solution generators and the corresponding R-matrix is found by bottom-up approach from $\mathcal{H}$. That leads to new integrable models or specific extensions and demonstrates universality of the method for periodic or infinite open systems described by a variety of symmetry algebras [22, 27].

[2] 16×16 S-/R-matrix in 2-particle representation that is embedded for $AdS_{\{2,3\}}$ backgrounds.

For the AdS integrability we considered a generic 8-vertex Ansatz of non-difference form and developed boost automorphism technique also for the string integrable sector, which resulted in differential nonlinear ODE problem. Generalised novel models were found, which also admitted embedding of known integrable models in $AdS_{\{2,3\}}$ space [24], and new constructions in AdS_5 . Such models obey all string integrable symmetry constraints and allow for further restrictions on their structure, as well as provide a proposal for the study of higher parametric σ-models, their scattering matrices and quantum limits [27].

Acknowledgements I would like to thank Marius de Leeuw, Alessandro Torrielli, Ana L. Retore, Chiara Paletta and Paul Ryan for their contribution and useful discussions. AP is supported by the Royal Society RGF\EA\180167.

References

1. C.N. Yang, C.P. Yang. One-dimensional chain of anisotropic spin-spin interactions. I. Proof of Bethe's hypothesis for ground state in a finite system. Phys. Rev. **150**(1), 321 (1966)
2. C.N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. Phys. Rev. Lett. **19**(23), 1312 (1967)
3. R.J. Baxter, Partition function of the eight-vertex lattice model. Ann. Phys. **70**(1), 193–228 (1972)
4. M. Tetelman, Lorentz group for two-dimensional integrable lattice systems. Sov. Phys. JETP **55**(2), 306–310 (1982)
5. M. Jimbo, Introduction to the Yang-Baxter equation. Int. J. Mod. Phys. A **4**(15), 3759–3777 (1989)
6. F.C. Alcaraz, M. Droz, M. Henkel, V. Rittenberg, Reaction - diffusion processes, critical dynamics and quantum chains. Ann. Phys. **230**, 250–302 (1994). https://doi.org/10.1006/aphy.1994.1026, [arXiv:hep-th/9302112 [hep-th]]
7. M.P. Grabowski, P. Mathieu, Integrability test for spin chains. J. Phys. A **28**, 4777–4798 (1995). https://doi.org/10.1088/0305-4470/28/17/013, [arXiv:hep-th/9412039 [hep-th]]
8. J.M. Maldacena, The Large N limit of superconformal field theories and supergravity. Adv. Theor. Math. Phys. **2**, 231–252 (1998). https://doi.org/10.1023/A:1026654312961, [arXiv:hep-th/9711200 [hep-th]]
9. I.R. Klebanov, J.M. Maldacena, Superconformal gauge theories and non-critical superstrings. Int. J. Mod. Phys. A **19**, 5003–5016 (2004). https://doi.org/10.1142/S0217751X04020865, [arXiv:hep-th/0409133 [hep-th]]
10. E. Witten, Anti-de Sitter space and holography. Adv. Theor. Math. Phys. **2**, 253–291 (1998). https://doi.org/10.4310/ATMP.1998.v2.n2.a2, [arXiv:hep-th/9802150 [hep-th]]
11. R.R. Metsaev, A.A. Tseytlin, Type IIB superstring action in $AdS_5 \times S^5$ background. Nucl. Phys. B **533**, 109–126 (1998). https://doi.org/10.1016/S0550-3213(98)00570-7, [arXiv:hep-th/9805028 [hep-th]]
12. N. Beisert, C. Ahn, L.F. Alday, Z. Bajnok, J.M. Drummond, L. Freyhult, N. Gromov, R.A. Janik, V. Kazakov, T. Klose, et al., Review of AdS/CFT integrability: an overview. Lett. Math. Phys. **99**, 3–32 (2012). https://doi.org/10.1007/s11005-011-0529-2, [arXiv:1012.3982 [hep-th]]
13. T. Bargheer, N. Beisert, F. Loebbert, Boosting nearest-neighbour to long-range integrable spin chains. J. Stat. Mech. **0811**, L11001 (2008). https://doi.org/10.1088/1742-5468/2008/11/L11001 ([arXiv:0807.5081 [hep-th]].)
14. B. Hoare, Towards a two-parameter q-deformation of $\mathrm{AdS}_3 \times S^3 \times M^4$ superstrings. Nucl. Phys. B **891**, 259–295 (2015). https://doi.org/10.1016/j.nuclphysb.2014.12.012, [arXiv:1411.1266 [hep-th]]

15. R. Borsato, O. Ohlsson Sax, A. Sfondrini, B. Stefański and A. Torrielli, The all-loop integrable spin-chain for strings on $AdS_3 \times S^3 \times T^4$: the massive sector. JHEP **08**, 043 (2013). https://doi.org/10.1007/JHEP08(2013)043, [arXiv:1303.5995 [hep-th]]
16. R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefanski, The complete $AdS_3 \times S^3 \times T^4$ worldsheet S matrix. JHEP **10**, 066 (2014). https://doi.org/10.1007/JHEP10(2014)066, [arXiv:1406.0453 [hep-th]]
17. A. Sfondrini, Towards integrability for AdS_3/CFT_2
18. R. Borsato, O. Ohlsson Sax, A. Sfondrini, B. Stefański, The $AdS_3 \times S^3 \times S^3 \times S^1$ worldsheet S matrix. J. Phys. A **48**(41), 415401 (2015). https://doi.org/10.1088/1751-8113/48/41/415401, [arXiv:1506.00218 [hep-th]]
19. B. Hoare, A. Pittelli, A. Torrielli, S-matrix algebra of the $AdS_2 \times S^2$ superstring. Phys. Rev. D **93**(6), 066006 (2016). https://doi.org/10.1103/PhysRevD.93.066006, [arXiv:1509.07587 [hep-th]]
20. F. Loebbert, Lectures on Yangian symmetry. J. Phys. A **49**(32), 323002 (2016). https://doi.org/10.1088/1751-8113/49/32/323002, [arXiv:1606.02947 [hep-th]]
21. M. De Leeuw, A. Pribytok, P. Ryan, Classifying two-dimensional integrable spin chains. J. Phys. A **52**(50), 505201 (2019). https://doi.org/10.1088/1751-8121/ab529f, [arXiv:1904.12005 [math-ph]]
22. M. De Leeuw, A. Pribytok, A.L. Retore, P. Ryan, New integrable 1D models of superconductivity. J. Phys. A **53**(38), 385201 (2020). https://doi.org/10.1088/1751-8121/aba860, [arXiv:1911.01439 [math-ph]]
23. M. de Leeuw, C. Paletta, A. Pribytok, A.L. Retore, P. Ryan, Yang-Baxter and the Boost: splitting the difference. SciPost Phys. **11**, 069 (2021). https://doi.org/10.21468/SciPostPhys.11.3.069 ([arXiv:2010.11231 [math-ph]].)
24. M. de Leeuw, C. Paletta, A. Pribytok, A.L. Retore, P. Ryan, Classifying nearest-neighbor interactions and deformations of AdS. Phys. Rev. Lett. **125**(3), 031604 (2020). https://doi.org/10.1103/PhysRevLett.125.031604, [arXiv:2003.04332 [hep-th]]
25. M. De Leeuw, C. Paletta, A. Pribytok, A.L. Retore, A. Torrielli, Free Fermions, vertex Hamiltonians, and lower-dimensional AdS/CFT. JHEP **02**, 191 (2021). https://doi.org/10.1007/JHEP02(2021)191 ([arXiv:2011.08217 [hep-th]].)
26. M. de Leeuw, C. Paletta, B. Pozsgay, Constructing integrable lindblad superoperators. Phys. Rev. Lett. **126**(24) (2021). https://doi.org/10.1103/physrevlett.126.240403, [arXiv:2101.08279v2 [cond-mat.stat-mech]]
27. M. de Leeuw, A. Pribytok, A.L. Retore, P. Ryan, Integrable deformations of AdS/CFT. [arXiv:2109.00017 [hep-th]]

Applications to Quantum Theory

The Conformal-Symmetry–Color-Neutrality Connection in Strong Interaction

Mariana Kirchbach, Todor Popov, and Jose Antonio Vallejo

Abstract The color neutrality of hadrons is interpreted as an expression of conformal symmetry of strong interaction, the latter being signaled through the detected "walking" at low transferred momenta, $\lim_{Q^2\to 0}\alpha_s(Q^2)/\pi \to 1$, of the strong coupling toward a fixed value (α_s "freezing"). The fact is that conformal symmetry admits quarks and gluons to reside on the compactified AdS_5 boundary, whose topology is $S^1 \times S^3$, a closed space that can bear exclusively color-charge neutral configurations, precisely as required by color confinement. The compactification radius, once employed as a second scale alongside with Λ_{QCD}, provides for an $\alpha_s(Q^2)$ "freezing" mechanism in the infrared regime of QCD, thus making the conformal-symmetry–color-neutrality connection at low energies evident. In this way, perturbative descriptions of processes in the infrared could acquire meaning. In consequence, it becomes possible to address QCD by quantum mechanics in terms of a conformal wave operator equation, which leads to an efficient description of a wide range of data on hadron spectra, electromagnetic form factors, and phase transitions.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_31

Keywords Conformal symmetry · Color confinement · Perturbative infrared regime · quantum mechanical limit of QCD

1 The Puzzle of Color Confinement

The non-observability of free quarks, spin-$1/2$ matter fields bound to hadrons, is puzzling scientists since the very beginning of their discovery in the 60ies of the past century. Contrary to traditional composite systems, such as molecules, atoms, and nuclei, hadrons can not be decomposed into their constituents through interactions with external probes. This peculiarity of strong interactions is related to the existence of three "strong" charges carried by the quarks, conditionally termed to as "colors", and hypothesized as the fundamental triplet of the gauge group, $SU(3)_c$, of strong interaction, a non-Abelian group giving rise to a highly non-linear dynamics among the messengers of strong interaction, the gluons. In one of the cases the dynamics can be such, that the interaction among the quarks grows with the increase of relative distances among them, frustrates their release, and also keeps the net charge neutral (color confinement). Meanwhile, it systematically weakens with the decrease of the relative distances, thus turning quarks almost non-interacting, though still trapping them in colorless configurations in the interior of hadrons (asymptotic freedom). Over the years, various insights into the mechanisms behind the non-observability of "color" could be gained on the basis of the elaborated fundamental gauge theory of strong interaction, the Quantum Chromodynamics (QCD), though the first principles provoking the color confinement remained so far as open problems. A further open problem of QCD is that contrary to Quantum Electrodynamics, it is still lacking a properly defined quantum mechanical limit. To progress in that regard, some fundamental symmetry principles should be emphasized, and employed in the construction of a quantum mechanical wave equation that describes quark systems interacting by a potential, whose magnitude would be entirely determined by the fundamental parameters furnishing QCD.

The symmetry underlying a quantum mechanical interaction problem is always reflected by the quantum numbers of the excited states (the spectrum) of the system under consideration. The spectra of hadrons are dominated, isospin by isospin, by the quantum numbers of the irreducible representations of $SO(4)$, much alike the levels of an electron bound within the Coulomb potential, though in the hadronic case the level splittings are moderately increasing with the energy, while in the H Atom they are notably decreasing. This observation, made by several authors [1–12], hints on a possible relevance of conformal symmetry not only in the electromagnetic but also in the strong interaction sector, at first glance a surprise, given the dependence of the strong coupling, α_s on the (negative) square of the transferred momentum, $(-q^2) := Q^2$. Nonetheless, the facts that (i) at high Q^2 values $\alpha_s(Q^2) \stackrel{Q^2\to\infty}{\longrightarrow} 0$ (asymptotic freedom in the ultraviolet), while (ii) at low Q^2 values $\alpha_s(Q^2)/\pi \stackrel{Q^2\to 0}{\longrightarrow} 1$ (conformal window in the infrared) point to the possibility that the dynamics in the two extreme regimes of QCD, the ultraviolet [13], and the infrared [14], might be governed by the conformal symmetry, though expected to be realized in different fashions. Specifically in the infrared, the role of the conformal symmetry turns out to be a pivotal one. Namely, as it should become clear in due course, it establishes a close link between color-neutrality and the opening of the conformal window. Such occurs upon identifying, in accord with the AdS_5/CFT_4 duality, the conformally compactified boundary of AdS_5, whose topology is $S^1 \times S^3$, as the space hosting the QCD degrees of freedom [15]. This space contains a closed space-like hyper-surface, on which no single charges can exist, so that systems residing on it are necessarily charge neutral (as required by confinement). As long as in addition the isometry of $S^1 \times S^3$ is determined by the conformal group, $SO(2, 4) \supset SO(4)$, the potential obtained from the fundamental solution of the respective Laplacian implements that very symmetry, and can be employed in the design of a conformally symmetric instantaneous potential. In this way, the conformal-symmetry (CS)–color-neutrality (CN) connection in the infrared regime of QCD can be established. It is the purpose of the present contribution to briefly review recent progress on that topic. The text is structured as follows. In the next section we briefly outline the genesis of the CS-CN connection concept in the infrared. In Sect. 3 we discuss the geometric aspects of color confinement and its relationship to conformal symmetry as introduced via the AdS_5/CFT_4 duality. Section 4 is devoted to an alternative interpretation of the CS-CN connection from the perspective of the Jordan algebra $\mathfrak{J}_2^C$, with the aim of hitting the road towards generalization to higher dimensions. The text closes by a concise summary section.

2 Spectroscopic Evidence for Conformal Symmetry of QCD

Besides the freezing of the strong coupling in the infrared, further evidences for the relevance of the conformal symmetry for hadron physics are independently provided by data on hadron scatterings and hadron spectra. In particular, relativistic two-body scattering amplitudes are well described in terms of exchange between the particles of physical entities that transform as irreducible representations of the Poincaré group [4]. Such representations inevitably emerge in the decomposition of the direct products of the representations describing the incoming and outgoing particle states. This relevance of conformal symmetry for hadrons has been noticed already in the early days of Regge's theory by [1–3], and Regge trajectories with $O(4)$ symmetric poles have been considered [5]. Furthermore, a dynamical conformal symmetry approach to the description of hadronic electromagnetic form factors has been developed for example in [6]. More hints come from the hadron spectra, both baryonic and mesonic, whose quantum numbers are markedly dominated by $SO(4)$ irreducible representations, easily recognizable as finite sequences of states consisting of K parity pairs of raising spins, terminating by a parity singlet state of highest spin, with K standing for the value of the four-dimensional angular momentum [7–11]. Specifically in [12] the possible relation of new data on light meson spectra reported by the

Crystal Barrel Collaboration to conformal symmetry of massless QCD has been first indicated. After the 70ies, the conformal symmetry of hadrons has been addressed in the literature sporadically and treated by purely algebraic means, while for the practical purposes of continued data evaluation, potential models based on different Lie symmetries and depending on a large number of free parameters, have been favored. Within this context, it appears important to find a method to design a potential model which implements the conformal symmetry, depends on same parameters as QCD, and which allows for an immediate evaluation of a variety of data. Such a model has been elaborated over the years in the series of articles [8, 16–21]. At first, in [8], a two-parameter empirical mass formula has been suggested which resulted quite adequate for the description of the excitation energies of all light flavor hadrons. This formula reads.

$$M_{\sigma'} - M_\sigma = m_1\left(\frac{1}{\sigma^2} - \frac{1}{\sigma'^2}\right) + \frac{1}{2}m_2\left(\frac{\sigma'^2-1}{2} - \frac{\sigma^2-1}{2}\right), \quad \sigma = K+1. \tag{1}$$

Here, σ plays a róle similar to the principal quantum number of the H Atom and describes the well known $(K+1)^2$-fold degeneracies of states in a level, viewed as $SO(4)$ irreps. The two parameters m_1 and m_2 take the values of $m_2 = 70$ MeV, $m_1 = 600$ MeV for both nucleon and Δ excitations. Later this formula has been reported in [16] to be interpretive in terms of the eigenvalues, $\varepsilon_{K\ell}$, of the trigonometric Rosen-Morse (tRM) potential (here in dimensionless units) and given by,

$$V_{tRM} = \frac{\ell(\ell+1)}{\sin^2\left(\frac{r}{d}\right)} - 2b\cot\left(\frac{r}{d}\right). \tag{2}$$

Here, d so far has been treated as a matching length parameter to the relative distance r, while the V_{tRM} spectrum reads,

$$\varepsilon_{K\ell} = -\frac{b^2}{(K+1)^2} + (K+1)^2, \quad \varepsilon_{K\ell} \sim M_\sigma. \tag{3}$$

Upon resolving the associated Schrödinger equation, be it with a linear, or quadratic energy, a surprising result has been obtained regarding the wave functions, which turned out to express in terms of some real orthogonal polynomials, which have been entirely absent from the standard mathematical physics literature available at that time. Later on, these polynomials have been identified in [17] with the Routh-Romanovski polynomials, scarcely covered by the specialized mathematical literature. Next, the case could be made in [18] that upon a suitable change of variables, the wave equation with V_{tRM} can be transformed to a quantum motion on the three dimensional hyper-sphere, S^3. In so doing, r acquires meaning of the arc length, $r \longrightarrow \widehat{r}$, of a great circle, measured from the North pole of S^3, d becomes the sphere's hyper-radius, R, and (r/d) takes the part of an angular variable χ, identical to the second polar angle parametrizing S^3 in global geodesic coordinates. Within this context, the $\ell(\ell+1)/\sin^2(r/d)$ term in (2) starts playing the part of the centrifugal term on S^3, while the cotangent potential solves the Laplace equation on S^3, much alike as the Coulomb potential solves the Laplace equation in the 3D flat space. As long as the Laplacian shares the symmetry of the isometry group of the manifold on which it acts, $SO(4)$ in our case, so do the solutions of the Laplace equation, which explains the $SO(2,4) \supset SO(4)$ patterns of the V_{tRM} eigenvalues. Finally, in [19] the constraint to neutrality imposed by the hyper-spherical manifold on the total charge of systems placed on it was also addressed. In effect, the conformal-symmetry–color-neutrality connection could be revealed. Moreover, via this connection the solution of the Laplace equation on S^3 could be related to a potential, obtained in [20] from Wilson loops with cusps at the North and South poles on S^3, which allowed us to express the potential magnitude, b, as $2b = \alpha_s N_c$, where N_c stands for the number of colors. In this fashion, the V_{tRM} potential parameters could be directly linked to the fundamental parameters of QCD. In this parametrization, the potential under discussion has been successfully used in the description of a variety of hadron physics phenomena, ranging from meson spectra [19], over nucleon electromagnetic form factors [21], and more recently, to heavy flavored mesons [22], and thermodynamic properties of quantum meson gases [23]. Especially in the latter case, a quantum gas of charmonium was shown to suffer a phase transition to a Bose-Einstein condensate at Hagedorn's value of the critical temperature. However, so far the question on the origin of the curved hyper-spherical manifold had remained pending. As a preliminary hypothesis, it has been seen as the hyper-sphere located at the waist of a four-dimensional hyperboloid, $\mathbf{H}_1^4$, of one sheet, emerging in the time-like foliation of space-time in dS_4 Special Relativity, whose space-like region has been conjectured in [19] as the internal space of hadrons. The aforementioned question has found a more assertive answer in [15], presented in the subsequent section.

3 The Geometric Foundations of the CS–CN Connection

Within the context of contemporary fundamental concepts, p-dimensional space-like spherical manifolds, S^p, appear at the compactified boundaries of $S^1 \times S^p$ anti-de Sitter spaces in $(p+2)$ dimensions [24]. Especially the S^3 sphere of our interest appears at the compactified AdS_5 boundary, a space of fundamental interest to QCD from the perspective of the AdS_5/CFT_4 gauge-gravity duality. In [15] it has been demonstrated in detail that this boundary is topologically equivalent to the compactified Minkowski space time [25]. Also there, the interest in the compactified Minkowski space as an internal space of the strong interaction degrees of freedom has been formulated for the first time. This interest is motivated by the observation that on such spaces, charges are forced to appear in pairs of vanishing total charge, much alike the appearance of color charges in mesons. The argument goes as follows. Be M a compact manifold with finite volume $\mathrm{vol}(M)$, which is equipped by any Riemannian metric g. Then a fundamental solution to the associated Laplacian Δ is any function $G : M \times M \to \mathbb{R}$ satisfying, for any fixed values of $y \in M$, the equation

$$\Delta_x G(x, y) = \delta_y - \frac{1}{\mathrm{vol}(M)} \,. \tag{4}$$

Alternatively, on S^3 another definition of fundamental solutions to the Laplacian can be given, namely, taking a base point and its antipodal at the same time, leads to [26]

$$\Delta G(x, x') = \delta(x, x') - \delta(-x, x') \,, \tag{5}$$

which is nothing but the equation for the well known dipole Green function. For the case of M chosen as S^3, and parametrized by global geodesic hyper-spherical coordinates (χ, θ, φ), one has $\mathrm{vol}(M) = 2\pi^2$, and a direct computation [27] shows that the fundamental solutions at the poles $\chi = 0$, and $\chi = \pi$, are given as,

$$G_0(\chi, \theta, \varphi) = \frac{1}{4\pi^2}(\pi - \chi)\cot\chi \,, \quad G_\pi(\rho, \theta, \chi) = -\frac{1}{4\pi^2}\chi \cot\chi \,, \tag{6}$$

their dipole combination being,

$$G_\pi(\chi, \theta, \varphi) - G_0(\chi, \theta, \varphi) = -\frac{1}{2\pi}\cot\chi \,, \tag{7}$$

and thus leading to the cotangent potential in (2). Therefore, the potential in (2) can be interpreted to be due to the charge neutral configuration of a 2^1 pole, residing on S^3, such as quark-anti-quark. The charge neutrality allows existence on S^3 of any 2^n poles. Furthermore, in employing the compactification radius, R, as another scale along Λ_{QCD}, and upon reparametrizing Q^2c^2 as $\left(Q^2c^2 + \hbar^2c^2/R^2\right)$, this in the spirit of [28], removes the logarithmic divergence of the strong coupling at origin according to,

$$\begin{aligned} \frac{\alpha_s(0)}{\pi} &= \lim_{Q^2 \to 0} 4\left(\beta_0 \ln\left(\frac{Q^2c^2}{\Lambda^2_{QCD}} + \frac{\hbar^2c^2}{R^2\Lambda^2_{QCD}}\right)\right)^{-1} \\ &\to 4\left(\beta_0 \ln\left(\frac{\hbar^2c^2}{R^2\Lambda^2_{QCD}}\right)\right)^{-1} \,. \end{aligned} \tag{8}$$

In this fashion, the conformal-symmetry–color-confinement connection at low energies becomes evident, and avenues open towards perturbative treatments of hadron processes in the infrared. The compactification radius extracted from data on light mesons [19] is $R = 0.58$ fm, while for the charmonium it is $R = 0.56$ fm [22], i.e. it seems maintains an universal meaning far beyond the infrared.

4 The Jordan Algebra View on Conformal Symmetry

So far we have emphasized on the conformal group as isometry of the $S^1 \times S^3$ space. Here we focus on an important aspect of its algebra, $so(2, 4)$, namely, on its property of being patterned after the 3-graded Lie algebra

$$\mathfrak{g}_{+1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \simeq so(2, 4), \tag{9}$$

where the Abelian sub-algebras $\mathfrak{g}_{-1}$, and $\mathfrak{g}_{+1}$ are defined in their turn by the commutators of the components P_μ, of the four-momentum operator, and by the components, K_μ of the operator of special conformal transformations, respectively. The P_μ and K_μ vector spaces are related to each other via the conformal inversion I as, $K^\mu = I P_\mu I := P_\mu^\dagger$, with $I(x^0, \boldsymbol{x}) = \left(\frac{x^0}{x^2}, -\frac{\boldsymbol{x}}{x^2}\right)$ acting as an involution, $I^2 = 1$. Moreover, the commutators of the $\mathfrak{g}_{+1}$ and $\mathfrak{g}_{-1}$ elements are given by, $\frac{1}{2}[K^\mu, P_\nu] = M^\mu{}_\nu - \delta^\mu_\nu D$, and recover the algebra of the Lorentz group, whose generators are $M^\mu{}_\nu$, with an associated to it grading operator, $D \in \mathfrak{g}_0$, which acts as a dilatation, $[D, g] = kg$, for any $g \in \mathfrak{g}_k$, with $k = \pm 1$. This property of the conformal group algebra allows one to link it to Jordan triple systems and pairs of Jordan triples as they appear in the special Jordan algebra, $\mathfrak{J}_2^C$, of the 2×2 Hermitian matrices. As a reminder [29], a special Jordan algebra is a non-associative algebra of a vector space $\mathfrak{J}$ over a field, whose multiplication, $\circ$, satisfies the commutative law, $x \circ y = y \circ x$, and the Jordan identity, $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$. A Jordan triple system (JTS) is a vector space $\mathfrak{J}$ endowed with a Jordan triple product, i.e., a trilinear map $\{\ ,\ ,\ \} : \mathfrak{J} \times \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}$, satisfying the symmetry condition, $\{u, v, w\} = \{w, v, u\}$, together with the identity

$$\{u, v, \{w, x, y\}\} = \{w, x, \{u, v, y\}\} + \{w\{u, v, x\}y\} - \{\{v, y, w\}x, y\}. \tag{10}$$

The latter relation implies that if a map, $S_{u,v} : \mathfrak{J} \to \mathfrak{J}$, is defined by $S_{u,v}(y) = \{u, v, y\}$, then one finds,

$$\left[S_{u,v}, S_{w,x}\right] = S_{w,\{u,v,x\}} - S_{\{v,u,w\},x}, \tag{11}$$

so that the space of the linear maps, $\mathrm{span}\{S_{u,v} : u, v \in V\}$, is closed under a commutator bracket, and hence represents a Lie algebra, $\mathfrak{str}(\mathfrak{J})$, termed to as “structure algebra”. Any Jordan algebra induces a Jordan Triple System when we define the Jordan Triple product through $\{u, v, w\} = u \circ (v \circ w) - v \circ (u \circ w) + (u \circ v) \circ w$. Moreover, introducing pairs, denoted by $\mathfrak{J}$, and $\mathfrak{J}^*$, of JTS, where $\mathfrak{J}$ and $\mathfrak{J}^*$ are dual to each other, a linear map can be constructed amounting to, $\mathfrak{J} \otimes \mathfrak{J}^* \longrightarrow \mathfrak{gl}(\mathfrak{J}) \oplus \mathfrak{gl}(\mathfrak{J}^*)$, whose image is a Lie sub-algebra $\mathfrak{str}(\mathfrak{J})$, and the Jordan identities imply the Jacobi identities for a graded Lie bracket on $\mathfrak{J} \oplus \mathfrak{str}(\mathfrak{J}) \oplus \mathfrak{J}^*$. Then, in a graded algebra as the one in (9), the pair $(\mathfrak{g}_{+1}, \mathfrak{g}_{-1})$ can be viewed as a Jordan pair, according to the correspondence, $\{x_\mp, y_\pm, z_\pm\}_\pm = \left[\left[x_\mp, y_\pm\right], z_\pm\right]$. The procedure outlined above, termed to as the Kantor-Koecher-Tits correspondence, when applied to the special

Jordan algebra, $\mathfrak{J}_2^C$, allows to interpret the $so(2,4)$ algebra in terms of Jordan triple systems and Jordan pairs and be cast as, $so(2,4) \simeq \mathfrak{co}(\mathfrak{J}) = \mathfrak{g}_{+1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} := \mathfrak{J}^* \oplus \mathfrak{str}(\mathfrak{J}) \oplus \mathfrak{J}^*$, meaning that the Abelian sub-algebra $\mathfrak{g}_{-1}(\mathfrak{g}_{+1})$ is generated in the space $\mathfrak{J}(\mathfrak{J}^*)$ (see [30] for details). The advantage of the Jordan algebra view on the conformal symmetry consists in the possibility of its straightforward generalization to higher symmetry groups. Specifically in [31, 32], attention has been drawn to the fact that the exceptional group F_4, hypothesized as internal space symmetry of a unified theory of strong and electroweak interactions, can be approached over the octonion Jordan algebra $\mathfrak{J}_3^O$.

5 Summary

We provided a concise review of recent progress on revealing the conformal-symmetry–color-neutrality connection in and near the infrared regime of QCD [15]. The work has been based on the assumption that the compactified boundary of the AdS_5 space, whose relevance to QCD follows from the AdS_5/CFT_4 duality conjecture, can be employed as space hosting the strong interaction degrees of freedom of QCD. As long as the topology of this space is $S^1 \times S^3$, it contains a closed space-like hyper-surface (S^3 in this case), on which only charge neutral configurations of the type 2^n poles, all necessarily neutral, can reside. In this way, the color-neutrality of hadrons finds a possible explanation. In addition, the space has the conformal group as isometry, a quality which allows to motivate the opening of the conformal window in the infrared by admitting the compactification radius as a second scale next to Λ_{QCD}. In consequence, the infrared regime acquires features of a perturbative one, permits for the approximation by Abelian color charges, and thus allows one to conclude on the quantum mechanical limit of QCD as represented by the following wave equation,

$$\left[\Box_{S^1\times S^3} - \alpha_s N_c \cot\chi + \frac{\alpha_s^2 N_c^2}{4(K+1)^2}\right]\psi = 0, \tag{12}$$

where $\Box_{S^1\times S^3}$ stands for the conformal wave operator on $S^1 \times S^3$. This wave equation describes quite realistically a broad range on hadron physics experiments from excitation spectra, over electromagnetic form factors, up to phase transitions.

Acknowledgements This work has been partly supported by the Bulgarian National Science Fund, research grant DN 18/3.

References

1. D.Z. Freedman, J.-M. Wang, Phys. Rev. Lett. **18**, 863–867 (1967)
2. G. Domokos, Phys. Rev. **159**, 1387–1403 (1967)
3. W.R. Frazer, H.M. Lipinski, D.R. Snider, Phys. Rev. **174**, 1932–1936 (1968)
4. M. Toller, Il Nuovo Cim. A **54**, 295–361 (1968)
5. H.M. Jones, Il Nuovo Cimento A **59**, 81–91 (1969)
6. A.O. Barut, H.M. Kleinert, Phys. Rev. **161**, 1464–1466 (1968)
7. F. Iachello, N.C. Mukhopadhyay, L. Zhang, Phys. Rev. D **44**, 898–914 (1991)
8. M. Kirchbach, Nucl. Phys. A **689**, 157–166 (2001)
9. J. Vijande, F. Fernandez, A. Valcarce, J. Phys. G: Nucl. Part. Phys. **31**, 481–506 (2005)
10. S.S. Afonin, Int. J. Mod. Phys. A **22**, 4537–4586 (2007)
11. S.S. Afonin, Int. J. Mod. Phys. A **23**, 4205–4217 (2008)
12. S.S. Afonin, Eur. Phys. J. A **29**, 327–335 (2006)
13. S.J. Brodski, Conformal symmetry as template for QCD. arXiv:hep-ph/0408069
14. A. Deur, V. Burkert, J.P. Chen, W. Korsch, Phys. Lett. **665**, 349–351 (2008)
15. M. Kirchbach, T. Popov, J.A. Vallejo, JHEP **09**, 171 (2021)
16. C.B. Compean, M. Kirchbach, Eur. Phys. J. A **33**, 1–4 (2007)
17. A. Raposo, H.J. Weber, D.E. Alvarez-Castillo, M. Kirchbach, C. Eur, J. Phys. **5**, 253–284 (2007)
18. M. Kirchbach, C.B. Compean, Phys. Rev. D **82**, 034008 (2010)
19. M. Kirchbach, C.B. Compean, Eur. Phys. J. A **52**, 210 (2016), Addendum: [Eur. Phys. J. A **53**, 65 (2017)]
20. A.V. Belitsky, A.S. Gorsky, G.P. Korchemsky, Nucl. Phys. B **67**, 3–54 (2003)
21. M. Kirchbach, C.B. Compean, Nucl. Phys. A **980**, 32–50 (2018)
22. A. Al-Jamel, Mod. Phys. Lett. A **34**, 1950307 (2019)
23. A. Bahroz Brzo, D. Alvarez-Castillo, Mod. Phys. Lett. A **36**, 2150095 (2021)
24. H. Nicolai, Nucl. Phys. B **305**(FS23), 483–496 (1988)
25. R. Penrose, W. Rindler, Spinors and Space-Time; vol. 2, Spinor and Twistor Methods in Space-Time Geometry (Cambridge University Press, Cambridge, 1986)
26. H.S. Cohl, SIGMA **7**, 108 (2011), eq. (117), Remark 4.1
27. B. Alertz, Ann. Inst. Henri Poincaré **53**, 319–342 (1990)
28. J.L. Richardson, Phys. Lett. B **82**, 272–274 (1979)
29. W. Beltram, *The Geometry of Jordan and Lie Structures*, Lecture Notes in Mathematics, vol. 1754. (Springer, Berlin, 2000)
30. T. Popov, in *Quantum Theory and Symmetries with Lie Theory and its Applications in Physics*. ed. by V. Dobrev. Springer Proceedings in Mathematics & Statistic. (Springer, Singapore, 2018), pp. 231–244
31. M. Dubois-Violette, I. Todorov, Nucl. Phys. B **912**, 426–449 (2016)
32. I. Todorov, S. Drenska, Adv. Appl. Clifford Algebras **28**, 82 (2018)

$s\ell(2)$ Gaudin Model with General Boundary Terms

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Abstract We study the $s\ell(2)$ Gaudin model with general boundary K-matrix in the framework of the algebraic Bethe ansatz. The off-shell action of the generating function of the $so(3)$ Gaudin Hamiltonians is determined without any restriction whatsoever on the boundary parameters.

Keywords Gaudin model · Generalized classical Yang-Baxter equation · Algebraic Bethe ansatz

1 Introduction

In the framework of the Bethe ansatz, Gaudin has studied the system obtained as the quasi-classical limit of the Heisenberg spin chain [1, 2]. This system has been recast in the framework of the quantum inverse scattering method with the help of the so-called Sklyanin linear bracket, corresponding to an $s\ell(2)$ invariant, unitary classical r-matrix [3]. This result enabled further generalizations based on other unitary solutions to the classical Yang-Baxter equation corresponding to higher-rank simple Lie algebras as well as Lie superalgebras [4–6] and the corresponding Jordanian deformation [7–9].

In our considerations of the non-periodic rational as well as trigonometric Gaudin model we have studied them as the quasi-classical limit, respectively, of the open XXX and XXZ Heisenberg spin chains [10, 11]. Also, we have shown how the

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_32

expansion of the XXZ transfer matrix, calculated at the special values of the spectral parameter, yields the Gaudin Hamiltonians in the trigonometric case [12] as well as for the Jordanian deformation of the ration $s\ell(2)$ Gaudin model with generic boundaries [13]. Returning back to the quasi-classical limit, following the Sklyanin proposal for the periodic boundary conditions [3, 14], we have derived the generating function of the Gaudin Hamiltonians both for the XXX [10], the XXZ chain [11] and for the Jordanian deformation of the XXX Heisenberg spin chain [15]. Moreover, we have shown [16] how, in the context of the quasi-classical limit, the solutions to the classical Yang-Baxter equation can be combined with the solutions to the classical reflection equation to yield solutions to the so-called generalized classical Yang-Baxter equation. These solutions are the non-unitary classical r-matrices [17]. In particular, the generic elliptic $s\ell(2)$ non-unitary r-matrix was studied in [18]. Also, we have developed an approach to the implementation of the algebraic Bethe ansatz for the rational as well as the trigonometric $s\ell(2)$ Gaudin model, in the case when the classical boundary K-matrix has a triangular form [19–22].

Following our approach to the generic $so(3)$ Gaudin [23, 24], here we study the non-periodic $s\ell(2)$ Gaudin model with the general boundary K-matrix. Namely, we show that it is possible to keep all of the K-matrix parameters arbitrary, though this requires a more complex form of the vacuum state.

The paper is organized as follows. In Sect. 2 we study the $s\ell(2)$ linear bracket which provides the algebraic framework for implementation of the Bethe ansatz. In the same section we propose the novel set of generators with simplified commutation relations. In Sect. 3 we will introduce an explicit form of the vacuum state which, together with the suitable choice of algebra generators, allows the implementation of the algebraic Bethe ansatz without reducing generality of the K-matrix. In this way, we will finally obtain the expression for the off-shell action of the generating function of the $s\ell(2)$ Gaudin Hamiltonians with general boundary terms.

2 The $s\ell(2)$ Linear Bracket

We consider the classical r-matrix

$$r(\lambda) = -\frac{\mathcal{P}}{\lambda}, \tag{1}$$

where $\mathcal{P}$ is the permutation matrix in $\mathbb{C}^2 \otimes \mathbb{C}^2$. This classical r-matrix (1) has the unitarity property

$$r_{21}(-\lambda) = -r_{12}(\lambda), \tag{2}$$

and it satisfies the classical Yang-Baxter equation

$$[r_{12}(\lambda-\mu), r_{13}(\lambda-\nu)] + [r_{12}(\lambda-\mu), r_{23}(\mu-\nu)] + [r_{13}(\lambda-\nu), r_{23}(\mu-\nu)] = 0. \tag{3}$$

The corresponding Gaudin Lax matrix is defined by [3]

$$L_0(\lambda) = \sum_{m=1}^{N} \frac{\boldsymbol{\sigma}_0 \cdot \mathbf{S}_m}{\lambda - \alpha_m} , \tag{4}$$

as usual, $\{\alpha_m, m = 1, \ldots, N\}$ are the inhomogeneous parameters, σ_0^α, with $\alpha = +, -, 3$, are the Pauli matrices

$$\sigma^\alpha = \begin{pmatrix} \delta_{\alpha 3} & 2\delta_{\alpha +} \\ 2\delta_{\alpha -} & -\delta_{\alpha 3} \end{pmatrix} , \tag{5}$$

in the auxiliary space $V_0 = \mathbb{C}^2$ and the spin $\frac{1}{2}$ operators $S^\alpha = \frac{1}{2}\sigma^\alpha$ are acting on the local space $V_m = \mathbb{C}^2$ at each site of the chain

$$S_m^\alpha = \mathbf{1} \otimes \cdots \otimes \underbrace{S^\alpha}_{m} \otimes \cdots \otimes \mathbf{1}, \tag{6}$$

with $\alpha = +, -, 3$ and $m = 1, 2, \ldots, N$. The Gaudin Lax matrix (4) and the classical r-matrix (1) obey the so-called Sklyanin linear bracket [3]

$$[L_1(\lambda), L_2(\mu)] = [r_{12}(\lambda - \mu), L_1(\lambda) + L_2(\mu)] \ . \tag{7}$$

The next step is the generalization of the model by introduction of the K-matrix, which must satisfy the reflection equation. The general, spectral parameter dependent solutions of the classical reflection equation [10]:

$$\begin{aligned} r_{12}(\lambda - \mu) K_1(\lambda) K_2(\mu) + K_1(\lambda) r_{21}(\lambda + \mu) K_2(\mu) = \\ = K_2(\mu) r_{12}(\lambda + \mu) K_1(\lambda) + K_2(\mu) K_1(\lambda) r_{21}(\lambda - \mu) , \end{aligned} \tag{8}$$

where the classical r-matrix is the one given in (1), can be written as follows [10]

$$K(\lambda) = \begin{pmatrix} \xi - \lambda & \psi\lambda \\ \phi\lambda & \xi + \lambda \end{pmatrix} . \tag{9}$$

Moreover, by introducing the non-unitary, classical r-matrix [16]

$$r_{12}^K(\lambda, \mu) = r_{12}(\lambda - \mu) - K_2(\mu) r_{12}(\lambda + \mu) K_2^{-1}(\mu) , \tag{10}$$

the two Eqs. (3) and (8) can be combined into the generalized classical Yang-Baxter equation [16]

$$\left[r_{32}^K(\nu, \mu), r_{13}^K(\lambda, \nu)\right] + \left[r_{12}^K(\lambda, \mu), r_{13}^K(\lambda, \nu)\right] + \left[r_{12}^K(\lambda, \mu), r_{23}^K(\mu, \nu)\right] = 0. \tag{11}$$

As we have shown [16], the appropriate Lax matrix is given by

$$\begin{aligned}\mathcal{L}_0(\lambda) &= L_0(\lambda) - K_0(\lambda)L_0(-\lambda)K_0^{-1}(\lambda) = \begin{pmatrix} H(\lambda) & F(\lambda) \\ E(\lambda) & -H(\lambda) \end{pmatrix} \\ &= \sum_{m=1}^{N} \left(\frac{\boldsymbol{\sigma}_0 \cdot \mathbf{S}_m}{\lambda - \alpha_m} + \frac{K_0(\lambda)\boldsymbol{\sigma}_0 K_0^{-1}(\lambda) \cdot \mathbf{S}_m}{\lambda + \alpha_m} \right) .\end{aligned} \tag{12}$$

The corresponding linear bracket based on the r-matrix $r^K(\lambda, \mu)$ (10) is defined by [16]

$$[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = \left[r^K_{00'}(\lambda, \mu), \mathcal{L}_0(\lambda)\right] - \left[r^K_{0'0}(\mu, \lambda), \mathcal{L}_{0'}(\mu)\right] . \tag{13}$$

This linear bracket is obviously anti-symmetric and it obeys the Jacobi identity because the r-matrix $r^K_{00'}(\lambda, \mu)$ (10) satisfies the generalized classical Yang-Baxter equation (11).

As it is well known [16], the linear bracket (13) yields the expression for the generating function of the $s\ell(2)$ Gaudin Hamiltonians with general boundary terms in terms of the Lax operator (12)

$$\tau(\lambda) = \frac{1}{2} \operatorname{tr}_0 \left(\mathcal{L}_0^2(\lambda)\right) = H^2(\lambda) + \frac{1}{2} \left(E(\lambda)F(\lambda) + F(\lambda)E(\lambda)\right) . \tag{14}$$

Using the bracket (13), it is straightforward to check that the operator $\tau(\lambda)$ commutes for different values of the spectral parameter

$$[\tau(\lambda), \tau(\mu)] = 0 . \tag{15}$$

Our aim here is to study the Gaudin system without any restriction whatsoever on the boundary parameters. Consequently, the commutation relations for the generators $H(\lambda)$, $E(\lambda)$ and $F(\lambda)$ turn out to be long and cumbersome and, thus, we will not present them here. Technically, these commutation relations are the principal difficulty in implementing the algebraic Bethe ansatz in this, fully general, case. To overcome this problem we propose the new set of generators

$$\mathcal{H}(\lambda) = \frac{1}{2\nu} \left(2H(\lambda) - \varphi F(\lambda) - \psi E(\lambda)\right) , \tag{16}$$

$$\mathcal{E}(\lambda) = -\frac{1-\nu}{2\nu} \left(2H(\lambda) - \frac{\varphi}{1+\nu} F(\lambda) - \frac{\psi}{1-\nu} E(\lambda)\right) , \tag{17}$$

$$\mathcal{F}(\lambda) = \frac{1+\nu}{2\nu} \left(2H(\lambda) - \frac{\varphi}{1-\nu} F(\lambda) - \frac{\psi}{1+\nu} E(\lambda)\right) . \tag{18}$$

The commutation relations for the new generators are substantially simpler than the relations for the initial generators. In particular,

$$[\mathcal{E}(\lambda), \mathcal{E}(\mu)] = [\mathcal{F}(\lambda), \mathcal{F}(\mu)] = [\mathcal{H}(\lambda), \mathcal{H}(\mu)] = 0 , \tag{19}$$

and the three non-trivial relations are

$$[\mathcal{H}(\lambda), \mathcal{E}(\mu)] = \frac{-2}{\lambda^2 - \mu^2} \left(\mu \, \frac{\xi - \lambda\nu}{\xi - \mu\nu} \, \mathcal{E}(\lambda) - \lambda \, \mathcal{E}(\mu) \right) , \tag{20}$$

$$[\mathcal{H}(\lambda), \mathcal{F}(\mu)] = \frac{2}{\lambda^2 - \mu^2} \left(\mu \, \frac{\xi + \lambda\nu}{\xi + \mu\nu} \, \mathcal{F}(\lambda) - \lambda \, \mathcal{F}(\mu) \right) , \tag{21}$$

$$[\mathcal{E}(\lambda), \mathcal{F}(\mu)] = \frac{-4}{\lambda^2 - \mu^2} \left(\mu \, \frac{\xi + \lambda\nu}{\xi + \mu\nu} \, \mathcal{H}(\lambda) - \lambda \, \frac{\xi - \mu\nu}{\xi - \lambda\nu} \, \mathcal{H}(\mu) \right) . \tag{22}$$

A straightforward but somewhat lengthy calculation shows that the generating function $\tau(\lambda)$ (14) has exactly the same form when expressed in terms of the new generators

$$\tau(\lambda) = \mathcal{H}^2(\lambda) + \frac{1}{2} \left(\mathcal{E}(\lambda)\mathcal{F}(\lambda) + \mathcal{F}(\lambda)\mathcal{E}(\lambda) \right) . \tag{23}$$

Our main result in this section are the new generators of the generalized $s\ell(2)$ Gaudin algebra (16)–(18). Due to their strikingly simple commutation relations (19)–(22) they provide a suitable framework for applying the algebraic Bethe ansatz without any restrictions on boundary parameters.

3 Implementation of the Algebraic Bethe Ansatz

A necessary prerequisite for the implementation of the algebraic Bethe ansatz is the existence of an appropriate vacuum vector $\Omega_+ \in \mathcal{H}$ in the Hilbert space

$$\mathcal{H} = \bigotimes_{m=1}^{N} V_m = (\mathbb{C}^2)^{\otimes N} . \tag{24}$$

The standard approach relies on its property to be annihilated by one of the algebra generators – usually: $\mathcal{E}(\lambda) \, \Omega_+ = 0$. To this end we obtain the local representation of the generators (16)–(18):

$$\mathcal{H}(\lambda) = \frac{\lambda}{\nu} \sum_{m=1}^{N} \frac{2S_m^3 - \psi S_m^+ - \varphi S_m^-}{(\lambda - \alpha_m)(\lambda + \alpha_m)} , \tag{25}$$

$$\mathcal{E}(\lambda) = \frac{-\lambda\,(1-\nu)}{\nu} \sum_{m=1}^{N} \frac{\xi - \alpha_m\nu}{\xi - \lambda\nu} \, \frac{2S_m^3 - \frac{\psi}{1-\nu}\, S_m^+ - \frac{\varphi}{1+\nu} S_m^-}{(\lambda - \alpha_m)(\lambda + \alpha_m)} , \tag{26}$$

$$\mathcal{F}(\lambda) = \frac{\lambda\,(1+\nu)}{\nu} \sum_{m=1}^{N} \frac{\xi + \alpha_m\nu}{\xi + \lambda\nu} \, \frac{2S_m^3 - \frac{\psi}{1+\nu}\, S_m^+ - \frac{\varphi}{1-\nu}\, S_m^-}{(\lambda - \alpha_m)(\lambda + \alpha_m)} . \tag{27}$$

A simple approach to ensure $\mathcal{E}(\lambda)\,\Omega_+ = 0$ is by fixing some of the K-matrix parameters – e.g. setting $\phi = 0$ and choosing the vacuum state as the tensorial product of highest spin states in each of the local spaces V_m [22]. Another approach, allowing for general boundary parameter in a set of special cases can be found in [25].

However, here we show that, for the $s\ell(2)$ Gaudin model, it is possible to retain the full generality of the K-matrix parameters by choosing a somewhat more complicated vacuum vector. To this purpose we observe that in every local space $V_m = \mathbb{C}^2$, $m \in \{1, \ldots, N\}$ there exists a vector $\omega_m \in V_m$ given by

$$\omega_m = \begin{pmatrix} \frac{\psi}{1-\sqrt{1+\psi\varphi}} \\ 1 \end{pmatrix} \in \mathbb{C}^2 = V_m \,, \tag{28}$$

where the parameters ν, ψ and φ are the parameters of the boundary K-matrix (9). Then it is easy to check that

$$\left(2S_m^3 - \frac{\psi}{1-\nu}\,S_m^+ - \frac{\varphi}{1+\nu}S_m^-\right)\omega_m = 0 \,, \tag{29}$$

$$\left(2S_m^3 - \psi S_m^+ - \varphi S_m^-\right)\omega_m = \nu\,\omega_m \,. \tag{30}$$

Therefore the vacuum vector Ω_+, defined as

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H} \tag{31}$$

is annihilated by the generator $\mathcal{E}(\lambda)$ (26) and, at the same time, it is an eigenvector of the generator $\mathcal{H}(\lambda)$ (25), that is

$$\mathcal{E}(\lambda)\,\Omega_+ = 0 \quad \text{and} \quad \mathcal{H}(\lambda)\,\Omega_+ = \rho(\lambda)\,\Omega_+ \quad \text{with} \quad \rho(\lambda) = \sum_{m=1}^{N} \frac{\lambda}{\lambda^2 - \alpha_m^2} \,. \tag{32}$$

Our next aim is to rewrite the formula for $\tau(\lambda)$ (23) in a more suitable way so that the action of the generating function $\tau(\lambda)$ on the vacuum vector Ω_+ (31) becomes more transparent. As it can be shown, the generating function $\tau(\lambda)$ (23) can be expressed as follows

$$\tau(\lambda) = \mathcal{H}^2(\lambda) + \frac{1}{\lambda}\,\frac{\xi^2 + \lambda^2\nu^2}{\xi^2 - \lambda^2\nu^2}\,\mathcal{H}(\lambda) - \mathcal{H}'(\lambda) + \mathcal{F}(\lambda)\mathcal{E}(\lambda) \,. \tag{33}$$

Taking into account (32) and (33), it is evident that the vacuum vector Ω_+ (31) is an eigenvector of the generating function

$$\tau(\lambda)\,\Omega_+ = \chi_0(\lambda)\,\Omega_+ \quad \text{with} \quad \chi_0(\lambda) = \rho^2(\lambda) + \frac{\xi^2 + \lambda^2\nu^2}{\xi^2 - \lambda^2\nu^2}\,\frac{\rho(\lambda)}{\lambda} - \rho'(\lambda) \,. \tag{34}$$

Now we can compute the commutator by a straightforward calculation, based on the formulae (33), (21) and (22)

$$
\begin{aligned}
[\tau(\lambda), \mathcal{F}(\mu)] = & -\frac{4}{\lambda^2-\mu^2}\,\mathcal{F}(\mu)\left(\lambda\mathcal{H}(\lambda)+\frac{\lambda^2\nu^2}{\xi^2-\lambda^2\nu^2}\right) \\
& + \frac{4}{\lambda^2-\mu^2}\,\frac{\lambda}{\mu}\,\frac{\xi-\mu\nu}{\xi-\lambda\nu}\,\mathcal{F}(\lambda)\left(\mu\mathcal{H}(\mu)+\frac{\mu^2\nu^2}{\xi^2-\mu^2\nu^2}\right) .
\end{aligned}
\tag{35}
$$

The relative simplicity of the right hand side of the equation above has encouraged us to seek the commutator between the operator $\tau(\lambda)$ and the product $\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)$ as the next step. In this case, an analogous direct calculation based on the previous formulae, leads to

$$
\begin{aligned}
[\tau(\lambda), \mathcal{F}(\mu_1)\mathcal{F}(\mu_2)] = & -\frac{4}{\lambda^2-\mu_1^2}\,\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\left(\lambda\mathcal{H}(\lambda)+\frac{\lambda^2\nu^2}{\xi^2-\lambda^2\nu^2}-\frac{\lambda^2}{\lambda^2-\mu_2^2}\right) \\
& -\frac{4}{\lambda^2-\mu_2^2}\,\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\left(\lambda\mathcal{H}(\lambda)+\frac{\lambda^2\nu^2}{\xi^2-\lambda^2\nu^2}-\frac{\lambda^2}{\lambda^2-\mu_1^2}\right) \\
& +\frac{4}{\lambda^2-\mu_1^2}\,\frac{\lambda}{\mu_1}\,\frac{\xi-\mu_1\nu}{\xi-\lambda\nu}\,\mathcal{F}(\lambda)\mathcal{F}(\mu_2)\left(\mu_1\mathcal{H}(\mu_1)+\frac{\mu_1^2\nu^2}{\xi^2-\mu_1^2\nu^2}-\frac{2\mu_1^2}{\mu_1^2-\mu_2^2}\right) \\
& +\frac{4}{\lambda^2-\mu_2^2}\,\frac{\lambda}{\mu_2}\,\frac{\xi-\mu_2\nu}{\xi-\lambda\nu}\,\mathcal{F}(\mu_1)\mathcal{F}(\lambda)\left(\mu_2\mathcal{H}(\mu_2)+\frac{\mu_2^2\nu^2}{\xi^2-\mu_2^2\nu^2}-\frac{2\mu_2^2}{\mu_2^2-\mu_1^2}\right) .
\end{aligned}
\tag{36}
$$

From these relations it is not difficult to infer, and to prove by mathematical induction, e.g. as in [22] that, for an arbitrary natural number M, the off-shell action of the generating function $\tau(\lambda)$ on the Bethe vectors takes the form:

$$
\Phi_M(\mu_1,\mu_2,\ldots,\mu_M) = \mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\Omega_+ , \tag{37}
$$

is given by

$$
\begin{aligned}
& \tau(\lambda)\Phi_M(\mu_1,\mu_2,\ldots,\mu_M) = \chi_M(\lambda,\mu_1,\mu_2,\ldots,\mu_M)\,\Phi_M(\mu_1,\mu_2,\ldots,\mu_M) \\
& +\sum_{j=1}^{M}\frac{4\lambda}{\lambda^2-\mu_j^2}\,\frac{\xi-\mu_j\nu}{\xi-\lambda\nu}\left(\rho(\mu_j)+\frac{\mu_j\nu^2}{\xi^2-\mu_j^2\nu^2}-\sum_{k\neq j}^{M}\frac{2\mu_j}{\mu_j^2-\mu_k^2}\right)\times \\
& \times\ \Phi_M(\lambda,\mu_1,\ldots,\widehat{\mu}_j,\ldots,\mu_M) ,
\end{aligned}
\tag{38}
$$

where the eigenvalue $\chi_M(\lambda,\mu_1,\mu_2,\ldots,\mu_M)$ is given by

$$\chi_M(\lambda, \mu_1, \mu_2, \ldots, \mu_M) = \chi_0(\lambda) - \sum_{j=1}^{M} \frac{4\lambda}{\lambda^2 - \mu_j^2} \times \\ \times \left(\rho(\lambda) + \frac{\lambda \nu^2}{\xi^2 - \lambda^2 \nu^2} - \sum_{k \neq j}^{M} \frac{\lambda}{\lambda^2 - \mu_k^2} \right) . \tag{39}$$

The unwanted terms on the right hand side of (38) are annihilated once the Bethe equations

$$\rho(\mu_j) + \frac{\mu_j \nu^2}{\xi^2 - \mu_j^2 \nu^2} - \sum_{k \neq j}^{M} \frac{2\mu_j}{\mu_j^2 - \mu_k^2} = 0 , \qquad j = 1, 2, \ldots, M , \tag{40}$$

are imposed on the parameters $\mu_1, \mu_2, \ldots, \mu_M$.

4 Conclusion

The usual approaches to nontrivial boundary conditions for the $s\ell(2)$ Gaudin model commonly require additional constraints on the K-matrix parameters [22], with the exception of some special cases for the trigonometric $s\ell(2)$ Gaudin model [25]. In this paper, we have demonstrated that, by the suitable choices of generators of the generalized $s\ell(2)$ Gaudin algebra and of the corresponding vacuum vector, it is possible to retain full generality of the K-matrix, i.e. without any restriction whatsoever on the boundary parameters. While here this was realized for fixed values of spin – $\frac{1}{2}$ at each node, we believe that the approach can be further generalized to the case of arbitrary spins.

Acknowledgements The cooperation between IS and NM was supported by the bilateral project "Symmetry and Quantization", 337-00-00227/2019-09/57. IS is supported by the Ministry of Education, Science and Technological Development (MPNTR) of the Republic of Serbia.

References

1. M. Gaudin, *La fonction d'onde de Bethe* (Masson, Paris, 1983)
2. M. Gaudin, *The Bethe Wavefunction* (Cambridge University Press, 2014)
3. E.K. Sklyanin, Zap. Nauchn. Semin. https://doi.org/10.1007/BF01840429
4. P.P. Kulish, N. Manojlovic, Lett. Math. Phys. https://doi.org/10.1023/A:1010913532054
5. P.P. Kulish, N. Manojlovic, J. Math. Phys. https://doi.org/10.1063/1.1398584
6. P.P. Kulish, N. Manojlovic, J. Math. Phys. https://doi.org/10.1063/1.1531250
7. N. Cirilo António, N. Manojlović, J. Math. Phys. https://doi.org/10.1063/1.2036932
8. P.P. Kulish, N. Manojlović, M. Samsonov, A. Stolin, Bethe Ansatz for deformed Gaudin model. (AGMF Tartu08 Workshop Proceedings) Proc. Est. Acad. Sci. **59**(4), 326–331 (2010)

9. N. Cirilo Antonio, N. Manojlovic, A. Stolin, J. Math. Phys. https://doi.org/10.1063/1.3644345
10. N. Cirilo António, N. Manojlović, I. Salom, Nucl. Phys. B. https://doi.org/10.1016/j.nuclphysb.2014.10.014
11. N. Manojlović, I. Salom, Nucl. Phys. B. https://doi.org/10.1016/j.nuclphysb.2017.07.017
12. N. Cirilo António, N. Manojlović, Z. Nagy, Rev. Math. Phys. https://doi.org/10.1142/S0129055X13430046
13. N. Cirilo António, N. Manojlović, Z. Nagy, Theor. Math. Phys. https://doi.org/10.1007/s11232-014-0155-0
14. N. Manojlović, Z. Nagy, I. Salom, *Derivation of the trigonometric Gaudin Hamiltonians*. SFIN **XXVIII** Series A: Conferences No. A1, 127–135 (2015). ISBN: 978-86-82441-43-4
15. N. Manojlović, N. Cirilo António, I. Salom, *Quasi-classical limit of the open Jordanian XXX spin chain*. SFIN **XXXI** Series A: Conferences No. A1, 259–266 (2018). ISBN: 978-86-82441-48-9
16. N. Cirilo António, N. Manojlović, E. Ragoucy, I. Salom, Nucl. Phys. B. https://doi.org/10.1016/j.nuclphysb.2015.02.011
17. T. Skrypnyk, N. Manojlović, Nucl. Phys. B. https://doi.org/10.1016/j.nuclphysb.2021.115424
18. N. Manojlovic, I. Salom, N. Cirilo António, *XYZ Gaudin model with boundary terms*. SFIN **XXXIII** Series A: Conferences, No. A1, 143–160 (2020). ISBN 978-86-82441-51-9
19. I. Salom, N. Manojlović, *Creation operators of the non-periodic $sl(2)$ Gaudin model*. SFIN **XXVIII** Series A: Conferences No. A1, 149–155 (2015). ISBN: 978-86-82441-43-4
20. I. Salom, N. Manojlović, N. Cirilo António, Nucl. Phys. B. https://doi.org/10.1016/j.nuclphysb.2018.12.025
21. N. Manojlović, I. Salom, Symmetry. https://doi.org/10.3390/sym12030352
22. I. Salom, N. Manojlović, Nucl. Phys. B. https://doi.org/10.1016/j.nuclphysb.2021.115462
23. I. Salom, N. Manojlovic, N. Cirilo António, *The spin 1 XXZ Gaudin model with boundary*. SFIN **XXXIII** Series A: Conferences, No. A1, 277–286 (2020). ISBN 978-86-82441-51-9
24. N. Manojlović, I. Salom, Rational $so(3)$ Gaudin model with general boundary terms, submitted to the Nuclear Physics **B**
25. N. Crampé, Symmetry, Integrability and Geometry: Methods and Applications. https://doi.org/10.3842/SIGMA.2017.094

Entanglement of Mixed States in Kähler Quantization

Tatyana Barron and Alexander Kazachek

Abstract Let M is a product of two integral compact Kähler manifolds. Fix a sufficiently large positive integer N. With a submanifold Λ of M one can associate a specific mixed state, and entanglement of formation $F_N(\Lambda)$ of this mixed state. We show that if Λ_1 and Λ_2 are two connected submanifolds of M such that $\Lambda_1 \cap \Lambda_2 = \emptyset$, then $F_N(\Lambda \cup \Lambda_2) \leq F_N(\Lambda_1) + F_N(\Lambda_2)$.

Keywords Submanifolds · Line bundles · Entanglement of formation

1 Preliminaries

To a compact integral Kähler manifold M geometric quantization associates a prequantum line bundle L. The prototypical case is when M is a coadjoint orbit of a compact Lie group with the Kirillov–Kostant–Souriau symplectic form. The most basic example is $M = \mathbb{C}P^1$ with the Fubini-Study form, a coadjoint orbit of $SU(2)$. Representation theorists typically deal with the infinite-dimensional Lie algebra $C^\infty(M)$ or with representations in the spaces of sections of tensor powers of L. From a somewhat different perspective, semiclassical analysis deals with L^N, $N \in \mathbb{N}$, as $N \to \infty$, and uses that to study the geometric aspects of M. In [1, 2] we attempted to bring entanglement entropy and entanglement of formation into this analytic setting. This note is a continuation of the same line of work.

Let (M_1, ω_1), (M_2, ω_2) be compact integral Kähler manifolds. W.l.o.g. $\dim M_1 \leq \dim M_2$. Let $L_1 \to M_1$ and $L_2 \to M_2$ be holomorphic hermitian line bundles such that for each $m \in \{1, 2\}$ the curvature of the Chern connection in L_m equals $-i\omega_m$. There is a sufficiently large $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, L_1^N and L_2^N are very ample,

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_33

$$\dim H^0(M_1, L_1^N) \leq \dim H^0(M_2, L_2^N)$$

and

$$H^0(M_1 \times M_2, L_1^N \boxtimes L_2^N) \cong H^0(M_1, L_1^N) \otimes H^0(M_2, L_2^N)$$

(isomorphism of Hilbert spaces) (see Lemma 3.1 [2]). Let N be sufficiently large (i.e. $N \geq N_0$).

In this paper, a mixed state will mean a (Hermitian) positive semidefinite linear operator of trace 1 on a finite-dimensional complex Hilbert space.

As in [2], we can associate with a connected submanifold Λ of $M_1 \times M_2$ a mixed state ρ_N as follows. Let

$$\mathcal{R}_N : H^0(M_1 \times M_2, L_1^N \boxtimes L_2^N) \to L^2(\Lambda, (L_1^N \boxtimes L_2^N)\Big|_{\Lambda})$$

$$s \mapsto s\Big|_{\Lambda}$$

be the restriction operator. Denote by V_N the image of $H^0(M_1 \times M_2, L_1^N \boxtimes L_2^N)$ under $\mathcal{R}_N$. Let Π_N^* be the Hilbert space adjoint of the operator

$$\Pi_N : H^0(M_1 \times M_2, L_1^N \boxtimes L_2^N) \to V_N$$

$$s \mapsto \mathcal{R}_N(s).$$

Define

$$\rho_N := \frac{1}{\mathrm{tr}(\Pi_N^* \Pi_N)} \Pi_N^* \Pi_N \in \mathrm{End}(H^0(M_1 \times M_2, L_1^N \boxtimes L_2^N)).$$

Now, denote $W_N = H^0(M_1, L_1^N) \otimes H^0(M_2, L_2^N)$ and define maps

$$\alpha : W_N \to W_N \oplus W_N$$

$$s \mapsto s \oplus s$$

$$\beta : W_N \oplus W_N \to W_N$$

$$s \oplus \tau \mapsto s + \tau.$$

Suppose X and Y are connected submanifolds of $M_1 \times M_2$ such that $X \cap Y = \emptyset$. For X, use the notations $\Pi_N^{(1)}$ and $\rho_N^{(1)}$ for the operators Π_N and ρ_N defined above, and for Y, denote these operators by $\Pi_N^{(2)}$ and $\rho_N^{(2)}$. Associate to $X \cup Y$ the map

$$\tilde{\rho}_N = \beta \circ (\Pi_N^{(1)} \oplus \Pi_N^{(2)})^* \circ (\Pi_N^{(1)} \oplus \Pi_N^{(2)}) \circ \alpha.$$

To unwrap this, it means that for each $s \in W_N$

$$\tilde{\rho}_N(s) = (\Pi_N^{(1)})^* \Pi_N^{(1)} s + (\Pi_N^{(2)})^* \Pi_N^{(2)} s.$$

Now define the mixed state

$$\rho_N = \frac{1}{\mathrm{tr}\tilde{\rho}_N} \tilde{\rho}_N.$$

Let $v \in W_N$ be a vector of norm 1. Denote by P_v the orthogonal projection onto the 1-dimensional $\mathbb{C}$-linear subspace of W_N spanned by v. Recall that the entanglement entropy of v is

$$E(v) = -\sum_{k=1}^{d} \lambda_k \log \lambda_k,$$

where $d = \dim H^0(M_1, L_1^N)$, $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of $Tr_2(P_v)$, by convention $0 \log 0 = 0$, and Tr_2 is the reduced trace map (see e.g. [3]). For a mixed state $A \in \mathrm{End}(W_N)$, its entanglement of formation [4] is

$$\inf \sum_i p_i E(v_i)$$

where the inf is taken over all finite decompositions $A = \sum_i p_i P_{v_i}$ with positive coefficients p_i such that $\sum_i p_i = 1$. Such a decomposition of A always exists. We note that when $A = P_v$ for some v, the entanglement of formation of A equals $E(v)$ (so, in this case, if v is decomposable, then entanglement of formation of A is zero, and if v is not decomposable, then entanglement of formation of A is strictly positive).

In the discussion above, for a submanifold Λ of $M_1 \times M_2$ with a corresponding mixed state ρ_N, we will denote by $F_N(\Lambda)$ the entanglement of formation of ρ_N.

2 Main Result

Theorem 1 *Let (M_1, ω_1), (M_2, ω_2) be compact integral Kähler manifolds. Let $L_1 \to M_1$ and $L_2 \to M_2$ be holomorphic hermitian line bundles such that for each $m \in \{1, 2\}$ the curvature of the Chern connection in L_m equals $-i\omega_m$. Let $N \in \mathbb{N}$ be sufficiently large, so that L_1^N and L_2^N are very ample. Suppose X and Y are connected submanifolds of $M_1 \times M_2$ such that $X \cap Y = \emptyset$. Then*

$$F_N(X \cup Y) \leq F_N(X) + F_N(Y).$$

Proof For every $\varepsilon > 0$ there are $k, m \in \mathbb{N}$, positive real numbers $p_1,\dots,p_k, q_1,\dots,q_m$ such that $p_1 + \dots + p_k = 1$ and $q_1 + \dots + q_m = 1$, and vectors $u_1,\dots,u_k, v_1,\dots,v_m$ in $H^0(M_1, L_1^N) \otimes H^0(M_2, L_2^N)$ of norm 1, such that

$$\frac{1}{\mathrm{tr}((\Pi_N^{(1)})^*\Pi_N^{(1)})}(\Pi_N^{(1)})^*\Pi_N^{(1)} = \sum_{i=1}^{k} p_i P_{u_i},$$

$$\frac{1}{\mathrm{tr}((\Pi_N^{(2)})^*\Pi_N^{(2)})}(\Pi_N^{(2)})^*\Pi_N^{(2)} = \sum_{j=1}^{m} q_j P_{v_j}$$

$$\sum_{i=1}^{k} p_i E(u_i) - F_N(X) < \varepsilon, \tag{1}$$

$$\sum_{j=1}^{m} q_j E(v_j) - F_N(Y) < \varepsilon. \tag{2}$$

We have:

$$\frac{1}{\mathrm{tr}(\tilde{\rho}_N)}\tilde{\rho}_N = \sum_{i=1}^{k} \tilde{p}_i P_{u_i} + \sum_{j=1}^{m} \tilde{q}_j P_{v_j}$$

where

$$\tilde{p}_i = \frac{\mathrm{tr}((\Pi_N^{(1)})^*\Pi_N^{(1)})}{\mathrm{tr}(\tilde{\rho}_N)} p_i > 0,\ 1 \leq i \leq k \tag{3}$$

and

$$\tilde{q}_j = \frac{\mathrm{tr}((\Pi_N^{(2)})^*\Pi_N^{(2)})}{\mathrm{tr}(\tilde{\rho}_N)} q_j > 0,\ 1 \leq j \leq m, \tag{4}$$

and we have:

$$\sum_{i=1}^{k} \tilde{p}_i + \sum_{j=1}^{m} \tilde{q}_j = 1.$$

Then

$$\sum_{i=1}^{k} \tilde{p}_i E(u_i) + \sum_{j=1}^{m} \tilde{q}_j E(v_j) \geq F_N(X \cup Y),$$

also because of (1), (2), (3), (4)

$$\sum_{i=1}^{k} \tilde{p}_i E(u_i) + \sum_{j=1}^{m} \tilde{q}_j E(v_j) - (F_N(X) + F_N(Y)) < 2\varepsilon.$$

Since there are ensembles of pure states as above for every $\varepsilon > 0$, the desired statement follows.

Remark 1 The statement of the Theorem extends from two disjoint connected submanifolds to finitely many pairwise disjoint connected submanifolds.

Acknowledgements We are thankful to the organizers of Lie Theory and Its Applications in Physics Workshop XIV. We acknowledge partial support from the Natural Sciences and Engineering Research Council of Canada. A.K. was a research student of T.B.

References

1. T. Barron, T. Pollock, Kähler quantization and entanglement. Rep. Math. Phys. **80**(2), 217–231 (2017)
2. T. Barron, N. Wheatley, *Entanglement and Products*. arxiv.org/abs/2104.10597
3. I. Bengtsson, K. Zyczkowski, *Geometry of Quantum States : An Introduction to Quantum Entanglement* (Cambridge University Press, Cambridge, 2017)
4. C. Bennett, D. DiVincenzo, J. Smolin, W. Wootters, Mixed-state entanglement and quantum error correction. Phys. Rev. A **54**(5), 3824–3851 (1996)

The Chirality-Flow Formalism for Standard Model Calculations

Joakim Alnefjord, Andrew Lifson, Christian Reuschle, and Malin Sjödahl

Abstract Scattering amplitudes are often split up into their color ($\mathfrak{su}(N)$) and kinematic components. Since the $\mathfrak{su}(N)$ gauge part can be described using flows of color, one may anticipate that the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ kinematic part can be described in terms of flows of chirality. In two recent papers we showed that this is indeed the case, introducing the chirality-flow formalism for standard model calculations. Using the chirality-flow method—which builds on and further simplifies the spinor-helicity formalism—Feynman diagrams can be directly written down in terms of Lorentz-invariant spinor inner products, allowing the simplest and most direct path from a Feynman diagram to a complex number. In this presentation, we introduce this method and show some examples.

Keywords Chirality flow · Feynman rules · Spinor-helicity formalism

1 Introduction

Since a few decades it is known that calculations in SU(3) color space can be elegantly simplified using a flow picture for color [1, 2]. In this talk we ask the question if we can similarly simplify the Lorentz structure, which at the algebra level is associated with a left and a right chiral $\mathfrak{su}(2)$.

More specifically, bearing in mind that for color, one can formulate color-flow Feynman rules, we ask whether we can analogously formulate a set of chirality-flow Feynman rules to simplify calculations of Lorentz structure. In this presentation we will answer this question affirmatively and show how Feynman rules can be recast into chirality flows and that this beautifully simplifies calculations [3–6].

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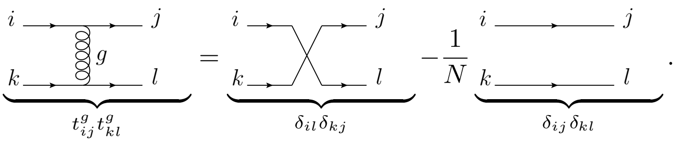
V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_34

On the QCD side, we can translate every color structure to flows of color using the $\mathfrak{su}(\mathrm{N})$ Fierz identity to remove adjoint indices



(1)

Similarly, external gluons can be rewritten in terms of color-anticolor pairs (with a color suppressed "U(1)" gluon contribution), and the color structure of triple-gluon vertices can be expressed in terms of traces, such that in the end, every amplitude is a linear combination of products of Kronecker deltas in color space [1, 2].

Before attempting the same procedure for the Lorentz structure, we recall that at the level of the (complexified) algebra, the Lorentz group consists of two copies of $\mathfrak{su}(2)$, $\mathfrak{su}(2)_{\text{left}} \oplus \mathfrak{su}(2)_{\text{right}}$, and that the Dirac spinor structure transforms under the direct sum representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. In the chiral (or Weyl) basis we have (for some conventions)

$$\begin{pmatrix} u_L \\ u_R \end{pmatrix} \to \begin{pmatrix} e^{-i\bar{\theta}\cdot\frac{\bar{\sigma}}{2}+\bar{\eta}\cdot\frac{\bar{\sigma}}{2}} & 0 \\ 0 & e^{-i\bar{\theta}\cdot\frac{\bar{\sigma}}{2}-\bar{\eta}\cdot\frac{\bar{\sigma}}{2}} \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix}, \tag{2}$$

i.e. we actually have two copies of SL(2,$\mathbb{C}$), generated by the complexified $\mathfrak{su}(2)$ algebra.

We will build heavily on the chiral representation and the spinor-helicity formalism [7–16], and start with considering the massless case, for which

$$u^+(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix}, u^-(p) = \begin{pmatrix} |p] \\ 0 \end{pmatrix}, \bar{u}^+(p) = \big([p|,\ 0\big), \bar{u}^-(p) = \big(0,\ \langle p|\big)\ . \tag{3}$$

From the spinor-helicity formalism we also borrow the expressions for the polarization vectors [13, 16], expressed in terms of the physical momentum p, and a reference momentum r

$$\epsilon_L{}^{\mu}(p,r) \to \frac{|r\rangle[p|}{\langle rp\rangle} \text{ or } \frac{|p]\langle r|}{\langle rp\rangle}, \qquad \epsilon_R^{\mu}(p,r) \to \frac{|r]\langle p|}{[pr]} \text{ or } \frac{|p\rangle[r|}{[pr]}, \tag{4}$$

where ϵ_L is for incoming negative helicity or outgoing positive helicity and ϵ_R is for incoming positive helicity or outgoing negative helicity.

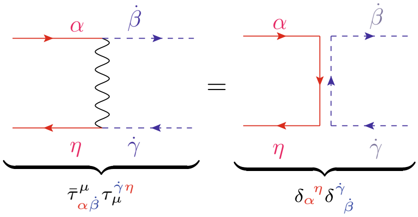
To construct Lorentz invariant amplitudes we build invariant spinor inner products using the only SL(2,$\mathbb{C}$) invariant tensor, $\epsilon^{\alpha\beta}$ ($\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1$). With $\langle i| = \langle p_i|$ etc., we have

$$\underbrace{\epsilon^{\alpha\beta}|i\rangle_\beta}_{\equiv\langle i|^\alpha}|j\rangle_\alpha = \langle i|^\alpha|j\rangle_\alpha = \langle ij\rangle, \quad \underbrace{\epsilon_{\dot\alpha\dot\beta}|i]^{\dot\beta}}_{\equiv[i|_{\dot\alpha}}|j]^{\dot\alpha} = [i|_{\dot\alpha}|j]^{\dot\alpha} = [ij]\,. \tag{5}$$

Amplitudes are thus built up out of contractions of the form $\langle ij\rangle$, $[ij] \sim \sqrt{s_{ij}}$, and if we manage to create a flow picture, the "flow" must contract left (dotted) and right (undotted) indices separately.

2 Towards Chirality Flow

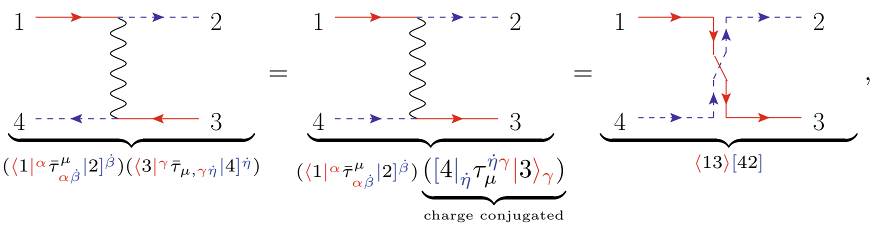
For the Lorentz structure, a fermion-photon vertex is associated with a factor $\gamma^\mu = \sqrt{2}\begin{pmatrix} 0 & \tau^\mu \\ \bar\tau^\mu & 0\end{pmatrix}$ ($\tau^\mu = \sigma^\mu/\sqrt{2}$ normalized in analogy with Eq. (1)). This can be split into two terms, and when a τ^μ from one vertex is contracted with a $\bar\tau^\mu$ from another vertex, we have (always reading indices along arrows),



(6)

We note that due to the presence of τ^0 there is no $1/N$-suppressed term. In this sense chirality flow is even simpler than color flow.

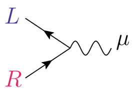
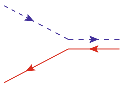
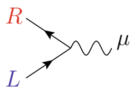
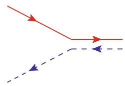
When a $\tau(\bar\tau)$ is contracted with a $\tau(\bar\tau)$ from the other vertex, the situation is more subtle, and we have to apply charge conjugation *at the level of expressions contracted with spinors* before removing the vector index



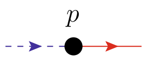
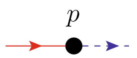
were we have implicitly used the identification of spinors and their graphical representation

$|j] =$  $, [i| =$  $, |j\rangle =$  $, \langle i| =$  .

In a similar way, charge conjugation can be applied when additional photons are attached to a quark-line [4]. A consistent arrow direction with opposing arrows for spin-1 particles can therefore always be chosen [4], and the fermion-photon vertex can be translated to

 $\rightarrow ie\sqrt{2}$  ,  $\rightarrow ie\sqrt{2}$  .

We also need to recast Fermion propagators to the flow picture. To this end, we split $p_\mu \gamma^\mu = p_\mu \sqrt{2} \begin{pmatrix} 0 & \tau^\mu \\ \bar{\tau}^\mu & 0 \end{pmatrix}$ into two terms

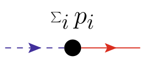
$$\not{p} \equiv \sqrt{2} p^\mu \tau_\mu^{\dot{\alpha}\beta} =$$  $$, \quad \bar{\not{p}} \equiv \sqrt{2} p_\mu \bar{\tau}^\mu_{\alpha\dot{\beta}} =$$  , (7)

where we have introduced a graphical "momentum-dot" notation for momenta slashed with σ or $\bar{\sigma}$.

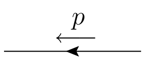
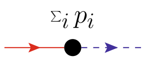
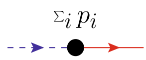
We further note that for massless momenta we have

$$\sqrt{2} p^\mu \tau_\mu \equiv \not{p} = |p]\langle p| \,, \quad \sqrt{2} p^\mu \bar{\tau}_\mu \equiv \bar{\not{p}} = |p\rangle [p| \,. \tag{8}$$

Thus any sum of light-like momenta, $p^\mu = \sum p_i^\mu \;,\; p_i^2 = 0$, can be written

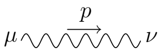
$\not{p} =$  $= \sum_i |i]^{\dot{\alpha}} \langle i|^{\beta}, \; \bar{\not{p}} =$  $= \sum_i |i\rangle_\alpha [i|_{\dot{\beta}}$ for $p_i^2 = 0$.

In particular, this gives for the fermion propagator

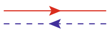
 $\rightarrow \frac{i}{p^2}$  or $\frac{i}{p^2}$  , (9)

where the momentum is read along the fermion arrow. (It may be aligned or anti-aligned with the chirality-flow arrows, any arrow assignment with opposing gauge boson arrows will do for massless tree-level QED and QCD since there is always an even number of spinor contractions [4]).

For the photon propagator we have [4]

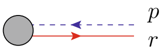


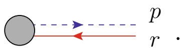
$$\to -\frac{i}{p^2} \quad \text{or} \quad -\frac{i}{p^2} \quad . \tag{10}$$



Finally, it is straightforward to translate the spinor structure of external gauge bosons to the flow picture, for example

$$\epsilon_L{}^{\mu}(p,r) \to \frac{1}{\langle rp\rangle} \quad \text{or} \quad \frac{1}{\langle rp\rangle} \quad . \tag{11}$$

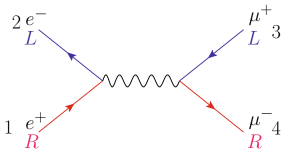




In a similar way, Feynman rules can be written down for massless QCD. The main complication is the introduction of a momentum-dot in the triple-gluon vertex, whereas the four-gluon vertex is just a linear combination of chirality-flows with one dotted and one undotted line for each metric factor [4].

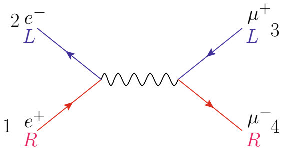
3 Examples

Equipped with the Feynman rules for QED, we consider the standard example of $e^+e^- \to \mu^+\mu^-$. For assigned helicities, it is not hard to calculate this amplitude within the spinor helicity formalism,

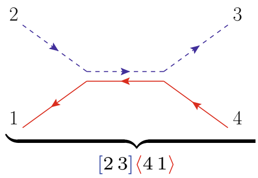


$$\begin{aligned} &= \frac{2ie^2}{s_{e^+e^-}}\left([2|_{\dot\alpha}\tau_\mu^{\dot\alpha\beta}|1\rangle_\beta\right)\left(\langle 4|^{\alpha}\bar\tau^{\mu}_{\alpha\dot\beta}|3]^{\dot\beta}\right) \\ &= \frac{2ie^2}{s_{e^+e^-}}[2|_{\dot\alpha}|3]^{\dot\alpha}\langle 4|^{\beta}|1\rangle_\beta = \frac{2ie^2}{s_{e^+e^-}}[2\,3]\langle 4\,1\rangle\ , \end{aligned} \tag{12}$$

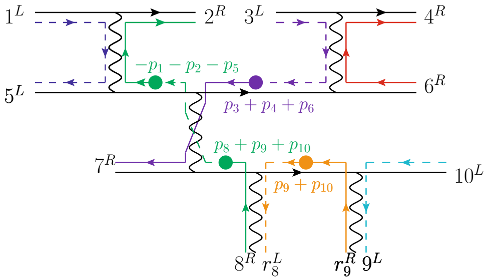
but with chirality flow the answer can directly be drawn



$$= \frac{2ie^2}{s_{e^+e^-}} \quad . \tag{13}$$



Similarly, the value of even a very complicated massless tree-level diagram can just be written down, for example (for reference vectors r_8 and r_9)



$$= \underbrace{(\sqrt{2}ei)^8}_{\text{vertices}} \frac{(-i)^3}{\underbrace{s_{1\,2}\; s_{3\,4}\; s_{7\,8\,9\,10}}_{\text{photon propagators}}} \times \frac{(i)^4}{\underbrace{s_{1\,2\,5}\; s_{3\,4\,6}\; s_{8\,9\,10}\; s_{9\,10}}_{\text{fermion propagators}}} \times \frac{1}{\underbrace{[8r_8]\langle r_9 9\rangle}_{\text{polarization vectors}}} \tag{14}$$

$$\times [15]\langle 64\rangle[10\;9]\Big(\langle r_9 9\rangle[9r_8] + \langle r_9 10\rangle[10r_8]\Big)\Big(\underbrace{[33]}_{0}\langle 37\rangle + [34]\langle 47\rangle + [36]\langle 67\rangle\Big)$$
$$\times\Big(-\langle 89\rangle[91]\langle 12\rangle - \langle 89\rangle[95]\langle 52\rangle - \langle 8\,10\rangle[10\;1]\langle 12\rangle - \langle 8\,10\rangle[10\;5]\langle 52\rangle\Big)\,.$$

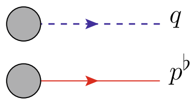
4 Massive Chirality Flow

To treat mass, we first note that a massive momentum p always can be written as a linear combination of two lightlike momenta, $p^\flat$ and q, $p^\mu = p^{\flat,\mu} + \alpha q^\mu$ where $\alpha = \frac{p^2}{2p\cdot q}$. This decomposition can be achieved in infinitely many ways—as is obvious from considering the system in its rest frame, where the momenta can be taken to have any opposing direction. Different decompositions correspond to different directions of measuring the spin [5, 17, 18], and in general the spin is measured along

$$s^\mu = \frac{1}{m}(p^{\flat,\mu} - \alpha q^\mu) = \frac{1}{m}(p^\mu - 2\alpha q^\mu) \tag{15}$$

for, for example, a u^+ spinor of the form [5]

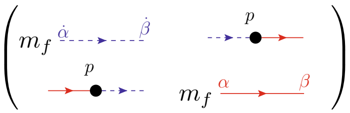
$$u^+(p) = \left(\; -\frac{m}{[qp^\flat]} \quad \right) . \tag{16}$$



The standard choice of measuring spin along the direction of motion (i.e. helicity) corresponds to decomposing p into a forward and backward direction, $s^\mu = \frac{1}{m}(p_f^\mu - p_b^\mu) = \frac{1}{m}(|\mathbf{p}|, p^0\hat{p})$.

Aside from massive spinors we need treat massive fermion propagators

$$\frac{i}{p^2-m_f^2}\begin{pmatrix} m_f\delta^{\dot\alpha}{}_{\dot\beta} & \sqrt{2}p^{\dot\alpha\beta} \\ \sqrt{2}\bar{p}_{\alpha\dot\beta} & m_f\delta_{\alpha}{}^{\beta}\end{pmatrix} = \frac{i}{p^2-m_f^2} \qquad (17)$$



The presence of Kronecker delta functions may give rise to an odd number of spinor inner products, implying that signs will have to be carefully tracked in the massive case. We also need to treat the third polarization degree of freedom for a massive vector boson, but other than that, the massive case follows quite straightforwardly from the massless case and using the above decompositions, all Feynman rules of the standard model can be written down [5].

5 Conclusion

Splitting Lorentz structure into $\mathfrak{su}(2)_{\text{left}}$ and $\mathfrak{su}(2)_{\text{right}}$, we have been able to recast all standard model Feynman rules to chirality-flow rules, giving a transparent and intuitive way of understanding the Lorentz inner products appearing in amplitudes.

If the ordinary spinor helicity method takes us from 4×4 Dirac matrices to 2×2 Pauli matrices, the chirality-flow method takes us from Pauli matrices to scalars. This significantly simplifies calculations with Feynman diagrams. Many processes are within range of quick pen and paper calculations, often without intermediate steps and the final result is transparent and intuitive.

More practically, we expect our method to be useful for event simulations with Monte Carlo event generators, in particular when sampling over helicity. Work towards consistent loop calculations is ongoing.

References

1. G. 't Hooft, A planar diagram theory for strong interactions. Nucl. Phys. B **72**, 461 (1974). [337 (1973)]
2. F. Maltoni, K. Paul, T. Stelzer, S. Willenbrock, Color flow decomposition of QCD amplitudes. Phys. Rev. D **67**, 014026 (2003)
3. A. Lifson, C. Reuschle, M. Sjödahl, Introducing the chirality-flow formalism. Acta Phys. Polon. B **51**(6), 1547–1557 (2020)
4. A. Lifson, C. Reuschle, M. Sjodahl, The chirality-flow formalism. Eur. Phys. J. C **80**(11), 1006 (2020)
5. J. Alnefjord, A. Lifson, C. Reuschle, M. Sjodahl, The chirality-flow formalism for the standard model. Eur. Phys. J. C **81**(4), 371 (2021)
6. J. Alnefjord, A. Lifson, C. Reuschle, M. Sjodahl, A brief look at the chirality-flow formalism for standard model amplitudes. PoS **LHCP2021**, 160 (2021)

7. P. De Causmaecker, R. Gastmans, W. Troost, T.T. Wu, Multiple bremsstrahlung in gauge theories at high-energies. 1. General formalism for quantum electrodynamics. Nucl. Phys. B **206**, 53–60 (1982)
8. F.A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, W. Troost, T.T. Wu, Multiple bremsstrahlung in gauge theories at high-energies. 2. Single bremsstrahlung. Nucl. Phys. B **206**, 61–89 (1982)
9. F.A. Berends, R. Kleiss, P. de Causmaecker, R. Gastmans, W. Troost, T.T. Wu, Multiple bremsstrahlung in gauge theories at high-energies. 3. Finite mass effects in collinear photon bremsstrahlung. Nucl. Phys. B **239**, 382–394 (1984)
10. R. Kleiss, The Cross-section for $e^+e^- \rightarrow e^+e^-e^+e^-$. Nucl. Phys. B **241**, 61 (1984)
11. F.A. Berends, P.H. Daverveldt, R. Kleiss, Complete lowest order calculations for four lepton final states in electron-positron collisions. Nucl. Phys. B **253**, 441–463 (1985)
12. J.F. Gunion, Z. Kunszt, Four jet processes: gluon-gluon scattering to nonidentical quark - anti-quark pairs. Phys. Lett. **159B**, 167 (1985)
13. J.F. Gunion, Z. Kunszt, Improved analytic techniques for tree graph calculations and the G g q anti-q lepton anti-lepton subprocess. Phys. Lett. **161B**, 333 (1985)
14. K. Hagiwara, D. Zeppenfeld, Helicity amplitudes for heavy lepton production in e+ e- annihilation. Nucl. Phys. B **274**, 1–32 (1986)
15. R. Kleiss, W.J. Stirling, Cross-sections for the production of an arbitrary number of photons in electron - positron annihilation. Phys. Lett. B **179**, 159–163 (1986)
16. Z. Xu, D.-H. Zhang, L. Chang, Helicity amplitudes for multiple bremsstrahlung in massless nonabelian gauge theories. Nucl. Phys. B **291**, 392–428 (1987)
17. S. Dittmaier, Weyl-van der Waerden formalism for helicity amplitudes of massive particles. Phys. Rev. D **59**, 016007 (1998)
18. H.K. Dreiner, H.E. Haber, S.P. Martin, Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry. Phys. Rept. **494**, 1–196 (2010)

Spacetime Stochasticity and Second Order Geometry

Folkert Kuipers

Abstract We discuss the Schwartz–Meyer second order geometry framework and its relevance to theories of quantum gravity that incorporate a notion of spacetime stochasticity or quantum foam. We illustrate the framework in the context of Nelson's stochastic quantization.

Keywords Second order geometry · Second order Lie derivative · Stochastic mechanics · Stochastic quantization · Quantum foam

1 Introduction

Since the introduction of the path integral formulation in quantum field theory, stochastic analysis has played a pivotal role in the mathematical construction of quantum field theories [7, 8, 13]. Closely related to these developments is the theory of stochastic mechanics which showed that several quantum theories are equivalent to a certain class of stochastic theories [6, 14]. In addition, the stochastic quantization framework used in this theory has proved to be a useful computational tool in the study of quantum field theories [3, 6, 15].

In this paper, we argue that the success of stochastic analysis in the study of quantum theories is not limited to flat spaces, but can help to elucidate the interplay between quantum theories and gravity, and could in the future provide handles in the formulation of a theory of quantum gravity.

The main argument for this statement is that the tools of stochastic analysis that provide a mathematical basis for Euclidean quantum theories can be extended to the context of pseudo-Riemannian manifolds using second order geometry as developed by Schwartz and Meyer [5, 12, 16]. Such extensions allow to construct and study physical theories on a fluctuating spacetime or quantum foam.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_35

2 Dynamics on Manifolds

We illustrate the framework by considering a particle moving on a d-dimensional Riemannian manifold $(\mathcal{M}, g)$. In classical physics, its trajectory is described by a map $x(t) : \mathcal{T} \to \mathcal{M}$, where $\mathcal{T} \subseteq \mathbb{R}$. The trajectory is the solution of the geodesic equation

$$\ddot{x}^j + \Gamma^j_{kl}(x)\,\dot{x}^k \dot{x}^l = 0, \tag{1}$$

which can be rewritten in a first order form for $(x, v)(t) : \mathcal{T} \to T\mathcal{M}$. Alternatively, the velocity can be treated as a vector field on the manifold. In this case the governing equations become

$$\begin{aligned} v^j(x)\,\nabla_j v^i(x) &= 0, \\ \dot{x}^i &= v^i(x). \end{aligned} \tag{2}$$

We will now introduce a notion of stochasticity in this trajectory. We must thus introduce a probability space $(\Omega, \Sigma, \mathbb{P})$ and promote the position x to a random variable $X : (\Omega, \Sigma, \mathbb{P}) \to (\mathcal{M}, \mathcal{B}(\mathcal{M}), \mu)$ with $\mu = \mathbb{P} \circ X^{-1}$. This allows to study continuous semi-martingale processes $\{X_t : t \in \mathcal{T}\}$, i.e. $X_t = C_t + M_t$ with C_t a càdlàg process and M_t a local martingale

In a stochastic theory, one would then like to derive a set of governing stochastic differential equations, similar to the set of ordinary differential equations in the deterministic theory. These stochastic differential equations should then be interpreted either in the sense of Itô or Stratonovich. However, the formulation of such systems on manifolds is complicated due to the presence of a non-vanishing quadratic variation

$$[[X^i, X^j]] = \lim_{k\to\infty} \sum_{[t_l, t_{l+1}]\in\pi_k} \big[X^i(t_{l+1}) - X^i(t_l)\big]\big[X^j(t_{l+1}) - X^j(l_l)\big]. \tag{3}$$

In the Itô formulation this quadratic variation leads to a violation of the Leibniz rule. Indeed for functions $f, g : \mathcal{M} \to \mathbb{R}$, one obtains a modified Leibniz rule of the form

$$d_2(f\,g) = f\,d_2 g + g\,d_2 f + 2\,df \cdot dg, \tag{4}$$

where

$$\begin{aligned} d_2 f &= \partial_i f\,dX^i + \frac{1}{2}\partial_i\partial_j f\,d[[X^i, X^j]], \\ df \cdot dg &= \frac{1}{2}\,\partial_i f\,\partial_j g\,d[[X^i, X^j]]. \end{aligned} \tag{5}$$

3 Second Order Geometry

The violation of Leibniz' rule implies that many notions from ordinary differential geometry are no longer applicable in a stochastic framework. However, this can be resolved by extending to second order geometry [5, 12, 16].

In second order geometry, first order tangent spaces $T\mathcal{M}$ are extended to second order tangent spaces $T_2\mathcal{M}$ such that a second order vector V can be represented in a local coordinate frame as $V = v^\mu \partial_\mu + v^{\mu\nu}\partial_\mu\partial_\nu$. Similarly one can construct second order forms $\Omega \in T_2^*\mathcal{M}$, which in a local coordinate system are given by $\Omega = \omega_\mu d_2 x^\mu + \omega_{\mu\nu} dx^\mu \cdot dx^\nu$.

The link between second order geometry and stochastic motion can now be made explicit by constructing second order vectors as

$$v^\mu(x) = \lim_{h\to 0} \frac{1}{h}\mathbb{E}\left[X^\mu(t+h) - X^\mu(t)\middle| X(t) = x\right], \tag{6}$$
$$v^{\nu\rho}(x) = \lim_{h\to 0} \frac{1}{2h}\mathbb{E}\left[\big(X^\mu(t+h) - X^\mu(t)\big)\big(X^\nu(t+h) - X^\nu(t)\big)\middle| X(t) = x\right].$$

Here, the first order part is constructed as usual, while the second order part reflects the non-vanishing quadratic variation of the stochastic process. It is important to note that when regarded as a second order vector, $v^{\mu}_{\nu\rho}$ does not transform covariantly. However, one can construct contravariant vectors $\hat{v}^{\mu}_{\nu\rho}$ such that

$$\begin{aligned} \hat{v}^\mu &:= v^\mu + \Gamma^\mu_{\nu\rho} v^{\nu\rho}, \\ \hat{v}^{\nu\rho} &:= v^{\nu\rho}. \end{aligned} \tag{7}$$

In a similar fashion, one can construct covariant forms $\hat{\omega}^{\mu}_{\nu\rho}$ by

$$\begin{aligned} \hat{\omega}_\mu &:= \omega_\mu, \\ \hat{\omega}_{\nu\rho} &:= \omega_{\nu\rho} - \Gamma^\mu_{\nu\rho}\omega_\mu. \end{aligned} \tag{8}$$

4 Lie Derivatives and Killing Vectors

It is possible to generalize many notions from first order geometry to the second order geometry framework, see e.g. [5, 9]. One way of doing so is by using the fact that a d-dimensional manifold equipped with a second order geometry can be mapped bijectively onto a d-dimensional brane embedded in a $\frac{d(d+3)}{2}$-dimensional manifold equipped with first order geometry [9].

Here, we focus on the constructions of a Lie derivative in second order geometry [9]. The second order Lie derivative of a scalar is simply given by

$$\mathcal{L}_V f = V f = v^\mu \partial_\mu f + v^{\mu\nu}\partial_\mu\partial_\nu f. \tag{9}$$

It is also possible to construct a Lie derivative of a second order vector U along a second order vector V. However, this requires that the second order parts of the vector fields are scalar multiples of each other. The Lie derivative is then given by the commutator

$$\mathcal{L}_V U = [V, U]. \tag{10}$$

Let us now turn to the construction of a Lie derivative of a first order (k, l)-tensor along a second order vector field. The result is a first order (k, l)-tensor given by

$$\mathcal{L}_V T = \mathcal{L}_{\mathcal{F}(V)} T + v^{\mu\nu} \left(\nabla_\mu \nabla_\nu + \mathcal{R}^{\cdot}{}_{\mu\cdot\nu}\right) T, \tag{11}$$

where the first part denotes an ordinary first order Lie derivative along the first order vector field, as $\mathcal{F} : T_2\mathcal{M} \to T\mathcal{M}$ s.t. $V \mapsto \hat{v}^\mu \partial_\mu$. Moreover,

$$\mathcal{R}^{\cdot}{}_{\alpha\cdot\beta} T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l} = \sum_{i=1}^{k} \mathcal{R}^{\mu_i}{}_{\alpha\lambda\beta} T^{\mu_1\ldots\mu_{i-1}\lambda\mu_{i+1}\ldots\mu_k}_{\nu_1\ldots\nu_l} - \sum_{j=1}^{l} \mathcal{R}^{\lambda}{}_{\alpha\nu_j\beta} T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_{j-1}\lambda\nu_{j+1}\ldots\nu_l}. \tag{12}$$

The construction of Lie derivatives of tensors along second order vector fields allows to construct a notion of a second order Killing vector. We find

$$\mathcal{L}_K g_{\mu\nu} = \nabla_\mu \hat{k}_\nu + \nabla_\nu \hat{k}_\mu - 2\,\hat{k}^{\rho\sigma} \mathcal{R}_{\mu\rho\nu\sigma}, \tag{13}$$

setting this to 0 leads to the second order Killing equation

$$\nabla_{(\mu} \hat{k}_{\nu)} = \hat{k}^{\rho\sigma} \mathcal{R}_{\mu\rho\nu\sigma}, \tag{14}$$

We thus find that a first order killing vector k^μ must be promoted to the covariant first order part of a second order vector $\hat{k}^\mu$. Secondly, a second order Killing vector has a non-vanishing divergence proportional to the curvature of space. A classical observer will interpret this deviation as a symmetry breaking of the classical spacetime due to the fluctuations.

5 Stochastic Dynamics on Manifolds

After setting up the machinery of second order geometry, one can derive stochastic differential equations of motions on a manifold. We will consider a Brownian motion for which the quadratic variation is well known to be

$$d[[X^i, X^j]]_t = \alpha\, g^{ij}(X_t)\, dt \tag{15}$$

with $\alpha \in [0, \infty)$. The system given in Eq. (2) now becomes [9, 11, 14]

$$\begin{aligned}\left[g_{ij}\hat{v}^k\nabla_k + \frac{\alpha}{2}\Big(g_{ij}\Box - \mathcal{R}_{ij}\Big)\right]\hat{v}^j &= \frac{\alpha^2}{12}\nabla_i\mathcal{R},\\ dX^i &= v^i\,dt + dM^i,\end{aligned} \tag{16}$$

which should be interpreted as a system of stochastic differential equations in the sense of Itô.

As we have only discussed non-relativistic processes on Riemannian manifolds, while the physical world is relativistic, we must extend our discussion to relativistic processes on Lorentzian manifolds. Extensions of second order geometry to Lorentzian manifolds are straightforward [9], as the framework is developed for any smooth manifold with a connection [5]. Furthermore, similar to a classical relativistic theory, the formulation of a relativistic theory on Lorentzian manifolds introduces a relativistic constraint equation [10, 11]. The the velocity field is then a solution of the system

$$\begin{aligned}\left[g_{\mu\nu}\hat{v}^\rho\nabla_\rho + \frac{\alpha}{2}\Big(g_{\mu\nu}\Box - \mathcal{R}_{\mu\nu}\Big)\right]\hat{v}^\nu &= \frac{\alpha^2}{12}\nabla_\mu\mathcal{R},\\ g_{\mu\nu}\hat{v}^\mu\hat{v}^\nu + \alpha\nabla_\mu\hat{v}^\mu - \frac{\alpha^2}{6}\mathcal{R} = \epsilon\end{aligned} \tag{17}$$

with $\epsilon \in \{-1, 0, +1\}$ for respectively time-like, light-like and space-like particles. Moreover, after splitting the manifold in time-like, light-like and space-like segments, one can construct a positive definite non-degenerate metric $g_{\text{Eucl.}}$, on these segments using a Wick rotation. The stochastic motion is then given by the solution of the Itô system [4, 10]

$$\begin{aligned}dX^\mu &= v^\mu\,d\tau + dM^\mu,\\ d[[X^\mu, X^\nu]] &= \alpha\, g^{\mu\nu}_{\text{Eucl.}}\,d\tau.\end{aligned} \tag{18}$$

One might object that we have only discussed a classical Brownian motion, which is not obviously related to quantum mechanics. However, one can analytically continue the manifold and define a process by

$$[[Z^\mu, Z^\nu]]_t = \alpha\, g^{\mu\nu}(Z_t) \tag{19}$$

with $Z = X + iY$ and $\alpha \in \mathbb{C}$. Then, for the choice $\alpha = i\,\hbar$, the real projection of this process is equivalent to quantum mechanics of a free scalar particle on the manifold [11].

6 Conclusions and Outlook

We have discussed the second order geometry framework and shown that it allows to describe stochastic dynamics on manifolds. Moreover we have discussed extensions to relativistic stochastic dynamics and discussed the relation between stochastic and quantum dynamics.

We should note that in this paper we have only described free particles moving in a fixed geometry. Although this picture can be extended to include external forces derived from scalar or vector potential and the notion of spin, see e.g. [14], a field theoretic formulation will be necessary to consider dynamical geometries, and to study quantum aspects of gravity. Stochastic field theories have been discussed in the context of Nelson's stochastic quantization, cf. e.g. Ref. [6], but the subject is not yet as mature as it is in the Paris-Wu formulation of stochastic quantization [3, 15].

Nevertheless, the discussion of point particles presented in this paper provides an indication of the geometrical structure that is necessary to formulate such theories. Indeed, the configuration space of a classical particle is the tangent bundle $T\mathcal{M}$, which, in the stochastic framework, is promoted to a second order tangent bundle $T_2\mathcal{M}$. The configuration space of a classical field theory, on the other hand, is a first order jet bundle $J^1\pi$ over the manifold $\mathcal{M}$. It is thus expected that the configuration space for a stochastic field theory is a second order jet bundle $J^2\pi$. We note that the possibility of constructing classical field theories on higher order jet bundles has already been discussed in the literature [1, 2].

Acknowledgements This work is supported by a doctoral studentship of the Science and Technology Facilities Council, award number: 2131791.

References

1. C.M. Campos, M. de Leon, D.M. de Diego, J. Vankerschaver, J. Phys. A **42**, 475207 (2009)
2. C.M. Campos, M. de Leon, D.M. de Diego, J. Phys. A **43**(45), 455206 (2010)
3. P.H. Damgaard, H. Huffel, Phys. Rept. **152**, 227 (1987)
4. D. Dohrn, F. Guerra, Phys. Rev. D **31**, 2521–2524 (1985)
5. M. Emery, *Stochastic Calculus in Manifolds* (Springer, Berlin, 1989)
6. F. Guerra, Phys. Rep. **77**(3), 263 (1981)
7. J. Glimm, A.M. Jaffe, *Quantum Physics: A Functional Integral Point of View* (Springer, New York, 1987)
8. M. Kac, Trans. Amer. Math. Soc. **65**, 1–13 (1949)
9. F. Kuipers, JHEP **05**, 028 (2021)
10. F. Kuipers, to appear in J. Math. Phys. **62**(12) (2021). arXiv:2103.02501 [gr-qc]
11. F. Kuipers, arXiv:2109.10710 [math-ph]
12. P.A. Meyer, in *Stochastic Integrals*, ed. by D. Williams. Lecture Notes in Mathematics vol. 851, (Springer, 1981), pp. 256–270
13. E. Nelson, J. Funct. Anal. **12**(1), 97 (1973)
14. E. Nelson, *Quantum Fluctuations* (Princeton University Press, Princeton, 1985)
15. G. Parisi, Y.S. Wu, Sci. Sin. **24**, 483 (1981)
16. L. Schwartz, *Semi-Martingales and their Stochastic Calculus on Manifolds* (Presses de l'Université de Montréal, Montréal, 1984)

Special Mathematical Results

Velocity Reciprocity in Flat and Curved Space-Time

Patrick Moylan

Abstract There is a widely held misconception that velocity reciprocity follows solely from the relativity of motion principle, where by velocity reciprocity is meant that the velocity of a reference frame S relative to another reference frame S' is equal and opposite to the velocity of S' relative to S. This misconception still persists even today among many physicists, in spite of efforts by others to explain why it is wrong. In view of this situation, we consider it worthwhile to give some interesting examples illustrating why this misconception is wrong. Our examples include ones in both curved and flat spacetimes.

Keywords Relativity principle · Velocity reciprocity · de Sitter space · Anisotropic space-time

1 Introduction

There is a longstanding and widespread misconception among many physicists concerning the relativity principle and one of its inferences *velocity reciprocity*. Velocity reciprocity just amounts to the fact that the inverse transformation relating a given inertial system S to another one S' is always obtained from the direct transformation from S to S' simply by replacing the velocity of S' relative to S by its negative. However, this need not always be the case and it is a misconception that velocity reciprocity is a consequence only of the relativity principle. This misconception comes from an improper understanding of the relativity principle and lack of appreciation for its universality. It has found its way into many monographs and articles on relativity [1–3], including some very recent ones [4, 5]. In view of the important role that the relativity of motion principle holds in physics, we consider a clarification of the matter to be of some importance.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_36

While velocity reciprocity is easily seen to be true for the Galilean and Lorentz transformations, we shall see that it does not always hold true in all situations. If we follow Poincaré and insist on universal validity to the relativity principle [6], then velocity reciprocity being a direct consequence only of the relativity principle would imply that it always must be true. However, velocity reciprocity need not always be true, even in relativistic theories where the relativity principle is assumed to hold [7]. Thus it cannot possibly follow solely from the relativity of motion.

Regarding necessary assumptions for velocity reciprocity to hold true, it is, in fact, much more complicated than most of us seem to be aware. Essentially the only thing the relativity principle implies is a group structure to the set of all inertial transformations, which are the transformations between various inertial systems, all equivalent regarding the form of natural law [8]. For two dimensional space-times with the topology of $\mathbb{R}^2$ and coordinates of points (events) being labelled by (t, x), with t the time and x the spatial location of the event, we have the following theorem:

Theorem 1 *Let G be the group of all inertial transformations, which, in addition to space-time translations, consists of linear transformations*[1] *of $\mathbb{R}^2$ that include the one parameter subgroup of inertial boosts $\{\Lambda_v | v \in \mathbb{R}\}$ where $v = \dfrac{dx}{dt}$ is the velocity of the origin of S' as measured in S. (Λ_v takes the coordinates (t, x) of an event in a reference frame S into the coordinates (t', x') for the same event in the inertial frame S'.) Reciprocally let $u = \dfrac{dx'}{dt'}$ be the velocity of S relative to S'. Suppose Λ_v is spatial orientation preserving and causality preserving. In addition, we assume spatial isotropy, which means that there are no preferred directions in space [7, 9]. Then, assuming the reciprocal velocity u is a continuous, real-valued, surjective mapping of its domain, the set Γ of all admissible velocities, and that Γ is an interval in $\mathbb{R}$, which is symmetric about zero, we have $u = -v$.*

For a proof of this important theorem we refer the reader to either Ref. [9] or Ref. [8]. By causality preserving transformation we mean that the transformation should not alter the causal order of events, and by spatial orientation preserving we mean that the inertial transformation Λ_v does not change the orientation of the space axis [7–9].

2 Examples in Flat Space-Times

Here we consider some examples of velocity reciprocity in space-times with the topology of $\mathbb{R}^n$, which is what we mean by flat space-time. Our examples include ones for which velocity reciprocity is true and also examples for which in it does not hold true.

[1] It was Einstein in his first paper on relativity, who, by an appeal to space-time homogeneity, first justified the linearity of the transformations fixing the origin [9].

2.1 Some Examples for which Velocity Reciprocity Holds

A. The Galilean transformation:

$$\begin{bmatrix} t' \\ x' \end{bmatrix} = \Lambda_v \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -v & 1 \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} \tag{1}$$

(y and z directions suppressed), and

$$\begin{bmatrix} t \\ x \end{bmatrix} = \Lambda_v^{-1} \begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix} \begin{bmatrix} t' \\ x' \end{bmatrix} = \Lambda_{-v} \begin{bmatrix} t' \\ x' \end{bmatrix}. \tag{1^{bis}}$$

So

$$\Lambda_v^{-1} = \Lambda_{-v}. \tag{2}$$

B. Velocity Dependent Scale-Extended Lorentz Transformation:

In $1 + 1$ dimensions enlarge the Lorentz transformation to include an arbitrary velocity dependent change of scale λ_v:

$$\begin{bmatrix} t' \\ x' \end{bmatrix} = \lambda_v \Lambda_v \begin{bmatrix} t \\ x \end{bmatrix} \tag{3}$$

and

$$\begin{bmatrix} t \\ x \end{bmatrix} = \lambda_v^{-1} \Lambda_v^{-1} \begin{bmatrix} t' \\ x' \end{bmatrix}. \tag{3^{bis}}$$

Here Λ_v is a pure Lorentz boost in the x direction with velocity v is (for $v < c$)

$$\Lambda_v = \gamma(v) \begin{bmatrix} 1 & -\dfrac{v}{c^2} \\ -v & 1 \end{bmatrix} \tag{4}$$

where $\gamma(v) = \dfrac{1}{\sqrt{1 - v^2/c^2}}$.

The requirement that the set $\{\lambda_v \Lambda_v | v \in \mathbb{R}\}$ has the structure of a group forces the set of all λ_v to be a one dimensional representation of the group of Lorentz boosts Λ_v which implies $\lambda_v \lambda_{v'} = \lambda_{v''}$ where $v'' = \dfrac{v + v'}{1 + \dfrac{vv'}{c^2}}$. A family of solutions to $\lambda_v \lambda_{v'} = \lambda_{v''}$, depending upon a parameter β, is

$$\lambda_v(\beta) = e^{\beta \operatorname{arctanh} v/c} = \left(\frac{1 + \dfrac{v}{c}}{1 - \dfrac{v}{c}} \right)^{\beta/2} \qquad (\beta \in \mathbb{R}). \tag{5}$$

For each β the set of all

$$\tilde{\Lambda}_v(\beta) = \lambda_v(\beta)\gamma(v)\begin{bmatrix} 1 & -\frac{v}{c^2} \\ -v & 1 \end{bmatrix} = \lambda_v(\beta)\Lambda_v, \tag{6}$$

as v varies over $\mathbb{R}$ ($v \neq \pm c$), forms a group as required by the relativity principle (at least for transformations which don't mix tachyonic and subluminal velocities). Thus in two space-time dimensions there is no obstacle to having a non-trivial velocity dependent scale factor, unlike the case where space is three dimensional [6]. Since $\lambda_v(\beta)^{-1} = \lambda_{-v}(\beta)$ and also $\Lambda_v^{-1} = \Lambda_{-v}$ we have

$$\tilde{\Lambda}_v(\beta)^{-1} = \tilde{\Lambda}_{-v}(\beta) \tag{7}$$

so that velocity reciprocity is true for this example. Notice that the Lorentz transformation is the special case $\beta = 0$.

2.2 Some Examples for which Velocity Reciprocity Does Not Hold

Doppler Group in 1 + 1 Dimensions: let $\alpha_v = \left(1 - \frac{v}{c}\right)$ so that

$$t' = \alpha_v\, t = \left(1 - \frac{v}{c}\right) t. \tag{8}$$

and

$$x' = x - vt. \tag{9}$$

The matrix of the transformation specified by Eqs. (8) and (9) is

$$\Lambda_v = \begin{bmatrix} \alpha_v & 0 \\ -v & 1 \end{bmatrix}. \tag{10}$$

Since $\alpha_v \alpha'_v = \alpha_{v+\alpha_v v'}$, then

$$\Lambda_{v''} = \Lambda_v \Lambda_{v'} \tag{11}$$

with $v'' = v + \alpha_v v'$. The inverse of Λ_v is $\Lambda_v^{-1} = \begin{bmatrix} \frac{1}{\alpha_v} & 0 \\ \frac{v}{\alpha_v} & 1 \end{bmatrix}$. To have a group we need to express Λ_v^{-1} as a product of inertial transformations. To this end we write $\frac{1}{1-x} = \left(1 - \left(\frac{x}{x-1}\right)\right)$ for $x \neq 1$. Thus with $x = \frac{v}{c}$ we obtain

$$\Lambda_v^{-1} = \begin{bmatrix} \frac{1}{\alpha_v} & 0 \\ \frac{v}{\alpha_v} & 1 \end{bmatrix} = \begin{bmatrix} \alpha_{\tilde{v}} & 0 \\ -\tilde{v} & 1 \end{bmatrix} = \Lambda_{\tilde{v}} \qquad (v \neq c). \tag{12}$$

where $\tilde{v} = -\frac{v}{1 - \frac{v}{c}}$. Since $\tilde{v} \neq -v$, velocity reciprocity does not hold.

An invariant line element for the flat space-time described by this example is

$$ds^2 = c^2 dt^2 - 2c dt dx + dx^2. \tag{13}$$

We can associate with this line element the degenerate symmetric bilinear form

$$g_{\mu\nu} = \begin{pmatrix} c^2 & -c \\ -c & 1 \end{pmatrix}. \tag{14}$$

For the infinitesimal generator of Λ_v (inertial boost) we have

$$\mathbf{L}_{01} = -\frac{x_0}{c}\left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}\right) = -\frac{x_0}{c}(\mathbf{P}_0 + \mathbf{P}_1) = -\frac{1}{c}(\mathbf{S}_0 + x_0\mathbf{P}_1) = \widehat{\mathbf{L}}_{01} - \frac{1}{c}\mathbf{S}_0 \tag{15}$$

where $\mathbf{P}_0 = \frac{\partial}{\partial x_0}$, $\mathbf{P}_1 = \frac{\partial}{\partial x_1}$ are the translation generators in time and space, respectively, and $\mathbf{S}_0 = x_0 \frac{\partial}{\partial x_0}$ is the scale generator in the time direction. $\widehat{\mathbf{L}}_{01} = -x_0 \frac{\partial}{\partial x_1}$ is the Galilean boost. The generators $\mathbf{S}_0, \widehat{\mathbf{L}}_{01}, \mathbf{P}_0, \mathbf{P}_1$ are a basis for a four dimensional real Lie algebra with commutation relations:

$$\underbrace{[\widehat{\mathbf{L}}_{01}, \mathbf{P}_0] = \mathbf{P}_1, [\widehat{\mathbf{L}}_{01}, \mathbf{P}_1] = 0, [\mathbf{P}_0, \mathbf{P}_1] = 0}_{\mathfrak{h} \cong \mathfrak{ig}_1} \tag{16a}$$

$$[\mathbf{S}_0, \widehat{\mathbf{L}}_{01}] = \widehat{\mathbf{L}}_{01}, [\mathbf{S}_0, \mathbf{P}_0] = -\mathbf{P}_0, [\mathbf{S}_0, \mathbf{P}_1] = 0 \tag{16b}$$

where $\mathfrak{h}$ is the Heisenberg Lie algebra and $\mathfrak{ig}_1$ is the inhomogeneous Galilei algebra in $1 + 1$ dimensions.

The generators $\mathbf{S}_0, \mathbf{L}_{01}, \mathbf{P}_0, \mathbf{P}_1$ are a basis for another four dimensional real Lie algebra with commutation relations:

$$[\mathbf{L}_{01}, \mathbf{P}_0] = \mathbf{P}_1 + \frac{1}{c}\mathbf{P}_0, [\mathbf{L}_{01}, \mathbf{P}_1] = 0, [\mathbf{P}_0, \mathbf{P}_1] = 0, \tag{17a}$$

$$[\mathbf{S}_0, \mathbf{L}_{01}] = \widehat{\mathbf{L}}_{01}, [\mathbf{S}_0, \mathbf{P}_0] = -\mathbf{P}_0, [\mathbf{S}_0, \mathbf{P}_1] = 0. \tag{17b}$$

Lemma 1 *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two finite dimensional Lie algebras. Suppose each has a basis with respect to which the structure constants are the same. Then $\mathfrak{g} \cong \mathfrak{g}'$.*

Since $\mathbf{L}_{01} = \widehat{\mathbf{L}}_{01} - \frac{1}{c}\mathbf{S}_0$, it follows from the Lemma that the two Lie algebras are isomorphic. This solvable Lie algebra is $\mathfrak{ig}_1$ extended by scaling in time and is $A_{4,8}^{b=-1}$ in Ref. [10] and is $\mathfrak{s}_{4,6}$ in Ref. [11].

For a description of the Doppler group in higher dimensions, specifically in the physically important case of three spatial dimensions, we refer the reader to the Supplementary Material in Ref. [7]. The two dimensional version presented here goes back to a 1911 *Annalen der Physik* paper by Frank and Rothe [12] and they point out that velocity reciprocity does not hold [12]. Their paper forms the basis for Lalan's 1937 classification of all possible two-dimensional linear kinematics compatible with the relativity principle [13]. There are many other examples of kinematical models in his classification where velocity reciprocity is not valid [13].

3 Examples in Curved Space-Time

Two dimensional de Sitter space

$$V^2 \;=\; \{\xi^a \in \mathbb{R}^3 \mid {\xi^0}^2 - {\xi^1}^2 - {\xi^2}^2 \;=\; -\,R^2\} \tag{18}$$

with R being the radius of the de Sitter space. Pseudo-spherical coordinates on V^2 are

$$\xi^0 = R\mathrm{sh}\alpha, \xi^1 = R\mathrm{ch}\alpha\,\cos\phi, \xi^1 = R\mathrm{ch}\alpha\,\sin\phi \tag{19}$$

with $-\infty < \alpha < \infty,\ 0 \le \phi < 2\pi$. Lemêitre or horospherical coordinates on V^2 are

$$\xi^0 = \frac{R}{2}(\lambda - \lambda^{-1}) - \frac{1}{2R}y^2\lambda^{-1} \quad , \quad \xi^1 = \lambda^{-1}y \quad , \quad \xi^4 = -\frac{R}{2}(\lambda + \lambda^{-1}) + \frac{1}{2R}y^2\lambda^{-1} \tag{20}$$

where $-\infty < y,\ \lambda < \infty \quad (\lambda \neq 0)$. One dimensional spaces of "equal time" on V^2 are the parabolas $\xi^0 + \xi^4 = -R\lambda^{-1} = \text{const}$. If we let $\lambda = \mp e^{\tau}$ we can rewrite Eq. (20) as

$$\xi^0 = \pm(R\mathrm{sh}\tau - \frac{1}{2R}y^2e^{-\tau})\ ,\ \xi^1 = \mp e^{-\tau}y\ ,\ \xi^2 = \mp(R\mathrm{ch}\tau - \frac{1}{2R}y^2e^{-\tau}) \tag{21}$$

where the upper signs are for $\lambda > 0$ and the lower signs for $\lambda < 0$. Let $V_+^2 = \{\xi^i \in V^2 | \xi^0 + \xi^2 > 0\}$ and $V_-^2 = \{\xi^i \in V^2 | \xi^0 + \xi^2 < 0\}$. We have

$$V^2 = V_+^2 \cup V_0^2 \cup V_-^2 \tag{22}$$

where $V_0 = \{\xi^i \in V^2 | \xi^0 + \xi^2 = 0\}$. The antipodal map $\mathcal{J}_0 : \xi^i \to -\xi^i$ permutes V_+^2 with V_-^2.

Let $\hat{\xi}^{\mu} = \dfrac{\xi^{\mu}}{R}$, $u = \dfrac{y}{R}$. Define the cross section ($\varepsilon = +1$) $V_{+}^{2} \ni \hat{\xi} \to \tau_{\hat{\xi}} = n(-u)\ a(-\tau)e_{+} \in SO_0(1,2)$ where

$$n(u) = \begin{bmatrix} 1+\dfrac{u^2}{2} & u & \dfrac{u^2}{2} \\ u & 1 & u \\ -\dfrac{u^2}{2} & -u & 1-\dfrac{u^2}{2} \end{bmatrix} \text{ and } a(\tau) = \begin{bmatrix} \cosh\tau & 0 & \sinh\tau \\ 0 & 1 & 0 \\ \sinh\tau & 0 & \cosh\tau \end{bmatrix}$$

with $e_{+}^{\dagger} = (0,0,-1)$. A cross section for V_{-}^{2}, the other half of de Sitter space, is similarly defined. The cross-sections specify the de Sitter parallelization for induced representations of $SO_0(2,2)$ (cf. Ref. [14]). Coordinates adapted to the de Sitter parallelization are ($\lambda = \mp e^{\tau}$): $x_0 = \dfrac{1}{\lambda}$, $x_1 = \dfrac{u}{\lambda}$.

Denote by $\mathbf{L}_{01}$ the infinitesimal generator of the subgroup of $SO_0(1,2)$ which corresponds to the hyperbolic rotation in (0, 1) plane of the embedding space $\mathbb{R}^3$ specified in Eq. (18) and similarly let $\mathbf{L}_{12}$ be the infinitesimal generator of a rotation in the (1, 2) plane of $\mathbb{R}^3$. The action of the inertial (Galilean) boost $\Lambda_v = e^{v\{\mathbf{L}_{01}-\mathbf{L}_{12}\}}$ in the (x_0, x_1) coordinates is $x_0' = x_0$, $x_1' = x_1 - vx_0$ [15] on both V_{+}^{2} and V_{-}^{2} which implies $\Lambda_v^{-1} = \Lambda_{-v}$ so that velocity reciprocity is valid.

Similarly we show for the action of time translations $e^{t\mathbf{L}_{02}}$, which consist of hyperbolic rotations in the (0, 2) plane of the embedding space, that $x_0' = e^{t}x_0$, $x_1' = x_1$ on V_{+}^{2}. Following Ref. [15] we are thus led to a representation of the Lie algebra $\mathfrak{so}(1,2)$ of the de Sitter group on functions of the (x_0, x_1) coordinates on V_{+}^{2}, the action of the basic generators being

$$L_{01} = S_0 + \frac{1}{2},\ L_{02} - L_{12} = -x_0\frac{\partial}{\partial x_1},\ L_{02} + L_{12} = \frac{1}{x_0}\left\{\frac{\partial}{\partial x_1} + x_1^2\frac{\partial}{\partial x_1} - 2x_1 S\right\} \tag{23}$$

where $S_0 = x_0\dfrac{\partial}{\partial x_0}$ and $S = x_0\dfrac{\partial}{\partial x_0} + x_1\dfrac{\partial}{\partial x_1}$. (For this representation we chose $ic + \dfrac{1}{2} = 0$ in Ref. [15].) It is easy to show that the L_{12}, L_{01}, and L_{02} satisfy the basic commutation relations of $\mathfrak{so}(1,2)$.

To get at an example in curved space where velocity reciprocity is not upheld we modify the above action of the Galilean boost in this representation in the same way as we modified it to obtain the Lie algebra defined by Eqs. (17a) and (17b). Specifically, the modification is

$$x_0' = \alpha_v x_0\ ,\ x_1' = x_1 - vx_0 \tag{24}$$

with α_v given by Eq. (8). The treatment is almost identical to that described above. Basis generators are S_0 together with

$$\hat{L}_{01} = L_{01}, \hat{L}_{02} - \hat{L}_{12} = L_{02} - L_{12} + \frac{1}{c} S_0, \hat{L}_{01} + \hat{L}_{12} = L_{01} + L_{12}. \tag{25}$$

$\hat{L}_{02} - \hat{L}_{12}$ is the infinitesimal generator for the modified Galilean boost specified by Eq. (24). It is easy to see that Eqs. (23) and (25) together with S_0 lead to isomorphic Lie algebras just as in the Doppler group case.

4 Conclusions

Regarding relevance of our investigation to current research, we mention interest in inhomogeneous and/or anisotropic models of the universe as an explanation of current cosmic acceleration without invoking dark energy or modified gravity [16]. In addition to clarifying by way of examples misunderstanding regarding velocity reciprocity, the above provides a starting framework for introducing inhomogeneity and anisotropy into cosmological models.

References

1. A.P. French, *Special Relativity* (W.W. Norton, New York, 1968), p.78
2. H.D. Young, R.A. Freedman, *Sears and Zemansky's University Physics with Modern Physics*, 12th edn. (Pearson, Addison Wesley, 2008), p. 1284, paragraph below Eq. (37.18). (Specifically, they write: " ...we note that the principle of relativity requires that the *form* of the transformation from S to S' be identical to that from S' to S. The only difference is a change in the sign of the relative velocity component u.)
3. Raymond A. Serway, John W. Jewett, *Physics for Scientists and Engineers with Modern Physics*, 9th edn. (Brooks/Cole Cengage, Boston, MA, 2012), p.1211
4. A. Pelissetto, M. Testa, Am. J. Phys. **83**(4), 38–340 (2015)
5. K.M. Browne, Am. J. Phys. **88**(3), 207–213 (2020)
6. H. Poincaré, Rendiconti del Circolo Matematico di Palermo **21**, 129–176 (1906). (See INTRODUCTION, paragraph 2 of this reference for Poincaré's assertion about the universal and urestricted validity of the relativity principle. Poincaré seems to have been the first person ever to insist that it should have universal validity.)
7. P. Moylan, Velocity reciprocity and the relativity principle. Am. J. Phys. **90**, 126–134 (2022)
8. J.M. Lévy-Leblond, Am. J. Phys. **44**(3), 1–6 (1976)
9. V. Berzi, V. Gorini, J. Math. Phys. **10**, 1518–1524 (1969)
10. R.O. Popovych et al., J. Phys. A. **36**(26), 73377360 (2003)
11. L. Šnobl, P. Winternitz, *Classification and Identification of Lie Alebras* CRM Monograph Series 33. (Am. Math. Soc., Providence, 2014), p. 228
12. P. Frank, H. Rothe, Ann. Phys., Lpz. **34**(IV), 823–853 (1911)
13. V. Lalan, Compte-rendus de l'Acad. Scien. **203**, 1491 (1936); V. Lalan, Bulletin Soc. Math. France **65**, 83–99 (1937)
14. P. Moylan, Parallelizations for induced representations of SO(2, q), in *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics*, ed. by V. Do-brev. Springer Proceedings in Mathematics & Statistics, vol 263 (Springer, Singapore, 2018)
15. M. Havliček, J. Kotrbatý, P. Moylan, S. Pošta, J. Math. Phys. **59**, 021702 (2018)
16. X.-P. Yan, D.-Z. Liu, H. Wei, Phys. Lett. B **742**, 149–159 (2015)

Meta-Schrödinger Transformations

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Abstract Meta-Schrödinger transformations are the dynamical symmetries of equations of combined ballistic and diffusive transport. Their Lie algebra is derived in $1+1$ space dimensions and the infinite-dimensional generalisation is constructed. Representations without time-translation-invariance are given and the co-variant two-point functions are computed.

Keywords Schrödinger algebra · Meta-conformal algebra · Critical phenomena · Non-equilibrium phase transition

1 Introduction

While $2D$ conformal invariance [4] is an essential ingredient in string theory [35] or equilibrium critical phenomena [19, 23], non-equilibrium statistical mechanics furnishes different examples where a naturally realised dilatation-symmetry can be extended to larger (eventually infinite-dimensional) symmetry algebra. Besides the conformal algebra $\mathfrak{vir} \oplus \mathfrak{vir}$ itself, see e.g. [5, 8, 9, 13] and also the example of generalised hydrodynamics after quantum quenches [6, 11, 14, 34, 36], the most simple example are systems described by an underlying simple diffusion equation whose dynamical symmetry was recognised by Jacobi [29] and Lie [30] and is called today *Schrödinger algebra* $\mathfrak{sch}(d)$ [33]. Its infinite-dimensional extension is the *Schrödinger-Virasoro algebra* $\mathfrak{sv}(d)$ [22]. *Contra* a wide-held belief, the Schrödinger

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_37

algebra is *not* the non-relativistic limit obtained from the conformal algebra by a Lie algebra contraction. Rather this procedure leads to the *conformal Galilean algebra*, e.g. [1, 16, 17, 21, 24, 32] (including its infinite-dimensional extension), which also arises independently in gravitational physics [2, 3, 7]. On the other hand, the dynamical symmetries of simple ballistic transport equations has recently been identified as the new *meta-conformal algebra* $\mathfrak{meta}(1, d)$ [28], whose infinite-dimensional extensions are isomorphic to the direct sum of two Virasoro algebras in $d = 1$ spatial dimensions and of three Virasoro algebras for $d = 2$. Again, the non-relativistic limit of these are the conformal Galilean algebras.

The physical context of these application is physical ageing: a many-body system is prepared in a disordered state and then quenched to either a critical point or else into the ordered phase where several equivalent equilibrium states coexist. After the quench, such systems (i) relax slowly, (ii) break time-translation-invariance and (iii) obey dynamical scaling, which are the three defining properties of *physical ageing* [26]. Schrödinger-invariance is realised in quenches to the ordered phase, see [26] for a detailed review. Meta-conformal invariance may be realised if the underlying microscopic dynamics has a directional bias [20, 28].

Here, we shall describe the new *'meta-Schrödinger algebra'* $\mathfrak{metasch}(1, 1)$ [38]. It arises for combined ballistic and diffusive transport in different spatial directions, see Eq. (2) below. Such equations also arise in the study of driven diffusive systems, see [37] for a classic review. Section 2 outlines the construction, including of the infinite-dimensional extension, called *meta-Schrödinger-Virasoro algebra* $\mathfrak{msv}(1, 1)$. We prove the semi-direct sum [38]

$$\mathfrak{msv}(1, 1) \cong \left(\mathfrak{vir} \oplus \mathfrak{vir}\right) \ltimes \mathfrak{gal}(1) \cong \mathfrak{vir} \ltimes \mathfrak{sv}(1) \tag{1}$$

(where $\mathfrak{gal}(1)$ is the infinite-dimensional algebra of generalised Galilei transformations in the y-direction). Subsection 2.4 considers the necessary generalisations for physical ageing, when the time-translation generator $-\partial_t$ must be modified. Section 3 lists the co-variant two-point functions of quasi-primary scaling operators.

2 Construction of the Meta-Schrödinger Algebra

Definition: *The* meta-Schrödinger algebra $\mathfrak{metasch}(1, 1)$ *acts as dynamical symmetry algebra of the following biased evolution equation*

$$\mathscr{S}\Phi(t, x, y) := \left(\partial_t - S_1\partial_x - S_2\partial_y^2\right)\Phi(t, x, y) = 0. \tag{2}$$

Following [33], infinitesimal symmetries of (2) are written in the form

$$X = -A(t, x, y)\partial_t - B(t, x, y)\partial_x - C(t, x, y)\partial_y - D(t, x, y), \tag{3}$$

where the functions A, B, C, D depend all on t, x, y and must satisfy

$$[\mathscr{S}, X]\Phi(t,x,y) = \lambda(t,x,y)\mathscr{S}\Phi(t,x,y) \tag{4}$$

for an arbitrary field $\Phi = \Phi(t,x,y)$. The functions A, B, C, D must obey the system[1]

$$\begin{array}{rl} S_2 A_{yy} + S_1 A_x - \dot{A} - \lambda = 0 \;\; , & S_2 B_{yy} + S_1 B_x - \dot{B} + \lambda S_1 = 0 \\ 2S_2 C_y + \lambda S_2 = 0 \;\; , & S_2 C_{yy} + S_1 C_x - \dot{C} + 2S_2 D_y = 0 \\ S_2 D_{yy} + S_1 D_x - \dot{D} = 0 \;\; , & 2S_2 A_y = 0 \quad , \quad 2S_2 B_y = 0 \end{array} \tag{5}$$

The solution of (5) is aided by considering two sub-algebras.

1. For fields $\Phi = \Phi(t,x)$, Eq. (2) becomes the ballistic transport equation in the spatial x-direction. Its dynamical symmetry is the meta-conformal algebra $\mathfrak{meta}(1,1)$.
2. For fields $\Phi = \Phi(t,y)$, Eq. (2) becomes the diffusion equation in the spatial y-direction. Its dynamical symmetry is the Schrödinger algebra $\mathfrak{sch}(1)$.

Both should be sub-algebras of the sought Lie algebra $\mathfrak{metasch}(1,1)$. A natural starting point for the construction of $\mathfrak{metasch}(1,1)$ will be representations of the meta-conformal algebra which obey the condition (4) with the Schrödinger operator (2). Recall the algebraic structure of the meta-conformal and Schrödinger algebras

$$\mathfrak{meta}(1,1) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}), \quad \mathfrak{sch}(1) \cong \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{hei}(1) \tag{6}$$

(the Heisenberg algebra $\mathfrak{hei}(1)$ includes the central extension). We start from
Ansatz: *Representations of* $\mathfrak{metasch}(1,1)$ *are given by semi-direct sums of representations of the meta-conformal algebra* $\mathfrak{meta}(1,1)$*, with known action in the spatial direction* x *and the Heisenberg algebra* $\mathfrak{hei}(1)$*, with known action in the spatial direction* y*.*
We write $X_{-1} = -\partial_t$, $Y_{-1}^x = -\partial_x$, $Y_{-\frac{1}{2}}^y = -\partial_y$, $M_0 = -\mathscr{M}$ and

$$\begin{aligned} X_0 &= -t\partial_t - x\partial_x - \frac{1}{2}y\partial_y - \delta \\ X_1 &= -(t^2+\alpha x^2)\partial_t - (2tx+\beta x^2)\partial_x - C^{X_1}(t,x,y)\partial_y - D^{X_1}(t,x,y) - 2\delta t - 2\gamma x \\ Y_0^x &= -\alpha x\partial_t - (t+\beta x)\partial_x - C^{Y_0}(t,x,y)\partial_y - D^{Y_0}(t,x,y) - \gamma \\ Y_1^x &= -\alpha(2tx+\beta x^2)\partial_t - ((t+\beta x)^2+\alpha x^2)\partial_x - C^{Y_1}(t,x,y)\partial_y - D^{Y_1}(t,x,y) \\ &\quad -2\gamma t - 2(\alpha\delta+\beta\gamma)x \\ Y_{\frac{1}{2}}^y &= -A^{Y_{\frac{1}{2}}}(t,x)\partial_t - B^{Y_{\frac{1}{2}}}(t,x)\partial_x - C^{Y_{\frac{1}{2}}}(t,x,y)\partial_y - D^{Y_{\frac{1}{2}}}(t,x,y) - \mathscr{M}y \end{aligned} \tag{7}$$

where α, β are constants. To find A, B, C, D of each generator, use the sub-algebras:

[1] We use the notations $\dot{A} = \partial_t A(t,x,y)$, $A_{xy} = \partial_x\partial_y A(t,x,y)$ etc.

- the meta-conformal algebra $\mathfrak{meta}(1,1) = \langle X_n, Y_n^x \rangle_{n\in\mathbb{Z}}$, with the commutators

$$
\begin{aligned}
&[X_n, X_m] = (n-m)X_{n+m}, \quad [X_n, Y_m^x] = (n-m)Y_{n+m}^x \\
&[Y_n^x, Y_m^x] = (n-m)\big(\alpha X_{n+m} + \beta Y_{n+m}^x\big)
\end{aligned} \tag{8}
$$

- the Schrödinger sub-algebra $\mathfrak{sch}(1) = \langle X_{0,\pm 1}, Y_{\pm\frac{1}{2}}^y, M_0 \rangle$ and

$$
\begin{aligned}
&[X_n, X_m] = (n-m)X_{n+m}, \quad [X_n, Y_p^y] = \left(\frac{n}{2} - p\right) Y_{n+p}^y, \\
&[Y_p^y, Y_q^y] = (p-q)M_{p+q}, \quad [X_n, M_m] - mM_{n+m}
\end{aligned} \tag{9}
$$

For $n \in \mathbb{Z}$ and $p \in \mathbb{Z} + \frac{1}{2}$, one has the infinite-dimensional algebras $\mathfrak{metav}(1,1) = \langle X_n, Y_n^x \rangle$ and $\mathfrak{sv}(1) = \langle X_n, Y_n^x, Y_p^y, M_n \rangle$.

We should find $\mathfrak{sl}(2,\mathbb{R}) = \langle X_{0,\pm 1} \rangle$ as sub-algebra acting on both x and y, but such that the commutator $[Y_n^x, Y_p^y]$ closes into the algebra. All unknown functions in the generators (7) are found from the above commutator relations (8), (9) and the Eq. (5). The first equation of the system (5) gives $\lambda^{X_1} = -2t + 2\alpha S_1 x$. Upon substitution into the second equation (5), this leads to a quadratic equation for S_1

$$
\alpha S_1^2 + \beta S_1 - 1 = 0. \tag{10}
$$

Set $c := -\alpha S_1$, then $\alpha = c(c-\beta) \neq 0$. It is enough to construct X_1 explicitly.

2.1 The General Case: $\alpha \neq 0$

The algebra $\mathfrak{metasch}(1,1)$ is spanned by the generators [38]:

$$
\begin{aligned}
X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - x\partial_x - \frac{y}{2}\partial_y - \delta \\
X_1 &= -(t^2 + \alpha x^2)\partial_t - (2tx + \beta x^2)\partial_x - (t + cx)y\partial_y - 2t\delta - 2\gamma x - \frac{\mathscr{M}}{2}y^2 \\
Y_{-1}^x &= -\partial_x, \quad Y_0^x = -\alpha x\partial_t - (t + \beta x)\partial_x - \frac{c}{2}y\partial_y - \gamma \\
Y_1^x &= -\alpha(2tx + \beta x^2)\partial_t - (t^2 + 2\beta tx + (\alpha + \beta^2)x^2)\partial_x - (ct + (\alpha + \beta c)x)y\partial_y \\
&\quad -2\gamma t - 2(\alpha\delta + \beta\gamma)x - \frac{c\mathscr{M}}{2}y^2 \\
Y_{\frac{1}{2}}^y &= -\partial_y, \quad Y_{\frac{1}{2}}^y = -(t + cx)\partial_y - \mathscr{M}y, \quad M_0 = -\mathscr{M},
\end{aligned} \tag{11}
$$

with the non-vanishing commutation relations, with $n, m = \pm 1, 0$ and $p = \pm\frac{1}{2}$

$$
\begin{aligned}
&[X_n, X_m] = (n-m)X_{n+m}, \quad [X_n, Y_m^x] = (n-m)Y_{n+m}^x \\
&[X_n, Y_p^y] = \left(\frac{n}{2} - p\right)Y_{n+p}^y, \qquad [Y_n^x, Y_m^x] = (n-m)\left(\alpha X_{n+m} + \beta Y_{n+m}^x\right) \\
&[Y_n^x, Y_p^y] = c\left(\frac{n}{2} - p\right)Y_{n+p}^y, \quad [Y_{1/2}^y, Y_{-1/2}^y] = M_0.
\end{aligned} \tag{12}
$$

Next, if we let

$$
S_1 = -\frac{c}{\alpha} = -\frac{1}{c-\beta} \quad , \quad S_2 = \frac{1}{2\mathscr{M}}\frac{2c-\beta}{c-\beta} \quad , \quad \text{and} \quad \gamma = \frac{2c-\beta}{4} + (\beta - c)\delta \tag{13}
$$

then the Schrödinger operator (2) becomes $\mathscr{S} = \partial_t + \frac{1}{c-\beta}\partial_x - \frac{1}{2\mathscr{M}}\frac{2c-\beta}{c-\beta}\partial_y^2$. It is readily checked that all symmetry conditions (4) are obeyed [38]. Notice that the representation (11) and all consequences are valid only for $c \neq \beta$ and $c \neq \beta/2$.

2.2 Infinite-Dimensional Extension

The infinite-dimensional extension of the representation (11) is constructed as follows [38]. First, in terms of the of the variable $\rho = t + cx$, the infinite-dimensional extension of the Heisenberg algebra is

$$
Y_p^y = -\rho^{p+1/2}\partial_y - (p+1/2)\mathscr{M}\rho^{p-1/2}y \quad , \quad M_n = -\mathscr{M}\rho^n \tag{14}
$$

such that for $p, q \in \mathbb{Z} + \frac{1}{2}$ we have the commutator $[Y_p^y, Y_q^y] = (p-q)M_{p+q}$. Next, following [28], define a new family of generators $\mathscr{Y}_n := \mathscr{N}\left(aX_n + Y_n^x\right)$ whose normalisation $\mathscr{N}$ will be fixed shortly. The new generators satisfy

$$
[\mathscr{Y}_n, \mathscr{Y}_m] = (n-m)(2a+\beta)\mathscr{N}\mathscr{Y}_{n+m} \tag{15}
$$

provided that a satisfies $a^2 + \beta a - c(c-\beta) = 0$. The two solutions $a_{1,2}$ of this quadratic equation, namely $a_1 = -c$ and $a_2 = c - \beta$, produce two distinct forms, denoted $\mathscr{Y}_n^{(1,2)}$, of the generators. We then obtain

$$
\mathscr{Y}_n^{(1)} = \mathscr{N}^{(1)}\left(-cX_n + Y_n^x\right) \quad , \quad \mathscr{Y}_n^{(2)} = \mathscr{N}^{(2)}\left((c-\beta)X_n + Y_n^x\right) \tag{16}
$$

which both satisfy the commutator (15). We now fix the normalisations from the requirements $(\beta - 2c)\mathscr{N}^{(1)} \stackrel{!}{=} 1 \stackrel{!}{=} (2c-\beta)\mathscr{N}^{(2)}$.

Analogously, we construct

$$
\mathscr{A}_n := X_n + b\mathscr{Y}_n \quad ; \quad \text{with } [\mathscr{A}_n, \mathscr{A}_m] = (n-m)\mathscr{A}_{n+m}. \tag{17}
$$

Since a has the admissible values $a_{1,2}$, it follows that either $b = 0$ or $b = -1$ [38]. Then three distinct forms of the $\mathscr{A}_n$ are possible, namely

$$\mathscr{A}_n^{(0)} = X_n \qquad\qquad \text{with } \left[\mathscr{A}_n^{(0)}, \mathscr{Y}_m^{(1,2)}\right] = (n-m)\mathscr{Y}_{n+m}^{(1,2)} \;\text{ if }\; b=0 \tag{18}$$

$$\left.\begin{array}{l} \mathscr{A}_n^{(1)} = \frac{(c-\beta)X_n+Y_n^x}{2c-\beta} = \mathscr{Y}_n^{(2)} \\ \mathscr{A}_n^{(2)} = -\frac{-cX_n+Y_n^x}{2c-\beta} = \mathscr{Y}_n^{(1)} \end{array}\right\} \text{with } \left[\mathscr{A}_n^{(0)}, \mathscr{Y}_m^{(1,2)}\right] = 0 \qquad \text{if }\; b=-1$$

This construction is valid[2] if $\beta \neq 2c$ and $\beta \neq c$. Then the maximal finite-dimensional sub-algebra (11) is the dynamical symmetry of the Eq. (2).

We write down the generators explicitly[3] for $b=-1$. Because of (18), we have $\mathscr{A}_n^{(1)} = \mathscr{Y}_n^{(2)}$, and $\mathscr{A}_n^{(2)} = \mathscr{Y}_n^{(1)}$. Hence, these possibilities are not independent. Let $\mathscr{A}_n = \mathscr{A}_n^{(1)}$ and $\mathscr{Y}_n = \mathscr{Y}_n^{(1)}$. They are readily obtained from the explicit expressions for X_n, Y_n^x in (11), but working with light-cone-like variables $\sigma = t + (\beta - c)x$ and $\rho = (t+cx)$ leads to the more elegant form

$$\mathscr{Y}_n = -\sigma^{n+1}\partial_\sigma + (n+1)\frac{c\delta-\gamma}{2c-\beta}\sigma^n \tag{19a}$$

$$\mathscr{A}_n = -\rho^{n+1}\partial_\rho - (n+1)\left(\frac{(c-\beta)\delta+\gamma}{2c-\beta} + \frac{y}{2}\partial_y\right)\rho^n - \frac{n(n+1)}{4}\mathscr{M}y^2\rho^{n-1} \tag{19b}$$

Hence the Lie algebra $\left\langle \mathscr{A}_n, \mathscr{Y}_n, Y_p^y, M_n \right\rangle$ has the non-vanishing commutators, for $n,m \in \mathbb{Z}$ and $p,q \in \mathbb{Z}+\frac{1}{2}$, using (14)

$$\begin{aligned} &[\mathscr{A}_n, \mathscr{A}_m] = (n-m)\mathscr{A}_{n+m}, \quad [\mathscr{Y}_n, \mathscr{Y}_m] = (n-m)\mathscr{Y}_{n+m} \\ &[\mathscr{A}_n, Y_p^y] = \left(\frac{n}{2} - p\right) Y_{n+p}^y, \quad [Y_p^y, Y_q^y] = (p-q)M_{p+q}, \quad [\mathscr{A}_n, M_m] = -mM_{n+m} \end{aligned} \tag{20}$$

which are those of the meta-Schrödinger-Virasoro Lie algebra (1). The Schrödinger-Virasoro algebra $\mathfrak{sv}(1) = \left\langle \mathscr{A}_n^B, Y_p^y, M_n \right\rangle \subset \mathfrak{msv}(1,1)$ is an obvious sub-algebra.

Since in light-cone coordinates, $\mathscr{S} = \left(\frac{2c-\beta}{c-\beta}\partial_\rho - \frac{1}{2\mathscr{M}}\partial_y^2\right)$, the dynamical symmetry follows from, if $\gamma = \frac{2c-\beta}{4} + (\beta-c)\delta$,

$$\begin{aligned} \left[\mathscr{S}, \mathscr{A}_n\right] &= \frac{2c-\beta}{c-\beta}\Big((n+1)\rho^n\mathscr{S} \\ &\qquad + n(n+1)\rho^{n-1}\left(\frac{(c-\beta)\delta+\gamma}{2c-\beta} - \frac{1}{4}\right) + \frac{n^3-n}{4}\rho^{n-2}\mathscr{M}y^2\Big) \\ \left[\mathscr{S}, Y_p^y\right] &= \frac{2c-\beta}{c-\beta}\mathscr{M}(p-\frac{1}{2})(p+\frac{1}{2})\rho^{p-3/2}y \quad ; \quad [\mathscr{S}, \mathscr{Y}_n] = [\mathscr{S}, M_0] = 0 \end{aligned} \tag{21}$$

[2] In the limit $\beta - 2c \to 0$, a Lie algebra contraction should lead to representations related to the conformal Galilean algebra.

[3] The case $b=0$ gives the same algebra, up to a change of basis [38].

such that either the maximal finite-dimensional sub-algebra $\mathfrak{metasch}(1,1)$ or else $\left\langle \mathscr{A}_{\pm 1,0}, \mathscr{Y}_n, Y_{\pm\frac{1}{2}}, M_0 \right\rangle$ leave the solution space of (2) invariant.

2.3 The Special Case $\alpha = 0$

If $\alpha = 0$, one has $\lambda^{X_1} = -2t$ and from (10) $S_1 = 1/\beta$. Since we had before $\alpha = c(c-\beta)$, the results for $\alpha = 0$ can be obtained from those of the previous sub-section by setting $c = 0$ (but not $c = \beta$!) in the generators (11) as well as in commutation relations (12). This directly produces an infinite-dimensional representation of $\mathfrak{msv}(1,1)$ whose commutators follow from (12) with $\alpha = c = 0$ In the light-cone variables $\tau = t$, $v = t + \beta x$, y, an elegant form is

$$\begin{aligned}
A_n &:= X_n - \frac{1}{\beta} Y_n^x = -\tau^{n+1}\partial_\tau - (n+1)\left(\frac{1}{2}\tau^n y \partial_y + \left(\delta - \frac{\gamma}{\beta}\right)\tau^n + \frac{n}{4}\mathscr{M}\tau^{n-1}y^2\right) \\
Y_n^x &= -\beta v^{n+1}\partial_v - (n+1)\gamma v^n, \\
Y_p^y &= -\tau^{p+\frac{1}{2}}\partial_y - (p+\frac{1}{2})\mathscr{M}\tau^{p-\frac{1}{2}}y, \quad M_n = -\mathscr{M}\tau^n,
\end{aligned} \tag{22}$$

whose non-vanishing commutators ($n, m \in \mathbb{Z}$ and $p, q \in \mathbb{Z} + \frac{1}{2}$)

$$\begin{aligned}
&[A_n, A_m] = (n-m)A_{n+m}, \qquad && [A_n, Y_p^y] = \left(\frac{n}{2} - p\right)Y_{n+p}^y, \\
&[Y_n^x, Y_m^x] = (n-m)\beta Y_{n+m}^x \qquad && [Y_p^y, Y_q^y] = (p-q)M_{p+q}
\end{aligned} \tag{23}$$

again reproduce (1). In light-cone variables the Schrödinger operator (2) simplifies into $\mathscr{S} = \partial_t - \frac{1}{\beta}\partial_x - \frac{1}{2\mathscr{M}}\partial_y^2 = \partial_\tau - \frac{1}{2\mathscr{M}}\partial_y^2$ and if $\frac{\gamma}{\beta} = \delta - \frac{1}{4}$, the dynamical symmetries are obvious, as before [38].

2.4 Representations Without Time-Translation-Invariance

A system undergoing physical ageing is brought out of equilibrium by a quench in its thermodynamic parameters. It cannot be at equilibrium which suggests that the generator $X_{-1} = -\partial_t$ of time-translations should not be part of the symmetry algebra [25]. However, for the meta-Schrödinger algebra, such a restrictive prescription is not adequate, because Eq. (12) shows that without the generator X_{-1} of time-translations, the conformal algebra of the generators Y_n^x does not close, especially

$$[Y_0^x, Y_{-1}^x] = \alpha X_{-1} + \beta Y_{-1}^x. \tag{24}$$

For the Schrödinger-Virasoro algebra, it can be shown how to generalise [31] the defining representation such that (a) time-translation-invariance is broken and (b) the full Lie algebra $\mathfrak{sv}(d)$ is kept [27]. This procedure can be applied to the meta-Schrödinger-Virasoro algebra as well. In the special case $\alpha = 0$, to which we restrict here, we find for the generator $A_n = X_n - \frac{1}{\beta} Y_n^x$, in light cone variables [38]

$$A_n = -\tau^{n+1}\partial_\tau - \frac{n+1}{2}\tau^n y \partial_y - (n+1)\big(\delta - \frac{\gamma}{\beta}\big)\tau^n - n\xi\tau^n - \frac{n(n+1)}{4}\mathscr{M}\tau^{n-1}y^2 \tag{25}$$

All other generators maintain their form stated in (23). The only new element is a 'second scaling dimension' ξ. Together with the 'first scaling dimension' δ and the 'rapidity' γ, it can be used to characterise scaling operators out of stationary states.

In light-cone variables, the invariant Schrödinger operator becomes $\mathscr{S} = \partial_\tau - \frac{1}{2\mathscr{M}}\partial_y^2 + \frac{\xi}{\tau}$ For checking the symmetry, all commutators of the time-translation-invariant case can be taken over. The only exception is

$$[\mathscr{S}, A_1] = -2t\mathscr{S} + \big(2\mathscr{M}S_2 - 1\big)y\partial_y - 2\big(\delta + \xi - \frac{\gamma}{\beta} - \frac{1}{4}\big) \tag{26}$$

Hence the conditions for a symmetry are now $S_2 = \frac{1}{2\mathscr{M}}$ and $\frac{\gamma}{\beta} = \delta + \xi - \frac{1}{4}$.

3 Covariant Two-Point Functions

As an application, we consider the form of the co-variant two point-function

$$F(t_1, t_2, x_1, x_2, y_1, y_1) := \big\langle \Phi_1(t_1, x_1, y_1)\Phi_2(t_2, x_2, y_2)\big\rangle \tag{27}$$

where Φ_1, Φ_2 are quasi-primary scaling operators of $\mathfrak{metasch}(1,1)$. From the representations constructed before, the scaling operators are characterised by the several parameters introduced. Physically, we distinguish between the stationary case, which is time-translation-invariant and the ageing case, which is not [38].

3.1 Stationary Case

In the time-translation-invariant case we consider scaling operators Φ_i $(i = 1, 2)$, characterised by the parameters $(\delta_i, \frac{\gamma_i}{\beta_i}, \mathscr{M}_i, \beta_i)$, transforming covariantly under the representation with $\alpha = 0$ (called case A). Lifting the single-body representation of Sect. 2 to a two-body representation, this produces a set of Ward identities which fix the form of $F(t, x, y) = G^{(A)}(t, t + \beta x, y)$. Because of the translation-invariance, F will only depend on the differences $t := t_1 - t_2$, $x := x_1 - x_2$ and $y := y_1 - y_2$ and it can also be shown that $\beta_1 = \beta_2 = \beta$. Letting $v = t + \beta x$, the final result is

$$G^{(A)}(t,v,y) = G_0\,\delta_{\mathscr{M}_1+\mathscr{M}_2,0}\,\delta_{\delta_1,\delta_2}\delta_{\gamma_1,\gamma_2}\,t^{-2\delta_1+2\gamma_1/\beta}\,v^{-2\gamma_1/\beta}\exp\left(-\frac{\mathscr{M}_1}{2}\frac{y^2}{t}\right), \tag{28}$$

where G_0 is a normalisation constant. This form combines aspects of Schrödinger-invariance in the transverse coordinate y and of meta-conformal invariance in the parallel coordinate x.

Similarly, if $\alpha \neq 0$ (called case B) one works with the coordinates $\rho = t + cx$ and $\sigma = t + (\beta - c)x$. Again, $\beta := \beta_1 = \beta_2$ and $c := c_1 = c_2$. Using the definitions $\Gamma := \frac{c\delta-\gamma}{2c-\beta}$ and $\Delta := \delta$ and writing $F(t,x,y) = G^{(B)}(\rho,\sigma,y)$, we find

$$G^{(B)}(\rho,\sigma,y) = G_0\,\delta_{\mathscr{M}_1+\mathscr{M}_2,0}\,\delta_{\Delta_1,\Delta_2}\delta_{\Gamma_1,\Gamma_2}\,\rho^{-2\Delta_1+2\Gamma_1}\,\sigma^{-2\Gamma_1}\exp\left(-\frac{\mathscr{M}_1}{2}\frac{y^2}{\rho}\right) \tag{29}$$

and where we also have $\rho = \rho_1 - \rho_2$, $\sigma = \sigma_1 - \sigma_2$ and $y = y_1 - y_2$.

3.2 Ageing Case

For non-equilibrium dynamics, as it occurs in ageing systems, time-translation-invariance does not hold and the scaling operators $\Phi_{1,2}$ are quasi-primary with respect to the representations derived in Sect. 3. The time variables are now $t = t_1 - t_2$ and $u = t_1/t_2$. We confine ourselves to the case $\alpha = 0$ (case A) and find, with $F(t,u,x,y) = G^{(A)}(t,u,v,y)$ and $v = t + \beta x$

$$\begin{aligned} G^{(A)}(t,u,v,y) = {} & G_0\delta_{\mathscr{M}_1+\mathscr{M}_2,0}\,\delta_{\gamma_1,\gamma_2}\delta_{\delta_1+\xi_1,\delta_2+\xi_2} \\ & \times t^{-\delta_1-\delta_2+2\gamma_1/\beta}u^{\xi_1}(u-1)^{-\xi_1-\xi_2}v^{-2\gamma_1/\beta}\exp\left(-\frac{\mathscr{M}_1}{2}\frac{y^2}{t}\right) \end{aligned} \tag{30}$$

Acknowledgements MH and SS thank the PHC Rila (KP-06-RILA/7) for financial support. SS is also supported by Bulgarian National Science Fund Grant KP-06-N28/6.

References

1. N. Aizawa, Z. Kuznetsova, F. Toppan, Invariant partial differential equations with two-dimensional exotic centrally extended conformal Galilei symmetry. J. Math. Phys. **57**, 041701 (2016). arXiv:1512.02290
2. G. Barnich, G. Compère, Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions. Class. Quant. Grav. **24**, F15 (2007); corrigendum **24**, 3139 (2007). arXiv:gr-qc/0610130
3. G. Barnich, A. Gomberoff, H.A. González, Three-dimensional Bondi-Metzner-Sachs invariant two-dimensional field-theories as the flat limit of Liouville theory. Phys. Rev. D **87**, 124032 (2007). arxiv:1210.0731

4. A.A. Belavin, A.M. Polykaov, A.B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field-theory. Nucl. Phys. B **241**, 333 (1984)
5. D. Bernard, B. Doyon, Conformal field theory out of equilibrium: a review, J. Stat. Mech. 064005 (2016). arxiv:1603.07765
6. B. Bertini, M. Collura, J. de Nardis, M. Fagotti, Transport in out-of-equilibrium XXZ chains: exact profiles of charges and currents, Phys. Rev. Lett. **117**, 207201 (2016), [arXiv:1605.09790]
7. H. Bondi, M.G.J. van der Burg, A.W.K. Metzner, Gravitational waves in general relativity. Proc. Roy. Soc. London **A269**, 21 (1962)
8. P. Calabrese, J.L. Cardy, Entanglement and correlation functions following a local quench: a conformal field theory approach. J. Stat. Mech. P10004 (2007). arXiv:0708.3750
9. P. Calabrese, J.L. Cardy, Quantum quenches in 1+1 dimensional conformal field theories. J. Stat. Mech. P064003 (2016). arXiv:1603.02889
10. A. Cappelli, G.V. Dunne, C.A. Trugenberger, G.R. Zemba, Conformal symmetry and universal properties of quantum Hall states. Nucl. Phys. B **398**, 531 (1993). ([arXiv:hep-th/9211071])
11. O.A. Castro-Alvaredo, B. Doyon, T. Yoshimura, Emergent hydrodynamics in integrable quantum systems out of equilibrium. Phys. Rev. **X6**, 041065 (2016). arXiv:1605.07331
12. R. Cherniha, V. Davydovych, *Nonlinear Reaction-diffusion Systems*, Springer Lecture Notes in Mathematics LNM, vol. 2196. (Springer, Heidelberg, 2017)
13. G. Delfino, Correlation spreading and properties of the quantum state in quench dynamics. Phys. Rev. **E97**, 062138 (2018). arXiv:1710.06275
14. B. Doyon, J. Dubail, R. Konik, T. Yoshimura, Large-scale description of interacting one-dimensional Bose gases. Phys. Rev. Lett. **119**, 195301 (2017). arXiv:1704.04151
15. A. Dutta, G. Aeppli, B.K. Chakrabarti, U. Divakaran, T.F. Rosenbaum, D. Sen, *Quantum Phase Transitions in Transverse-Field Spin Models* (Cambridge University Press, Cambridge 2015)
16. C. Duval, P.A. Hórvathy, Non-relativistic conformal symmetries and Newton-Cartan structures. J. Phys. A: Math. Theor. **42**, 465206 (2009). arXiv:0904.0531
17. C. Duval, G.W. Gibbons, P.A. Hórvathy, Conformal Carroll groups. J. Phys. A: Math. Theor. **47**, 335204 (2014). arXiv:1403.4213
18. C. Duval, G.W. Gibbons, P.A. Hórvathy, Conformal Carroll groups and BMS symmetry. Class. Quantum Gravity **34**, 092001 (2014). arXiv:1402.5894
19. P. di Francesco, P. Mathieu, D. Sénéchal, *Conformal Field Theory* (Springer, Heidelberg, 1997)
20. C. Godrèche, J.-M. Luck, Single spin-flip dynamics of the Ising chain. J. Stat. Mech. P05033 (2015). arxiv:1503.01661
21. P. Havas, J. Plebanski, Conformal extensions of the Galilei group and their relation to the Schrödinger group. J. Math. Phys. **19**, 482 (1978)
22. M. Henkel, Schrödinger-invariance and strongly anisotropic critical systems. J. Stat. Phys. **75**, 1023 (1994). arxiv:hep-th/9310081
23. M. Henkel, *Conformal Invariance and Critical Phenomena* (Springer, Heidelberg, 1999)
24. M. Henkel, J. Unterberger, Schrödinger invariance and space-time symmetries. Nucl. Phys. B **660**, 407 (2003). ([hep-th/0302187])
25. M.Henkel, T. Enss, M. Pleimling, On the identification of quasiprimary operators in local scale-invariance. J. Phys. **39**, L589 (2006). arxiv:cond-mat/0605211
26. M. Henkel, M. Pleimling, *Non-equilibrium Phase Transitions Vol. 2: Ageing and Dynamical Scaling Far from Equilibrium* (Springer, Heidelberg, Germany, 2010)
27. M. Henkel, Dynamical symmetries and causality in non-equilibrium phase transitions. Symmetry **7**, 2108 (2015). arxiv:1509.03669
28. M. Henkel, S. Stoimenov, Infinite-dimensional meta-conformal Lie algebras in one and two spatial dimensions. J. Stat. Mech. 084009 (2020). arxiv:1810.09855v3
29. C.G. Jacobi, Vorlesungen über Dynamik, 4. Vorlesung (Königsberg 1842/43), in *Gesammelte Werke von C.G. Jacobi*. ed. by A. Clebsch, A. Lottner (Akademie der Wissenschaften, Berlin, 1866/1884)
30. S. Lie, Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen. Arch. Math. Naturvidenskap. (Kristiania) **6**, 328 (1881); S. Lie, *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*, Teubner (Leipzig 1891)

31. D. Minic, D. Vaman, C. Wu, Three-point functions of ageing dynamics and the AdS-CFT correspondence. Phys. Rev. Lett. **109**, 131601 (2012). arxiv:1207.0243
32. D. Martelli, Y. Tachikawa, Comments on Galilean conformal field theories and their geometric realization. JHEP **1005**, 091 (2010). arXiv:0903.5184
33. U. Niederer, The maximal kinematical invariance group of the free Schrödinger equation. Helv. Phys. Acta **45**, 802 (1972)
34. L. Piroli, J. de Nardis, M. Collura, B. Bertini, M. Fagotti, Transport in out-of-equilibrium XXZ chains. Phys. Rev. **B96**, 115124 (2017). arXiv:1706.00413
35. J. Polchinski, *String Theory*, vol. 2 (Cambridge University Press, Cambridge, 2001)
36. P. Ruggiero, P. Calabrese, B. Doyon, J. Dubail, Quantum generalized hydrodynamics. Phys. Rev. Lett. **124**, 140603 (2020). arxiv:1810.07170
37. B. Schmittmann, R.K.P. Zia, Statistical mechanics of driven diffusive systems, in *Phase Transitions and Critical Phenomena*, vol. 17, ed. by C. Domb, J.L. Lebowitz (Academic Press, London, 1995)
38. S. Stoimenov, M. Henkel, Meta-Schrödinger invariance (2021). arxiv:2112.14143

The Quantum Mirror to the Quartic del Pezzo Surface

Hülya Argüz

Abstract A log Calabi–Yau surface (X, D) is given by a smooth projective surface X, together with an anti-canonical cycle of rational curves $D \subset X$. The homogeneous coordinate ring of the mirror to such a surface—or to the complement $X \setminus D$—is constructed using wall structures and is generated by theta functions [6, 7]. In [1], we provide a recipe to concretely compute these theta functions from a combinatorially constructed wall structure in $\mathbf{R}^2$, called the heart of the canonical wall structure. In this paper, we first apply this recipe to obtain the mirror to the quartic del Pezzo surface, denoted by dP_4, together with an anti-canonical cycle of 4 rational curves. We afterwards describe the deformation quantization of this coordinate ring, following [4]. This gives a non-commutative algebra, generated by quantum theta functions. There is a totally different approach to construct deformation quantizations using the realization of the mirror as the monodromy manifold of the Painlevé IV equation [5, 8]. We show that these two approaches agree.

Keywords Mirror symmetry · del Pezzo surfaces · Quantization

1 The Mirror to (dP_4, D)

We construct the mirror to (dP_4, D), following the recipe in [1]. To obtain the theta functions generating the coordinate ring of the mirror, we define an initial wall structure associated to (X, D), using the following data:

- A choice of a *toric model*, that is, a birational morphism $(X, D) \to (\overline{X}, \overline{D})$ to a smooth toric surface $\overline{X}$ with its toric boundary $\bar{D}$ such that $D \to \overline{D}$ is an isomorphism. For the quartic del Pezzo surface $X = dP_4$, obtained by blowing up 5 general points in $\mathbf{P}^2$, we consider the toric model given by a toric blow-up of $\mathbf{P}^2$, illustrated as in Fig. 1.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_38

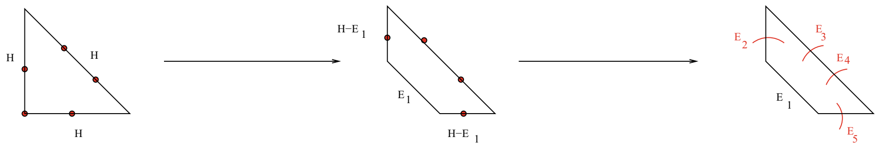

Fig. 1 On the left hand figure H is the class of a general line on the projective plane $\mathbf{P}^2$, which is illustrated by its momentum map image. In the middle $\overline{X}$ is the blow up of $\mathbf{P}^2$ at a toric point, E_1 is the class of the exceptional curve, and by abuse of notation H denotes the class of the pull-back of a general line. On the right hand figure we illustrate dP_4 together with the exceptional divisors obtained by 4 further non-toric blow-ups

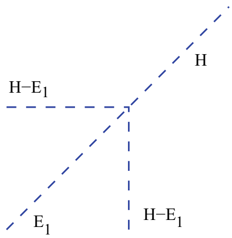

Fig. 2 The kinks of the MVPL function φ on the rays of the fan $\Sigma_{\overline{X}}$ associated to the toric model $\overline{X}$, drawn in dashed blue lines to distinguish them in what follows from walls

We choose $\overline{D}$ to be the toric boundary divisor in $\overline{X}$, and let D be the strict transform of $\overline{D}$. We note that the equations of the mirror will be independent of the choice of the toric model [1, 6].
This choice defines a natural subdivision of $M_{\mathbf{R}}$, where $M \cong \mathbf{Z}^2$ is a fixed lattice, and $M_{\mathbf{R}} = M \otimes \mathbf{R} \cong \mathbf{R}^2$, given by the toric fan $\Sigma_{\overline{X}} \subset M_{\mathbf{R}}$ of $\overline{X}$. We denote $M_{\mathbf{R}}$ together with this subdivision by $(M_{\mathbf{R}}, \Sigma_{\overline{X}})$.

- A *multi-valued piecewise linear* (MVPL) function φ on $(M_{\mathbf{R}}, \Sigma_{\overline{X}})$ with values in the monoid of integral points of the cone of effective curves on X, denoted by $\mathrm{NE}(X)$. Up to a linear function, we uniquely define φ by specifying its kinks along each ray of $\Sigma_{\overline{X}}$ to be the pullback of the class of the curve in $\overline{X}$ corresponding to this ray, as illustrated in Fig. 2.

Definition 1 A wall structure on $(M_{\mathbf{R}}, \Sigma_{\overline{X}})$ is a collection of pairs (ρ, f_ρ), called walls, consisting of rays $\rho \subset M_{\mathbf{R}}$, together with functions $f_\rho \in \mathbf{C}[NE(X)^{\mathrm{gp}}][M]$, referred to as wall-crossing functions. Each wall crossing function defines a wall-crossing transformation

$$\theta_{\gamma,\rho} : \quad z^v \longmapsto f_\rho^{\langle n_\rho, v\rangle} z^v , \tag{1}$$

prescribing how a monomial z^v changes when crossing the wall (ρ, f_ρ). Here, n_ρ is the normal vector to ρ chosen with a sign convention as in [2, Sect. 2.2.1].

We note that in general one might need to work with infinitely many walls, and then to work modulo an ideal of $NE(X)$ while defining the wall crossing functions. However, in the case of the quartic del Pezzo surface we will only need finitely many

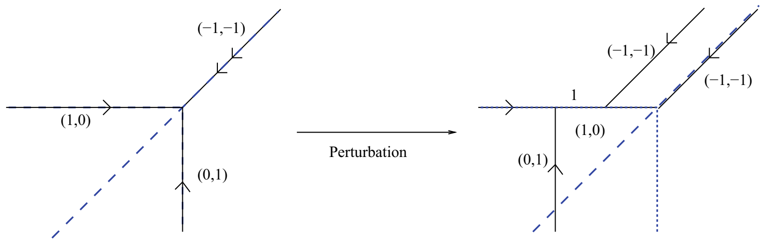

Fig. 3 Walls of the initial wall structure associated to (dP_4, D), and their perturbations obtained by translating some walls, so that each intersection is formed by only two walls meeting at a point. The dashed blue lines indicate where the kinks of the MVPL function φ are, the black rays with arrows are walls. On the left hand figure the function attached to wall with direction $(1, 0)$ is given by $1 + t^{-E_2}x^{-1}$, the function on the wall wit direction $(0, 1)$ is $1 + t^{-E_5}y^{-1}$, and the one on the wall with direction $(-1, -1)$ is $(1 + t^{-E_3}xy)(1 + t^{-E_4}xy)$, as there are two walls on top of each other. On the right hand figure the functions on the walls with $(1, 0)$ and $(0, 1)$ remain same. However, we have now two separate walls in direction $(-1, -1)$, with attached functions $(1 + t^{-E_3}xy)$ and $(1 + t^{-E_4}xy)$ respectively—we choose to attach the latter to the most right ray

walls, and the wall crossing functions will be elements of a polynomial ring, as in Definition 1.

To obtain the equation of the mirror to (X, D), we first define an initial wall structure associated to (X, D). To do this, we first define an initial set of walls in $(M_{\mathbf{R}}, \Sigma_{\overline{X}})$. For every non-toric blow-up in the toric model, we include a wall (ρ, f_ρ), where ρ is the ray in $\Sigma_{\overline{X}}$ corresponding to the divisor on which we do the non-toric blow-up, and

$$f_\rho = 1 + t^{-E_i}z^{-v_\rho} = 1 + t^{-E_i}x^{-a}y^{-b} \,,$$

where $E_i \in NE(X)$ is the class of the exceptional curve, z^{v_ρ} denotes the element in the monoid ring $\mathbf{C}[M]$, corresponding to the primitive direction $v_\rho \in M$ of ρ pointing towards 0 referred to as the direction vector, and we use the convention $x = z^{(1,0)}$ and $y = z^{(0,1)}$. The negative signs on the powers of t and z are chosen following the sign conventions of [2].

The walls of the initial wall structure do not generally intersect only pairwise, but there can be triple or more complicated intersections as illustrated on the left hand Fig. 3, where we have 4 walls intersecting - two of these walls lie on top of each other and have direction vector $(-1, -1)$, and the other two have direction vectors $(1, 0)$ and $(0, 1)$. However, we can always move these walls, so that any of the intersection points of the initial walls will be formed by only two walls intersecting. The initial wall structure after a choice of such a perturbation is illustrated on the right hand side of Fig. 3. In this situation, only when two walls intersect each time, we can easily describe a *consistent wall structure* obtained from the initial wall structure by inserting new walls to it, so that the composition of the wall crossing transformations around each intersection point is identity. This diagram is referred as the heart of the canonical wall structure in [1].

Roughly put, each time a wall with support on a ray ρ_i with direction v_i and with attached function $1 + t^{-E_i} z^{-v_i}$ intersects another wall with support on a ray ρ_j with direction v_j and with attached function $1 + t^{-E_j} z^{-v_j}$, we first extend these rays to lines, so that on the new rays we form while doing this we attach the functions $1 + t^{\kappa_i - E_i} z^{-v_i}$ and $1 + t^{\kappa_j - E_j} z^{-v_j}$ respectively. Here κ_i and κ_j denote the sums of the kinks of the MVPL function φ, lying on rays that intersect the initial rays. We furthermore insert an additional wall with support on a ray ρ_{i+j} with direction $v_i + v_j$ and with attached function

$$1 + t^{(\kappa_i+\kappa_j)-(E_i+E_j)} z^{-(v_i+v_j)} .$$

Note that this simple prescription describes the consistent wall structure, because each time two walls with direction vectors say v_i and v_j intersect, the determinant of v_i and v_i is ± 1.

The coordinate ring of the mirror to (X, D) is generated by *theta functions*, determined by keeping track of how a set of initial monomials change under wall crossing in this consistent wall structure [1, Sect. 3]. We have as many theta functions as the number of rays of the toric model associated to (dP_4, D), whose fan is illustrated in Fig. 2. The direction vectors of these rays are given by $(-1, -1)$, $(-1, 0)$, $(1, 1)$, $(0, -1)$, which correspond to the initial set of monomials $x^{-1}y^{-1}$, x^{-1}, xy, y^{-1}. The corresponding 4 theta functions, respectively denoted by $\vartheta_1, \ldots, \vartheta_4$, are determined by tracing how these monomials change as the corresponding rays cross walls in the consistent wall structure associated to (dP_4, D) illustrated in Fig. 4. To determine the wall crossings we first fix a general point P as in Fig. 4 , and look at the rays coming from the directions $(-1, -1)$, $(-1, 0)$, $(1, 1)$, $(0, -1)$ and stopping at this point—note that, we made a choice of the point P so that each ray will cross exactly two walls and the situation will be symmetric, and it will make it easier to calculate the theta functions.

Each of the 4 theta functions is then obtained by the following wall crossings: The red ray labelled with I crosses the two walls with attached functions $1 + t^{H-E_4-E_5}x$ and $1 + t^{E_1-E_5}y^{-1}$, the functions on the two walls crossed by the red ray labelled by II are $1 + t^{E_1-E_5}y^{-1}$ and $1 + t^{H-E_1-E_3}xy$, the functions on the walls crossed by the red ray labelled by III are $1 + t^{-E_2}x^{-1}$ (in this case there is also the kink of the MVPL function $H - E_1$ we take into account in the computation of the theta functions) and $1 + t^{H-E_1-E_2-E_3}y$, and finally the red ray labelled by IV crosses the walls with attached functions $1 + t^{2H-E_1-E_2-E_3-E_4}xy^2$ and $1 + t^{H-E_1-E_4}xy$ (and in this case there is also a kink of the MVPL function given by E_1). Hence, we obtain;

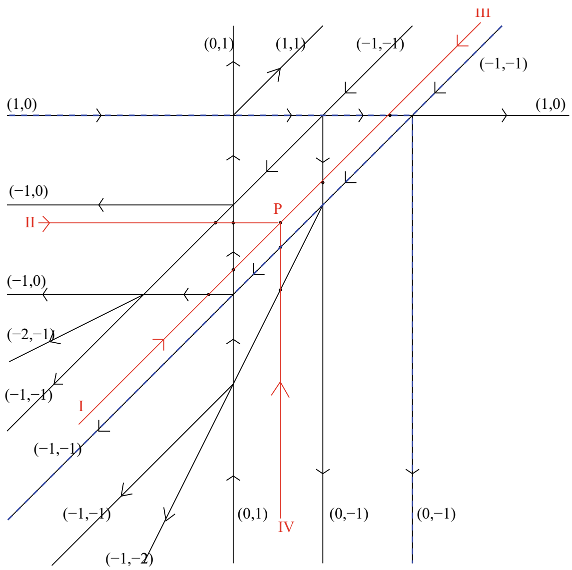

Fig. 4 The consistent wall structure obtained from the perturbed initial wall structure with 4 incoming rays, illustrated in Fig. 3. The kinks of the MVPL function φ are on the 4 blued dashed rays, whereas all the black rays are walls. Tracing the red rays labelled by I, II, III, and IV, and the walls crossed by them while they reach the point P, we determine the theta 4 functions $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$

$$
\begin{aligned}
& x^{-1}y^{-1} \mapsto x^{-1}y^{-1} + t^{H-E_4-E_5}y^{-1} \mapsto x^{-1}y^{-1} + t^{E_1-E_5}x^{-1}y^{-2} + \\
& \quad + \; t^{H-E_4-E_5}y^{-1} \; =: \; \vartheta_1 \; ; \\
& x^{-1} \mapsto x^{-1} + t^{E_1-E_5}x^{-1}y^{-1} \mapsto x^{-1} + t^{H-E_1-E_3}y + t^{E_1-E_5}x^{-1}y^{-1} \; =: \; \vartheta_2 \; ; \\
& xy \mapsto t^{H-E_1}xy + t^{H-E_1-E_2}y \mapsto t^{H-E_1}xy + t^{2H-2E_1-E_2-E_3}xy^2 + \\
& \quad + t^{H-E_1-E_2}y \; =: \; \vartheta_3 \; ; \\
& y^{-1} \mapsto y^{-1} + t^{2H-E_1-E_2-E_3-E_4}xy \mapsto t^{E_1}y^{-1} + t^{H-E_4}x + \\
& \quad + \; t^{2H-E_1-E_2-E_3-E_4}xy \; =: \; \vartheta_4 \; ;
\end{aligned}
$$

The above theta functions generate the mirror to (dP_4, D), which is obtained as a family of log Calabi–Yau surfaces (dP_4, D) given as Spec of the quotient of $\mathbf{C}[\mathrm{NE}(X)][\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4]$ by the following two quadratic equations:

$$
\begin{aligned}
\vartheta_1\vartheta_3 &= C_1 + t^{H-E_1-E_2}\vartheta_2 + t^{H-E_1-E_5}\vartheta_4 \\
\vartheta_2\vartheta_4 &= C_2 + t^{E_1}\vartheta_1 + t^{H-E_3-E_4}\vartheta_3
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
C_1 &= t^{H-E_1} + t^{2H-E_1-E_2-E_3-E_5} + t^{2H-E_1-E_2-E_4-E_5} \\
C_2 &= t^{H-E_4} + t^{H-E_3} + t^{2H-E_2-E_3-E_4-E_5}.
\end{aligned}
\tag{3}
$$

Note that the resulting equations for the mirror given above, agrees with the one obtained in [3] using computer algebra.

2 The Quantum Mirror to (dP_4, D)

A general recipe to construct a deformation quantization of mirrors of log Calabi–Yau surfaces is given in [4]. For (dP_4, D), as the consistent wall structure consists of only finitely many walls, the general recipe reduces to the following simple prescription: the quantum theta functions, denoted by $\hat{\vartheta}_1, \ldots, \hat{\vartheta}_4$, are obtained from the theta functions above by replacing the monomials $z^v \in \mathbf{C}[M]$, by quantum variables, denoted by $\hat{z}^v$, which are elements of the quantum torus, that is such that $\hat{z}^v \hat{z}^{v'} = q^{\frac{1}{2}\det(v,v')} \hat{z}^{v+v'}$, where q is the quantum deformation parameter. Hence, from the equations for the theta functions above, we obtain:

$$\begin{aligned}
\hat{\vartheta}_1 &= \hat{z}^{(-1,-1)} + t^{E_1-E_5}\hat{z}^{(-1,-2)} + t^{H-E_4-E_5}\hat{z}^{(0,-1)} \\
\hat{\vartheta}_2 &= \hat{z}^{(-1,0)} + t^{H-E_1-E_3}\hat{z}^{(0,1)} + t^{E_1-E_5}\hat{z}^{(-1,-1)} \\
\hat{\vartheta}_3 &= t^{H-E_1}\hat{z}^{(1,1)} + t^{2H-2E_1-E_2-E_3}\hat{z}^{(1,2)} + t^{H-E_1-E_2}\hat{z}^{(0,1)} \\
\hat{\vartheta}_4 &= t^{E_1}\hat{z}^{(0,-1)} + t^{H-E_4}\hat{z}^{(1,0)} + t^{2H-E_1-E_2-E_3-E_4}\hat{z}^{(1,1)}
\end{aligned} \tag{4}$$

These quantum theta functions satisfy the following 8 relations—first two obtained as deformations of the two quadric equations in (2), and the latter 6 equations determining the non-commutativity of the products of each two among four quantum theta functions.

$$\begin{aligned}
\hat{\vartheta}_1\hat{\vartheta}_3 &= C_1 + q^{-\frac{1}{2}}t^{H-E_1-E_2}\hat{\vartheta}_2 + q^{\frac{1}{2}}t^{H-E_1-E_5}\hat{\vartheta}_4 \\
\hat{\vartheta}_2\hat{\vartheta}_4 &= C_2 + q^{\frac{1}{2}}t^{E_1}\hat{\vartheta}_1 + q^{-\frac{1}{2}}t^{H-E_3-E_4}\hat{\vartheta}_3 \\
q^{\frac{1}{2}}\hat{\vartheta}_1\hat{\vartheta}_3 - q^{-\frac{1}{2}}\hat{\vartheta}_3\hat{\vartheta}_1 &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})C_1 + (q - q^{-1})t^{H-E_1-E_5}\hat{\vartheta}_4 \\
q^{\frac{1}{2}}\hat{\vartheta}_2\hat{\vartheta}_4 - q^{-\frac{1}{2}}\hat{\vartheta}_4\hat{\vartheta}_2 &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})C_2 + (q - q^{-1})t^{E_1}\hat{\vartheta}_1 \\
q^{\frac{1}{2}}\hat{\vartheta}_1\hat{\vartheta}_2 - q^{-\frac{1}{2}}\hat{\vartheta}_2\hat{\vartheta}_1 &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})t^{2H-E_1-E_3-E_4-E_5} \\
q^{\frac{1}{2}}\hat{\vartheta}_2\hat{\vartheta}_3 - q^{-\frac{1}{2}}\hat{\vartheta}_3\hat{\vartheta}_2 &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})t^{H-E_5} \\
q^{\frac{1}{2}}\hat{\vartheta}_3\hat{\vartheta}_4 - q^{-\frac{1}{2}}\hat{\vartheta}_4\hat{\vartheta}_3 &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})t^{H-E_2} \\
q^{\frac{1}{2}}\hat{\vartheta}_4\hat{\vartheta}_1 - q^{-\frac{1}{2}}\hat{\vartheta}_1\hat{\vartheta}_4 &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})t^{2H-E_1-E_2-E_3-E_4}
\end{aligned} \tag{5}$$

Eliminating $\hat{\vartheta}_4$ in (5), we obtain the same equations as in [8, Corollary 4.6], obtained there using a totally different approach based on the fact that the cubic equation obtained from (2) by eliminating ϑ_4 is the defining equation of the wild character variety arising as the monodromy manifold of the Painlevé IV equation.

References

1. H. Argüz, Equations of mirrors to log Calabi–Yau pairs via the heart of canonical wall structures (2021). arXiv:2109.08664
2. H. Argüz, M. Gross, The higher dimensional tropical vertex (2020), to appear in Geometry & Topology. arXiv:2007.08347
3. L. Barrott, Explicit equations for mirror families to log Calabi-Yau surfaces. Bull. Korean Math. Soc. (2018)
4. P. Bousseau, Quantum mirrors of log Calabi-Yau surfaces and higher-genus curve counting. Compos. Math. **156**(2), 360–411 (2020)
5. L. Chekhov, M. Mazzocco, V. Rubtsov, Quantised Painlevé monodromy manifolds, Sklyanin and Calabi-Yau algebras. Adv. Math. **376**, 107442 (2021)
6. M Gross, P. Hacking, S. Keel, Mirror symmetry for log Calabi-Yau surfaces I. Publications Math.ématiques de l'IHES **122**(1), 65–168 (2015)
7. M Gross, P. Hacking, B. Siebert, Theta functions on varieties with effective anti- canonical class (2016), to appear in Memoirs of the AMS. arxiv:1601.07081
8. M. Mazzocco, Confluences of the Painlevé equations, Cherednik algebras and q-Askey scheme. Nonlinearity **29**(9), 2565 (2016)

Bidirectional Processes—In Category Theory, Physics, Engineering, …

Alexander Ganchev

Abstract This is a brief on the categorical view of bidirectional processes (aka categorical optics) which appear in such diverse areas as learning, quantum physics, dynamical systems and control, etc. We note that both quantum processes and transmission are also examples of categorical optics.

Keywords Bidirectional processes · Categorical optics · CPM construction of Selinger · Linear transport described by transfer matrices

1 Introduction

Functions are ubiquitous in mathematics. With functions we can model one-directional flow—from the source to the target. But the examples of bidirectional flow of information are abundant. Bidirectionality is key in cybernetics, control theory, and systems theory where the actions of the environment on a system cause the back reaction of the environment on the system. Learning is also bidirectional—assessment is the back reaction of the environment on the learner and one learns from mistakes. Functions can be composed, i.e., we can form categories of certain functions. Also bidirectional processes can be composed and form appropriate categories. A clean mathematical formulation of bidirectionality is achieved in category theory based on the notion of categorical lenses and more generally categorical optics. By "bending wires" one can turn a bidirectional process into a super-process and vice versa.

After introducing the necessary notions to define categorical optics we will give two examples. We note that quantum processes, obtained by doubling of pure quantum processes and partial tracing, are given by optics and thus the CPM construction of Selinger is a particular case of categorical optics. We also give the example, prob-

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_39

ably the simplest one of a bidirectional process, of linear transport described by a transfer matrix which factorizes into two optics. For this example we have to very briefly recall graphical linear algebra (the presentation of matrix theory in terms of generators and relations, i.e., as a prop). We end with a short discussion.

2 String Diagrams, (co)ends, Optics

String diagrams are a very convenient graphical presentation of monoidal categories where the coherence rules are implicit in the invariance of the diagrams under topological moves. Objects are represented by strings/wires. Morphisms are depicted by boxes/nodes with wires coming in and out. Placing wires in parallel depicts the monoidal product. Bending strings around depicts dualities. String diagrams are the Poincare dual of commutative diagrams. For a survey on string diagrams see [26, 33] or the nLab entry.

Yoneda introduced (co)ends in 1960 [38]. They appear peripherally in textbooks such as MacLane [25] or Bourceux [8] but take central stage in [24]. Given a functor

$$P : C^{op} \times C \to D$$

the coend $\int^{c \in C} P(c, c)$ is the coequilizer:

$$\coprod_{a,b \in C} P(a, b) \rightrightarrows \coprod_{c \in C} P(c, c) \to \int^{c \in C} P(c, c)$$

where the elements of the coend are equivalence classes of the coproduct $\coprod_{c \in C} P(c, c)$ given by $u \sim v$, for $u \in P(a, a)$ and $v \in P(b, b)$, if there exists an $f \in P(a, b)$ such that $v \circ f = f \circ u$; that is, two arrows u and v are equivalent if we can 'slide' some f between them. An example is given by the tensor product of a two R modules where R is a commutative ring and the "sliding" above is the sliding of scalars through the tensor product. Another example is Tannaka reconstruction. The application of coends in Conformal Field Theory are described in [17–19].

Categorical optics was first introduced by Pasto and Street [27] where they are called doubles of monoidal categories. Boisseau and Gibbons [5] and Riley [29] gave it the name of categorical optics showing how it covers all kinds of different data accessories from computer science, in particular lenses. The building blocks of a deep neural networks are lenses [14, 15, 21] and more generally optics. A foundation for cybernetics will also be based on optics [9, 34]. String diagrams for optics were introduced in [6, 30]. Another recent paper on categorical optics is [11]. The blog of Bartosz Milewski is a excellent source for profunctor optics, the blog of Jules

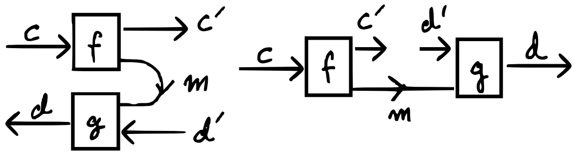


Fig. 1 String diagrams for optics in Boisseau style on the left and Riley-Roman style on the right

Hedges gives a quick tour of lenses, and of course the *n-category cafe* has several entries on optics.[1]

Let M be a autonomous[2] category, C is an M-category if there is an action of M on C, i.e., a functor $\bullet : M \times C \to C$ with the necessary natural isomorphisms. Let C and D be two M-categories. (To simplify we can assume $C = D = M$ acting on itself by the monoidal product.) The category $\mathbf{Optic}_{C,D}$ has objects $(c, d) \in C \times D$ and hom-set of arrows from (c, d) to (c', d') given by

$$\mathbf{Optic}_{C,D}((c, d), (c', d')) = \int^{m \in M} C(c, m \bullet c') \times D(m \bullet d', d)$$

where for m an object in M and two arrows $f \in C(c, m \bullet c')$ and $g \in D(m \bullet d', d)$, the arrow $\langle f|g\rangle_m : (c, d) \to (c', d')$ is obtained by imposing the equivalence $\langle(\mu \times c') \circ f|g\rangle_n = \langle f|g \circ (\mu \times d')\rangle_m$ for any $\mu \in M(m, n)$.

In the simpler case when $C = D = V = \mathbf{Set}$ with the cartesian product as monoidal product and action it is known that optics are lenses.

String diagrams for optics (Fig. 1) appear in [30] as diagrams with 'holes' (on the right) and in [6] as bidirectional transformation (on the left). The intuition for optics as data accessories is the following: the object c we want to split into a part of interest c' and the rest or the residual m. Updating the part c' to d' we combine it back with the rest m obtaining d.

3 Quantum Mechanics

Abramsky and Coecke [1] introduced dagger compact closed categories in abstracting the standard axiomatization of quantum mechanics in terms of Hilbert spaces and pure states. On the other hand Selinger approached quantum mechanical axiomatization in terms of mixed states, density matrices, and completely positive maps [32]. In this paper Selinger introduced the CPM construction, that associates to any dagger

[1] https://bartoszmilewski.com/2016/01/21/tambara-modules/
https://julesh.com/2018/08/16/lenses-for-philosophers/
https://golem.ph.utexas.edu/category/2019/11/doubles_for_monoidal_categorie.html
https://golem.ph.utexas.edu/category/2020/01/profunctor_optics_the_categori.html.

[2] For the applications bellow we would like to "bend" wires so we ask that the category is rigid/autonomous. Generalizations are possible where this is relaxed [6] but then one has to go to the presheaf category of optics/monoidal-doubles which is the category of Tambara modules.

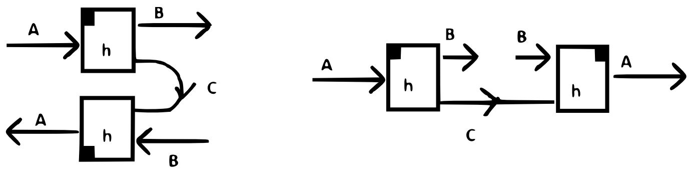


Fig. 2 Completely positive map on the left and super-operator on the right

compact closed category its category of completely positive maps thus relating the two approaches (Fig. 2).

In fact the CPM construction is an optic. Corollary 4.13 (d) in [32] can be rephrased as follows. For objects A and B in a dagger compact closed category a map $f: A^* \otimes A \to B^* \otimes B$ is a completely positive map if it is an optic. Corollary 4.13 (d) in [32] appears as Proposition 6.46 in the book on categorical quantum mechanics [12] and as exercise 7.7 in [22]. The string diagram for a completely positive map is on the left in the figure above. The string diagram on the right side depicting an optic as a diagram with a hole describes a super-operator (Remark 6.50 of [12]). (N.B. the term super-operator indicates that it is a map taking an operator as input and producing another operator as output.) Super-operators are examples of combs introduced in [10]. Combs and generalizations from the point of view of optics are the subject of [31].

All completely positive maps are obtained by purification, i.e., by discarding, or tracing out, the environment part of a pure map, i.e., a applying the doubling to a Hilbert space morphism. In the language of optics the environment is the residual and the discarding is taking the coend. According to [10] purification is the axiom that distinguishes quantum mechanics.

4 Graphical Linear Algebra

To proceed further we give pointers to the literature on graphical linear algebra (GLA). A starting point could be the blog of Sobocinski.[3] These developments are grounded in the compositional approach of networks by Baez et al. [2–4] and Bonci-Sobocinski-Zanasi et al. [7, 39].

The goal is to present the category of matrices, i.e., the skeleton of the category of finite dimensional vector spaces, in terms of generators and relations. A PROP (a PROduct and Permutation category) is a strict symmetric monoidal category generated on one object by the monoidal product, i.e., whose objects are the natural numbers $0, 1, 2, \ldots,$ the monoidal product is addition, and 0 is the monoidal unit.

One starts with 'white' generators standing for addition and the zero and 'black' generators standing for copying and deleting. (This black/white convention is used

[3] https://graphicallinearalgebra.net/.

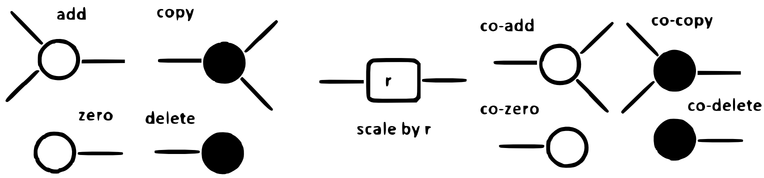


Fig. 3 Generators for graphical linear algebra

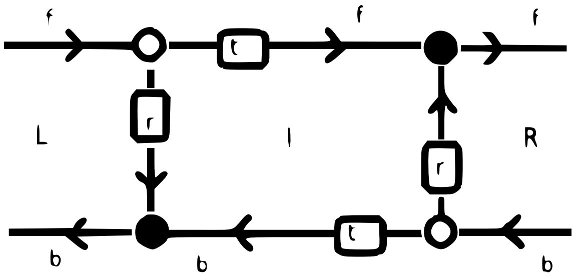
in Bonchi et al. while in the ZX-calculus [12, 22] another coloring style is used.) The white structure satisfies relations for commutativity, associativity, and the zero (unit) thus forms a commutative monoid. The black structure forms a cocommutative comonoid. One also needs the adjoints of the white and black generators, co-addition and co-zero and co-copy and co-delete which are depicted by turning the pictures to run backwards but not changing the color. The generators and the co-generators of the same color form a Frobenius algebra of that color. On the other hand the generators of opposite colors form a bialgebra and similarly for the co-generators of opposite colors. One also has to introduce scaling, the action of the field or PID of scalars. The multiplication by a scalar r is depicted by putting an oriented box containing the scalar on the corresponding wire. Lack's theory [23] of composing PROPs is used in putting together the different structures. The first main result in GLA is that the bialgebra of addition, copy, and scaling is a presentation of the prop of matrices (Fig. 3).

A linear relation between two vector spaces is a linear subspace in the direct sum of the two spaces. The second main result in GLA is that the prop of interacting Hopf algebra provides a presentation for the prop of linear relations [7, 39].

5 Scattering and Transmission

In a process of scattering the on-coming waves/particles/radiation/… enter a region of obstacles and are reflected, transmitted, and/or absorbed. We will follow the paper of Redheffer [28]. As a simple example one can consider the wave function solving the Schroedinger equation in one spacial dimension where the potential is zero but for a bounded region of obstacles where it is positive. In the left free region the solutions are free waves propagating to the right/left (forward/backward) with amplitudes A^f_L and A^b_L respectively. In the free region to the right of the obstacle denote the amplitudes $A^{f/b}_R$. The scattering matrix S transforms the on-coming $A_{in} = (A^f_L, A^b_R)^T$ into the out-going $A_{out} = (A^f_R, A^b_L)^T$. One has $S = \begin{pmatrix} t^f & r^b \\ r^f & t^b \end{pmatrix}$. Here t and r are the transmission and reflection coefficients and the superscript indicates forward versus backward. The problem with the S-matrix is that it does

Fig. 4 Factorization of a transfer matrix into two optics



not compose.[4] The composable matrix is the transfer matrix T which transforms $A_R = T\, A_L$ where $A_X = (A_X^f,\ A_X^b)^T$ and X indicates the region L, I, or R. (I is the intermediate region, the region of the obstacle.) We will build T by factoring it $T = T_R^{-1}\, T_L$ where $A_I = T_L\, A_L$ and $A_I = T_R\, A_R$. Using the notation from graphical linear algebra the situation is illustrated in Fig. 4. (The white dot describes copying while the black—addition.)

It is straight forward to find $T_L = \begin{pmatrix} t^f & 0 \\ -r^f & 1 \end{pmatrix}$ and $T_R = \begin{pmatrix} 1 & -r^b \\ 0 & t^b \end{pmatrix}$. The two transfer matrices T_L and T_R can be interpreted as optics in the prop of matrices. I cannot resist the temptation to mention that the factorization $T = T_R^{-1}\, T_L$ looks like a Wiener-Hopf factorization.

6 Discussion

The numerous examples of bidirectional processes indicate that categorical optics could be viewed as the "Lego" elements out of which any bidirectional process can be built. Besides the well known examples we have also noted that CMP construction of Selinger is an optic and that transmission processes can be described as optics.

One area of active study is optics in machine learning. It is known that supervised learning can be viewed as a problem of optimal control (Pontryagin's maximum principle). One should be able to formulate optimal control in the language of categorical optics. A possible route is to formulate port-Hamiltonian systems in categorical language. The work of Willems (e.g., [37]) was an inspiration both for the development of the port-Hamiltonian approach [36] and the categorical approach to dynamical systems (Baez, Spivak, Fong, et al. [3, 16, 35]) but interestingly these two approaches have been practically disjoint. Port-Hamiltonian systems are interconnected by Dirac structures. The closely related Lagrangian relations have been formulated [13] in the language of GLA and one can show that they are instances of categorical optics. Thus the first step in the categorical formulation of port-Hamiltonian systems is to express Dirac structures as categorical optics. This will be the subject of future work.

[4] To compose two S-matrices instead of matrix multiplication one should use the Redheffer star product, i.e., pass to the transfer matrix.

Acknowledgements The work was partial supported by NFSR grant KP-06-N58/5 (ФНИ договор КП-06-Н58/5). AUBG is acknowledged for a much needed sabbatical leave and INRNE-BAS for hospitality.

References

1. S. Abramsky, B. Coecke, A categorical semantics of quantum protocols, in *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, LICS 2004* (IEEE Computer Society Press, 2004) pp. 415–425
2. J.C. Baez, B. Coya, F. Rebro, Props in network theory. Theory Appl. Categ. **33**(25), 727–783 (2018) (2017)
3. J.C. Baez, J. Erbele, Categories in control. Theory Appl. Categ. **30**(24), 836–881 (2015) (2014)
4. J.C. Baez, B. Fong, A compositional framework for passive linear networks. Theory Appl. Categ. **33**(38), 1158–1222 (2018) (2015)
5. G. Boisseau, J. Gibbons, What you need a know about Yoneda: profunctor optics and the Yoneda Lemma (functional pearl), in *Proceedings of the ACM on Programming Languages*, 2(ICFP) (2018), pp. 1–27
6. G. Boisseau, String diagrams for optics (2020). arXiv:2002.11480
7. F. Bonchi, P. Sobociński, F. Zanasi, Interacting hopf algebras. J. Pure Appl. Algebra **221**(1), 144–184 (2017)
8. F. Borceux, *Handbook of Categorical Algebra* (Cambridge University Press, 1994)
9. M. Capucci, B. Gavranović, J. Hedges, E.F. Rischel, Towards foundations of categorical cybernetics (2021). arXiv:2105.06332
10. G. Chiribella, G.M. D'Ariano, P. Perinotti, Theoretical framework for quantum networks. Phys. Rev. A **80**(2), 022339 (2009)
11. B. Clarke, D. Elkins, J. Gibbons, F. Loregian, B. Milewski, E. Pillmore, M. Roman, Profunctor optics, a categorical update (2020). arXiv:2001.07488
12. B. Coecke, A. Kissinger, *Picturing Quantum Processes* (Cambridge University Press, 2017)
13. C. Comfort, A. Kissinger, A graphical calculus for Lagrangian relations (2021). arXiv:2105.06244
14. B. Fong, D. Spivak, R. Tuyeras, Backprop as a functor: a compositional perspective on supervised learning, in *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pp. 1–13
15. B. Fong, M. Johnson, Lenses and Learners. arXiv:1903.03671
16. B. Fong, D. Spivak, *An Invitation to Applied Category Theory: Seven Sketches in Compositionality* (Cambridge University Press, 2019)
17. J. Fuchs, C. Schweigert, Coends in conformal field theory, in *Contemporary Mathematics*, vol. 695 (2017)
18. J. Fuchs, C. Schweigert, Bulk from boundary in finite CFT by means of pivotal module categories. Nucl. Phys. B **967**, 115392 (2021)
19. J. Fuchs, C. Schweigert, Y. Yang, String-net construction of RCFT correlators (2021). arXiv:2112.12708
20. J. Hedges, Coherence for lenses and open games 2017). arXiv:1704.02230
21. J. Hedges, From open learners to open games. arXiv:1902.08666
22. C. Heunen, J. Vicary, *Categories for Quantum Theory* (Oxford University Press, 2019)
23. S. Lack, Composing Props. Theory Appl. Categ. **13**(9), 147–163 (2004)
24. F. Loregian, *Coend Calculus* (Cambridge University Press, 2021) arXiv:1501.02503
25. S. Mac Lane, *Categories for the Working Mathematician* (Springer, 1998)
26. D. Marsden, Category theory using string diagrams (2014). arXiv:1401.7220
27. C. Pastro, R. Street, Doubles for monoidal categories. Theory Appl. Categ. **21**(2008), 61–75 (2007)

28. R. Redheffer, On the relation of transmission-line theory to scattering and transfer. J. Math. Phys. **41** (1962)
29. M. Riley, Categories of optics. arXiv:1809.00738
30. M. Roman, Open diagrams via coend calculus (2020). arXiv:2004.04526
31. M. Roman, Comb diagrams for discrete-time feedback (2020). arXiv:2003.06214
32. P. Selinger, Dagger compact closed categories and completely positive maps. Electron. Notes Theor. Comput. Sci. **170**, 139–163 (2007)
33. P. Selinger, A survey of graphical languages for monoidal categories. Springer Lect. Notes Phys. **813**, 289–355 (2011)
34. T.S.C. Smithe, Cyber kittens, or some first steps towards categorical cybernetics (2021). arXiv:2101.10483
35. D. Vagner, D. Spivak, E. Lerman, Algebras of open dynamical systems on the operad of wiring diagrams. Theory Appl. Categ. **30**(51), 1793–1822 (2015)
36. A. Van Der Schaft, D. Jeltsema, Port-Hamiltonian systems theory: an introductory overview. Found. Trends Syst. Control **1**(2–3), 173–378 (2014)
37. J. Willems, The behavioral approach to open and interconnected systems. IEEE Control Syst. Mag. **27**, 46–99 (2007)
38. N. Yoneda, On Ext and exact sequences. J. Fac. Sci. Univ. Tokyo Sect. **I**(8), 507–576 (1960)
39. F. Zanasi, Interacting Hopf Algebras: the theory of linear systems. Ph.D. thesis (2018). arXiv:1805.03032 (2018)

Gauge Theories and Applications

Nonholomorphic Superpotentials in Heterotic Landau-Ginzburg Models

Richard S. Garavuso

Abstract The aim of this talk is to derive two constraints imposed by supersymmetry for a class of heterotic Landau-Ginzburg models with nonholomorphic superpotentials. One of these constraints relates the nonholomorphic parameters of the superpotential to the Hermitian curvature. Various special cases of this constraint have been used to establish properties of Mathai-Quillen form analogues which arise in the corresponding heterotic Landau-Ginzburg models. The other constraint was not anticipated from studies of Mathai-Quillen form analogues.

Keywords Superstrings and heterotic strings · Supersymmetry and duality · Topological field theories

1 Introduction

A Landau-Ginzburg model is a nonlinear sigma model with a superpotential. For a *heterotic* Landau-Ginzburg model [1–8], the nonlinear sigma model possesses only $(0, 2)$ supersymmetry and the superpotential is a Grassmann-odd function of the superfields which may or may not be holomorphic. It was claimed in [7] that, for various heterotic Landau-Ginzburg models with nonholomorphic superpotentials, supersymmetry imposes a constraint which relates the nonholomorphic parameters of the superpotential to the Hermitian curvature. Details supporting that claim were worked out in [8]. The analysis revealed an additional constraint imposed by supersymmetry which was not anticipated in [7]. This talk will summarize the analysis found in [8].

This talk is organized as follows: In Sect. 2, we will write down the action for the class of heterotic Landau-Ginzburg models that we are considering. In Sect. 3, for the case of a holomorphic superpotential, we demonstrate that the action is invariant

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_40

on-shell under supersymmetry transformations up to a total derivative. Finally, in Sect. 4, we will extend the analysis to the case in which the superpotential is not holomorphic. In this case, we obtain two constraints imposed by supersymmetry.

2 Action

Heterotic Landau-Ginzburg models have field content consisting of (0, 2) bosonic chiral superfields $\Phi^i = (\phi^i, \psi^i_+)$ and (0, 2) fermionic chiral superfields $\Lambda^a = \left(\lambda^a_-, H^a, E^a\right)$, along with their conjugate antichiral superfields $\Phi^{\overline{\imath}} = \left(\phi^{\overline{\imath}}, \psi^{\overline{\imath}}_+\right)$ and $\Lambda^{\overline{a}} = \left(\lambda^{\overline{a}}_-, \overline{H}^{\overline{a}}, \overline{E}^{\overline{a}}\right)$. The ϕ^i are local complex coordinates on a Kähler manifold X. The E^a are local smooth sections of a Hermitian vector bundle $\mathcal{E}$ over X, i.e. $E^a \in \Gamma(X, \mathcal{E})$. The H^a are nonpropagating auxiliary fields. The fermions couple to bundles as follows:

$$\psi^i_+ \in \Gamma\left(K^{1/2}_\Sigma \otimes \Phi^*\left(T^{1,0}X\right)\right), \qquad \lambda^a_- \in \Gamma\left(\overline{K}^{1/2}_\Sigma \otimes \left(\Phi^*\overline{\mathcal{E}}\right)^\vee\right),$$
$$\psi^{\overline{\imath}}_+ \in \Gamma\left(K^{1/2}_\Sigma \otimes \left(\Phi^*\left(T^{1,0}X\right)\right)^\vee\right), \qquad \lambda^{\overline{a}}_- \in \Gamma\left(\overline{K}^{1/2}_\Sigma \otimes \Phi^*\overline{\mathcal{E}}\right),$$

where $\Phi : \Sigma \to X$ and K_Σ is the canonical bundle on the worldsheet Σ.

In [5], heterotic Landau-Ginzburg models with superpotential of the form

$$W = \Lambda^a F_a\,, \tag{2.1}$$

where $F_a \in \Gamma\left(X, \mathcal{E}^\vee\right)$, were considered. In this talk, we will study supersymmetry in these heterotic Landau-Ginzburg models with $E^a = 0$. Assume that X has metric g, antisymmetric tensor B, and local real coordinates ϕ^μ. Furthermore, assume that $\mathcal{E}$ has Hermitian fiber metric h. Then the action of a Landau-Ginzburg model on X with gauge bundle $\mathcal{E}$ and $E^a = 0$ can be written [8]

$$\begin{aligned} S = 2t \int_\Sigma d^2z \Bigg[&\frac{1}{2}\left(g_{\mu\nu} + iB_{\mu\nu}\right)\partial_z\phi^\mu\partial_{\overline{z}}\phi^\nu \\ &+ i g_{\overline{\imath}i}\psi^{\overline{\imath}}_+\overline{D}_{\overline{z}}\psi^i_+ + i h_{a\overline{a}}\lambda^a_- D_z\lambda^{\overline{a}}_- + F_{i\overline{\imath}a\overline{a}}\,\psi^i_+\psi^{\overline{\imath}}_+\lambda^a_-\lambda^{\overline{a}}_- \\ &+ h^{a\overline{a}}F_a\overline{F}_{\overline{a}} + \psi^i_+\lambda^a_- D_i F_a + \psi^{\overline{\imath}}_+\lambda^{\overline{a}}_-\overline{D}_{\overline{\imath}}\overline{F}_{\overline{a}}\Bigg]. \end{aligned} \tag{2.2}$$

Here, t is a coupling constant, $d^2z = -i\, dz \wedge d\overline{z}$, and

$$
\begin{aligned}
\overline{D}_{\bar z}\,\psi^i_+ &= \overline{\partial}_{\bar z}\,\psi^i_+ + \overline{\partial}_{\bar z}\,\phi^j\,\Gamma^i_{jk}\psi^k_+\,, &\qquad D_z\lambda^{\bar a}_- &= \partial_z\lambda^{\bar a}_- + \partial_z\phi^{\bar\imath}A^{\bar a}_{\bar\imath\bar b}\,\lambda^{\bar b}_-\,,\\
D_i F_a &= \partial_i F_a - A^b_{ia}F_b\,, & \overline{D}_{\bar\imath}\overline{F}_{\bar a} &= \overline{\partial}_{\bar\imath}\,\overline{F}_{\bar a} - A^{\bar b}_{\bar\imath\bar a}\,\overline{F}_{\bar b}\,,\\
A^b_{ia} &= h^{b\bar b}\,h_{\bar b a,i}\,, & A^{\bar b}_{\bar\imath\bar a} &= h^{\bar b b}\,h_{b\bar a,\bar\imath}\,,\\
\Gamma^i_{jk} &= g^{i\bar\imath}\,g_{\bar\imath k,j}\,, & F_{i\bar\imath a\bar a} &= h_{a\bar b}\,A^{\bar b}_{\bar\imath\bar a,i}\,.
\end{aligned}
$$

The action (2.2) is invariant on-shell under the supersymmetry transformations

$$
\begin{aligned}
\delta\phi^i &= i\alpha_-\psi^i_+\,,\\
\delta\phi^{\bar\imath} &= i\tilde\alpha_-\psi^{\bar\imath}_+\,,\\
\delta\psi^i_+ &= -\tilde\alpha_-\overline{\partial}_{\bar z}\phi^i\,,\\
\delta\psi^{\bar\imath}_+ &= -\alpha_-\partial_z\phi^{\bar\imath}\,,\\
\delta\lambda^a_- &= -i\alpha_-\psi^j_+\,A^a_{jb}\,\lambda^b_- + i\alpha_- h^{a\bar a}\,\overline{F}_{\bar a}\,,\\
\delta\lambda^{\bar a}_- &= -i\tilde\alpha_-\psi^{\bar\jmath}_+\,A^{\bar a}_{\bar\jmath\bar b}\,\lambda^{\bar b}_- + i\tilde\alpha_- h^{\bar a a}\,F_a
\end{aligned}
\tag{2.3}
$$

up to a total derivative.

3 Supersymmetry Invariance for Holomorphic Superpotential

In this section, we will show that, when the superpotential is holomorphic, the action (2.2) is invariant on shell under the supersymmetry transformations (2.3) up to a total derivative. For this purpose, it is sufficient to set $\tilde\alpha_- = 0$,[1] yielding

$$
\begin{aligned}
\delta\phi^i &= i\alpha_-\psi^i_+\,, &\qquad \delta\phi^{\bar\imath} &= 0\,,\\
\delta\psi^i_+ &= 0\,, & \delta\psi^{\bar\imath}_+ &= -\alpha_-\partial_z\phi^{\bar\imath}\,,\\
\delta\lambda^a_- &= -i\alpha_-\psi^j_+\,A^a_{jb}\,\lambda^b_- + i\alpha_- h^{a\bar a}\,\overline{F}_{\bar a}\,, & \delta\lambda^{\bar a}_- &= 0\,.
\end{aligned}
\tag{3.1}
$$

With this simplification, using the λ^a_- equation of motion,[2] the action (2.2) can be written [8]

[1] The calculations for the case in which $\alpha_- = 0$ and $\tilde\alpha_- \neq 0$ are analogous to those we will perform explicitly for the case in which $\alpha_- \neq 0$ and $\tilde\alpha_- = 0$. The general case, i.e. α_- and $\tilde\alpha_-$ both nonzero, is obtained by combining the above two cases.

[2] This is valid because we have integrated out the auxiliary fields H^a.

$$S = it\int_\Sigma d^2z\,\{Q,V\} + t\int_\Sigma \Phi^*(K) + 2t\int_\Sigma d^2z\left(\psi_+^{\bar\imath}\lambda_-^{\bar a}\overline{D}_{\bar\imath}\overline{F}_{\bar a} - \psi_+^i\lambda_-^a D_a F_a\right), \tag{3.2}$$

where Q is the BRST operator,

$$V = 2\left(g_{\bar\imath i}\psi_+^{\bar\imath}\overline{\partial}_{\bar z}\phi^i + i\lambda_-^a F_a\right), \tag{3.3}$$

and

$$\int_\Sigma \Phi^*(K) = \int_\Sigma d^2z\,(g_{i\bar\imath} + iB_{i\bar\imath})\left(\partial_z\phi^i\,\overline{\partial}_{\bar z}\phi^{\bar\imath} - \overline{\partial}_{\bar z}\phi^i\,\partial_z\phi^{\bar\imath}\right) \tag{3.4}$$

is the integral over the worldsheet Σ of the pullback to Σ of the complexified Kähler form $K = -i\,(g_{i\bar\imath} + iB_{i\bar\imath})\,d\phi^i\wedge d\phi^{\bar\imath}$.

Since $\delta f = -i\alpha_-\{Q,f\}$, where f is any field, the Q-exact part of (3.2) is δ-exact and hence δ-closed. We will now establish that the remaining terms of (3.2) are δ-closed on shell up to a total derivative. For the non-exact term of (3.2) involving $\Phi^*(K)$, note that

$$\int_\Sigma \Phi^*(K) = \int_{\Phi(\Sigma)} K = \int_{\Phi(\Sigma)} \left[-i\,(g_{i\bar\imath} + iB_{i\bar\imath})\right] d\phi^i\wedge d\phi^{\bar\imath}$$

and K satisfies

$$\partial K = -i\,\partial_k\,(g_{i\bar\imath} + iB_{i\bar\imath})\,d\phi^k\wedge d\phi^i\wedge d\phi^{\bar\imath} = 0\,.$$

Thus,

$$\delta\left[\Phi^*(K)\right] = \left[\Phi^*(K)\right]_k\,\delta\phi^k = 0\,. \tag{3.5}$$

It remains to consider the non-exact expression of (3.2) involving

$$\psi_+^{\bar\imath}\lambda_-^{\bar a}\overline{D}_{\bar\imath}\overline{F}_{\bar a} - \psi_+^i\lambda_-^a D_i F_a\,.$$

First, we compute

$$\begin{aligned}
\delta\left(\psi_+^{\bar\imath}\lambda_-^{\bar a}\overline{D}_{\bar\imath}\overline{F}_{\bar a}\right) &= \left(\delta\psi_+^{\bar\imath}\right)\lambda_-^{\bar a}\,\overline{D}_{\bar\imath}\,\overline{F}_{\bar a} + \psi_+^{\bar\imath}\left(\delta\lambda_-^{\bar a}\right)\overline{D}_{\bar\imath}\,\overline{F}_{\bar a} + \psi_+^{\bar\imath}\lambda_i^{\bar a}\left[\delta\left(\overline{D}_{\bar\imath}\,\overline{F}_{\bar a}\right)\right] \\
&= \left(-\alpha_-\partial_z\phi^{\bar\imath}\right)\lambda_-^{\bar a}\,\overline{D}_{\bar\imath}\,\overline{F}_{\bar a} + \psi_+^{\bar\imath}\lambda_-^{\bar a}\left[\delta\left(\overline{\partial}_{\bar\imath}\overline{F}_{\bar a} - A_{\bar\imath\,\bar a}^{\bar b}\,\overline{F}_{\bar b}\right)\right] \\
&= \left(-\alpha_-\partial_z\phi^{\bar\imath}\right)\lambda_-^{\bar a}\,\overline{D}_{\bar\imath}\,\overline{F}_{\bar a} + \psi_+^{\bar\imath}\lambda_-^{\bar a}\left\{\overline{\partial}_{\bar\imath}\left[\,\overline{F}_{\bar a,k}\left(\delta\phi^k\right)\right]\right. \\
&\quad \left. -\left[A_{\bar\imath\,\bar a,k}^{\bar b}\left(\delta\phi^k\right)\right]\overline{F}_{\bar b} - A_{\bar\imath\,\bar a}^{\bar b}\left[\overline{F}_{\bar b,k}\left(\delta\phi^k\right)\right]\right\} \\
&= \left(-\alpha_-\partial_z\phi^{\bar\imath}\right)\lambda_-^{\bar a}\,\overline{D}_{\bar\imath}\,\overline{F}_{\bar a} - \psi_+^{\bar\imath}\lambda_-^{\bar a}A_{\bar\imath\,\bar a,k}^{\bar b}\left(i\alpha_-\psi_+^k\right)\overline{F}_{\bar b}\,,
\end{aligned} \tag{3.6}$$

where we have used $\overline{F}_{\overline{a},k} = 0$ in the last step. Now, we compute

$$\begin{aligned}\delta\left(-\psi_+^i \lambda_-^a D_i F_a\right) &= -\left(\delta\psi_+^i\right)\lambda_-^a D_i F_a - \psi_+^i\left(\delta\lambda_-^a\right) D_i F_a \\ &\quad - \psi_+^i \lambda_-^a\left[\delta(D_i F_a)\right] \\ &= -\alpha_- \overline{F}_{\overline{a}} D_z \lambda_-^{\overline{a}} + \left(i\alpha_- h^{a\overline{b}}\,\overline{F}_{\overline{b}}\right) F_{i\overline{\imath}a\overline{a}}\,\psi_+^i \psi_+^{\overline{\imath}} \lambda_-^{\overline{a}}\,, \end{aligned} \tag{3.7}$$

where we have used the λ_-^a equation of motion. Note that the first term on the right-hand side of (3.7) cancels the first term on the right-hand side of (3.6) up to a total derivative:

$$\begin{aligned} -\alpha_- \overline{F}_{\overline{a}}\, D_z \lambda_-^{\overline{a}} &= -\alpha_- \overline{F}_{\overline{a}}\left(\partial_z \lambda_-^{\overline{a}} + \partial_z \phi^{\overline{\imath}}\, A_{\overline{\imath}\,\overline{b}}^{\overline{a}}\,\lambda_-^{\overline{b}}\right) \\ &= \alpha_-\left(\overline{F}_{\overline{a},k}\,\partial_z\phi^k + \overline{F}_{\overline{a},\overline{k}}\,\partial_z\phi^{\overline{k}}\right)\lambda_-^{\overline{a}} - \alpha_-\partial_z\left(\overline{F}_{\overline{a}}\,\lambda_-^{\overline{a}}\right) \\ &\quad - \alpha_- \overline{F}_{\overline{a}}\,\partial_z\phi^{\overline{\imath}}\, A_{\overline{\imath}\,\overline{b}}^{\overline{a}}\,\lambda_-^{\overline{b}} \\ &= \left(\alpha_-\partial_z\phi^{\overline{\imath}}\right)\lambda_-^{\overline{a}}\left(\overline{\partial}_{\overline{\imath}}\overline{F}_{\overline{a}} - A_{\overline{\imath}\,\overline{a}}^{\overline{b}}\,\overline{F}_{\overline{b}}\right) - \alpha_-\partial_z\left(\overline{F}_{\overline{a}}\,\lambda_-^{\overline{a}}\right) \\ &= \left(\alpha_-\partial_z\phi^{\overline{\imath}}\right)\lambda_-^{\overline{a}}\,\overline{D}_{\overline{\imath}}\,\overline{F}_{\overline{a}} - \alpha_-\partial_z\left(\overline{F}_{\overline{a}}\,\lambda_-^{\overline{a}}\right), \end{aligned} \tag{3.8}$$

where we used $\overline{F}_{\overline{a},k} = 0$ in the third step. Furthermore, the second term on the right-hand side of (3.7) cancels the second term on the right-hand side of (3.6):

$$\begin{aligned}\left(i\alpha_- h^{a\overline{b}}\,\overline{F}_{\overline{b}}\right) F_{i\overline{\imath}a\overline{a}}\,\psi_+^i\psi_+^{\overline{\imath}}\lambda_-^{\overline{a}} &= \left(i\alpha_- h^{a\overline{c}}\,\overline{F}_{\overline{c}}\right)\left(h_{a\overline{b}}\,A_{\overline{\imath}\,\overline{a},i}^{\overline{b}}\right)\psi_+^{\overline{\imath}}\lambda_-^{\overline{a}}\psi_+^i \\ &= \psi_+^{\overline{\imath}}\lambda_-^{\overline{a}}\,A_{\overline{\imath}\,\overline{a},k}^{\overline{b}}\,\left(i\alpha_-\psi_+^k\right)\overline{F}_{\overline{b}}\,, \end{aligned} \tag{3.9}$$

where we have used $F_{i\overline{\imath}a\overline{a}} = h_{a\overline{b}}\,A_{\overline{\imath}\,\overline{a},i}^{\overline{b}}$ in the first step. It follows that (3.7) cancels (3.6) up to a total derivative, i.e.

$$\delta\left(-\psi_+^i\lambda_-^a D_i F_a\right) = -\,\delta\left(\psi_+^{\overline{\imath}}\lambda_-^{\overline{a}}\,\overline{D}_{\overline{\imath}}\overline{F}_{\overline{a}}\right) - \alpha_-\partial_z\left(\overline{F}_{\overline{a}}\,\lambda_-^{\overline{a}}\right). \tag{3.10}$$

This completes our argument for the case of a holomorphic superpotential.

4 Supersymmetry Invariance for Nonholomorphic Superpotential

In this section, we will extend the analysis of Sect. 3 to the case in which the superpotential is not holomorphic. This requires revisiting the steps in (3.6) and (3.8) where we used $\overline{F}_{\overline{a},k} = 0$. Allowing for $\overline{F}_{\overline{a},k} \neq 0$, (3.8) becomes

$$-\alpha_-\overline{F}_{\overline{a}}\,D_z\lambda_-^{\overline{a}} = \left(\alpha_-\partial_z\phi^{\overline{\imath}}\right)\lambda_-^{\overline{a}}\,\overline{D}_{\overline{\imath}}\,\overline{F}_{\overline{a}} - \alpha_-\partial_z\left(\overline{F}_{\overline{a}}\,\lambda_-^{\overline{a}}\right) + \alpha_-\overline{F}_{\overline{a},k}\,\partial_z\phi^k\lambda_-^{\overline{a}}\,.$$

It follows that the cancellation described by (3.8) will still apply provided that the constraint

$$\overline{F}_{\overline{a},k}\,\partial_z\phi^k\lambda_-^{\overline{a}} = 0 \tag{4.1}$$

is satisfied. Furthermore, in the next to last step of (3.6), we now have

$$\begin{aligned}
&\psi_+^{\overline{\imath}}\lambda_-^{\overline{a}}\left\{\overline{\partial}_{\overline{\imath}}\left[\,\overline{F}_{\overline{a},k}\left(\delta\phi^k\right)\right]-A_{\overline{\imath}\,\overline{a}}^{\overline{b}}\left[\overline{F}_{\overline{b},k}\left(\delta\phi^k\right)\right]\right\}\\
&\quad=\psi_+^{\overline{\imath}}\lambda_-^{\overline{a}}\left\{\overline{\partial}_{\overline{\imath}}\left[\overline{F}_{\overline{a},k}\left(i\alpha_-\psi_+^k\right)\right]-A_{\overline{\imath}\,\overline{a}}^{\overline{b}}\,\overline{F}_{\overline{b},k}\left(i\alpha_-\psi_+^k\right)\right\}\\
&\quad=\psi_+^{\overline{\imath}}\lambda_-^{\overline{a}}\left(\overline{\partial}_{\overline{\imath}}\overline{F}_{\overline{a},i}-A_{\overline{\imath}\,\overline{a}}^{\overline{b}}\,\overline{F}_{\overline{b},i}\right)\left(i\alpha_-\psi_+^i\right)\\
&\quad=\psi_+^{\overline{\imath}}\lambda_-^{\overline{a}}\left[\overline{\partial}_{\overline{\imath}}\overline{F}_{\overline{a},i}+A_{\overline{\imath}\,\overline{a},i}^{\overline{b}}\,\overline{F}_{\overline{b}}-\partial_i\left(A_{\overline{\imath}\,\overline{a}}^{\overline{b}}\,\overline{F}_{\overline{b}}\right)\right]\left(i\alpha_-\psi_+^i\right)\\
&\quad=\psi_+^{\overline{\imath}}\lambda_-^{\overline{a}}\left[\partial_i\left(\overline{\partial}_{\overline{\imath}}\overline{F}_{\overline{a}}-A_{\overline{\imath}\,\overline{a}}^{\overline{b}}\,\overline{F}_{\overline{b}}\right)+A_{\overline{\imath}\,\overline{a},i}^{\overline{b}}\,\overline{F}_{\overline{b}}\right]\left(i\alpha_-\psi_+^i\right)\\
&\quad=\psi_+^{\overline{\imath}}\lambda_-^{\overline{a}}\left(\partial_i\overline{D}_{\overline{\imath}}\overline{F}_{\overline{a}}+F_{i\overline{\imath}\,a\overline{a}}\,h^{a\overline{b}}\,\overline{F}_{\overline{b}}\right)\left(i\alpha_-\psi_+^i\right),
\end{aligned} \tag{4.2}$$

where we have used $A_{\overline{\imath}\,\overline{a},i}^{\overline{b}} = h^{a\overline{b}}\,F_{i\overline{\imath}\,a\overline{a}}$ in the last step. It follows that, in addition to requiring (4.1), supersymmetry imposes the constraint

$$\partial_i\overline{D}_{\overline{\imath}}\overline{F}_{\overline{a}}+F_{i\overline{\imath}\,a\overline{a}}\,h^{a\overline{b}}\,\overline{F}_{\overline{b}} = 0\,. \tag{4.3}$$

Various special cases of (4.3) were used in [7] to establish properties of Mathai-Quillen form analogues which arise in the corresponding heterotic Landau-Ginzburg models. In that paper, it was claimed that supersymmetry imposes those constraints. In this talk, we have presented details worked out in [8] supporting that claim. It would be interesting to see what constraints are imposed by supersymmetry in other models when the superpotential is nonholomorphic.

References

1. E. Witten, Phases of $N = 2$ theories in two dimensions. Nucl. Phys. **B 403**, 159 (1993). arXiv:hep-th/9301042
2. J. Distler, S. Kachru, $(0, 2)$ Landau-Ginzburg theory. Nucl. Phys. **B 413**, 213 (1994). arXiv:hep-th/9309110
3. A. Adams, A. Basu, S. Sethi, $(0, 2)$ Duality. Adv. Theor. Math. Phys. **7**, 865 (2004). arXiv:hep-th/0309226
4. I. Melnikov, S. Sethi, Half-twisted $(0, 2)$ Landau-Ginzburg models. JHEP **03**, 040 (2008). arXiv:0712.1058
5. J. Guffin, E. Sharpe, A-twisted heterotic Landau-Ginzburg models. J. Geom. Phys. **59**, 1581 (2009). arXiv:0801.3955

6. I. Melnikov, S. Sethi, E. Sharpe, Recent developments in $(0, 2)$ mirror symmetry. SIGMA **8**, 068 (2012). arXiv:1209.1134
7. R.S. Garavuso, E. Sharpe, Analogues of Mathai-Quillen forms in sheaf cohomology and applications to topological field theory. J. Geom. Phys. **92**, 1 (2015). arXiv:1310.5754
8. R.S. Garavuso, Curvature constraints in heterotic Landau-Ginzburg models. JHEP **11**, 019 (2020). arXiv:1912.01552

Automorphic Forms and Fermion Masses

Ferruccio Feruglio

Abstract Symmetry principles have long been applied to the flavour puzzle. In a bottom-up approach, the variety of possible symmetry groups and symmetry breaking sectors is huge, the predictability is reduced and the number of allowed models diverges. A relatively well-motivated and more constrained framework is provided by supersymmetric theories where a discrete subgroup Γ of a non-compact Lie group G plays the role of flavour symmetry and the symmetry breaking sector spans a coset space G/K, K being a compact subgroup of G. For a general choice of G, K, Γ and a generic matter content, we show how to construct a minimal Kähler potential and a general superpotential, for both rigid and local $N = 1$ supersymmetric theories.

Keywords Flavour symmetries · Symplectic modular invariance · Automorphic forms

1 A Fresh Look into an Old Matter

Traditional (linearly realized) flavour symmetries act in generation space. In their simplest implementation the flavour group G_f commutes with both the Poincaré and the gauge groups, but it can be also combined with CP resulting in an additional non-trivial action in flavour space. The most important fact about this type of symmetries is that in any realistic construction they need to be broken [1]. Broken symmetries are well understood and ubiquitous in particle physics and, at first sight, do not represent a problem in their implementation. Why should we be worried about them? A first unpleasant aspect is that the freedom is huge: the flavour group can be Abelian or not, continuous or discrete, global or local. There is no accepted baseline model in a bottom-up approach and in most of the existing constructions the predictability is very limited. Usually the symmetry breaking sector consists of a set $\{\tau_\alpha\}$ of dimensionless, gauge-invariant fields charged under G_f. When space-time coordinates are varied,

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_41

these fields span a moduli space $\mathcal{M}$ describing the possible vacua of the system. We can expand a fermion mass matrix $m_{ij}(\tau)$ in powers of τ_α[1]:

$$m_{ij}(\tau) = m_{ij}^{(0)} + m_{ij}^{(1)\ \alpha}\tau_\alpha + m_{ij}^{(1)\ \bar{\alpha}}\bar{\tau}_{\bar{\alpha}} + m_{ij}^{(2)\ \alpha\beta}\tau_\alpha\tau_\beta + \cdots \tag{1}$$

A realistic model requires at least few terms in the series (1). Additional parameters are brought in by the renormalization group evolution needed to translate the high-energy predictions into low-energy physical parameters. If the theory is supersymmetric, extra parameters associated to supersymmetry breaking are needed. Most of realistic models depend on a large number of free parameters to the detriment of predictability.

On top of that, a very unattractive feature is the need of a mechanism producing τ_α with appropriate size and orientation in flavour space. This alignment problem is typically solved at the expenses of enlarging both the symmetry group, including additional "shaping" factors in G_f, and the symmetry breaking sector, including a pletora of driving fields, not directly entering the expression (1). In model building the usual path proceeds from the choice of G_f and its representations $\rho^{(f)}(g)$ in field space, to an ad hoc and often baroque construction of the symmetry breaking sector $\{\tau_\alpha\}$. In this way the central ingredient of the whole construction is relegated to the very last step.

Can we reverse the logic? If the symmetry breaking sector is so crucial, why not look for physically and/or mathematically motivated symmetry breaking sectors and inspect their symmetry properties? Consider the following simple example. Imagine that the moduli space $\mathcal{M}$ describes the (non-oriented) lines of the plane passing through the origin. To parametrize this set we can choose points lying on the unit circle centered at the origin of the complex plane: $\mathcal{M} = \{\tau \in C, |\tau| = 1\}$, with the agreement that τ and $\gamma\tau = -\tau$ should be identified, since they describe the same line. The $\Gamma \equiv Z_2$ parity symmetry $\tau \to \gamma\tau$ is a gauge symmetry, since it reflects the redundancy of the adopted parametrization. The moduli space $\mathcal{M}$ is "too large" and a one-to-one correspondence with the lines of the plane is obtained by considering the quotient $\mathcal{M}/\Gamma$.[2] In a putative field theory where τ is a scalar field, we should also assign matter fields $\Psi(x)$ to (possibly non-linear and projective) Γ representations. By consistency, the low-energy EFT should satisfy the gauge symmetry under Γ. Following this procedure, the flavour group Γ and its representations are derived from the moduli space $\mathcal{M}$, that in turns describes the allowed vacua. We also notice that the gauge symmetry Γ is always realized in the broken phase, since there is no point on the unit circle that is left invariant by Γ.

A less trivial example is that of a theory where the physically inequivalent vacua are in a one-to-one correspondence with classes of conformally equivalent metrics on the torus [2]. The moduli space is the upper half-plane $\mathcal{M} = SL(2, \mathbb{R})/SO(2) = \{\tau |\ \Im(\tau) > 0\}$. Since tori related by a transformation γ of $\Gamma = SL(2, \mathbb{Z})$ are confor-

[1] We make no distinction between a field τ_α and its VEV.

[2] In the string terminology, the moduli space is $\mathcal{M}/\Gamma$. Here we remain closer to the QFT dictionary: we distinguish $\mathcal{M}$ and Γ, call moduli space the whole $\mathcal{M}$ and interpret Γ as a gauge symmetry.

mally equivalent, we can adopt as candidate flavour symmetry $SL(2,\mathbb{Z})$. Indeed the most general transformation of matter fields under this group is:

$$\Psi(x) \xrightarrow{\gamma} (c\tau + d)^{k_\Psi} \rho_\Psi(\gamma)\Psi(x) \quad , \tag{2}$$

where $\rho_\Psi(\gamma)$ is a unitary representation of a finite modular group $SL(2,\mathbb{Z}_N)$,[3] k_Ψ is the weight and N is the level of the representation. When non-vanishing weights are present, Yukawa couplings should be functions of the modulus τ with the appropriate transformation property to enforce invariance under Γ. In a supersymmetric construction Yukawa couplings $Y(\tau)$ are modular forms of given weight k_Y and level N_Y. Since such forms span a finite-dimensional linear space, we have a limited number of allowed couplings and mass/mixing parameters are sharply constrained.

To generalize the above framework we can adopt an Hermitian Symmetric Space (HSS) as moduli space $\mathcal{M}$ [3]. HSS have several attractive features. They have been completely classified. They are Kähler and therefore support supersymmetric realizations. Non-compact HSS naturally arise as moduli space in supergravity and string compactifications. They are nicely related to the theory of automorphic forms, a generalization of modular forms, that are the building blocks of Yukawa couplings. Every HSS is a coset space of the type $\mathcal{M} = G/K$ for some connected Lie group G and a compact subgroup K of G.[4] The generic element τ of $\mathcal{M}$ can be obtained by performing a generic G transformation on an element τ_0 left invariant by K:

$$\tau = g\,\tau_0 \qquad g \in G \quad \& \quad h\,\tau_0 = \tau_0 \qquad \text{for any } h \in K. \tag{3}$$

We choose as flavour symmetry group a discrete subgroup Γ of G, whose action on τ is given by

$$\tau \xrightarrow{\gamma} \gamma\tau \equiv (\gamma g)\tau_0 \qquad \gamma \in \Gamma \quad . \tag{4}$$

To build a (supersymmetric) model incorporating a local symmetry under Γ and possessing physically inequivalent vacua described by $\mathcal{M}/\Gamma$, we need the transformation laws of the matter fields $\Psi(x)$ under Γ. To this purpose we introduce an automorphic factor $j(g,\tau)$ $(g \in G)$ with the property:

[3] The group $SL(2,\mathbb{Z}_N)$, N being an integer, can be view as an unfaithful finite copy of $SL(2,\mathbb{Z})$. While the latter is infinite and does not possess finite unitary representations, the former is finite and its representations are unitary and finite dimensional.

[4] The Lie algebra $\mathcal{G}$ of G decomposes as $\mathcal{G} = \mathcal{V} \oplus \mathcal{A}$, $\mathcal{V}$ being the Lie algebra of K. The algebra $\mathcal{G}$ is invariant under $V + A \to V - A$ and satisfies $[\mathcal{V},\mathcal{V}] \subset \mathcal{V}$, $[\mathcal{V},\mathcal{A}] \subset \mathcal{A}$ and $[\mathcal{A},\mathcal{A}] \subset \mathcal{V}$. An hermitian symmetric space M can be of compact type, of noncompact type or of Euclidean type. In general none of these cases applies and M decomposes as a product $M = M_c \times M_{nc} \times M_e$, where the three factors are hermitian symmetric spaces of compact, noncompact and Euclidean type, respectively. A hermitian symmetric space is irreducible if it is not the product of two hermitian symmetric spaces of lower dimension. Irreducible hermitian symmetric spaces of compact type can be obtained from the noncompact ones, by means of a transformation on the generators of the Lie algebra $\mathcal{G}$: $(V, A) \to (V, iA)$.

$$j(g_1g_2,\tau)=j(g_1,g_2\tau)j(g_2,\tau) \quad . \tag{5}$$

In general Γ is an infinite group and does not admit finite unitary representations. These can be recovered by building an unfaithful finite copy of Γ. Given a normal subgroup G_n of Γ with finite index, we define the finite group $\Gamma_n = \Gamma/G_n$. A general transformation law for matter fields under Γ reads:

$$\begin{cases} \tau \xrightarrow{\gamma} \gamma\tau \\ \Psi^{(I)}(x) \xrightarrow{\gamma} j(\gamma,\tau)^{k_I}\rho^{(I)}(\gamma)\Psi^{(I)}(x) \end{cases} , \tag{6}$$

where we have separated the matter fields $\{\Psi^{(I)}(x)\}$ in subsets with a common weight k_I and $\rho^{(I)}(\gamma)$ is a unitary representation of Γ_n. The property (5) guarantees that the transformation is a (non-linear) realization of Γ.

We consider the case of rigid $\mathcal{N}=1$ supersymmetry and collect all chiral superfields in a multiplet $\Phi=(\tau,\Psi^{(I)})$. Focussing on the Yukawa interactions, the action $\mathcal{S}$ is defined in terms of a Kähler potential $K(\Phi,\bar{\Phi})$ and a superpotential $w(\Phi)$:

$$\mathcal{S}=\int d^4x d^2\theta d^2\bar{\theta}\; K(\Phi,\bar{\Phi})+\Big[\int d^4x d^2\theta\; w(\Phi)+\text{h.c.}\Big], \tag{7}$$

where the Kähler potential $K(\Phi,\bar{\Phi})$, is a real gauge-invariant function of the chiral superfields Φ and their conjugates and the superpotential $w(\Phi)$ is a holomorphic gauge-invariant function of the chiral superfields Φ. The invariance of the action $\mathcal{S}$ under Eq. (6) requires the invariance of the superpotential $w(\Phi)$ and the invariance of the Kähler potential up to a Kähler transformation[5]

$$\begin{cases} w(\Phi)\to w(\Phi) \\ K(\Phi,\bar{\Phi})\to K(\Phi,\bar{\Phi})+f(\Phi)+\bar{f}(\bar{\Phi}) \end{cases} . \tag{8}$$

A candidate minimal Kähler potential is given by:

$$K_{\mathtt{min}}(\Phi,\bar{\Phi})=-c\log Z(\tau,\bar{\tau})+\sum_I Z(\tau,\bar{\tau})^{k_I}|\Psi^{(I)}|^2 \quad . \tag{9}$$

Here

$$Z(\tau,\bar{\tau})\equiv[j^\dagger(g,\tau_0)j(g,\tau_0)]^{-1} \quad , \tag{10}$$

where the dependence on τ is through the element g via the correspondence in Eq. (3) and c is a real constant whose sign is chosen to guarantee local positivity of the metric for the moduli τ. By construction, the above potential is invariant under Γ up to a

[5] In $\mathcal{N}=1$ local supersymmetry these requirements are relaxed and replaced by the invariance of the real gauge-invariant function $\mathcal{G}=K+\log|w|^2$. The superpotential is not necessarily invariant and its variation under Γ can be compensated by the transformation of K [3].

Kähler transformation for a general choice of G, K, Γ, G_n and $j(g,\tau)$. This is not the most general Kähler potential invariant under Γ. Invariance under Γ allows to add to $K_{\mathtt{min}}(\Phi,\bar{\Phi})$ other terms, that cannot be excluded or constrained in a pure bottom-up approach. In general these terms can modify the flavour properties of the theory such as physical fermion masses and mixing angles. Additional assumptions or inputs from a top-down approach are needed in order to reduce the arbitrariness of the predictions.

The conditions for the invariance of the superpotential under Γ can be deduced by expanding $w(\Phi)$ in powers of the supermultiplets $\Psi^{(I)}$:

$$w(\Phi) = \sum_p Y_{I_1 \dots I_p}(\tau)\ \Psi^{(I_1)} \dots \Psi^{(I_p)} \quad . \tag{11}$$

The p-th order term is invariant provided the functions $Y_{I_1 \dots I_p}(\tau)$ obey:

$$Y_{I_1 \dots I_p}(\gamma\tau) = j(\gamma,\tau)^{k_Y(p)} \rho^{(Y)}(\gamma)\ Y_{I_1 \dots I_p}(\tau) \quad , \tag{12}$$

with $k_Y(p)$ and $\rho^{(Y)}$ such that:

(i) The weight $k_Y(p)$ compensates the total weight of the product $\Psi^{(I_1)} \dots \Psi^{(I_p)}$:

$$k_Y(p) + k_{I_1} + \dots + k_{I_p} = 0 \quad . \tag{13}$$

(ii) The product $\rho^{(Y)} \times \rho^{(I_1)} \times \dots \times \rho^{(I_p)}$ contains an invariant singlet.

The field-dependent Yukawa couplings $Y_{I_1 \dots I_n}(\tau)$ are closely related to automorphic forms. Indeed when we restrict to transformations γ of the group G_n in Eq. (12), we obtain:

$$Y_{I_1 \dots I_n}(\gamma\tau) = j(\gamma,\tau)^{k_Y(n)}\ Y_{I_1 \dots I_n}(\tau)\,, \qquad \gamma \in G_n \quad , \tag{14}$$

Thus the function

$$\mathcal{A}(g) \equiv j(g,\tau_0)^{-k_Y(n)} Y_{I_1 \dots I_n}(g\tau_0) \tag{15}$$

is an automorphic form for G, K and G_n, that is a smooth complex function $\mathcal{A}(g)$ that is invariant under the action of the discrete group G_n:

$$\mathcal{A}(\gamma g) = \mathcal{A}(g), \qquad \gamma \in G_n \quad , \tag{16}$$

and that under K transforms as

$$\mathcal{A}(gh) = j(h,\tau_0)^{-1}\ \mathcal{A}(g)\,, \qquad h \in K \quad . \tag{17}$$

Moreover $\mathcal{A}(g)$ is required to be an eigenfunction of the algebra $\mathcal{D}$ of invariant differential operators on G, that is an eigenfunction of all the Casimir operators of G. The definition is completed by suitable growth conditions [4].

As an example of the general framework outlined above, we analyze the case $G = Sp(2m, \mathbb{R})$, $K = U(m)$ and $\Gamma = Sp(2m, \mathbb{Z})$. The related automorphic forms are provided by Siegel modular forms. The elements of the symplectic group $Sp(2m, \mathbb{R})$ are $2m \times 2m$ real matrices of the type:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad g^T\, J\, g = J \qquad J \equiv \begin{pmatrix} 0 & \mathbb{1}_m \\ -\mathbb{1}_m & 0 \end{pmatrix} \quad . \tag{18}$$

The symplectic group $Sp(2m, \mathbb{R})$ has a maximal compact subgroup, $K = U(m)$. An element g of $Sp(2m, \mathbb{R})$ can be uniquely decomposed as:

$$g = \begin{pmatrix} \sqrt{Y} & X\sqrt{Y^{-1}} \\ 0 & \sqrt{Y^{-1}} \end{pmatrix} h \quad , \tag{19}$$

where X and Y are real symmetric $m \times m$ matrices, Y is positive definite ($Y > 0$) and h is an element of K. We see that the moduli space $\mathcal{M} = G/K$, of complex dimension $m(m+1)/2$, can be parametrized by a symmetric complex $m \times m$ matrix τ with positive definite imaginary part, $\tau = X + iY$. This space is called Siegel upper half-plane, $\mathcal{H}_m$, a natural generalization of the complex upper half-plane. The integer m is the genus. The action of $Sp(2m, \mathbb{R})$ on τ is given by:

$$\tau \to g\tau = (A\tau + B)(C\tau + D)^{-1} \quad . \tag{20}$$

As automorphy factor, satisfying the cocycle condition of Eq. (5), we can choose:

$$j(g, \tau) = \mathtt{det}(C\tau + D) \quad . \tag{21}$$

A natural candidate for the discrete gauge group is the Siegel modular group $\Gamma_m = Sp(2m, \mathbb{Z})$. Other discrete subgroups of $G = Sp(2m, \mathbb{R})$ relevant to our purposes are the principal congruence subgroups $\Gamma_m(n)$ of level n, defined as:

$$\Gamma_m(n) = \left\{\gamma \in \Gamma_m \,\middle|\, \gamma \equiv \mathbb{1}_{2m}\,(\mathtt{mod}\, n)\right\}, \tag{22}$$

where n is a generic positive integer, and $\Gamma_m(1) = \Gamma_m$. The group $\Gamma_m(n)$ is a normal subgroup of Γ_m, and the quotient group $\Gamma_{m,n} = \Gamma_m/\Gamma_m(n)$, which is known as finite Siegel modular group, has finite order [5]. By keeping both the genus m and the level n fixed throughout our construction, the supermultiplets $\Psi^{(I)}$ of each sector I are assumed to transform in a representation $\rho^{(I)}(\gamma)$ of the finite Siegel modular group $\Gamma_{m,n}$, with a weight k_I. Under a discrete gauge transformation $\gamma \in \Gamma_m$ we have:

$$\begin{cases} \tau \xrightarrow{\gamma} \gamma\tau = (A\tau + B)(C\tau + D)^{-1} \,, \\ \Psi^{(I)} \xrightarrow{\gamma} [\det(C\tau + D)]^{k_I} \rho^{(I)}(\gamma)\, \Psi^{(I)} \,, \end{cases} \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_m \quad . \tag{23}$$

Due to the cocycle condition in Eq. (5) and the properties of $\rho^{(I)}(\gamma)$, the above definition satisfies the group law. A minimal Kähler potential is given by:

$$K = -c\ \Lambda^2 \log\det(-i\tau + i\tau^\dagger) + \sum_I [\det(-i\tau + i\tau^\dagger)]^{k_I} |\Psi^{(I)}|^2 \quad c > 0. \tag{24}$$

For the p-th order term of the expansion (11) to be modular invariant, the functions $Y_{I_1 \ldots I_p}(\tau)$ should transform as Siegel modular forms with weight $k_Y(p)$ in the representation $\rho^{(Y)}(\gamma)$ of $\Gamma_{m,n}$:

$$Y_{I_1 \ldots I_p}(\gamma\tau) = [\det(C\tau + D)]^{k_Y(p)} \rho^{(Y)}(\gamma)\ Y_{I_1 \ldots I_p}(\tau) \quad , \tag{25}$$

with $k_Y(p)$ and $\rho^{(Y)}(\gamma)$ satisfying the conditions $i)$ and $ii)$ seen above.

In a generic point τ of the moduli space $\mathcal{H}_m$ the discrete symmetry Γ_m is completely broken (i.e. $\gamma\tau = \gamma$ has no solution for $\gamma \in \Gamma_m$), but there can be regions where a part of Γ_m is preserved. The invariant locus Ω_H is a region of $\mathcal{H}_m$ whose points τ are individually left invariant by some subgroup H of Γ_m. The group that, as a whole, leaves the region Ω_H invariant is the normalizer $N(H)$ of H, whose elements γ_N satisfy $\gamma_N^{-1} H \gamma_N = H$. As a consequence, in our supersymmetric action we can restrict the moduli τ to the region Ω_H, which supersedes the full moduli space $\mathcal{H}_m$, and replace the group Γ_m with $N(H)$. Consistent CP transformations can be defined on moduli, matter multiplets and modular forms [6]. These tools allow the construction of viable and predictive models of lepton masses and mixing angles.

Acknowledgements I would like to thank prof. Dobrev for the kind invitation to "Lie Theory and Its Applications in Physics XIV". I am very grateful to Gui-Jun Ding and Xian-Gang Liu, the real key actors in this project.

References

1. F. Feruglio, A. Romanino, Rev. Mod. Phys. **93**(1), 015007 (2021)
2. F. Feruglio, in *From My Vast Repertoire ...: Guido Altarelli's Legacy*, ed. by A. Levy, S. Forte, G. Ridolfi, pp. 227–266. arXiv:1706.08749 [hep-ph]
3. G.J. Ding, F. Feruglio, X.G. Liu, JHEP **01**, 037 (2021)
4. A. Borel, Automorphic forms on reductive groups. in *Automorphic Forms and Applications*. IAS/Park City Mathematics Series, vol. 12, pp. 5–40 (2007)
5. M. Koecher, Zur theorie der modulfunktionen n-ten grades. Matematische Zeitschrift **59** (1954)
6. G.J. Ding, F. Feruglio, X.G. Liu, SciPost Phys. **10**(6), 133 (2021)

Wilson Lines and Their Laurent Positivity

Tsukasa Ishibashi

Abstract Wilson loops are functions of particular interest on the moduli space of G-local systems associated with closed curves. In this article, we study an analogous functions associated with open arcs connecting boundary intervals of a marked surface, which we call the *Wilson lines*. We see that they have nice multiplicative properties for concatenation, and positive Laurent expressions in certain cluster Poisson charts on the moduli space. The contents are based on the joint work [6] with Hironori Oya.

Keywords Moduli space of G-local systems · Wilson lines · Cluster algebra · Laurent positivity

1 Wilson Loops

Let us begin with the moduli space $\mathrm{Loc}_{G,\Sigma}$ of G-local systems (or equivalently, flat G-bundles) on a closed surface Σ. Here G is any semisimple algebraic group. This moduli space can be identified with the character stack:

$$\mathrm{Loc}_{G,\Sigma} = [\mathrm{Hom}(\pi_1(\Sigma), G)/G],$$

where the right-hand side describes the monodromy homomorphisms of local systems. For any conjugacy class $[\gamma]$ in $\pi_1(\Sigma)$ (i.e., free homotopy class of a loop on Σ), associated is a function

$$\rho_{[\gamma]} : \mathrm{Loc}_{G,\Sigma} \to [G/\operatorname{Ad} G]$$

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_42

obtained by evaluating the loop γ by the monodromy homomorphism, which we call the *Wilson loop* along $[\gamma]$. It is a classical fact that the trace functions $\mathrm{Tr}_V(\rho_{[\gamma]})$, when V runs over all finite-dimensional representations of G and $[\gamma]$ runs over all conjugacy classes, generate the function ring $\mathcal{O}(\mathrm{Loc}_{G,\Sigma})$.

2 Cluster Varieties Related to the Moduli Space $\mathrm{Loc}_{G,\Sigma}$

When G is a semisimple algebraic group with trivial center (e.g., $G = PGL_n$), Fock–Goncharov–Shen [1, 5] introduced two kinds of extensions $\mathcal{A}_{\widetilde{G},\Sigma}$, $\mathcal{P}_{G,\Sigma}$ of the moduli space $\mathrm{Loc}_{G,\Sigma}$. Here $\widetilde{G}$ denotes the universal cover of G. These moduli spaces parametrize local systems on Σ equipped with additional decoration data, and fit into the following diagram:

$$\begin{array}{ccc} \mathcal{A}_{\widetilde{G},\Sigma} & & \\ \downarrow & \searrow^{p} & \\ \mathrm{Loc}_{G,\Sigma} & \hookrightarrow & \mathcal{P}_{G,\Sigma} \end{array}$$

Now let us assume that our surface Σ is a marked surface, which is a compact oriented surface equipped with a finite set of marked points. In this slightly variated setting, the extra data of decorations upgrade the moduli spaces $\mathcal{A}_{\widetilde{G},\Sigma}$ and $\mathcal{P}_{G,\Sigma}$ into a *cluster K_2-variety* and a *cluster Poisson variety*, respectively. Namely, they are equipped with a distinguished collection of coordinate charts (birational isomorphisms with algebraic tori), whose transitions have two kinds of particular forms called the *cluster K_2-/Poisson transformations*. These cluster structures automatically lead to the following additional structures:

- The *positive real points* of these moduli spaces, which form contractible real-analytic manifolds. In particular, we get the manifold $\mathrm{Loc}^+_{G,\Sigma}$ of positive real points of $\mathrm{Loc}_{G,\Sigma}$. When $G = PGL_n$, $\mathrm{Loc}^+_{PGL_n,\Sigma}$ reproduces the Hitchin component of the $PSL_n(\mathbb{R})$-character variety [13], which is also regarded as the phase space of the Toda CFT of type A_{n-1} (Liouville CFT for $n = 2$).
- Quantization of these moduli spaces [2, 5]. In particular, the cluster Poisson algebra $\mathcal{O}_{\mathrm{cl}}(\mathcal{P}_{G,\Sigma})$ is naturally accompanied with a non-commutative deformation $\mathcal{O}_q(\mathcal{P}_{G,\Sigma})$, and the latter is represented as positive operators on a certain Hilbert space $\mathcal{H}^+_{G,\Sigma}$. The Hilbert space $\mathcal{H}^+_{G,\Sigma}$ is expected to be identified with the space of conformal blocks in the Toda CFT (Modular Functor Conjecture: see [2, 5, 11, 12]).
- Fock–Goncharov duality [3, 4]. This is a duality between $\mathcal{A}_{\widetilde{G},\Sigma}$ and $\mathcal{P}_{G,\Sigma}$, which expects a "canonical basis" of the function ring of each one, parametrized by the *integral tropical points* of the other. The basis is required to have positive structure constants and Laurent expressions with positive coefficients in each cluster charts, among other axioms.

3 Wilson Lines

Let G be a semisimple algebraic group with trivial center. Let Σ be a marked surface, which is a compact oriented surface with boundary equipped with a finite set $\mathbb{M}$ of marked points (see Figs. 1 and 2). An interior marked point is called a puncture. Let $\Sigma^* := \Sigma \setminus \{\text{punctures}\}$ be the punctured surface. A connected component of the punctured boundary $\partial^\times \Sigma := \partial \Sigma \setminus \mathbb{M}$ is called a *boundary interval*. The moduli space $\mathcal{P}_{G,\Sigma}$ introduced by Goncharov–Shen [5] parametrizes G-local systems $\mathcal{L}$ on Σ^* equipped with some additional data assigned to the marked points and boundary intervals. In particular, it contains a local trivialization datum p_E of $\mathcal{L}$ assigned to each boundary interval E called the *pinning*. The data of pinnings allow one to glue the G-local systems along boundary intervals in an unambiguous way. Thus we have the *gluing morphism* [5]

$$q_{E_1,E_2} : \mathcal{P}_{G,\Sigma_1} \times \mathcal{P}_{G,\Sigma_2} \to \mathcal{P}_{G,\Sigma},$$

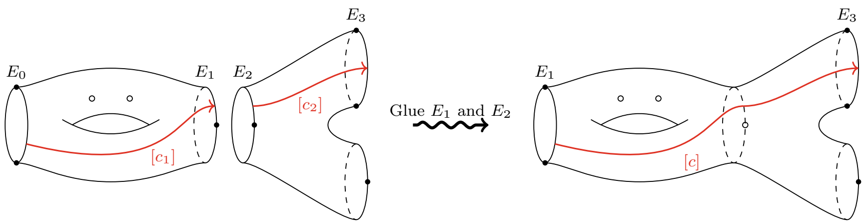

Fig. 1 Gluing marked surfaces: the case where E_1 and E_2 belong to distinct connected components

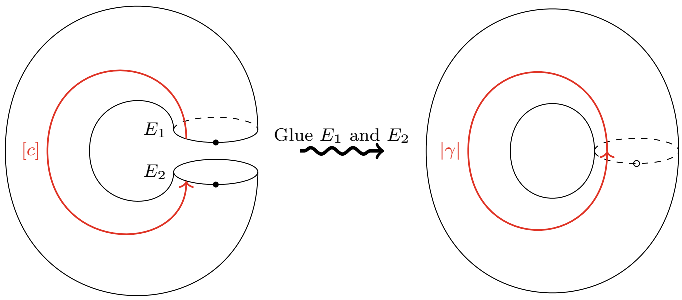

Fig. 2 Gluing marked surfaces: the case where E_1 and E_2 belong to a common connected component

where Σ is obtained by gluing two marked surfaces Σ_1 and Σ_2 along their boundary intervals E_1 and E_2.

The data of pinnings also allow us to introduce the Wilson lines. Let E_{in}, E_{out} be two boundary intervals (which may be identical), and $[c] : E_{\mathrm{in}} \to E_{\mathrm{out}}$ an *arc class* on Σ, by which we mean a homotopy class of continuous paths c running from a point on E_{in} to a point on E_{out}. Then the *Wilson line* along the arc class $[c]$ is the morphism $g_{[c]} : \mathcal{P}_{G,\Sigma} \to G$ defined as the "comparison element" of two pinnings assigned to E_{in} and E_{out} via the parallel-transport along c. Precisely, given a point $m \in \mathcal{P}_{G,\Sigma}$ that contains a G-local system $\mathcal{L}$ with pinnings $(p_E)_E$, the element $g_{[c]}(m) \in G$ is defined as follows.

- The pinning $p_{E_{\mathrm{in}}}$ assigned to E_{in} determines a local trivialization of $\mathcal{L}$ near E_{in}. Extend this local trivialization by the parallel-transport along c until the terminal point.
- Under this extended trivialization, we have a relation $p_{E_{\mathrm{out}}} = g.p^*_{E_{\mathrm{in}}}$ with a unique element $g \in G$. Here for a pinning p, the symbol p^* refers to the *opposite pinning* corresponding to the reversal of orientation of the boundary intervals (see [5] for a detail).

The last element is the Wilson line: $g_{[c]}(m) := g$.

4 Multiplicativity of Wilson Lines

The following multiplicative properties of Wilson lines immediately follow from the definition.

1. Internal multiplicativity. Let $[c_1] : E_1 \to E_2$ and $[c_2] : E_2 \to E_3$ be arc classes on a marked surface Σ, and $[c] = [c_1] * [c_2] : E_1 \to E_3$ the arc class obtained by their concatenation. Then we have

$$g_{[c]} = g_{[c_1]}\overline{w_0}g_{[c_2]},$$

where $\overline{w_0} \in N_G(H)$ is a lift of the longest element in the Weyl group $W(G) = N_G(H)/H$. For instance, $\overline{w_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for $G = PGL_2$.

2. Gluing multiplicativity. Let Σ_1, Σ_2 be marked surfaces. Let $[c_1] : E_0 \to E_1$ and $[c_2] : E_2 \to E_3$ be arc classes of Σ_1 and Σ_2, respectively. Gluing Σ_1 and Σ_2 along the boundary intervals E_1 and E_2, we get a new marked surface Σ, on which the concatenation $[c] := [c_1] * [c_2] : E_0 \to E_3$ defines an arc class. In this setting, we have

$$q^*_{E_1,E_2} g_{[c]} = g_{[c_1]} \cdot g_{[c_2]},$$

where $q_{E_1,E_2} : P_{G,\Sigma_1} \times \mathcal{P}_{G,\Sigma_2} \to \mathcal{P}_{G,\Sigma}$ is the gluing morphism.

Similarly, we may glue the initial and terminal intervals of an arc class $[c] : E_1 \to E_2$ (see Fig. 2). In this case, the arc class $[c]$ naturally closes up to produce a free loop $[\gamma]$ on the new marked surface. Then the Wilson line $g_{[c]}$ represents the conjugacy class $q^*_{E_1,E_2}\rho_{[\gamma]}$ (i.e., Wilson loop).

The internal multiplicativity can be restated that the *twisted Wilson line* $g^{\mathrm{tw}}_{[c]} := g_{[c]}\overline{w_0}$ satisfy a simpler relation $g^{\mathrm{tw}}_{[c]} = g^{\mathrm{tw}}_{[c_1]}g^{\mathrm{tw}}_{[c_2]}$. However, it turns out that the Wilson lines have a nice positivity property while the twisted ones lose this property. The gluing multiplicativity allows us to compute the Wilson loops from Wilson lines.

5 Positivity of Wilson Lines

Since the Wilson lines $g_{[c]} : \mathcal{P}_{G,\Sigma} \to G$ are morphisms of stacks, they induces ring homomorphisms

$$g^*_{[c]} : \mathcal{O}(G) \to \mathcal{O}(\mathcal{P}_{G,\Sigma})$$

between the function rings. The ring $\mathcal{O}(G)$ is generated by matrix coefficients $c^V_{f,v}(g) := \langle f, g.v\rangle_V$ in finite dimensional representations V, where $v \in V$ and $f \in V^*$. The images $c^V_{f,v}(g_{[c]}) := g^*_{[c]}(c^V_{f,v}) \in \mathcal{O}(\mathcal{P}_{G,\Sigma})$ are called the matrix coefficients of the Wilson line $g_{[c]}$.

Proposition 1 ([6, Corollary 3.32]) *When Σ has no punctures, the function ring $\mathcal{O}(\mathcal{P}_{G,\Sigma})$ is generated by the matrix coefficients of the Wilson lines.*

Moreover, the relations among the generators can be computed from the multiplicative relations among Wilson lines. When Σ has punctures, a similar result holds for a subalgebra of $\mathcal{O}(\mathcal{P}_{G,\Sigma})$ of finite index.

As briefly mentioned in Sect. 2, the moduli space $\mathcal{P}_{G,\Sigma}$ admits a natural cluster Poisson structure [1, 5, 9]. Moreover, Shen [10] proved that the function ring $\mathcal{O}(\mathcal{P}_{G,\Sigma})$ is isomorphic to the *cluster Poisson algebra*:

$$\mathcal{O}(\mathcal{P}_{G,\Sigma}) \cong \mathcal{O}_{\mathrm{cl}}(\mathcal{P}_{G,\Sigma}) = \bigcap_{\text{cluster chart}} \mathbb{Z}[X_1^{\pm 1}, \ldots, X_N^{\pm 1}].$$

It ensures that any global functions on $\mathcal{P}_{G,\Sigma}$ are expressed as Laurent polynomials in each cluster chart.

Our main result concerns the positivity of coefficients of these Laurent polynomials. We remark that in the construction of cluster Poisson structure, we begin with an explicit chart associated with an ideal triangulation of Σ, and then generate other charts by cluster Poisson transformations. Hence we typically have infinitely many “unknown charts” whose geometric description is not yet obtained. Let us call the charts associated with triangulations the *Goncharov–Shen charts* (*GS charts* for short).

Theorem 1 ([6, Theorem 6.2]) *For any finite-dimensional representation V of G, there exists a basis $\mathbb{B} \subset V$ (with the dual basis $\mathbb{B}^* \subset V^*$) such that for any arc class $[c] : E_{\mathrm{in}} \to E_{\mathrm{out}}$ on Σ, the matrix coefficients*

$$c^V_{f,v}(g_{[c]}) \in \mathcal{O}(\mathcal{P}_{G,\Sigma})$$

for $v \in \mathbb{B}$ and $f \in \mathbb{B}^*$ are expressed as Laurent polynomials with positive coefficients in any GS chart.

The existence of such a nice basis $\mathbb{B} \subset V$ comes from the deep theory on *categorification of quiver Hecke algebras* (see, for instance, [7, 8]). We expect that the Laurent expressions of the matrix coefficients in Theorem 1 have positive coefficients in any cluster charts.

Acknowledgements The author is grateful to the organizers of LT-14 for giving him this opportunity. This work is partially supported by JSPS KAKENHI Grant Numbers 18J13304 and 20K22304, and the Program for Leading Graduate Schools, MEXT, Japan.

References

1. V.V. Fock, A.B. Goncharov, Publ. Math. Inst. Hautes Études Sci. **103**, 1–211 (2006)
2. V.V. Fock, A.B. Goncharov, Invent. Math. **175**(2), 223–286 (2009)
3. V.V. Fock, A.B. Goncharov, Ann. Sci. Éc. Norm. Supér. **42**(6), 865–930 (2009)
4. M. Gross, P. Hacking, S. Keel, M. Kontsevich, J. Amer. Math. Soc. **31**(2), 497–608 (2018)
5. A.B. Goncharov, L. Shen, arXiv:1904.10491
6. T. Ishibashi, H. Oya, arXiv:2011.14260
7. S.-J. Kang, M. Kashiwara, Invent. Math. **190**(3), 699–742 (2012)
8. M. Khovanov, A. Lauda, Represent. Theory **13**, 309–347 (2009)
9. I. Le, Forum Math. Sigma **7**, e13, 165 (2019)
10. L. Shen, https://doi.org/10.1093/imrn/rnab094
11. G. Schrader, A. Shapiro, arXiv:1708.08107
12. J. Teschner, Handbook of Teichmüller Theory, vol. I, pp. 685–760. Math. Soc., Zürich (2007)
13. A. Wienhard, Proceedings of the International Congress of Mathematicians-Rio de Janeiro 2018, vol. II. Invited lectures (World Sci. Publ., Hackensack, NJ, 2018), pp. 1013–1039

Gauging Higher-Spin-Like Symmetries Using the Moyal Product

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Abstract Well established approaches to gauging U(1) transformations or space-time translations lead to theories of interacting bosons of spin 1 or spin 2. We describe a novel approach to gauging their higher derivative generalizations (i.e. higher-spin-like symmetries), leading to a Yang-Mills like theory defined over a symplectic manifold dubbed "master space". The theory incorporates the starting symmetries by using the Moyal product, has a weakly non-local action functional and it is perturbatively stable. Coupling to matter in various representations is displayed. In the spin-2 sector we find a geometric description reminiscent of teleparallelism, with the induced linear connection related to Weitzenböck's.

Keywords Gauge symmetry · Higher spin symmetry · Non-commutative geometry · Higher spin gravity

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_43

1 Introduction

Theoretical descriptions of free massless fields of any integer spin[1] are well known in the literature, while their complete interacting counterparts in flat spacetime are known only for the "low-spins" $s = 1, 2$. One naturally wonders whether a complete interacting higher spin theory could provide us with more tools for describing nature and hopefully teach us more about field theory.

Apart from pure curiosity, a more concrete motivation comes from attempts to explain dark matter and approaches to a quantum theory of gravity [2–5].

One of the most direct ways of establishing the research program to study interacting higher spin theory was formulated by Fronsdal [6]. Even though the problem seems to be well- posed, solving it in a direct way is a very hard task and one must also be careful to take into account the "no-go" theorems which severely constrain[2] some properties of the sought-for theory [7].

Some progress can be made, as is evidenced by current contemporary research. For instance, the famous Vasiliev's theory [8] describes interacting higher spin fields on AdS spacetime. In flat spacetime, a closed solution to the deformation program was found in the "Chiral Higher Spin Gravity" [9]. Since the obtained Hamiltonian is complex it is not completely clear if unitarity is present.

We propose to use symmetries of matter as a proxy, building on the research in [10–14]. The aim of our construction is to explore the gauging procedure for a whole tower of higher-derivative symmetries dubbed higher spin-like symmetries built as an extension of the lower spin cases and approach the construction of a gauge field model. To make this analysis and construction consistent, it proves fruitful to use a symplectic manifold as a domain and realize the Lie algebra of symmetries with a Moyal-commutator of functions on this manifold.

2 Moyal-Higher-Spin Theory

Moyal-Higher-Spin (MHS) symmetries were first realized in [14] and the gauge field models were analyzed in a different context in [15, 16]. A novel perspective on the MHS gauge field, with its stability, symmetries and geometric interpretations was provided in [17]. Coupling to matter, spacetime content and simple scattering amplitudes were explored in [18].

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[1] Contemporary terminology is degenerate between spin and helicity. These numbers find their origin in Wigner's classification [1].

[2] Strict locality, finite number of particle species, existence of higher spin states in the asymptotic regions and a flat background are not altogether compatible with higher-spin particles interacting non-trivially.

2.1 Gauging Free Field Symmetries

The matter model, a complex scalar field $S[\phi] = \frac{1}{2} \int d^d x \, \left(\partial_\mu \phi \, \partial^\mu \phi^\dagger - m^2 \phi \phi^\dagger\right)$, is symmetric under the following field transformation

$$\delta_\varepsilon \phi(x) = \sum_{n=0}^{\infty} (-i)^{n+1} \varepsilon^{\mu_1 \dots \mu_n} \, \partial_{\mu_1} \cdots \partial_{\mu_n} \phi(x) \,. \tag{1}$$

By itself, the $n = 0$ term leads to $U(1)$ transformations whose gauging produces a massless spin 1 field, while the $n = 1$ term leads to spacetime translations whose gauging is related to a massless spin 2 field. The generalization to higher derivative terms led to the name higher spin-like symmetries.

To promote the transformation parameters of the whole tower from rigid tensors to functions on spacetime, following [13] we rewrite both the matter action and (1) using an auxiliary space with coordinates u_μ, which makes the symplectic manifold $\mathcal{M} \times \mathcal{U}$ the domain. We use the non-commutative Moyal product [19] denoted by $\star$.[3] The matter action becomes

$$S[\phi] = \int d^d x d^d u \, (\eta^{ab} u_a u_b - m^2) \star W_\phi(x, u) \tag{2}$$

where $W_\phi = \phi(x) \star \delta(u) \star \phi^\dagger(x)$ is formally the Wigner's function. The symmetry parameters can be represented by $\varepsilon(u) = \sum_{n=0}^{\infty} \varepsilon^{\mu_1 \dots \mu_n} u_{\mu_1} \dots u_{\mu_n}$ and it can be shown [17] that the transformation (1) is now realized as

$$\delta_\varepsilon W_\phi(x, u) = i [W_\phi(x, u) \overset{\star}{,} \varepsilon(u)] \,. \tag{3}$$

This setup allows us to view $\varepsilon(u)$ as a function and promote $\varepsilon(u) \to \varepsilon(x, u)$. If we demand that the localized parameter still creates a symmetry transformation it proves necessary to introduce a compensating field:

$$u_a \to u_a + h_a(x, u), \qquad \delta h_a(x, u) = \partial_a \varepsilon(x, u) + i [h_a(x, u) \overset{\star}{,} \varepsilon(x, u)] \,. \tag{4}$$

The infinite-dimensional Lie algebra of symmetries is thus realized through the Moyal commutator

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \delta_{i[\varepsilon_1 \overset{\star}{,} \varepsilon_2]} \,. \tag{5}$$

[3] Note that the non-commutativity is realized only between spacetime and auxiliary space coordinates. Spacetime in itself remains commutative.

2.2 Mimicking the Yang-Mills Construction

We now define the covariant derivative as $\mathcal{D}^\star_a \equiv \partial^x_a + i[h_a(x,u) \stackrel{\star}{,} \ ]$, and build the curvature tensor

$$F_{ab}(x,u) = \partial^x_a h_b(x,u) - \partial^x_b h_a(x,u) + i\,[h_a(x,u) \stackrel{\star}{,} h_b(x,u)] \tag{6}$$

which transforms in a covariant way $\delta_\varepsilon F_{ab}(x,u) = i\,[F_{ab}(x,u) \stackrel{\star}{,} \varepsilon(x,u)]$. The action is defined as

$$S_{\text{ym}} = -\frac{1}{4g^2_{\text{ym}}} \int d^dx\, d^du\, F^{ab}(x,u) \star F_{ab}(x,u) \tag{7}$$

and an important criterion for the physical viability of the theory is positivity of energy in the linear regime

$$U \approx \frac{1}{2g^2_{\text{ym}}} \int d^{d-1}\mathbf{x} \int d^du \Big(\sum_j F_{0j}(x,u)^2 + \sum_{j<k} F_{jk}(x,u)^2 \Big) . \tag{8}$$

2.3 Spacetime Content

The conventional approach to understanding the spacetime content of our model would be to expand $h_a(x,u)$ in a Taylor series in the auxiliary space[4]

$$h_a(x,u) = \sum_{n=0}^{\infty} h_a^{(n)\mu_1\cdots\mu_n}(x)\, u_{\mu_1} \ldots u_{\mu_n} . \tag{9}$$

Even though this expansion provides coefficient functions with familiar behavior under Lorentz transformations, it can lead to some manifestly divergent expressions when used in integrals. The novelty in our approach [17, 18] is to impose suitable fall-off conditions for $h_a(x,u)$ in all directions of u, ensuring integrability. A good choice are multidimensional Hermite functions, and the expansion becomes

$$h_a(x,u) = \sum_{n_0,\cdots,n_{d-1}\in\mathbf{N}_0} {h_a}^{n_0\cdots n_{d-1}}(x)\, f_{n_0}(u_0) \cdots f_{n_{d-1}}(u_{d-1}) \tag{10}$$

with $f_{n_i}(u_i) = (2^{n_i} n_i! \sqrt{\pi})^{-1/2} H_{n_i}(u_i) e^{-(u_i^2)/2}$ and $H_{n_i}(u_i)$ being a Hermite polynomial of order n_i. The chosen basis furnishes a unitary infinite-dimensional representation of the Lorentz group [18], which in turn guarantees that we can build a unitary

[4] We emphasize that not all indices are symmetrized by using a Latin index outside the expansion. This has a further interpretation in the geometric picture [17].

Hilbert space for the linearized theory. MHS Yang-Mills theory does not contain perturbative ghosts.

It is possible to show that the helicity content in a single coefficient function ${h_a}^{n_0 \cdots n_d}(x)$ is equal to $\pm r, \pm r - 2, \ldots, 0$ for r even or ending with ± 1 for r odd, with $r = \sum_{i=1}^{d} n_i$.

2.4 Coupling to Matter

It proves useful to consider $e_a(x, u) = u_a + h_a(x, u)$, dubbed MHS vielbein as, a more fundamental object in the MHS theory. The gauge field can be coupled to matter in the minimal way used in the gauging procedure. For instance, coupling to the Dirac field is given with $W_\psi = \psi(x) \star \delta^{(d)}(u) \star \bar{\psi}(x)$

$$S_m[\phi, e] = \int d^d x\, d^d u\, W_\psi(x, u) \star K(e(x, u)), \quad K(x, u) = -\gamma^0 \big(\gamma^a e_a(x, u) + M\big)\,. \tag{11}$$

We can calculate a simple 4-point tree level amplitude [18], and obtain a non trivial scattering only for equal sets of momenta in the incoming and outgoing states, which might point to compatibility with the no-go theorems

$$\mathcal{M} = \frac{1}{2}\mathcal{M}_t^{(\mathrm{QED})} \delta^d\big(\ell_h(p_1 - p_2')\big) - \frac{1}{2}\mathcal{M}_u^{(\mathrm{QED})} \delta^d\big(\ell_h(p_1 - p_1')\big)\,. \tag{12}$$

Besides the minimal description, we can model matter actions in the master space by defining them as master fields. In case a matter field transforms in the fundamental representation as $\delta_\varepsilon \phi(x, u) = -i\varepsilon(x, u) \star \psi(x, u)$, we define the covariant derivative as $D_a\phi(x, u) = i e_a(x, u) \star \psi(x, u)$ and the action for a master space Dirac field as

$$S_D[\psi, e] = \int d^d x d^d u \bar{\psi}(x, u) \star \big(i\gamma^a D_a^\star - M\big)\, \psi(x, u)\,. \tag{13}$$

Fundamental matter tree level scattering amplitudes show a softer UV behavior when compared to QED [18].

2.5 Geometric Interpretation

The Taylor expansion (9), though of limited validity, can nevertheless be used to examine some properties of our theory. The linear terms in the expansion of the MHS vielbein $e_a(x, u) \approx e_a^{(0)}(x) + {e_a}^{(1)\mu}(x) u_\mu + \ldots$ and the quadratic terms in the expansion of the MHS metric $g(x, u) \equiv e_a(x, u) \star e^a(x, u) \approx g^{(0)} + g^{(1)\mu} u_\mu +$

$g^{(2)\mu\nu}(x)u_\mu u_\nu + \ldots$ transform under MHS transformations as a vielbein or a metric would under diffeomorphisms.

To explore a possible induced geometric picture [17], we define a linear connection by demanding compatibility between the linear part in the expansion of the MHS covariant derivative and the induced geometric covariant derivative

$$(\mathcal{D}_a^\star V)^\mu_{(1)}(x) = \left(e_a{}^{(1)\nu}\partial_\nu V^\mu_{(1)} - V^\nu_{(1)}\partial_\nu e_a{}^{(1)\mu}\right) \equiv e_a{}^{(1)\nu}\nabla_\nu V^\mu \,. \tag{14}$$

The resulting connection

$$\Gamma^\mu{}_{\rho\nu} = e_a{}^{(1)\mu}\partial_\rho e^{(1)a}{}_\nu \tag{15}$$

is very interesting as it has a non-vanishing torsion, Riemann curvature and non-metricity tensors. It can be related to the Weitzenböck connection by simply adding the respective torsion tensor $\Gamma^\mu{}_{\nu\rho} = \Gamma^\mu_{W\rho\nu} = T^\mu{}_{\nu\rho} + \Gamma^\mu_{W\nu\rho}$.

3 Summary

We have displayed the construction and the most important properties of the MHS Yang Mills theory, along with possible couplings to matter. The spacetime content was explained in terms of an expansion in a suitable basis of Hermite functions, which by construction compels our theory to respect unitarity. Further knowledge of the behavior of our basis under Lorentz transformations will enable classifying our field in terms of Wigner's classification.

Acknowledgements This research has been supported by the University of Rijeka under the project uniri- prirod-18-256. The research of S.G. has been supported by the Israel Science Foundation (ISF), grant No. 244/17.

References

1. E.P. Wigner, Ann. Math. **40**, 149–204 (1939)
2. X.O. Camanho, J.D. Edelstein, J. Maldacena, A. Zhiboedov, JHEP **02**, 020 (2016)
3. A. Falkowski, G. Isabella, C.S. Machado, SciPost Phys. **10**(5), 101 (2021)
4. I.R. Klebanov, A.M. Polyakov, Phys. Lett. B **550**, 213–219 (2002)
5. G. Bonelli, Nucl. Phys. B **669**, 159–172 (2003)
6. C. Fronsdal, Phys. Rev. D **18**, 3624 (1978)
7. X. Bekaert, N. Boulanger, P. Sundell, Rev. Mod. Phys. **84**, 987–1009 (2012)
8. M.A. Vasiliev, Phys. Lett. B **243**, 378–382 (1990)
9. D. Ponomarev, E.D. Skvortsov, J. Phys. A **50**(9), 095401 (2017)
10. X. Bekaert, Ann. U. Craiova Phys. **16**(II), 58–65 (2006)
11. X. Bekaert, Int. J. Geom. Meth. Mod. Phys. **6**, 285–342 (2009)
12. X. Bekaert, E. Joung, J. Mourad, JHEP **05**, 126 (2009)
13. X. Bekaert, E. Joung, J. Mourad, JHEP **02**, 048 (2011)

14. L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari, M. Paulišić, T. Štemberga, JHEP **04**, 095 (2018)
15. L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari, T. Štemberga, Eur. Phys. J. C **79**(3), 258 (2019)
16. L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari, T. Stemberga, arXiv:1812.05030 [hep-th]
17. M. Cvitan, P. Dominis Prester, S. Giaccari, M. Paulišić, I. Vuković, JHEP **06**, 144 (2021)
18. M. Cvitan, P.D. Prester, S.G. Giaccari, M. Paulišić, I. Vuković, Symmetry **13**(9), 1581 (2021)
19. H.J. Groenewold, Physica **12**, 405–460 (1946)

Integration of Double Field Theory Algebroids and Pre-rackoid in Doubled Geometry

Noriaki Ikeda and Shin Sasaki

Abstract We study the integration problem of the D-bracket and a Vaisman (metric, pre-DFT) algebroid which are geometric structures of double field theory (DFT). We introduce a notion of a pre-rackoid as a global group-like object for the infinitesimal algebroid structure. The pre-rackoid is defined by cotangent paths along doubled foliations in a para-Hermitian manifold. We show that when the strong constraint of DFT is imposed, the self-distributivity of the rack action is recovered and the pre-rackoid reduces to a rackoid that is an integration of the Courant algebroid.

Keywords Double field theory · Algebroid · Rack

1 Introduction

Double field theory (DFT) is a supergravity where T-duality is realized manifestly [3, 8]. DFT is defined in a $2D$-dimensional doubled space $\mathcal{M}$ which is characterized by the space-time coordinate x^μ together with its T-dual counterpart $\tilde{x}_\mu$. The NSNS sector of type II supergravities, namely, the space-time metric $g_{\mu\nu}$, the B-field $B_{\mu\nu}$ and the dilation ϕ are packaged into the generalized metric $\mathcal{H}_{MN}(x, \tilde{x})$ and the generalized dilation $d(x, \tilde{x})$. T-duality is implemented as a global $O(D, D)$ symmetry in the doubled space $\mathcal{M}$. The geometry of $\mathcal{M}$ that incorporates the DFT structure is given by a para-Hermitian manifold [10, 11]. Due to the para-complex structure $K : T\mathcal{M} \to T\mathcal{M}$ satisfying $K^2 = 1$, the tangent bundle $T\mathcal{M}$ is decomposed into the $K = \pm 1$ eigenbundles $T\mathcal{M}_+$, $T\mathcal{M}_-$. There are doubled foliations $\mathcal{F}$, $\tilde{\mathcal{F}}$ of $\mathcal{M}$ associated with the integrability of $T\mathcal{M}_+$, $T\mathcal{M}_-$. The decomposition of the local

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_44

coordinates $x^M = (x^\mu, \tilde{x}_\mu)$ follows from this double foliation structure of the para-Hermitian manifold.

In order to define the D-dimensional physical space-time in the doubled space $\mathcal{M}$, it is necessary to impose the constraints on any quantities in DFT. This is known as the strong constraint. Once the strong constraint is imposed, DFT has the gauge symmetry that encompasses the diffeomorphism invariance and the B-field gauge symmetry. The D-dimensional physical space-time is realized as a leaf in the doubled foliations on which the strong constraint is satisfied trivially [2].

The infinitesimal gauge transformation in DFT is governed by the C-bracket that defines the Vaisman (metric, pre-DFT) algebroids on $T\mathcal{M}$ [1, 9, 11]. This is a generalization of the Courant algebroid based on the Courant bracket. In this contribution, we introduce an integration problem of the Vaisman algebroid which uncovers the global group-like structures of the gauge symmetry of DFT. This is a kind of *coquecigrue problem* [7]. The details are found in our published paper [4].

2 Leibniz, Courant and Vaisman Algebroids

We first introduce the notion of the Courant algebroid.

Definition 1 (*Courant algebroid*) Let $E \xrightarrow{\pi} M$ be a vector bundle over a manifold M. A *Courant algebroid* is a quadruple $(E, [\cdot,\cdot], \rho, (\cdot,\cdot))$ where $[\cdot,\cdot]$ is a bilinear bracket on $\Gamma(E)$, $\rho : E \to TM$ is an anchor map, and $(\cdot,\cdot)$ is a non-degenerate bilinear form on $\Gamma(E)$. They satisfy the following axioms for any $e_i \in \Gamma(E)$ and $f \in C^\infty(M)$:

1. $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$.
2. $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]_{TM}$.
3. $[e_1, f e_2] = f[e_1, e_2] + (\rho(e_1) \cdot f)e_2$.
4. $[e, e] = \frac{1}{2}\mathcal{D}(e, e)$.
5. $\rho(e_1) \cdot (e_2, e_3) = ([e_1, e_2], e_3) + (e_2, [e_1, e_3])$.

Here $\mathcal{D}$ is a generalized exterior derivative on $\Gamma(E)$ and $[\cdot,\cdot]_{TM}$ is the Lie bracket of vector fields on TM.

We note that $[\cdot,\cdot]$ is not skew-symmetric in general. We also note that the axioms 1, 2, 3 define the Leibniz algebroid. Therefore any Courant algebroids are Leibniz algebroids. When M is a point $M = \{\mathrm{pt}\}$ and $\rho = 0$, the Leibniz algebroid becomes a Leibniz algebra. We next define the Vaisman algebroid.

Definition 2 (*Vaisman algebroid*) A *Vaisman algebroid* is a quadruple $(E, [\![\cdot,\ \cdot]\!]_D, \rho, (\cdot,\cdot))$ which satisfies the axioms of 2 and 5 for the Courant algebroid.

The bracket $[\![\cdot,\ \cdot]\!]_D$ is called a D-bracket. The Vaisman algebroid is a generalization of the Courant algebroid. There is an alternative but equivalent definition based on a skew-symmetric bracket. The skew symmetrization of a D-bracket is called a C-bracket. The skew-symmetric bracket of the Vaisman algebroid on a para-Hermitian manifold is nothing but the C-bracket that governs the gauge symmetry of DFT. When the strong constraint is imposed on the gauge parameters, the D-bracket reduces to the Dorfman bracket of generalized geometry. In this case, the Vaisman algebroid reduces to the Courant algebroid.

3 Racks and Rackoids

In this section, we focus on the integration problem of algebroids in DFT. As a first step, we introduce a rack.

Definition 3 (*Rack*) The set S together with a binary operation $(x, y) \mapsto x \triangleright y$ for any $x, y \in S$ is called a *rack* if the map $y \mapsto x \triangleright y$ is bijective and the operation $\triangleright$ satisfies the following left *self-distributivity*:

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z), \tag{1}$$

for any $x, y, z \in S$. $x \triangleright y$ and the map $y \mapsto x \triangleright y$ are called the rack product and the rack action of x on y, respectively.

One finds that the differentiation of the self-distributivity (1) gives the Leibniz identity [5]. Therefore a rack embodies an integration of Leibniz algebra. In order to incorporate algebroid structure, we generalize the notion of the rack. Before that, we define the bisection.

Definition 4 (*Bisection*) Let $\mathcal{G} \rightrightarrows M$ be a semi-precategory. A *bisection* of $\mathcal{G}$ is defined by the following equivalent data:

1. A subset $\Sigma \subset \mathcal{G}$ such that the restricted source and the target maps $s, t : \Sigma \to M$ are bijection.
2. A map $\underline{\Sigma} = t \circ \Sigma : M \to M$ that is bijection. Here $\Sigma : M \to \mathcal{G}$ is a right inverse of s, namely, it is defined by $s \circ \Sigma = \mathrm{id}_M$.

Now we define the notion of rackoids.

Definition 5 (*Rackoid*) For a semi-precategory $\mathcal{G} \rightrightarrows M$, a bisection $\Sigma \subset \mathcal{G}$ and $g \in \mathcal{G}_x^y$, one defines an action of Σ on g

$$\triangleright : (\Sigma, g) \mapsto \Sigma \triangleright g \in \mathcal{G}_{\underline{\Sigma}(x)}^{\underline{\Sigma}(y)}. \tag{2}$$

For bisections $\Sigma, T \subset \mathcal{G}$, we define $\Sigma \triangleright T$ as the image of an assignment $\Sigma \triangleright (\cdot)$ on T. When the action $\triangleright$ satisfies the following properties, this becomes a rack action:

1. For any bisections Σ, an assignment $\Sigma \triangleright (\cdot) : \mathcal{G} \to \mathcal{G}$ is bijective.
2. For any bisections Σ, T and any $g \in \mathcal{G}$, the action $\triangleright$ satisfies the following self-distributivity,
$$\Sigma \triangleright (T \triangleright g) = (\Sigma \triangleright T) \triangleright (\Sigma \triangleright g). \tag{3}$$

Here $\mathcal{G}_x^y$ is a set of all the morphisms $g \in \mathcal{G}$ that satisfy $s(g) = x$, $t(g) = y$ for all $x, y \in M$.

Then, $(\mathcal{G} \rightrightarrows M, \triangleright)$ is called a non-unital rackoid. In addition, for any $x \in M$, $g \in \mathcal{G}$, when there exists $\epsilon(x) = 1_x \in \mathcal{G}$ such that

$$1_M \triangleright g = g, \quad \Sigma(x) \triangleright 1_x = 1_{\underline{\Sigma}(x)}, \tag{4}$$

then $(\mathcal{G} \rightrightarrows M, \triangleright)$ is called a unital (or pointed) rackoid. Here 1_M stands for the bisection $\epsilon(M)$, namely, the collection of 1_x for all $x \in M$. When all the structures defined above are smooth, $(\mathcal{G} \rightrightarrows M, \triangleright)$ is called a Lie rackoid.

It was shown that a rackoid is an integration of a Courant algebroid [6]. We next generalize the notion of rackoids and find an example of integration of the Vaisman algebroid.

4 Pre-rackoids and Doubled Cotangent Paths

The D-bracket on the doubled space $\mathcal{M}$ is defined by

$$\begin{aligned} [\![e_1, e_2]\!]_D = &[X_1, X_2]_{T\mathcal{M}_+} + \mathcal{L}_{\xi_1} X_2 - \iota_{\xi_2} \mathrm{d}^* X_1 \\ &+ [\xi_1, \xi_2]_{T\mathcal{M}_-} + \mathcal{L}_{X_1} \xi_2 - \iota_{X_2} \mathrm{d}\xi_1. \end{aligned} \tag{5}$$

Here $e_i = X_i + \xi_i$ and X_i, ξ_i are vectors and dual vectors, d^*, d are exterior derivatives on the vector and its dual vector spaces. $[\cdot, \cdot]_{T\mathcal{M}_+}, [\cdot, \cdot]_{T\mathcal{M}_-}$ are Lie brackets on vector and dual vector spaces and $\mathcal{L}_X, \mathcal{L}_\xi$ are Lie derivatives associated with the vectors and their duals. We now examine an structure on which the differentiation reproduces (5). We note that the D-bracket (5) violates the Leibniz identity. This means that the rackoid is not an integration of the Vaisman algebroid. We define the notion of the pre-rackoids.

Definition 6 (*Pre-rackoid*) Let $\mathcal{G} \rightrightarrows M$ be a semi-precategory. Bisections of $\mathcal{G}$ are defined as in the definition 4. For any bisection Σ and $g \in \mathcal{G}_x^y$, we define an action of Σ on g

$$\trianglerighteq : (\Sigma, g) \mapsto \Sigma \trianglerighteq g \in \mathcal{G}_{\Sigma \cdot x}^{\Sigma \cdot y}. \tag{6}$$

Here $\Sigma \cdot x$ stands for a smooth action of Σ on $x \in M$. When the assignment $\Sigma \trianglerighteq (\cdot) : \mathcal{G} \to \mathcal{G}$ is bijective, we call this the pre-rack action (product). We then call $(\mathcal{G} \rightrightarrows M, \trianglerighteq)$ the pre-rackoid.

We now exhibit an explicit realization of the pre-rackoid product which is an integration of the D-bracket (5). Let us consider a path $(\phi, \eta) \subset PT^*\mathcal{M} = C^\infty([0, 1], T^*\mathcal{M})$ in a leaf $\mathcal{F}_{x,[\tilde{x}]}$ given by $\tilde{x} = \text{const}$. Here $\phi = \pi \circ \eta$ on $T^*\mathcal{M} \xrightarrow{\pi} \mathcal{M}$ is a path in the base space and η is the associated path in the cotangent bundle. One defines a cotangent path rackoid $(PT^*\mathcal{M} \rightrightarrows \mathcal{F}_{x,[\tilde{x}]}, \triangleright)$ whose rack action

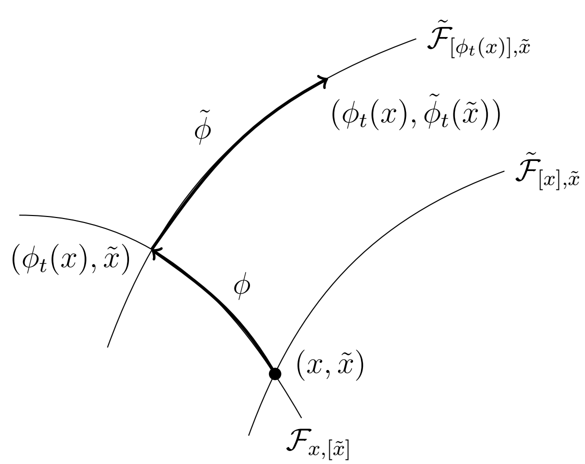

Fig. 1 Leaves for the doubled foliations of $\mathcal{M}$ (thin lines). The doubled path is the concatenation of the paths along the leaves $\mathcal{F}_{x,[\tilde{x}]}$ and $\tilde{\mathcal{F}}_{[\phi_t(x)],\tilde{x}}$ (bold lines)

is defined by the adjoint action on paths $(\psi \triangleright \varphi)_{t\in[0,1]} = \psi_1 \circ \varphi_t \circ \psi_1^{-1}$. One also defines another cotangent path rackoid $(\widetilde{PT^*\mathcal{M}} \rightrightarrows \tilde{\mathcal{F}}_{[x],\tilde{x}}, \triangleright)$ based on the cotangent path $(\tilde{\phi}, \tilde{\eta}) \subset \widetilde{PT^*\mathcal{M}}$ on the leaf $\tilde{\mathcal{F}}_{[x],\tilde{x}}$ at $x = \text{const}$. Now we introduce a new path based on a pair of paths $(\phi, \eta) \subset PT^*\mathcal{M}$, $(\tilde{\phi}, \tilde{\eta}) \subset \widetilde{PT^*\mathcal{M}}$ in the doubled foliations on $\mathcal{M}$ and the cotangent bundle $T^*\mathcal{M}$. The new path on the base space $\mathcal{M}$ is defined by the concatenation of the paths $\phi : [0, 1] \to \mathcal{F}_{x,[\tilde{x}]}$ and $\tilde{\phi} : [0, 1] \to \tilde{\mathcal{F}}_{[\phi_t(x)],\tilde{x}}$ along the leaves $\mathcal{F}_{x,[\tilde{x}]}$ and $\tilde{\mathcal{F}}_{\phi_t(x),[\tilde{x}]}$, respectively (see Fig. 1). The paths in the cotangent space is defined similarly by the concatenation of η and $\tilde{\eta}$ on $(T\mathcal{M}_+)^*$ and $(T\mathcal{M}_-)^*$. We call this the doubled cotangent path and denote it $PT^*\mathcal{M} \diamond \widetilde{PT^*\mathcal{M}} \equiv \mathbf{PT}^*\mathcal{M}$. We define the source and the target maps $\mathbf{PT}^*\mathcal{M} \to \mathcal{M}$ as s and $\tilde{t}$. Here $s, \tilde{t}$ are the source and the target maps of $PT^*\mathcal{M} \rightrightarrows \mathcal{F}_{x,[\tilde{x}]}$ and $\widetilde{PT^*\mathcal{M}} \rightrightarrows \tilde{\mathcal{F}}_{[\phi_t(x)],\tilde{x}}$. Then, $\mathbf{PT}^*\mathcal{M} \rightrightarrows \mathcal{M}$ becomes a semi-precategory. If we employ the pair of the unit maps $(\epsilon, \tilde{\epsilon})$ of $PT^*\mathcal{M}$ and $\widetilde{PT^*\mathcal{M}}$ as the unit map of $\mathbf{PT}^*\mathcal{M} \rightrightarrows \mathcal{M}$, it becomes a smooth precategory. Bisections of $\mathbf{PT}^*\mathcal{M}$ are defined similarly through the ones in the precategories $PT^*\mathcal{M}$, $\widetilde{PT^*\mathcal{M}}$. For bisections $\varSigma = (\phi, \eta)$, $\tilde{\varSigma} = (\tilde{\phi}, \tilde{\eta})$ of $PT^*\mathcal{M}$ and $\widetilde{PT^*\mathcal{M}}$, a bisection Σ of $\mathbf{PT}^*\mathcal{M}$ is given by $\Sigma = \varSigma \diamond \tilde{\varSigma}$.

We then define a pre-rack product $\trianglerighteq$ in the precategory $\mathbf{PT}^*\mathcal{M} \rightrightarrows \mathcal{M}$. A product of bisections between $\Sigma = \tilde{\varSigma} \diamond \varSigma$ and $\mathbf{T} = \tilde{T} \diamond T$ of $\mathbf{PT}^*\mathcal{M} \rightrightarrows \mathcal{M}$ is defined by

$$\begin{aligned}\Sigma \trianglerighteq \mathbf{T} = \Big(&\phi_1 \circ \psi_t \circ \phi_1^{-1} \diamond \tilde{\phi}_1 \circ \tilde{\psi}_t \circ \tilde{\phi}_1^{-1}, \\ &((\phi_1^{-1})^*(\zeta_t) - (\phi_1^{-1})^* \iota_{\dot{\psi}} \phi_1^* \mathrm{d}\beta_\varSigma) + ((\tilde{\phi}_1^{-1})^*(\tilde{\zeta}_t) - (\tilde{\phi}^{-1})_1^* \tilde{\iota}_{\dot{\tilde{\psi}}} \tilde{\phi}_1^* \tilde{\mathrm{d}} \tilde{\beta}_{\tilde{\varSigma}})\Big).\end{aligned} \tag{7}$$

Here $\beta_\varSigma = \int_0^1 ds\ \phi_s^* \eta_s$, $\tilde{\beta}_{\tilde{\varSigma}} = \int_0^1 ds\ \tilde{\phi}_s \tilde{\eta}_s$ are 1-forms associated with the bisections $\varSigma, \tilde{\varSigma}$. The product (7) does not satisfy the self-distributivity in general and

$(\mathbf{PT}^*\mathcal{M} \rightrightarrows \mathcal{M}, \trianglerighteq)$ defines a pre-rackoid. One can show that by differentiating the pre-rack product (7), the structure of the D-bracket (5) is recovered. The other structures of algebroids are also obtained from the pre-rackoid [4] and it provides an integration of the Vaisman algebroid.

5 Summary

In this contribution, we studied a global aspect of the doubled geometry in DFT through the coquecigrue problem, namely, an integration of the Vaisman algebroid.

We first introduce an integration of the Leibniz algebroid. An integrated structure corresponding to the Leibniz algebroid is a rackoid, which is a groupoid-like generalization of a rack. Since the Courant algebroid is a Leibniz algebroid, its integration is a rackoid with additional structures. This is given by the cotangent path rackoid proposed in [6].

It is obvious that the Vaisman algebroid fail to satisfy the Leibniz identity in the Courant algebroid. Since the Leibniz identity is encoded into the self-distributivity of the rack product, an integration of the Vaisman algebroid is given by a rackoid type structure without the self-distributivity. We called this structure the pre-rackoid. We showed that this structure is encoded in the doubled cotangent path on the doubled foliations of the para-Hermitian manifold $\mathcal{M}$. Due to the intermediate shift between different leaves of the doubled foliations, the self-distributivity of the pre-rack product is explicitly broken. When the strong constraint is imposed, the pre-rack product is restricted only on a leaf (physical space-time) and the self-distributivity is trivially recovered. This indicates that the pre-rackoid becomes a rackoid and the Vaisman algebroid becomes a Courant algebroid.

Acknowledgements The work of S. S. was supported by Grant-in-Aid for Scientific Research (C), JSPS KAKENHI Grant Number JP20K03952.

References

1. A. Chatzistavrakidis, L. Jonke, F.S. Khoo, R.J. Szabo, JHEP **1807**, 015 (2018)
2. L. Freidel, F.J. Rudolph, D. Svoboda, JHEP **1711**, 175 (2017)
3. C. Hull, B. Zwiebach, JHEP **0909**, 099 (2009)
4. N. Ikeda, S. Sasaki, J. Math. Phys. **62**(3), 032306 (2021)
5. M.K. Kinyon, J. Lie Theory **17**, 99–114 (2007)
6. C. Laurent-Gengoux, F. Wagemann, arXiv:1807.05891 [math.AT]
7. J.L. Loday, Enseign. Math. **39**, 269–293 (1993)
8. W. Siegel, Phys. Rev. D **48**, 2826–2837 (1993)
9. D. Svoboda, J. Math. Phys. **59**(12), 122302 (2018)
10. I. Vaisman, J. Math. Phys. **53**, 033509 (2012)
11. I. Vaisman, J. Math. Phys. **54**, 123507 (2013)

Doubled Aspects of Algebroids and Gauge Symmetry in Double Field Theory

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Abstract The metric algebroid proposed by Vaisman (the Vaisman algebroid) governs the gauge symmetry algebra generated by the C-bracket in Double Field Theory (DFT). We show that the Vaisman algebroid is obtained by an analogue of the Drinfel'd double of Lie algebroids. We examine geometric implementations of this algebroid in the para-Hermitian manifold which is a realization of the doubled space-time in DFT.

Keywords Double Field Theory · T-duality · Algebroid · Para-Hermitian manifold

1 Introduction

T-duality in string theory is important to understand the nature of space-time. It has been studied in the framework of Hitchin's generalized geometry [1]. A Courant algebroid naturally appears as an algebraic structure in the geometry [2].

Double Field Theory (DFT) [3] is one of the effective theories of superstring theory where T-duality is realized manifestly. DFT has a T-duality covariantized gauge symmetry. It originates from the diffeomorphism and the $U(1)$ gauge symmetry of the NSNS B-field. This gauge symmetry in DFT is described by a C-bracket. It does not satisfy the Jacobi identity. Therefore, the algebraic structure of the gauge symmetry in DFT is not Lie algebra but a metric (Vaisman) algebroid [4]. The Vaisman algebroid is a generalization of a Courant algebroid. The Vaisman algebroid is also known as a pre-DFT algebroid [5]. In this proceeding, we show that the Vaisman algebroid has

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_45

a doubled structure. This algebroid is constructed by a pair of Lie algebroids. This doubled structure is closely related with a Drinfel'd double of the Lie bialgebroid [6].

The geometry of DFT is called "doubled geometry." This is related to the para-Hermitian geometry [8] and the Born geometry [9]. In the latter part of our paper [7], we construct the Vaisman algebroid with the doubled structure in the para-Hermitian geometry, and we discuss the mathematical origin of the strong constraint which is the physical constraint of DFT. In this proceeding, we give a brief outline of the results.

2 Lie Bialgebroid

A Lie algebroid is defined as a generalization of a Lie algebra. It is defined by a vector bundle E on a manifold M, an anchor map $\rho : E \to TM$, a section of E, and a Lie algebroid bracket $[\cdot, \cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$. The Lie algebroid bracket satisfies the Jacobi identity. In addition, the following two conditions are required:

$$[X, fY]_E = (\rho(X) \cdot f)Y + f[X, Y]_E, \tag{1}$$

$$\rho([X, Y]_E) = [\rho(X), \rho(Y)], \tag{2}$$

where $f \in C^\infty(M)$ and the $[\cdot, \cdot]$ is the Lie bracket of $\Gamma(TM)$. Considering the dual bundle E^* for E, we can also define a dual Lie algebroid $(E^*, \rho_*, [\cdot, \cdot]_{E^*})$. The inner product $\langle \cdot, \cdot \rangle$ is naturally introduced between E and E^*.

If we consider the multi vector $\Gamma(\wedge^\bullet E)$ and the multi form $\Gamma(\wedge^\bullet E^*)$, we can introduce exterior derivatives d and d* for $\Gamma(\wedge^\bullet E)$ and $\Gamma(\wedge^\bullet E^*)$ respectively. It acts for $\xi \in \Gamma(\wedge^p E^*)$ as follows [10]:

$$\begin{aligned} \mathrm{d}\xi(X_1, \ldots, X_{p+1}) = & \sum_{i=1}^{p+1} (-)^{i+1} \rho(X_i) \cdot \Big(\xi(X_1, \ldots, \check{X}_i, \ldots, X_{p+1}) \Big) \\ & + \sum_{i<j} (-)^{i+j} \xi([X_i, X_j]_E, X_1, \ldots, \check{X}_i, \ldots, \check{X}_j, \ldots, X_{p+1}). \end{aligned} \tag{3}$$

Here $X_i \in \Gamma(E)$ and $\check{X}_i$ stands for that the term is omitted in the expression. Similarly, we can also define the Lie derivative $\mathscr{L}$ and the interior product ι.

The Schouten-Nijenhuis bracket $[\cdot, \cdot]_{\mathrm{S}}$ is a generalization of the Lie algebroid bracket for multi vectors (forms). The bracket $[\cdot, \cdot]_{\mathrm{S}}$ has the following properties:

(i) $[X, Y]_{\mathrm{S}} = -(-)^{pq}[Y, X]_{\mathrm{S}}$.
(ii) $[X, f]_{\mathrm{S}} = \rho(X) \cdot f$ for $X \in \Gamma(E)$.
(iii) For $X \in \Gamma(\wedge^{p+1} E)$, the bracket $[X, \cdot]_{\mathrm{S}}$ acts on $\Gamma(\wedge^q E)$ as a degree-p derivation.

If the following derivation condition

$$d_*[X, Y]_S = [d_* X, Y]_S + [X, d_* Y]_S \tag{4}$$

is satisfied between the algebroids E and E^*, this Lie algebroid pair becomes a Lie bialgebroid [10].

3 Doubled Structure of Vaisman Algebroid

A Vaisman algebroid $(\mathscr{V}, \rho, (\cdot,\cdot), [\cdot,\cdot]_V)$ is defined by a vector bundle $\mathscr{V}$ on a manifold M, an anchor map $\rho : \mathscr{V} \to TM$, a nondegenerate bilinear form $(\cdot,\cdot)$ and an antisymmetric bracket (Vaisman bracket) $[\cdot,\cdot]_V : \Gamma(\mathscr{V}) \times \Gamma(\mathscr{V}) \to \Gamma(\mathscr{V})$. The quadruple $(\mathscr{V}, \rho, (\cdot,\cdot), [\cdot,\cdot]_V)$ satisfies these two axioms

Axiom V1 $[e_1, f e_2]_V = f[e_1, e_2]_V + (\rho(e_1) \cdot f) e_2 - (e_1, e_2)\mathscr{D} f,$

Axiom V2 $\rho(e_1) \cdot (e_2, e_3) = ([e_1, e_2]_V + \mathscr{D}(e_1, e_2), e_3) + (e_2, [e_1, e_3]_V + \mathscr{D}(e_1, e_3)),$

where $e_1, e_2, e_3 \in \Gamma(\mathscr{V})$ and $\mathscr{D}$ is an exterior derivative for $f \in C^\infty(M)$.

In our paper [7], we obtain the Vaisman algebroid by considering a double of two Lie algebroids. In the following, we give a brief outline of the results. We consider $\mathscr{V} = E \oplus E^*$ and examine the Axioms V1 and V2 using only the properties of a Lie algebroid. In the following, we denote $X \in \Gamma(E)$ and $\xi \in \Gamma(E^*)$. The Vaisman bracket is defined by

$$\begin{aligned}[e_1, e_2]_V = {} & [X_1, X_2]_E + \mathscr{L}_{\xi_1} X_2 - \mathscr{L}_{\xi_2} X_1 - d_*(e_1, e_2)_- \\ & + [\xi_1, \xi_2]_{E^*} + \mathscr{L}_{X_1}\xi_2 - \mathscr{L}_{X_2}\xi_1 + d(e_1, e_2)_-,\end{aligned} \tag{5}$$

where

$$(e_1, e_2)_\pm = \frac{1}{2}\Big(\langle \xi_1, X_2\rangle \pm \langle \xi_2, X_1\rangle\Big). \tag{6}$$

As a nondegenerate symmetric bilinear form $(\cdot,\cdot)$, we take $(\cdot,\cdot)_+$.

Axiom V1 is the Leibniz rule of the Vaisman bracket. The left-hand side of Axiom V1 becomes

$$[e_1, f e_2]_V = [X_1, f X_2]_V + [X_1, f\xi_2]_V + [\xi_1, f X_2]_V + [\xi_1, f\xi_2]_V. \tag{7}$$

If we calculate the right-hand side of (7), we obtain

$$
\begin{aligned}
&[X_1, f\xi_2]_{\mathrm{V}} = f[X_1, \xi_2]_{\mathrm{V}} + (\rho(X_1)\cdot f)\xi_2 - \frac{1}{2}\mathscr{D}g\langle \xi_2, X_1\rangle,\\
&[\xi_1, fX_2]_{\mathrm{V}} = f[\xi_1, X_2]_{\mathrm{V}} + (\rho_*(\xi_1)\cdot f)X_2 - \frac{1}{2}\mathscr{D}f\langle \xi_1, X_2\rangle,\\
&[X_1, fX_2]_{\mathrm{V}} = f[X_1, X_2]_{\mathrm{V}} + (\rho(X_1)\cdot f)X_2,\\
&[\xi_1, f\xi_2]_{\mathrm{V}} = f[\xi_1, \xi_2]_{\mathrm{V}} + (\rho_*(\xi_1)\cdot f)\xi_2.
\end{aligned} \tag{8}
$$

Adding up all the terms on the right-hand side of (8), we obtain

$$
[e_1, fe_2]_{\mathrm{V}} = f[e_1, e_2]_{\mathrm{V}} + (\rho(e_1)\cdot f)e_2 - (e_1, e_2)_+\mathscr{D}f. \tag{9}
$$

Therefore, Axiom V1 holds.

Axiom V2 is the compatibility condition with $(\cdot,\cdot)$ and $\mathscr{D}$. From the property of $(\cdot,\cdot)$, we can calculate the left-hand side of Axiom V2 as follows:

$$
\begin{aligned}
\rho_{\mathrm{V}}(e)\cdot(e_1, e_2)_+ &= ([e, e_1]_{\mathrm{V}}, e_2)_+ + (e_1, [e, e_2]_{\mathrm{V}})_+\\
&\quad + \frac{1}{2}\rho_{\mathrm{V}}(e_1)\cdot(e, e_2) + \frac{1}{2}\rho_{\mathrm{V}}(e_2)\cdot(e, e_1),
\end{aligned} \tag{10}
$$

and

$$
\frac{1}{2}\rho_{\mathrm{V}}(e_1)\cdot(e, e_2) = (\mathscr{D}(e, e_2)_+, e_1)_+, \tag{11}
$$

$$
\frac{1}{2}\rho_{\mathrm{V}}(e_2)\cdot(e, e_1) = (\mathscr{D}(e, e_1)_+, e_2)_+. \tag{12}
$$

Then, the left-hand side of Axiom V2 becomes

$$
\rho(e_1)\cdot(e_2, e_3)_+ = ([e_1, e_2]_{\mathrm{V}} + \mathscr{D}(e_1, e_2), e_3)_+ + (e_2, [e_1, e_3]_{\mathrm{V}} + \mathscr{D}(e_1, e_3))_+. \tag{13}
$$

Therefore, Axiom V2 is satisfied.

4 The Vaisman Algebroid on a Para-Hermitian Geometry

A flat para-Hermitian geometry is a geometric realization of doubled geometry. In this section, we construct the Vaisman algebroid with the doubled structure on the para-Hermitian manifold. We also explain that the derivation condition (4) corresponds to the strong constraint in the context of DFT. Here is a brief outline of the results. See [7] for details.

Let $\mathscr{M}$ be a $2D$ dimensional flat para-Hermitian manifold. In other words, $\mathscr{M}$ has an inner product η,

$$\eta(\Xi_1, \Xi_2) = \eta_{MN} \Xi_1{}^M \Xi_2{}^N, \qquad \Xi_1, \Xi_2 \in \Gamma(T\mathscr{M}), \tag{14}$$

and a para-complex structure $K^2 = 1$. If we consider the projection operator with K

$$P = \frac{1}{2}(1 + K), \qquad \tilde{P} = \frac{1}{2}(1 - K), \tag{15}$$

then the tangent bundle of $\mathscr{M}$ can be decomposed into $P(T\mathscr{M}) = L$ and $\tilde{P}(T\mathscr{M}) = L^*$. The vector fields on L and L^* are defined as follows respectively:

$$X = X^\mu \partial_\mu \in \Gamma(L), \qquad \xi = \xi_\mu \tilde{\partial}^\mu \in \Gamma(L^*). \tag{16}$$

Therefore, the Lie bracket on L, L^* are given as

$$[X_1, X_2]_L = (X_1{}^\nu \partial_\nu X_2{}^\mu - X_2{}^\nu \partial_\nu X_1{}^\mu)\partial_\mu, \tag{17}$$

$$[\xi_1, \xi_2]_{L^*} = (\xi_{1\nu} \tilde{\partial}^\mu \xi_{1\mu} - \xi_{2\nu} \tilde{\partial}^\mu \xi_{1\mu})\tilde{\partial}^\nu. \tag{18}$$

By defining the anchor maps L and L^* appropriately, L and L^* become a Lie algebroid respectively. We define the k-vector $A \in \Gamma(\wedge^k L)$ and the k-form $\alpha \in \Gamma(\wedge^k L^*)$ as

$$A = \frac{1}{k!} A^{\mu_1 \cdots \mu_k} \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_k}, \tag{19}$$

$$\alpha = \frac{1}{k!} \alpha_{\mu_1 \cdots \mu_k} \tilde{\partial}^{\mu_1} \wedge \cdots \wedge \tilde{\partial}^{\mu_k}. \tag{20}$$

The differential operators on L and L^* are given as

$$\mathrm{d}_* A = \frac{1}{k!} \tilde{\partial}^\mu A^{\nu_1 \cdots \nu_k} \partial_\mu \wedge \partial_{\nu_1} \wedge \cdots \wedge \partial_{\nu_k}, \tag{21}$$

$$\mathrm{d}\alpha = \frac{1}{k!} \partial_\mu \alpha_{\nu_1 \cdots \nu_k} \tilde{\partial}^\mu \wedge \tilde{\partial}^{\nu_1} \wedge \cdots \wedge \tilde{\partial}^{\mu_k}. \tag{22}$$

The d and d* become a para-Dolbeault operator because they have the following properties.

$$\mathrm{d}^2 = 0, \qquad \mathrm{d}\mathrm{d}_* + \mathrm{d}_*\mathrm{d} = 0, \qquad \mathrm{d}_*^2 = 0. \tag{23}$$

We now examine the derivation condition on $\mathscr{M}$. In order to make the calculation be apparent, we introduce the Grassmannian coordinates $\zeta_\mu = \partial_\mu$ and replace the basis of the k-vector on L:

$$A = \frac{1}{k!} A^{\mu_1 \cdots \mu_k} \zeta_{\mu_1} \cdots \zeta_{\mu_k}. \tag{24}$$

The Schouten-Nijenhuis bracket is given by

$$[A, B]_{\mathrm{S}} = \left(\frac{\partial}{\partial \zeta_\mu}\right) \partial_\mu B - (-1)^{(p-1)(q-1)} \left(\frac{\partial}{\partial \zeta_\mu}\right) \partial_\mu A, \tag{25}$$

where the zeta derivative is defined as the right derivative. We calculate the three terms constituting the derivation condition (4). The left-hand side of (4) becomes

$$\mathrm{d}_*[A, B]_{\mathrm{S}} = (\tilde{\partial}^\mu A^\rho \partial_\rho B^\nu + A^\rho \partial_\rho \tilde{\partial}^\mu B^\nu - \tilde{\partial}^\mu B^\rho \partial_\rho A^\nu - B^\rho \partial_\rho \tilde{\partial}^\mu A^\nu) \partial_\mu \wedge \partial_\nu. \tag{26}$$

Similarly, we can calculate the right-hand side of (4):

$$[\mathrm{d}_* A, B]_{\mathrm{S}} = (\tilde{\partial}^\mu A^\rho \partial_\rho B^\nu - \tilde{\partial}^\rho A^\mu \partial_\rho B^\nu - B^\rho \partial_\rho \tilde{\partial}^\mu A^\nu) \partial_\mu \wedge \partial_\nu, \tag{27}$$

$$[A, \mathrm{d}_* B]_{\mathrm{S}} = -(A^\rho \partial_\rho \tilde{\partial}^\mu B^\nu - \tilde{\partial}^\rho A^\mu \partial_\rho A^\nu - \tilde{\partial}^\mu B^\rho \partial_\rho A^\nu) \partial_\mu \wedge \partial_\nu. \tag{28}$$

Therefore, we obtain the following result:

$$\tilde{\mathrm{d}}[A, B]_{\mathrm{S}} = [\tilde{\mathrm{d}} A, B]_{\mathrm{S}} + [A, \tilde{\mathrm{d}} B]_{\mathrm{S}} + (\partial^M A^\mu \partial_M B^\nu) \partial_\mu \wedge \partial_\nu. \tag{29}$$

The third term on the right-hand side is the violation of the derivation condition. In order to hold the derivation condition on the para-Hermitian manifold $\mathscr{M}$, we need to impose an additional condition:

$$\partial^M A^\mu \partial_M B^\nu = 0. \tag{30}$$

This is exactly the same form as the strong constraint in DFT. Therefore, the algebraic origin of the strong constraint is the derivation condition.

5 Conclusion and Discussion

In this proceeding, we consider the Vaisman algebroid. It appears as a gauge symmetry algebra in DFT. We showed that the Vaisman algebroid has a doubled structure. From a physical viewpoint, the doubled structure of the gauge symmetry in DFT consists of the Kaluza-Klein modes and the winding modes of strings. On the other hand, mathematically, it can be interpreted as an analogue of the Drinfel'd double for Lie algebroids. We also see the Vaisman algebroid based on a doubled structure which appears naturally on a para-Hermitian manifold $\mathscr{M}$. Then, we check the violation of the derivation condition on $\mathscr{M}$ and we show that it is the algebraic origin of the strong constraint in DFT.

Moreover, there is a family of algebroids described by the C-bracket—pre- and ante-Courant algebroids and so on. We discuss these algebroids focusing on the doubled structure in [11]. The algebroid structures discussed in this proceeding provide the local structure in a para-Hermitian manifold. The global nature of the symmetry is discussed in [12], it is closely related to a rackoid structure.

Acknowledgements The work of H.M. is supported by the Japanese Association of University Women (JAUW). The work of S.S. is supported by Grant-in-Aid for Scientific Research (C), JSPS KAKENHI Grant Number JP20K03952. The work of K.S. is supported by Grant-in-Aid for JSPS Research Fellow, JSPS KAKENHI Grant Number JP20J13957.

References

1. N. Hitchin, Quart. J. Math. **54**, 281–308 (2003)
2. T.J. Courant, Trans. Am. Math. Soc. **319**, 631–661 (1990)
3. C. Hull, B. Zwiebach, JHEP **0909**, 099 (2009)
4. I. Vaisman, J. Math. Phys. **53**, 033509 (2012)
5. A. Chatzistavrakidis, L. Jonke, F.S. Khoo, R.J. Szabo, JHEP **1807**, 015 (2018)
6. Z.-J. Liu, A. Weinstein, P. Xu, J. Differ. Geom. **45**, 547–574 (1997)
7. H. Mori, S. Sasaki, K. Shiozawa, J. Math. Phys. **61**, 013505 (2020)
8. D. Svoboda, J. Math. Phys. **59**, 122302 (2018)
9. L. Freidel, F.J. Rudolph, D. Svoboda, Commun. Math. Phys. **372**, 119–150 (2019)
10. K.C.H. Mackenzie, P. Xu, Duke Math. J. **73**, 415–452 (1994)
11. H. Mori, S. Sasaki, J. Math. Phys. **61**, 123504 (2020)
12. N. Ikeda, S. Sasaki, J. Math. Phys. **62**, 032306 (2021)

Lie Algebroids and Weight Systems

Cristian Anghel and Dorin Cheptea

Abstract We put the Rozansky-Witten weight systems obtained from Lie algebroids by Voglaire & Xu, into the general machine provided by Kontsevich in the context of foliations and formal geometry.

Keywords Weight systems · Lie algebroids · Rozansky-Witten invariants · Gelfand-Fuks cohomology · Atiyah class

1 Introduction

Weight systems appeared first time in Kontsevich (1993), in the context of Vassiliev's theory of finite type invariants. The first concrete examples come from classical Lie algebras, and recover the Reshetikhin-Turaev invariants constructed from the corresponding quantum group.

A few years later, Rozansky & Witten (1997) discovered a totally different source of weight systems, namely one coming from the world of complex geometry; for any hyperkahler manifold X they associate a canonical weight system using the curvature tensor of X. Their work was formalized in (1999) by Kapranov, using the Atiyah class and simultaneously, vastly generalized, by Kontsevich for the context of symplectic foliations, using Gelfand-Fuks cohomology, characteristic classes of foliations, the Lie algebra of formal Hamiltonian vector fields and the graph complex. More recently, Voglaire & Xu, using the formalism of Atiyah classes, constructed RW-type weight systems using Lie algebroids.

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Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_46

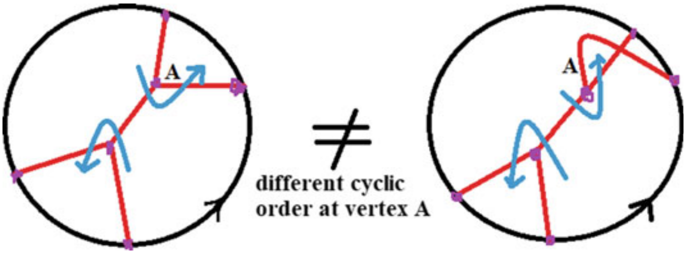
Our aim is to put the Voglaire-Xu machine into the general context of Kontsevich and to suggest, along the same ideas, some other possible sources of weight systems coming from holomorphic foliations, holomorphic Poisson and generalized complex geometry.

2 Weight Systems

2.1 What Is a Weight System

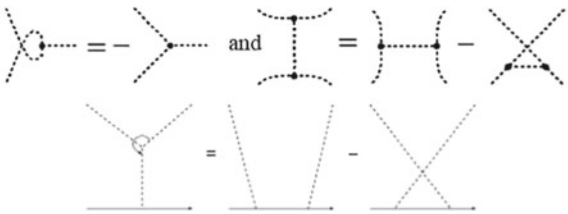
Roughly speaking, a weight system is a linear map defined on the space $\mathcal{A}$ of Jacobi diagrams (defined in the sequel), valued in a finite dimensional vector space, for example **Q** or **C**. A diagram D is a trivalent graph with a distinguished oriented cycle and a specified cyclic order at each vertex.

Example:



Definition 1 **A** is the **C**-vector space generated by all diagrams.

Definition 2 The space $\mathcal{A}$ of Jacobi diagrams is the quotient of **A** by the subspace generated by the following AS, IHX and STU local relations:



In fact $\mathcal{A}$ is a graded vector space, the grading being by half the total number of vertices. It is also an algebra under connected sum of the specified oriented cycles.

Definition 3 A weight system is a linear map $w : \mathcal{A} \to V$, where V is a finite dimensional vector space.

2.2 Motivation: 3-Manifold Invariants from Weight Systems

Why would we be interested in weight systems? The fundamental reason is the Kontsevich integral

$$\mathcal{K} : \{framed\ knots \in \mathbf{S}^3 mod\ isotopy\} \to \hat{\mathcal{A}}$$

where $\hat{\mathcal{A}}$ is the grading completion of the algebra $\mathcal{A}$. Although it is unknown if $\mathcal{K}$ is a complete knot invariant, it is equivalent with all Vassiliev's finite type invariants and has been vastly generalized by Le, Murakami and Ohtsuki to links and 3-manifolds. One of the main inconvenience of $\mathcal{K}$ is that its values are infinite series and consequently it is very hard to compute. In this respect, the idea of weight systems gives a major simplification; if a weight system w vanishes in high degree, then the composition $w \circ \mathcal{K}$ is a coarser, but a much more tractable knot invariant.

At this moment, there are 4 main sources of weight systems:

– the "classical" introduced by Kontsevich [4] and coming from Lie algebras
– the Rozansky-Witten ones [6], coming from hyperkahler manifolds
– the Kontsevich ones [5], coming from symplectic foliations
– and the Voglaire-Xu ones [7], coming from Lie algebroids.

In the sequel we will briefly describe all of them. Then, we will put the Voglaire-Xu one, into the general context of symplectic foliations. Finally, we will give some other potential sources of weight systems coming from many areas of complex geometry.

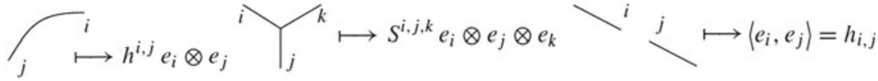
2.3 Weight Systems from Classical Lie Algebras

The basic input of this construction is a metric Lie algebra g, i.e. a finite dimensional one, with a symmetric, non-degenerate, Ad-invariant bilinear form $h := <\cdot,\cdot>$: $g \times g \to k$. g alone, with its metric h induces the universal g-weight system

$$\mathcal{T}_g : \mathcal{A} \to U(g)$$

to the enveloping algebra $U(g)$. The strategy is to cut every Jacobi diagram into elementary pieces and to assign to these, canonical tensors in $T(g)$, the tensor algebra of g. By projecting then onto $U(g)$, one gets a well defined map modulo AS, IHX and STU (independent of the cutting of the diagram). The last step, is to introduce into the picture a g-module V. By composing $\mathcal{T}_g$ with the canonical action $U(g) \to End(V)$ and taking the trace, one obtains the final $\mathbf{C}$-valued weight system $w_{g,V}$.

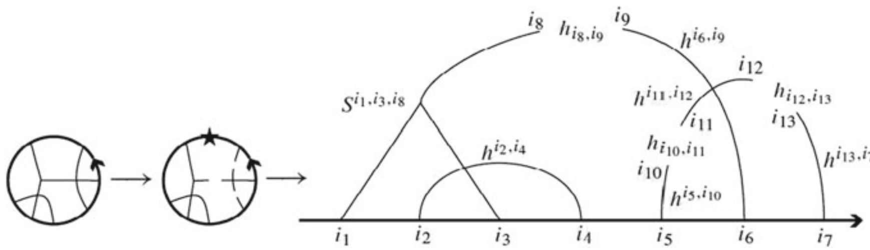
More precisely denote the metric as an isomorphism $g \to g^\star$, $x \to <\cdot, x>$; take $e_1, e_2, \dots$ a basis in g, $e^1, e^2, \dots$ its dual basis and denote: the bracket $[e_i, e_j] = \sum_k \gamma_{i,j}^k e_k$, the metric $<\cdot,\cdot>$ $e_i \to \sum_j h_{i,j} e^j$, its inverse $e^i \to \sum_j h^{i,j} e_j$ and $S^{a,b,c} = -\sum_{i,j} \gamma_{i,j}^c h^{i,a} h^{j,b}$. Then, decompose the diagram D into elementary pieces, label the endpoints by indices of the g-basis and make the following assignments and contractions:



$$\longmapsto h^{i,j}\, e_i \otimes e_j \qquad \longmapsto S^{i,j,k}\, e_i \otimes e_j \otimes e_k \qquad \longmapsto \langle e_i, e_j\rangle = h_{i,j}$$

The resulting element $\mathcal{T}_g(D) \in U(g)$ is what we are searching for.

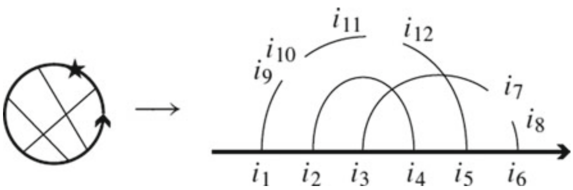
Example 1, cf. [2]: for the diagram and the cutting



we have the following value in $U(g)$:

$$T_{\mathfrak{g}}(D) = \sum_{i_1,\dots,i_{13}\in\{1,\dots,n\}} S^{i_1,i_3,i_8} h^{i_2,i_4} h^{i_5,i_{10}} h^{i_6,i_9} h^{i_{11},i_{12}} h^{i_7,i_{13}} h_{i_8,i_9} h_{i_{10},i_{11}} h_{i_{12},i_{13}} \cdot e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_7}$$

Example 2, cf. [2]: for the diagram and the cutting



we have the following value in $U(g)$:

$$T_{\mathfrak{g}}(D) = \sum_{i_1,\dots,i_{12}\in\{1,2,\dots,n\}} h^{i_1,i_9} h^{i_2,i_4} h^{i_3,i_7} h^{i_5,i_{12}} h^{i_6,i_8} h^{i_{10},i_{11}} h_{i_7,i_8} h_{i_9,i_{10}} h_{i_{11} i_{12}} \cdot e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4} \otimes e_{i_5} \otimes e_{i_6}$$

3 The Rozansky-Witten Weight System

3.1 Hyperkahler Manifolds

The second series of examples of weight systems comes from the world of complex geometry. They appeared in'97 in the work of Rozansky & Witten, coming from physics and was formalised by Kapranov in'99 from the mathematical viewpoint. The main characters in this story are the hyperkahler manifolds and their cousins the holomorphic symplectic manifolds:

Definition 4 A Riemannian manifold (M, g) is hyperkahler if it satisfies one of the following equivalent conditions:

1. is of dimension $4k$ and the Riemannian holonomy is contained in $Sp(k)$.
2. it has three complex Kahler structures w.r.t. g, I, J, K which satisfy the quaternion relations.

A related class of manifolds is the one of holomorphic symplectic ones:

Definition 5 A Kahler manifold is holomorphic symplectic if it has a nowhere degenerated holomorphic two form.

There is a close relation between the two:

Proposition 1 *1. Any hyperkahler manifold is holomorphic symplectic. 2. If compact, any holomorphic symplectic manifold is hyperkahler. (The proof uses the Calabi-Yau theorem)*

Despite the simplicity of the definition, in the compact case there are only a few known hyperkahler examples:

- in every even dimension, the Hilbert schemes of K3 or abelian algebraic surfaces,
- two sporadic examples in dimension 6 and 10 constructed by O'Grady.

However, in the noncompact case, the examples abound, a good source being for example various moduli spaces of Higgs bundles. In what follows, it is essential the hyperkahler manifold to be compact, for integration on it to obtain **C**-valued weight systems. Otherwise, the values are only in the Dolbeault cohomology of the variety.

3.2 RW-Weight Systems from Hyperkahler Manifolds

The main point in Rozansky-Witten theory is to use the curvature tensor R on a hyperkahler manifold M, in the vertices of a Jacobi diagram instead of the structure constants of a Lie algebra.

In general, on any complex manifold, R satisfies an IHX type relation modulo $\bar{\partial}$-co-boundaries. However, in general we don't know the same thing for the AS type relation.

In the hyperkahler setting, or in fact even only on holomorphic symplectic one, the "magic" is the fact that also the AS type relation is a $\bar{\partial}$-co-boundary. As a consequence, in the presence of a holomorphic symplectic structure, we obtain a weight system with values in the Dolbeault cohomology of the manifold M. If additionally M is compact, we can integrate top degree forms, obtaining Rozansky-Witten numbers associated to the couple M and the initial 3-manifold.

4 The Kontsevich Weight System

4.1 *The Lie Algebra of Formal Hamiltonian Vector Fields*

The previous Rozansky-Witten construction was vastly generalized by Kontsevich using two main ingredients:

1. the cohomology of the Lie algebra of formal hamiltonian vector fields [1]
2. characteristic classes of foliations with transversal symplectic structure.

Formal vector fields on $\mathbf{R}^{\mathbf{n}}$ for n even, are usual sums $\Sigma a_i \partial_i$, where the a_i's are formal series in the n co-ordinates $x_1, \ldots, x_n$. They form an infinite dimensional Lie algebra under the usual bracket and $\mathcal{H}$ is its Lie-subalgebra of hamiltonian formal vector fields.

4.2 *Foliations and the Kontsevich Weight System*

The main point in the Kontsevich construction is the fact that the graph cohomology (more or less the space of Jacobi diagrams) is a subspace in the cohomology of $\mathcal{H}$ if we work with hamiltonian vector fields:

$$\mathcal{A} \subset H^*(\mathcal{H}).$$

If we compose this with the characteristic class morphism

$$H^*(\mathcal{H}) \to H^*(M)$$

for a symplectic foliation, we arrive at the Kontsevich general weight system associated with any symplectic foliation. In the particular case of the anti-holomorphic foliation on a holomorphic symplectic manifold, one obtains the Rozansky-Witten weight system.

5 Lie Algebroids and Voglaire-Xu Weight System

In 2015, Voglaire and Xu extended the Rozansky-Witten construction in the context of Lie algebroids. Roughly speaking, a Lie algebroid L on a manifold M is a vector bundle, with a bracket on the space of sections, and a morphism to T_M, the tangent bundle of M, satisfying a Leibniz rule.

A Lie pair (L, A) is composed of two algebroids $A \subset L$. There is a notion of symplectic structure on a pair, and any holomorphic symplectic manifold gives rise to a symplectic Lie pair using the anti-holomorphic bundle in the complexified tangent bundle. Using a substitute of the curvature tensor in the case of a symplectic Lie pair (L, A), Voglaire and Xu arrive at a weight system with values in the cohomology of A.

Our main result mimics Kontsevich idea: We factorize the Voglaire-Xu morphism $\mathcal{A} \to H^*(A)$ into $\mathcal{A} \to H^*(\mathcal{H}) \to H^*(A)$, where the second one $H^*(\mathcal{H}) \to H^*(A)$ is a universal characteristic class morphism associated to any symplectic Lie pair (L, A).

6 Conclusions and Future Directions

We overviewed the 4 main sources of weight systems known in literature. We sketched the idea to view the Voglaire-Xu weight system through Kontsevich's general machine using the characteristic class morphism for symplectic Lie pairs. For the future, we intend to extend this idea to other cases coming from the geometry of the symplectic world: holomorphic foliations, holomorphic Poisson manifolds and generalized complex geometry.

References

1. D. Fuks, *Cohomology of Infinite-Dimensional Lie Algebras* (Consultants Bureau, New York, 1986)
2. D.M. Jackson, I. Moffatt, *An Introduction to Quantum and Vassiliev Knot Invariants* (CBS Books in Mathematics, Springer, 2019)
3. M. Kapranov, Compositio Math. **115**, 71–113 (1999)
4. M. Kontsevich, Vassiliev's knot invariants. Adv. Sov. Math. **16** (1993)
5. M. Kontsevich, Compositio Math. **115**, 115–127 (1999)
6. L. Rozansky, E. Witten, Selecta Math. New Ser. **3**, 401–458 (1997)
7. Y. Voglaire, P. Xu, Comm. Math. Phys. **336**(1), 217–241 (2015)

Structures on Lie Groups and Lie Algebras

Visible Actions of Certain Affine Transformation Groups of a Siegel Domain of the Second Kind

Koichi Arashi

Abstract In this contribution we give results concerning visible and strongly visible actions introduced by T. Kobayashi for a uniform treatment of multiplicity-free representations of Lie groups. We are concerned with the actions of certain affine transformation groups of a Siegel domain of the second kind, which is a generalization of a non-compact Hermitian symmetric space of non-tube type. More precisely, we show that an action of a two-step nilpotent Lie group on a Siegel domain is visible, and for a Siegel domain and an affine transformation group we give a sufficient condition that the action is strongly visible. Moreover, we discuss some examples to illustrate the latter result.

Keywords Visible action · Coisotropic action · Polar action · Siegel domain · Hermitian symmetric space · Multiplicity-free representation

1 Introduction

Multiplicity-free representations of Lie groups are intimately related to some geometric notions, such as coisotropic action [3], polar action [11], spherical variety [16], and visible action [8]. Especially, multiplicity-free representations for various Lie groups can be understood from the perspective of visible action and propagation of multiplicity-freeness property for holomorphic vector bundles [7, 9, 13]. On the other hand, visible action itself has been extensively studied, especially for real reductive groups and complex spherical varieties (see [15] and references therein).

Motivated by these results, we consider visible and strongly visible actions of certain affine transformation groups of a Siegel domain of the second kind, which is a generalization of a non-compact Hermitian symmetric space of non-tube type. For complex analysis related to such affine automorphism groups, we refer to [1, 4]. Our main result can be considered as a variant of the following result: For a

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Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_47

Hermitian symmetric space $D = G/K$ without compact factor, the action of a maximal unipotent subgroup of G is strongly visible [8, Theorem 1.10].

Let $\Omega \subset \mathbb{R}^N$ be a regular cone, that is, a nonempty open convex cone which contains no entire straight line. Let $Q : \mathbb{C}^M \times \mathbb{C}^M \to \mathbb{C}^N$ be an Ω-positive Hermitian map, i.e. a sesquilinear map satisfying $Q(u, u) \in \overline{\Omega}\backslash\{0\}$ $(u \in \mathbb{C}^M\backslash\{0\})$, where $\overline{\Omega}$ is the closure of Ω. Then we call the following domain a *Siegel domain of the second kind*:

$$\mathscr{D} := \{(z, u) \in \mathbb{C}^N \times \mathbb{C}^M \mid \operatorname{Im} z - Q(u, u) \in \Omega\}.$$

For $x_0 \in \mathbb{R}^N$ and $u_0 \in \mathbb{C}^M$, we denote by $n(x_0, u_0)$ the following affine transformation of $\mathbb{C}^N \times \mathbb{C}^M$:

$$(z, u) \mapsto (z + x_0 + 2\sqrt{-1}Q(u, u_0) + \sqrt{-1}Q(u_0, u_0), u + u_0) \quad ((z, u) \in \mathscr{D}).$$

The set $\mathbf{N} := \{n(x, u) \mid x \in \mathbb{R}^N, u \in \mathbb{C}^M\}$ is closed under composition, and has the natural structure of a group. Let $G(\Omega) := \{g \in GL(\mathbb{R}^N) \mid g(\Omega) = \Omega\}$. We can extend a map $g \in G(\Omega)$ to a unique complex-linear map $g : \mathbb{C}^N \to \mathbb{C}^N$. Let

$$\mathrm{Aff}_0(\mathscr{D}) := \left\{(g, l) \in G(\Omega) \times GL(\mathbb{C}^M) \mid \begin{array}{r} gQ(u_1, u_2) = Q(lu_1, lu_2) \\ \text{for all } u_1, u_2 \in \mathbb{C}^M \end{array}\right\}.$$

Then it is known [5, 10, 12] that the group $\mathrm{Aff}(\mathscr{D})$ of affine transformations of $\mathbb{C}^N \times \mathbb{C}^M$ which leave $\mathscr{D}$ fixed admits the natural structure of a Lie group and is the semidirect product of $\mathrm{Aff}_0(\mathscr{D})$ and $\mathbf{N}$.

We will show that the action of $\mathbf{N}$ on $\mathscr{D}$ is visible, as well as coisotropic and polar with respect to the Kähler structure defined by the Bergman metric on $\mathscr{D}$. Moreover, for a Siegel domain $\mathscr{D}$ of the second kind and a subgroup $G_0 \subset \mathrm{Aff}_0(\mathscr{D})$ we will give a sufficient condition that the action of $G := G_0\mathbf{N}$ on $\mathscr{D}$ is strongly visible (see Theorem 4).

2 Coisotropic Action, Polar Action, and Visible Action

Let us briefly recall the definitions of coisotropic, polar, visible and strongly visible actions [3, 7, 11].

Suppose that a Lie group H acts on a connected complex manifold D by holomorphic automorphisms. We call the action *previsible* if there exist a totally real submanifold S in D and a (non-empty) H-invariant open subset D' of D such that S meets every H-orbit in D'. A previsible action is called *visible* if $J_x(T_xS) \subset T_x(H \cdot x)$ for all $x \in S$, where $J_x \in \mathrm{End}(T_xD)$ denotes the complex structure. In addition, a previsible action is called *strongly visible* if there exist an anti-holomorphic diffeomorphism σ of an H-invariant open subset D' and a submanifold S of D' such that

$$\sigma|_S = \mathrm{id},$$
$$\sigma \text{ preserves each } H\text{-orbit in } D'.$$

Next, suppose a Lie group H acts on a symplectic manifold (D, ω) by symplectic automorphisms. The action is called *coisotropic* if $T_x(H \cdot x)^{\perp,\omega} \subset T_x(H \cdot x)$ for every principal orbit $H \cdot x$. Last, suppose a Lie group H acts on a Riemannian manifold D by isometries. The action is called *polar* if there exist a closed connected submanifold S (called a *section*) that meets every H-orbit orthogonally.

Now we shall see an integral expression for the Bergman kernel of $\mathscr{D}$ [2]. By using the expression, we will show that the action of $\mathbf{N}$ is visible, as well as coisotropic and polar with respect to the Kähler structure defined by the Bergman metric, which is positive-definite since the domain $\mathscr{D}$ is biholomorphic to a bounded domain [10]. Let

$$\Omega^* := \{\xi \in (\mathbb{R}^N)^* \mid \langle \xi, y \rangle > 0 \text{ for all } y \in \overline{\Omega} \backslash \{0\}\}.$$

For $\xi \in \Omega^*$, let $I(\xi) := \int_\Omega e^{-2\langle \xi, y \rangle}\, dy$ and $I_Q(\xi) := \int_{\mathbb{C}^M} e^{-2\langle \xi, Q(u,u) \rangle}\, du$.

Theorem 1 ([2, Theorem 5.4]) *The Bergman kernel K of $\mathscr{D}$ is given by*

$$K(z, u, w, v) := \frac{1}{(2\pi)^N} \int_{\Omega^*} e^{\sqrt{-1}\langle \xi, z - \overline{w} - 2\sqrt{-1} Q(u,v) \rangle} I(\xi)^{-1} I_Q(\xi)^{-1}\, d\xi$$
$$((z, u), (w, v) \in \mathscr{D}).$$

By Theorem 1, we get the following theorem.

Theorem 2 *The action of* $\mathbf{N}$ *on $\mathscr{D}$ is polar with a section $\sqrt{-1}\Omega \subset \mathscr{D}$.*

Proof For $(z, u) \in \mathscr{D}$, we have

$$n(-\operatorname{Re} z, -u)(z, u) = (\sqrt{-1}(\operatorname{Im} z - Q(u, u)), 0) \in \sqrt{-1}\Omega, \tag{1}$$

which implies that $\sqrt{-1}\Omega$ meets every $\mathbf{N}$-orbit. Next, let us denote the standard bases of $\mathbb{R}^N$ and $\mathbb{C}^M$ by e_i $(1 \le i \le N)$ and e'_j $(1 \le j \le M)$, respectively. For $(z, u) \in \mathscr{D}$ we have

$$\begin{aligned}
\frac{\partial^2}{\partial z_i \partial \overline{u_j}} \log K(z, u, z, u) &= \frac{1}{K(z, u, z, u)^2} (K(z, u, z, u) \\
&\cdot \int_{\Omega^*} 2\sqrt{-1}\langle \xi, e_i \rangle \langle \xi, Q(u, e'_j) \rangle e^{\sqrt{-1}\langle \xi, z - \overline{z} - 2\sqrt{-1} Q(u,u) \rangle} I(\xi)^{-1} I_Q(\xi)^{-1}\, d\xi \\
&- \int_{\Omega^*} \sqrt{-1}\langle \xi, e_i \rangle e^{\sqrt{-1}\langle \xi, z - \overline{z} - 2\sqrt{-1} Q(u,u) \rangle} I(\xi)^{-1} I_Q(\xi)^{-1}\, d\xi \\
&\cdot \int_{\Omega^*} 2\langle \xi, Q(u, e'_j) \rangle e^{\sqrt{-1}\langle \xi, z - \overline{z} - 2\sqrt{-1} Q(u,u) \rangle} I(\xi)^{-1} I_Q(\xi)^{-1}\, d\xi),
\end{aligned}$$

from which we can see that

$$\frac{\partial^2}{\partial z_i \partial \overline{u_j}} \log K(z, 0, z, 0) = 0 \quad ((z, 0) \in \mathscr{D}). \tag{2}$$

Since the tangent space $T_{\sqrt{-1}y}(\mathbf{N} \cdot \sqrt{-1}y) \subset T_{\sqrt{-1}y}(\mathscr{D})$ for any $y \in \Omega$ can be naturally identified with $\mathbb{R}^N \oplus \mathbb{C}^M$, it follows from the equality (2) that every $\mathbf{N}$-orbit meets $\sqrt{-1}\Omega$ orthogonally. □

The following result can be applied to the action of $\mathbf{N}$ on $\mathscr{D}$.

Theorem 3 ([7, Theorems 6, 8]) *Suppose that a connected Lie group acts on a connected Kähler manifold D by holomorphic isometries. If the action is polar with a totally real section $S \subset D$, then the action is visible and coisotropic.*

Combining Theorems 2 and 3, we obtain the following result.

Corollary 1 *The action of* $\mathbf{N}$ *on* $\mathscr{D}$ *is visible and coisotropic.*

3 Strongly Visible Action

Before proceeding with a sufficient condition for a strongly visible action, we shall take a look at anti-holomorphic diffeomorphisms of $\mathscr{D}$.

Suppose that a linear bijective map $\sigma_1 : \mathbb{R}^N \to \mathbb{R}^N$ and an antilinear bijective map $\sigma_2 : \mathbb{C}^M \to \mathbb{C}^M$ satisfy the following condition:

$$\begin{aligned} &-\sigma_1 \in G(\Omega), \\ &-\sigma_1(Q(u, u)) = Q(\sigma_2(u), \sigma_2(u)) \quad (u \in \mathbb{C}^M). \end{aligned} \tag{3}$$

If we extend σ_1 to an antilinear map $\sigma_1 : \mathbb{C}^N \to \mathbb{C}^N$, then we have

$$\operatorname{Im}(\sigma_1(z)) - Q(\sigma_2(u), \sigma_2(u)) = -\sigma_1(\operatorname{Im} z - Q(u, u)) \in \Omega \quad ((z, u) \in \mathscr{D}). \tag{4}$$

Hence the map $(z, u) \mapsto (\sigma_1(z), \sigma_2(u))$ induces an anti-holomorphic diffeomorphism σ of $\mathscr{D}$ (cf. [6, Theorem 11]). A closed subgroup $G_0 \subset \mathrm{Aff}_0(\mathscr{D})$ acts on Ω by $(g, l)y := gy$ $((g, l) \in G_0, y \in \Omega)$. Let $\Omega^{-\sigma_1} := \{y \in \Omega \mid -\sigma_1(y) = y\}$. We can consider the following condition (C):

$$-\sigma_1 : \Omega \to \Omega \text{ preserves each } G_0\text{-orbit in } \Omega, \tag{C-1}$$

$$\Omega^{-\sigma_1} \text{ meets every } G_0\text{-orbit in } \Omega. \tag{C-2}$$

Let $G := G_0\mathbf{N} \subset \mathrm{Aff}(\mathscr{D})$. Our main result can be stated as follows.

Theorem 4 *The condition* (C-1) *implies that σ preserves each G-orbit in $\mathscr{D}$. In addition, the condition* (C) *implies that the action of G on $\mathscr{D}$ is strongly visible.*

Proof Let $(z, u) \in \mathscr{D}$. Suppose that the condition (C-1) holds. Then we can find $(g, l) \in G_0$ such that

$$(-\sigma_1(\operatorname{Im} z - Q(u, u)), 0) = (g(\operatorname{Im} z - Q(u, u)), 0).$$

For $(g, l) \in \mathrm{Aff}_0(\mathscr{D})$, let $R(g, l)$ denote the map $(z, u) \mapsto (gz, lu)$. By (1) and (4), we have

$$\begin{aligned}
&n(-\operatorname{Re}\sigma_1(z), -\sigma_2(u))(\sigma_1(z), \sigma_2(u))\\
&\quad= (\sqrt{-1}(\operatorname{Im}(\sigma_1(z) - Q(\sigma_2(u), \sigma_2(u))), 0)\\
&\quad= (-\sqrt{-1}\sigma_1(\operatorname{Im} z - Q(u, u)), 0) = (\sqrt{-1}g(\operatorname{Im} z - Q(u, u)), 0)\\
&\quad= R(g, l)(\sqrt{-1}(\operatorname{Im} z - Q(u, u)), 0).
\end{aligned}$$

Therefore, we obtain

$$(\sigma_1(z), \sigma_2(u)) = n(\operatorname{Re}\sigma_1(z), \sigma_2(u))R(g, l)n(-\operatorname{Re} z, -u)(z, u)$$

with $n(\operatorname{Re}\sigma_1(z), \sigma_2(u))R(g, l)n(-\operatorname{Re} z, -u) \in G$. Hence σ preserves each G-orbit in $\mathscr{D}$. The latter assertion is immediate from (C-2). □

We shall give some examples of anti-holomorphic diffeomorphisms σ and groups G_0 which satisfy the conditions (C-1) and (C). In the examples below, ν and r stand for natural numbers. For a complex matrix X we denote its transpose and conjugate by ${}^tX$ and $\overline{X}$, respectively.

- Let $\nu \geq 2$ and $r \geq 1$. Let $\mathrm{Herm}_\nu(\mathbb{C})$ be the set of ν-by-ν Hermitian matrices and $\mathcal{P}_\nu(\mathbb{C})$ the subset of positive-definite matrices. When $N = \nu^2$ and $M = \nu r$, we can identify $\mathrm{Herm}_\nu(\mathbb{C})$ with $\mathbb{R}^N$, $\mathrm{Mat}_\nu(\mathbb{C})$ with $\mathbb{C}^N$, and $\mathrm{Mat}_{\nu,r}(\mathbb{C})$ with $\mathbb{C}^M$. The regular cone $\mathcal{P}_\nu(\mathbb{C}) \subset \mathrm{Herm}_\nu(\mathbb{C})$ and the Hermitian map Q given by $Q(U, V) := \frac{1}{2}U{}^t\overline{V}$ $(U, V \in \mathrm{Mat}_{\nu,r}(\mathbb{C}))$ define a Hermitian symmetric space $\mathscr{D}$ of type $\mathrm{I}_{\nu+r,\nu}$. The pair of maps (σ_1, σ_2) defined by $\sigma_1(X) := -{}^tX$ $(X \in \mathrm{Herm}_\nu(\mathbb{C}))$ and $\sigma_2(U) := \overline{U}$ $(U \in \mathrm{Mat}_{\nu,r}(\mathbb{C}))$ satisfy the condition (3). In this case, we have a natural injection $\iota : GL_\nu(\mathbb{C}) \hookrightarrow \mathrm{Aff}_0(\mathscr{D})$ defined by

  $$R(\iota(g))(Z, U) = (gZ{}^t\overline{g}, gU) \quad (g \in GL_\nu(\mathbb{C}), (Z, U) \in \mathscr{D}).$$

  The following subgroups of $GL_\nu(\mathbb{C})$ are examples of G_0 in Theorem 4: The general linear group $GL_\nu(\mathbb{R})$ (which satisfies only (C-1)), the unitary group $U(\nu)$, and the lower-triangular group L with 1 in each diagonal entries. Note that $G = \iota(L)\mathbf{N}$ is isomorphic to a maximal unipotent subgroup of $SU(\nu + r, \nu)$.

- Let $r, \nu \geq 1$. Let $\mathrm{Herm}_\nu(\mathbb{R})$ be the set of ν-by-ν real symmetric matrices, and $\mathcal{P}_\nu(\mathbb{R})$ the subset of positive-definite matrices. The regular cone $\mathcal{P}_\nu(\mathbb{R}) \subset \mathrm{Herm}_\nu(\mathbb{R})$ and the Hermitian map $Q(U, V) := \frac{1}{4}(U^t\overline{V} + \overline{V}^tU)$ $(U, V \in \mathrm{Mat}_{\nu,r}(\mathbb{C}))$ define a quasi-symmetric Siegel domain $\mathscr{D}$ of type $\mathrm{III}_{\nu,r}$ (see [14]). The pair of maps (σ_1, σ_2) defined by $\sigma_1(X) := -X$ $(X \in \mathrm{Herm}_r(\mathbb{R}))$ and $\sigma_2(U) := \overline{U}$ $(U \in \mathrm{Mat}_{\nu,r}(\mathbb{C}))$ satisfy the condition (3). In this case, the trivial subgroup of $\mathrm{Aff}_0(\mathscr{D})$ satisfies the condition (C).

Acknowledgements The author is supported by the Foundation of Research Fellows, The Mathematical Society of Japan.

References

1. J. Faraut, E.G.F. Thomas, Invariant Hilbert spaces of holomorphic functions. J. Lie Theory **9**, 383–402 (1999)
2. S.G. Gindikin, Analysis in homogeneous domains. Russ. Math. Surv. **19**, 1–89 (1964)
3. A. Huckleberry, T. Wurzbacher, Multiplicity-free complex manifolds. Math. Ann. **286**, 261–280 (1990)
4. H. Ishi, Representations of the affine transformation groups acting simply transitively on Siegel domains. J. Funct. Anal. **167**, 425–462 (1999)
5. S. Kaneyuki, *Homogeneous Bounded Domains and Siegel Domains*. Lecture Notes in Mathematics, vol. 241. (Springer, Berlin, 1971), p. v+89
6. W. Kaup, Y. Matsushima, T. Ochiai, On the automorphisms and equivalences of generalized Siegel domains. Amer. J. Math. **92**, 475–498 (1970)
7. T. Kobayashi, Multiplicity-free representations and visible actions on complex manifolds. Publ. Res. Inst. Math. Sci. **41**, 497–549 (2005)
8. T. Kobayashi, Visible actions on symmetric spaces. Transform. Groups **12**, 671–694 (2007)
9. T. Kobayashi, Multiplicity-free theorems of the restrictions of unitary highest weight modules with respect to reductive symmetric pairs, in *Representation Theory and Automorphic Forms*. Progr. Math., vol. 255. (Birkhäuser, Boston, 2008), pp. 45–109
10. I.I. Piatetski-Shapiro, *Automorphic Functions and the Geometry of Classical Domains*. Mathematics and its Applications, vol. 8. (Gordon and Breach, New York, 1969), p. 264
11. F. Podestà, G. Thorbergsson, Coisotropic actions on compact homogeneous Kähler manifolds. Math. Z. **243**, 471–490 (2003)
12. H. Rossi, M. Vergne, Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions and the application to the holomorphic discrete series of a semisimple Lie group. J. Funct. Anal. **13**, 324–389 (1973)
13. A. Sasaki, Admissible representations, multiplicity-free representations and visible actions on non-tube type Hermitian symmetric spaces. Proc. Japan Acad. Ser. A Math. Sci. **91**, 70–75 (2015)
14. I. Satake, *Algebraic Structures of Symmetric Domains*. Kanô Memorial Lectures, 4. (Iwanami Shoten, Tokyo; Princeton University Press, Princeton, 1980), p. xvi+321
15. Y. Tanaka, Visible actions of reductive algebraic groups on complex algebraic varieties. Thesis (Ph.D.)–The University of Tokyo (2015)
16. È.B. Vinberg, B.N. Kimelfeld, Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups. Funct. Anal. Appl. **12**, 168–174 (1978)

Quantum Particle on Lattices in Weyl Alcoves

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Abstract The application of the generalized discrete Fourier–Weyl transforms to a quantum particle propagation on the lattice fragments inside Weyl alcoves is summarized. The rescaled dual weight and dual root lattices intersected with the signed fundamental domains of the affine Weyl groups induce the position bases of the associated Hilbert spaces. The generalized dual-weight and dual-root Fourier–Weyl transforms provide unitary transition matrices between the position and momentum bases. The vectors of the momentum bases satisfy the time-independent Schrödinger equations and the corresponding eigenenergies are determined as sums of the symmetric Weyl orbit functions. The matrix forms of the Hamiltonians together with the eigenenergies of the A_3 dual-weight lattice model are exemplified.

Keywords Quantum dot · Weyl group · Fourier–Weyl transform

1 Position and Momentum Bases

The current notation stems from articles [1–4]. The irreducible root systems Π of the four series A_n $(n \geq 1)$, B_n $(n \geq 3)$, C_n $(n \geq 2)$, D_n $(n \geq 4)$ together with exceptional systems E_6, E_7, E_8, F_4 and G_2 contain the set $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset \Pi$ of the simple roots [5]. The linear span of the set Δ forms the Euclidean space $\mathbb{R}^n$ that is equipped with the scalar product $\langle\, , \,\rangle$. The sets Δ containing two different root-lengths decompose into the corresponding sets of short simple roots Δ_s and sets of long simple

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_48

roots Δ_l. The set of vectors $\Delta^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$, with $\alpha_i^\vee = 2\alpha_i/\langle \alpha_i, \alpha_i \rangle$, forms the set of dual simple roots of the dual root system $\Pi^\vee$.

The roots and dual roots $\mathbb{Z}$-span the root lattice $Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n$ and dual root lattice $Q^\vee = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee$. The bases of the fundamental weights $\omega_1, \ldots, \omega_n$ and dual fundamental weights $\omega_1^\vee, \ldots, \omega_n^\vee$, determined by the relations $\langle \alpha_i^\vee, \omega_j \rangle = \langle \alpha_i, \omega_j^\vee \rangle = \delta_{ij}$, span the weight lattice $P = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_j$ and dual weight lattice $P^\vee = \mathbb{Z}\omega_1^\vee + \cdots + \mathbb{Z}\omega_n^\vee$. The cone of the dual weights is given as

$$P^{\vee +} = \mathbb{Z}^{\geq 0}\omega_1^\vee + \cdots + \mathbb{Z}^{\geq 0}\omega_n^\vee.$$

The determinant of the Cartan matrix $c \in \mathbb{N}$ provides the orders of the lattice quotient groups, $|P^\vee/Q^\vee| = |P/Q| = c$.

Generated by reflections with respect to the simple roots $r_{\alpha_1}, \ldots, r_{\alpha_n} \in O(\mathbb{R}^n)$, the Weyl group $W \subset O(\mathbb{R}^n)$ together with the affine Weyl group $W^{\text{aff}} = Q^\vee \rtimes W \subset \mathbb{R}^n \rtimes O(\mathbb{R}^n)$ induce the map $\tau : W^{\text{aff}} \to Q^\vee$ and standard retraction homomorphism $\psi : W^{\text{aff}} \to W$ given for $z = T(q^\vee)w \in W^{\text{aff}}$ by the relations $\tau(z) = q^\vee$ and $\psi(z) = w$. The extended dual affine Weyl group $W_P^{\text{aff}} = P \rtimes W$ along with the dual affine Weyl group $W_Q^{\text{aff}} = Q \rtimes W \subset W_P^{\text{aff}}$ induce the map $\widehat{\tau} : W_P^{\text{aff}} \to P$ and dual retraction homomorphism $\widehat{\psi} : W_P^{\text{aff}} \to W$ determined for $y = T(p)w \in W_P^{\text{aff}}$ by the relations $\widehat{\tau}(y) = p$ and $\widehat{\psi}(y) = w$.

The admissible shifts $\nu, \nu^\vee, \rho, \rho^\vee \in \mathbb{R}^n$ of the root, dual root, weight and dual weight lattices preserve the W-invariance of the shifted lattices [3, 4],

$$\begin{aligned} W(\nu + Q) &= \nu + Q, \\ W(\nu^\vee + Q^\vee) &= \nu^\vee + Q^\vee, \\ W(\rho + P) &= \rho + P, \\ W(\rho^\vee + P^\vee) &= \rho^\vee + P^\vee. \end{aligned}$$

The fundamental domains $F, F_Q, F_P \subset \mathbb{R}^n$ of the standard action of the affine groups W^{aff}, W_Q^{aff} and W_P^{aff} on $\mathbb{R}^n$, respectively, contain exactly one point from the corresponding group orbit [6]. The discrete counting functions ε, $h_{Q,M}$ and $h_{P,M}$ are defined for points $a, b \in \mathbb{R}^n$ and any magnifying factor $M \in \mathbb{N}$ via the corresponding stabilizers as

$$\begin{aligned} \varepsilon(a) &= |W| \cdot |\mathrm{Stab}_{W^{\text{aff}}}(a)|^{-1}, \\ h_{Q,M}(b) &= \left|\mathrm{Stab}_{W_Q^{\text{aff}}}(\tfrac{b}{M})\right|, \\ h_{P,M}(b) &= \left|\mathrm{Stab}_{W_P^{\text{aff}}}(\tfrac{b}{M})\right|. \end{aligned}$$

The four sign homomorphisms $\imath, \sigma^e, \sigma^s, \sigma^l : W \to \{\pm 1\}$ are given on the generators of the Weyl group W [7] by the defining relations:

$$\imath(r_\alpha) = 1, \quad \sigma^e(r_\alpha) = -1,$$
$$\sigma^s(r_\alpha) = \begin{cases} 1, & \alpha \in \Delta_l, \\ -1, & \alpha \in \Delta_s, \end{cases} \qquad \sigma^l(r_\alpha) = \begin{cases} 1, & \alpha \in \Delta_s, \\ -1, & \alpha \in \Delta_l. \end{cases}$$

For any sign homomorphism $\sigma \in \{\imath, \sigma^e, \sigma^s, \sigma^l\}$, any point $a \in \mathbb{R}^n$ and label $b \in \rho + P$, the Weyl orbit functions $\varphi_b^\sigma : \mathbb{R}^n \to \mathbb{C}$ are given by the general relation [3, 4],

$$\varphi_b^\sigma(a) = \sum_{w \in W} \sigma(w)\, e^{2\pi \mathrm{i} \langle wb,\, a\rangle}.$$

The symmetric orbit C-functions [8] are determined by the renormalization

$$C_b = |\mathrm{Stab}_W(b)|^{-1}\, \varphi_b^{\imath}.$$

Employing the multiplicative group of c-th roots of unity U_c, the three γ-homomorphisms $\gamma_\rho^\sigma : W^{\mathrm{aff}} \to U_2, \widehat{\gamma}_{\rho^\vee}^\sigma : W_Q^{\mathrm{aff}} \to U_2$ and $\widehat{\gamma}_{\nu^\vee}^\sigma : W_P^{\mathrm{aff}} \to U_{2c}$ are given for $w \in W^{\mathrm{aff}}$, $y \in W_Q^{\mathrm{aff}}$ and $z \in W_P^{\mathrm{aff}}$ by the defining formulas [3, 4],

$$\begin{aligned} \gamma_\rho\,(w) &= e^{2\pi \mathrm{i} \langle \tau(w),\, \rho\rangle}[\sigma \circ \psi(w)], \\ \widehat{\gamma}_{\rho^\vee}(y) &= e^{2\pi \mathrm{i} \langle \widehat{\tau}(y),\, \rho^\vee\rangle}[\sigma \circ \widehat{\psi}(y)], \\ \widehat{\gamma}_{\nu^\vee}(z) &= e^{2\pi \mathrm{i} \langle \widehat{\tau}(z),\, \nu^\vee\rangle}[\sigma \circ \widehat{\psi}(z)]. \end{aligned}$$

For any $a \in \mathbb{R}^n$ there exists $a' \in F$ and $w[a] \in W^{\mathrm{aff}}$ such that $a = w[a]a'$. The function $\chi_\rho^\sigma : \mathbb{R}^n \to \{-1, 0, 1\}$ is introduced via the relation [1, 2],

$$\chi_\rho^\sigma(a) = \begin{cases} \gamma_\rho^\sigma\,(w[a])\,, & \gamma_\rho^\sigma\,(\mathrm{Stab}_{W^{\mathrm{aff}}}(a)) = 1, \\ 0, & \gamma_\rho^\sigma\,(\mathrm{Stab}_{W^{\mathrm{aff}}}(a)) = U_2. \end{cases}$$

The signed fundamental domains $F^\sigma(\rho) \subset F$, $F_Q^\sigma(\rho^\vee) \subset F_Q$ and $F_P^\sigma(\nu^\vee) \subset F_P$ are defined as [3, 4]

$$\begin{aligned} F^\sigma(\rho) &= \left\{a \in F \;\middle|\; \gamma_\rho^\sigma\,(\mathrm{Stab}_{W^{\mathrm{aff}}}(a)) = 1\right\}, \\ F_Q^\sigma(\rho^\vee) &= \left\{b \in F_Q \;\middle|\; \widehat{\gamma}_{\rho^\vee}^\sigma\left(\mathrm{Stab}_{W_Q^{\mathrm{aff}}}(b)\right) = 1\right\}, \\ F_P^\sigma(\nu^\vee) &= \left\{b \in F_P \,|\, \widehat{\gamma}_{\nu^\vee}^\sigma\left(\mathrm{Stab}_{W_P^{\mathrm{aff}}}(b)\right) = 1\right\}. \end{aligned}$$

The finite sets of points $F_{P^\vee,M}^\sigma(\rho, \rho^\vee)$ and $F_{Q^\vee,M}^\sigma(\rho, \nu^\vee)$ from refined shifted dual root and weight lattices in Weyl alcove F are introduced via the signed fundamental domain $F^\sigma(\rho) \subset F$ as

$$F^{\sigma}_{P^\vee,M}(\rho,\rho^\vee)=\tfrac{1}{M}(\rho^\vee+P^\vee)\cap F^{\sigma}(\rho), \tag{1}$$
$$F^{\sigma}_{Q^\vee,M}(\rho,\nu^\vee)=\tfrac{1}{M}(\nu^\vee+Q^\vee)\cap F^{\sigma}(\rho).$$

The corresponding finite sets of labels $\Lambda^{\sigma}_{Q,M}(\rho,\rho^\vee)$ and $\Lambda^{\sigma}_{P,M}(\rho,\nu^\vee)$ from shifted weight lattice are determined as

$$\Lambda^{\sigma}_{Q,M}(\rho,\rho^\vee)=(\rho+P)\cap MF^{\sigma}_{Q}(\rho^\vee), \tag{2}$$
$$\Lambda^{\sigma}_{P,M}(\rho,\nu^\vee)=(\rho+P)\cap MF^{\sigma}_{P}(\nu^\vee).$$

The ordered sets of points $F^{\sigma}_{P^\vee,M}(\rho,\rho^\vee)$ and $F^{\sigma}_{Q^\vee,M}(\rho,\nu^\vee)$ induce the orthonormal position bases $\left|a;\,P^\vee\right\rangle, a\in F^{\sigma}_{P^\vee,M}(\rho,\rho^\vee)$ and $\left|b;\,Q^\vee\right\rangle, b\in F^{\sigma}_{Q^\vee,M}(\rho,\nu^\vee)$ of the finite-dimensional complex Hilbert spaces $\mathcal{H}^{\sigma}_{P^\vee,M}(\rho,\rho^\vee)$ and $\mathcal{H}^{\sigma}_{Q^\vee,M}(\rho,\nu^\vee)$, respectively. Considering a length factor $l\in\mathbb{R}$, the states determined by the vectors $\left|a;\,P^\vee\right\rangle\in\mathcal{H}^{\sigma}_{P^\vee,M}(\rho,\rho^\vee)$ and $\left|b;\,Q^\vee\right\rangle\in\mathcal{H}^{\sigma}_{Q^\vee,M}(\rho,\nu^\vee)$ represent a quantum particle positioned inside the rescaled Weyl alcove lF at points la and lb, respectively.

The orthonormal momentum bases $|\lambda;\,Q\rangle\in\mathcal{H}^{\sigma}_{P^\vee,M}(\rho,\rho^\vee)$, $\lambda\in\Lambda^{\sigma}_{Q,M}(\rho,\rho^\vee)$ and $|\mu;\,P\rangle\in\mathcal{H}^{\sigma}_{Q^\vee,M}(\rho,\nu^\vee)$, $\mu\in\Lambda^{\sigma}_{P,M}(\rho,\nu^\vee)$ are defined with respect to the position bases by defining the unitary transition matrices elements via the generalized dual-weight and dual-root Fourier–Weyl transforms [1, 2],

$$\left\langle a;\,P^\vee\right|\,\lambda;\,Q\rangle=\varepsilon^{\frac{1}{2}}(a)\left(c\,|W|\,M^n h_{Q,M}(\lambda)\right)^{-\frac{1}{2}}\varphi^{\sigma}_{\lambda}(a), \tag{3}$$
$$\left\langle b;\,Q^\vee\right|\,\mu;\,P\rangle=\varepsilon^{\frac{1}{2}}(b)\left(|W|\,M^n h_{P,M}(\mu)\right)^{-\frac{1}{2}}\varphi^{\sigma}_{\mu}(b).$$

2 Hamiltonians and Energy Spectra

The dual-weight and dual-root hopping functions $\mathcal{P}^\vee: P^\vee\to\mathbb{C}$ and $\mathcal{Q}^\vee: Q^\vee\to\mathbb{C}$ of finite support are W-invariant and Hermitian [1, 2], i.e. for any $p^\vee\in P^\vee$, $q^\vee\in Q^\vee$ and $w\in W$ it holds that

$$\mathcal{P}^\vee(wp^\vee)=\mathcal{P}^\vee(p^\vee),\quad \mathcal{P}^\vee(-p^\vee)=\mathcal{P}^{\vee*}(p^\vee),$$
$$\mathcal{Q}^\vee(wq^\vee)=\mathcal{Q}^\vee(q^\vee),\quad \mathcal{Q}^\vee(-q^\vee)=\mathcal{Q}^{\vee*}(q^\vee).$$

The finite sets $\mathrm{supp}^+(\mathcal{Q}^\vee)$ and $\mathrm{supp}^+(\mathcal{P}^\vee)$ are determined as intersections of the supports of the hopping functions with the cone of the dual weights $P^{\vee+}$, i.e. $\mathrm{supp}^+(\mathcal{Q}^\vee)=\mathrm{supp}(\mathcal{Q}^\vee)\cap P^{\vee+}$ and $\mathrm{supp}^+(\mathcal{P}^\vee)=\mathrm{supp}(\mathcal{P}^\vee)\cap P^{\vee+}$.

The dual-weight and dual-root coupling sets $N_{p^\vee,M}(a,a')$, $p^\vee\in\mathrm{supp}^+(\mathcal{P}^\vee)$, $a,a'\in F^{\sigma}_{P^\vee,M}(\rho,\rho^\vee)$ and $N_{q^\vee,M}(b,b')$, $q^\vee\in\mathrm{supp}^+(\mathcal{Q}^\vee)$, $b,b'\in F^{\sigma}_{Q^\vee,M}(\rho,\nu^\vee)$ are introduced as

$$N_{p^\vee,M}(a,a') = W^{\mathrm{aff}} a' \cap \left(a + \tfrac{1}{M} W p^\vee\right),$$
$$N_{q^\vee,M}(b,b') = W^{\mathrm{aff}} b' \cap \left(b + \tfrac{1}{M} W q^\vee\right).$$

The dual-weight and dual-root hopping operators $\widehat{A}^{\sigma}_{p^\vee,M}(\rho,\rho^\vee)$, $p^\vee \in \mathrm{supp}^+(\mathcal{P}^\vee)$ and $A^{\sigma}_{q^\vee,M}(\rho,\nu^\vee)$, $q^\vee \in \mathrm{supp}^+(\mathcal{Q}^\vee)$ are defined via the following relations,

$$\left\langle a;\, P^\vee \,\middle|\, \widehat{A}^{\sigma}_{p^\vee,M}(\rho,\rho^\vee) \,\middle|\, a';\, P^\vee \right\rangle = -\varepsilon^{\frac{1}{2}}(a)\varepsilon^{-\frac{1}{2}}(a')\mathcal{P}^\vee\left(p^\vee\right) \sum_{d \in N_{p^\vee,M}(a,a')} \chi^{\sigma}_{\rho}(d)\,,$$
$$\left\langle b;\, Q^\vee \,\middle|\, \widehat{A}^{\sigma}_{q^\vee,M}(\rho,\nu^\vee) \,\middle|\, b';\, Q^\vee \right\rangle = -\varepsilon^{\frac{1}{2}}(b)\varepsilon^{-\frac{1}{2}}(b')\mathcal{Q}^\vee\left(q^\vee\right) \sum_{d \in N_{q^\vee,M}(b,b')} \chi^{\sigma}_{\rho}(d)\,.$$

The dual-weight and dual-root Hamiltonian operators which describe the quantum particle propagating on the rescaled dual-weight and dual-root lattice fragments inside rescaled Weyl alcoves, are given via the corresponding hopping operators as

$$\widehat{H}^{\sigma}_{P^\vee,M}(\rho,\rho^\vee) = \sum_{p^\vee \in \mathrm{supp}^+(\mathcal{P}^\vee)} \widehat{A}^{\sigma}_{p^\vee,M}(\rho,\rho^\vee),$$
$$\widehat{H}^{\sigma}_{Q^\vee,M}(\rho,\nu^\vee) = \sum_{q^\vee \in \mathrm{supp}^+(\mathcal{Q}^\vee)} \widehat{A}^{\sigma}_{q^\vee,M}(\rho,\nu^\vee).$$

The vectors of the momentum bases $|\lambda;\, Q\rangle \in \mathcal{H}^{\sigma}_{P^\vee,M}(\rho,\rho^\vee)$, $\lambda \in \Lambda^{\sigma}_{Q,M}(\rho,\rho^\vee)$ and $|\mu;\, P\rangle \in \mathcal{H}^{\sigma}_{Q^\vee,M}(\rho,\nu^\vee)$, $\mu \in \Lambda^{\sigma}_{P,M}(\rho,\nu^\vee)$ satisfy the time-independent Schrödinger equations

$$\widehat{H}^{\sigma}_{P^\vee,M}(\rho,\rho^\vee)\,|\lambda;\, Q\rangle = E^{\sigma}_{P^\vee,\lambda,M}(\rho,\rho^\vee)\ |\lambda;\, Q\rangle\,,$$
$$\widehat{H}^{\sigma}_{Q^\vee,M}(\rho,\nu^\vee)\,|\mu;\, P\rangle = E^{\sigma}_{Q^\vee,\mu,M}(\rho,\nu^\vee)\ |\mu;\, P\rangle\,,$$

with the eigenenergies given by relations

$$E^{\sigma}_{P^\vee,\lambda,M}(\rho,\rho^\vee) = -\sum_{p^\vee \in \mathrm{supp}^+(\mathcal{P}^\vee)} \mathcal{P}^\vee(p^\vee)\, C_{p^\vee}\left(\tfrac{\lambda}{M}\right), \tag{4}$$
$$E^{\sigma}_{Q^\vee,\mu,M}(\rho,\nu^\vee) = -\sum_{q^\vee \in \mathrm{supp}^+(\mathcal{Q}^\vee)} \mathcal{Q}^\vee(q^\vee)\, C_{q^\vee}\left(\tfrac{\mu}{M}\right).$$

3 Dual-Weight Models of A_3

For the root system A_3, the fundamental weights and dual fundamental weights coincide and only trivial shifts $\rho = \rho^\vee = 0$ of the weight lattice exist [3]. The identity sign homomorphism weight lattice point and label sets (1) and (2) are for $M = 2$ given in ω-basis as

$$
\begin{aligned}
F^{\mathbb{1}}_{P^\vee,2}(0,0) = \big\{ & (0,0,0)\,, \left(\tfrac{1}{2},0,0\right), \left(0,\tfrac{1}{2},0\right), \left(0,0,\tfrac{1}{2}\right), (1,0,0)\,, \left(\tfrac{1}{2},\tfrac{1}{2},0\right), \\
& \left(\tfrac{1}{2},0,\tfrac{1}{2}\right), (0,1,0)\,, \left(0,\tfrac{1}{2},\tfrac{1}{2}\right), (0,0,1) \big\}, \\
\Lambda^{\mathbb{1}}_{Q,2}(0,0) = \big\{ & (0,0,0)\,, (1,0,0)\,, (0,1,0)\,, (0,0,1)\,, (2,0,0)\,, (1,1,0)\,, \\
& (1,0,1)\,, (0,2,0)\,, (0,1,1)\,, (0,0,2) \big\}.
\end{aligned}
$$

Chosen to characterize the nearest and next-to-nearest neighbour couplings, the dual-weight hopping function $\mathcal{P}^\vee$ is determined by the following three values: $\mathcal{P}^\vee(\omega_1) = \mathcal{P}^{\vee *}(\omega_3) = I = B + D\mathrm{i}$, $\mathcal{P}^\vee(\omega_2) = A$ with $A, B, D \in \mathbb{R}$. The corresponding three identity sign homomorphism hopping operators, evaluated in the position basis ordered as the point set $F^{\mathbb{1}}_{P^\vee,2}(0,0)$, are given in matrix form as

$$
\widehat{A}^{\mathbb{1}}_{\omega_1,2}(0,0) = \left[\widehat{A}^{\mathbb{1}}_{\omega_3,2}(0,0)\right]^\dagger = -I \begin{pmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{6} & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{6} & 0 & \sqrt{6} & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 2 & 0 & 0 \\
0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

and

$$
\widehat{A}^{\mathbb{1}}_{\omega_2,2}(0,0) = -A \begin{pmatrix}
0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 \\
\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 4 & \sqrt{6} & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 4 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 & \sqrt{6} \\
0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0
\end{pmatrix}.
$$

The eigenenergies, ordered as the labels from the set $\Lambda^{\mathbb{1}}_{Q,2}(0,0)$, are calculated exactly using C-function expression (4) as

$$
\begin{aligned}
\left(E^{\mathbb{1}}_{P^\vee,\lambda,2}(0,0),\, \lambda \in \Lambda^{\mathbb{1}}_{Q,2}(0,0)\right) = & A\,(-6, 0, -2, 0, 6, 0, 2, -6, 0, 6) \\
& + B\left(-8, -2\sqrt{2}, 0, -2\sqrt{2}, 0, 2\sqrt{2}, 0, 8, 2\sqrt{2}, 0\right) \\
& + D\left(0, -2\sqrt{2}, 0, 2\sqrt{2}, -8, -2\sqrt{2}, 0, 0, 2\sqrt{2}, 8\right).
\end{aligned}
$$

The exact forms of the wave functions together with the inherent discrete probability densities can be computed similarly from analytic relation (3).

Acknowledgements The authors acknowledge support from the Czech Science Foundation (GAČR), Grant No. 19-19535S. A.B. is also grateful for partial support by the student grant of the Grant Agency of the Czech Technical University in Prague, grant number SGS19/183/OHK4/3T/14.

References

1. A. Brus, J. Hrivnák, L. Motlochová, Symmetry **13**, 1338 (2021)
2. A. Brus, J. Hrivnák, L. Motlochová, J. Phys. A: Math. Theor. **54**, 095202 (2021)
3. T. Czyżycki, J. Hrivnák, J. Math. Phys. **55**, 113508 (2014)
4. T. Czyżycki, J. Hrivnák, L. Motlochová, Symmetry **12**, 1018 (2020)
5. R. Kane, *Reflection Groups and Invariant Theory* (Springer, New York, 2001)
6. J. Hrivnák, L. Motlochová, J. Fourier Anal. Appl. **25**, 2521–2569 (2019)
7. J. Hrivnák, L. Motlochová, J. Patera, J. Phys. A **45**, 255201 (2012)
8. A.U. Klimyk, J. Patera, SIGMA **2**, 006 (2006)

Abelian J-Invariant Ideals on Nilpotent Lie Algebras

Adela Latorre and Luis Ugarte

Abstract We study the existence of non-trivial Abelian J-invariant ideals $\mathfrak{f}$ in nilpotent Lie algebras $\mathfrak{g}$ endowed with a complex structure J. This condition appears as one of the hypotheses in a recent theorem by A. Fino, S. Rollenske and J. Ruppenthal on the Dolbeault cohomology of complex nilmanifolds. Among other results, we find a pair $(\mathfrak{g}, J)$ for which the only Abelian J-invariant ideal $\mathfrak{f}$ in $\mathfrak{g}$ is the zero one.

Keywords Nilpotent Lie algebra · Complex structure · Abelian J-invariant ideal

1 Introduction

A nilmanifold is a compact quotient $M = \Gamma\backslash G$, where G is a real simply connected nilpotent Lie group and Γ is a discrete cocompact subgroup of G. Let us suppose that $\mathfrak{g}$, the Lie algebra of G, admits a complex structure J (see Sect. 2 for definitions). Then, J determines a left-invariant complex structure on G and one has a compact complex manifold $X = (M, J)$ given by the nilmanifold M endowed with the induced complex structure J.

The natural map $\iota\colon H^{\bullet,\bullet}(\mathfrak{g}, J) \longrightarrow H^{\bullet,\bullet}_{\bar\partial}(X)$ from the Dolbeault cohomology of left-invariant forms to the usual Dolbeault cohomology of $X = (M, J)$ is always injective. Moreover, the map ι has been proved to be an isomorphism in the following cases:

- J is complex parallelizable [11];
- $\mathfrak{g}$ admits a principal torus bundle series with respect to J [2, 3];

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_49

- $\mathfrak{g}$ admits a stable torus bundle series [7–9];
- the complex nilmanifold X is suitably foliated in toroidal groups [4];
- the complex structure J is nilpotent [10] (more generally, the holomorphic tangent bundle of X has a complex nilpotent frame).

These results require that $(\mathfrak{g}, J)$ satisfies some special algebraic properties, which vary for the different cases above. In this short note we focus on one of the hypotheses in the main result of Fino, Rollenske and Ruppenthal [4]: the existence of a non-trivial Abelian J-invariant ideal $\mathfrak{f}$ in the nilpotent Lie algebra $\mathfrak{g}$. Concretely, we prove the following two propositions.

Proposition 1 *Let $\mathfrak{g}$ be a nilpotent Lie algebra endowed with a complex structure J. Then, $\mathfrak{g}$ has a non-trivial Abelian J-invariant ideal in the following cases:*

- *the complex structure J is quasi-nilpotent;*
- *the Lie algebra $\mathfrak{g}$ has dimension ≤ 6.*

This result implies that there exist many nilpotent Lie algebras with complex structures having such an ideal. In particular, a complex structure on any 2-step nilpotent Lie algebra is (quasi-)nilpotent [7, Proposition 3.3]. However, we have

Proposition 2 *There is an 8-dimensional nilpotent Lie algebra $\mathfrak{g}$ such that, for any complex structure J on $\mathfrak{g}$, the only Abelian J-invariant ideal in $\mathfrak{g}$ is the trivial one.*

To our knowledge, this seems to be the first example in the literature of such behaviour. In view of these propositions and the role that Abelian J-invariant ideals play in [4], we ask the following question:

Which nilpotent Lie algebras $\mathfrak{g}$ admit complex structures J having a non-trivial Abelian J-invariant ideal?

This note is structured as follows. In Sect. 2 we recall the main definitions about Lie algebras with complex structures and prove Proposition 1. Section 3 is devoted to the proof of Proposition 2.

2 Complex Structures on Nilpotent Lie Algebras

Let $\mathfrak{g}$ be a real Lie algebra of dimension $2n$. An almost complex structure on $\mathfrak{g}$ is an endomorphism $J\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $J^2 = -\mathrm{Id}$. If J additionally satisfies the integrability condition

$$N_J(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0, \tag{1}$$

for all $X, Y \in \mathfrak{g}$, then J is called a complex structure.

Let $\mathfrak{g}_{\mathbb{C}}{}^*$ be the dual of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$, and denote by $d\colon \bigwedge^* \mathfrak{g}_{\mathbb{C}}{}^* \longrightarrow \bigwedge^{*+1} \mathfrak{g}_{\mathbb{C}}{}^*$ the extension of the usual Chevalley-Eilenberg differential to the complexified exterior algebra. Given an almost complex structure J on $\mathfrak{g}$, there is a natural bigraduation induced by J on $\bigwedge^* \mathfrak{g}_{\mathbb{C}}{}^* = \bigoplus_{p,q} \bigwedge_J^{p,q}(\mathfrak{g}^*)$, where the space $\bigwedge_J^{1,0}(\mathfrak{g}^*)$, resp. $\bigwedge_J^{0,1}(\mathfrak{g}^*)$, is the eigenspace of the eigenvalue i, resp. $-i$, of J as an endomorphism of $\mathfrak{g}_{\mathbb{C}}{}^*$. The integrability condition (1) is equivalent to

$$\pi_{0,2} \circ d|_{\bigwedge_J^{1,0}(\mathfrak{g}^*)} \equiv 0, \tag{2}$$

where $\pi_{0,2}\colon \bigwedge^2 \mathfrak{g}_{\mathbb{C}}{}^* \longrightarrow \bigwedge_J^{0,2}(\mathfrak{g}^*)$ denotes the canonical projection.

Hence, a pair $(\mathfrak{g}, J)$ consisting of a Lie algebra $\mathfrak{g}$ and a complex structure J on $\mathfrak{g}$ can be defined just by taking equations of the form

$$d\omega^k = \sum_{1\le r<s\le n} A^k_{rs}\,\omega^{rs} + \sum_{1\le r,s\le n} B^k_{r\bar{s}}\,\omega^{r\bar{s}}, \quad 1 \le k \le n, \tag{3}$$

where we declare $\{\omega^k\}_{k=1}^n$ to be a basis of $\bigwedge_J^{1,0}(\mathfrak{g}^*)$. Here, $A^k_{r\,s}$, $B^k_{r\,\bar{s}}$ are complex coefficients and the Eqs. (3) satisfy $d \circ d = 0$ (which is equivalent to the Jacobi identity). Moreover, ω^{rs}, resp. $\omega^{r\bar{s}}$, denotes the wedge product $\omega^r \wedge \omega^s$, resp. $\omega^r \wedge \omega^{\bar{s}}$, where $\omega^{\bar{s}}$ is the complex conjugate of ω^s.

When $\mathfrak{g}$ is a *nilpotent* Lie algebra (NLA for short), there is a stronger characterization provided by Salamon in [12]: J is a complex structure on $\mathfrak{g}$ if and only if $\bigwedge_J^{1,0}(\mathfrak{g}^*)$ has a basis $\{\omega^k\}_{k=1}^n$ such that

$$d\omega^1 = 0 \quad \text{and} \quad d\omega^k \in \mathcal{I}(\omega^1, \ldots, \omega^{k-1}), \quad \text{for } k = 2, \ldots, n,$$

where $\mathcal{I}(\omega^1, \ldots, \omega^{k-1})$ is the ideal in $\bigwedge^* \mathfrak{g}_{\mathbb{C}}{}^*$ generated by $\{\omega^1, \ldots, \omega^{k-1}\}$. Moreover, if there exists a basis $\{\omega^k\}_{k=1}^n$ for $\bigwedge_J^{1,0}(\mathfrak{g}^*)$ satisfying $d\omega^1 = 0$ and

$$d\omega^k \in \bigwedge^2 \langle \omega^1, \ldots, \omega^{k-1}, \omega^{\bar{1}}, \ldots, \omega^{\overline{k-1}} \rangle, \quad \text{for } k = 2, \ldots, n,$$

then the complex structure J is called *nilpotent*. These complex structures were first introduced and studied in [3]. They are characterized by the fact that the series $\{\mathfrak{a}_l(J)\}_{l\ge 0}$, defined by $\mathfrak{a}_0(J) = \{0\}$ and

$$\mathfrak{a}_l(J) = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \mathfrak{a}_{l-1}(J) \text{ and } [JX, \mathfrak{g}] \subseteq \mathfrak{a}_{l-1}(J)\}, \text{ for } l \ge 1,$$

satisfies $\mathfrak{a}_l(J) = \mathfrak{g}$ for some positive integer l.

Observe that in general every $\mathfrak{a}_l(J)$ is a J-invariant ideal in $\mathfrak{g}$, possibly zero if the complex structure J is not nilpotent. In fact, the ideal $\mathfrak{a}_1(J)$ is the largest subspace of the center of the NLA $\mathfrak{g}$ that is J-invariant.

Definition 1 ([5]) A complex structure J is said to be *quasi-nilpotent* if $\mathfrak{a}_1(J) \neq \{0\}$. Otherwise, J is called *strongly non-nilpotent* (*SnN* for short).

It is clear from the definitions above that any *nilpotent* complex structure is quasi-nilpotent.

Let $\{Z_k, \bar{Z}_k\}_{k=1}^n$ be the complex basis for $\mathfrak{g}_{\mathbb{C}}$ dual to the basis $\{\omega^k, \omega^{\bar{k}}\}_{k=1}^n$. The brackets $[Z_r, Z_s]$, $[Z_r, \bar{Z}_s]$ and $[\bar{Z}_r, \bar{Z}_s]$ can be directly computed from (3) and their complex conjugate equations by using the well-known formula

$$\psi([W, T]) = -d\psi(W, T), \tag{4}$$

for any $W, T \in \mathfrak{g}_{\mathbb{C}}$ and $\psi \in \mathfrak{g}_{\mathbb{C}}{}^*$. We will also consider the *J-adapted real basis* for $\mathfrak{g}$ given by $\{X_k = Z_k + \bar{Z}_k,\ JX_k = i\,(Z_k - \bar{Z}_k)\}_{k=1}^n$.

Proof of Proposition 1 For the first part, it is clear by definition that if J is quasi-nilpotent then $\mathfrak{f} = \mathfrak{a}_1(J)$ is a non-trivial J-invariant ideal in $\mathfrak{g}$. To get the result, it suffices to recall that $\mathfrak{a}_1(J)$ is a subspace of the center of $\mathfrak{g}$.

For the second part, we first observe that any complex structure J on an NLA $\mathfrak{g}$ of dimension ≤ 4 is nilpotent. In real dimension 6 all the complex structures are either nilpotent or SnN. Indeed, the SnN complex structures only exist on the NLAs labelled as $\mathfrak{h}_{19}^-$ and $\mathfrak{h}_{26}^+$ in [13]. Furthermore, there are exactly four complex structures (up to isomorphism) on these two NLAs (see for instance [1]), parametrized by the complex equations:

$(\mathfrak{g}_\delta, J_\pm):\quad d\omega^1 = 0,\quad d\omega^2 = \omega^{13} + \omega^{1\bar{3}},\quad d\omega^3 = i\,\delta\,\omega^{1\bar{1}} \pm i(\omega^{1\bar{2}} - \omega^{2\bar{1}}),$

where $\delta \in \{0, 1\}$. For $\delta = 0$ the Lie algebra $\mathfrak{g}_0$ is isomorphic to $\mathfrak{h}_{19}^-$, and for $\delta = 1$ the Lie algebra $\mathfrak{g}_1$ is isomorphic to $\mathfrak{h}_{26}^+$.

Let $\{Z_k, \bar{Z}_k\}_{k=1}^3$ be the complex basis for $(\mathfrak{g}_\delta)_{\mathbb{C}}$ dual to $\{\omega^k, \omega^{\bar{k}}\}_{k=1}^3$, and consider the $J_\pm$-adapted real basis $\{Y_k = Z_k + \bar{Z}_k, J_\pm Y_k = i\,(Z_k - \bar{Z}_k)\}_{k=1}^3$ for the NLA $\mathfrak{g}_\delta$. From the complex structure equations above and (4), we get the following non-zero brackets:

$$\begin{aligned}&[Y_1, Y_2] = \mp J_\pm Y_3,\quad [Y_1, Y_3] = -2\,Y_2,\quad [Y_1, J_\pm Y_1] = -2\delta\,Y_3,\\ &[Y_3, J_\pm Y_1] = 2\,J_\pm Y_2,\quad [J_\pm Y_1, J_\pm Y_2] = \mp 2\,J_\pm Y_3.\end{aligned}$$

Therefore, we have that

$$\mathfrak{f} = \langle Y_2, J_\pm Y_2, Y_3, J_\pm Y_3\rangle$$

is a $J_\pm$-invariant ideal in the corresponding Lie algebra $\mathfrak{g}_\delta$. Note that $\mathfrak{f}$ is clearly Abelian. □

Remark 1 In [7, Theorem 4.4], it is proved that the NLAs $\mathfrak{h}_{19}^-$ and $\mathfrak{h}_{26}^+$ admit a stable torus bundle series; in particular, it is proved that there are certain J-invariant ideals for any J. One can check that those ideals are Abelian.

3 Proof of Proposition 2

According to Sect. 2, the complex structure equations

$$d\omega^1 = 0,\ d\omega^2 = \omega^{14} + \omega^{1\bar{4}},\ d\omega^3 = i\,(\omega^{24} + \omega^{2\bar{4}}),\ d\omega^4 = -\omega^{2\bar{2}} + i\,(\omega^{1\bar{3}} - \omega^{3\bar{1}}),$$

define a pair $(\mathfrak{g}, J)$ consisting of a Lie algebra $\mathfrak{g}$ endowed with a complex structure J. In fact, these equations satisfy (2) as well as the Jacobi identity condition $d \circ d = 0$. Let $\{Z_k, \bar{Z}_k\}_{k=1}^4$ be the complex basis for $\mathfrak{g}_{\mathbb{C}}$ dual to $\{\omega^k, \omega^{\bar{k}}\}_{k=1}^4$, and let $\{X_k = Z_k + \bar{Z}_k,\ JX_k = i\,(Z_k - \bar{Z}_k)\}_{k=1}^4$ be the J-adapted real basis for $\mathfrak{g}$. Using (4) one can check that the non-zero Lie brackets of the elements in this basis are

$$\begin{aligned}
&[X_1, X_3] = -2\,JX_4, \quad [X_2, JX_2] = -2\,JX_4, \quad [X_4, JX_2] = -2\,X_3,\\
&[X_1, X_4] = -2\,X_2, \quad [X_4, JX_1] = 2\,JX_2, \quad [JX_1, JX_3] = -2\,JX_4,\\
&[X_2, X_4] = -2\,JX_3.
\end{aligned} \tag{5}$$

We look for Abelian J-invariant ideals $\mathfrak{f}$ in the Lie algebra $\mathfrak{g}$. Any element $Y \in \mathfrak{g}$ can be written, in terms of the real basis above, as

$$Y = \sum_{k=1}^{4} \big(a_k\, X_k + b_k\, JX_k\big), \text{ for some } a_k, b_k \in \mathbb{R}. \tag{6}$$

Using (5), one can easily compute

$$\begin{aligned}
&[Y, X_1] = 2\big(a_4\, X_2 + a_3\, JX_4\big), && [Y, JX_1] = 2\big(a_4\, JX_2 + b_3\, JX_4\big),\\
&[Y, X_2] = 2\big(a_4\, JX_3 + b_2\, JX_4\big), && [Y, JX_2] = -2\big(a_4\, X_3 + a_2\, JX_4\big),\\
&[Y, X_3] = -2\,a_1\, JX_4, && [Y, JX_3] = -2\,b_1\, JX_4,\\
&[Y, X_4] = -2\big(a_1\, X_2 - b_2\, X_3 + b_1\, JX_2 + a_2\, JX_3\big), && [Y, JX_4] = 0.
\end{aligned} \tag{7}$$

We first observe that any non-zero J-invariant ideal $\mathfrak{I}$ in $\mathfrak{g}$ (not necessarily Abelian) must contain X_4 and JX_4. Indeed, let us suppose that there exists a J-invariant ideal $\mathfrak{I}$ in $\mathfrak{g}$ such that $X_4,\ JX_4 \notin \mathfrak{I}$. Then, any element $Y \in \mathfrak{I}$ is given by (6) with $a_4 = b_4 = 0$. Since $[Y, U] \in \mathfrak{I}$ for any $U \in \mathfrak{g}$, the brackets (7) immediately give $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$. Therefore, $\mathfrak{I} = \{0\}$.

Now, if $\mathfrak{I}$ is a non-zero J-invariant ideal in $\mathfrak{g}$ then $X_4,\ JX_4 \in \mathfrak{I}$, and thus $[X_4, U]$ and $[JX_4, U] \in \mathfrak{I}$ for any $U \in \mathfrak{g}$. Consequently, one has from (5) that $X_2,\ JX_2,\ X_3,\ JX_3 \in \mathfrak{I}$. Note that X_1 and JX_1 could also belong to $\mathfrak{I}$, but then $\mathfrak{I} = \mathfrak{g}$. Therefore, the only non-trivial J-invariant ideal in $\mathfrak{g}$ is

$$\mathfrak{I} = \langle X_2,\ JX_2,\ X_3,\ JX_3,\ X_4,\ JX_4\rangle.$$

This ideal is clearly non Abelian due to (5). Hence, we conclude that the only Abelian J-invariant ideal in $\mathfrak{g}$ is the trivial one, i.e. $\mathfrak{f} = \{0\}$.

It remains to show that the same property holds for *any* complex structure on the NLA $\mathfrak{g}$. For this, we first consider the real basis $\{E_1, \ldots, E_8\}$ for $\mathfrak{g}$ obtained from the J-adapted real basis above by reordering and rescaling its generators as follows:

$$E_1 = 2X_1, \quad E_2 = 2JX_1, \quad E_3 = \tfrac{1}{2}X_4, \quad E_4 = 2X_2,$$
$$E_5 = 2JX_2, \quad E_6 = 2X_3, \quad E_7 = 2JX_3, \quad E_8 = 8JX_4.$$

By (5), the non-zero brackets of the elements in this basis are

$$[E_1, E_3] = -E_4, \quad [E_2, E_3] = -E_5, \quad [E_3, E_5] = -E_6, \quad [E_3, E_4] = E_7,$$
$$[E_1, E_6] = [E_2, E_7] = [E_4, E_5] = -E_8.$$

Note that the Lie algebra is rational, and the center of $\mathfrak{g}$ is generated by E_8, so any complex structure on $\mathfrak{g}$ is SnN. Complex structures on 8-dimensional nilpotent Lie algebras with one-dimensional center are classified in [6]. Using (4) one can see that $\mathfrak{g}$ is isomorphic to the Lie algebra labelled as $\mathfrak{g}_9^0$ in [6, Theorem 1.2]. Moreover, it is proved that there is only one (up to isomorphism) complex structure on $\mathfrak{g}_9^0$ (see Appendix B in [6] for details).

In conclusion, for any complex structure J on $\mathfrak{g}$ the only Abelian J-invariant ideal in $\mathfrak{g}$ is the zero one. This completes the proof of Proposition 2.

Acknowledgements This work has been partially supported by the projects PID2020-115652GB-I00, MCIN/AEI/10.13039/501100011033, and E22-17R "Algebra y Geometría" (Gobierno de Aragón/FEDER).

References

1. M. Ceballos, A. Otal, L. Ugarte, R. Villacampa, J. Geom. Anal. **26**, 252–286 (2016)
2. S. Console, A. Fino, Transform. Groups **6**, 111–124 (2001)
3. L.A. Cordero, M. Fernández, A. Gray, L. Ugarte, Trans. Am. Math. Soc. **352**, 5405–5433 (2000)
4. A. Fino, S. Rollenske, J. Ruppenthal, Q. J. Math. **70**, 1265–1279 (2019)
5. A. Latorre, L. Ugarte, R. Villacampa, Trans. Am. Math. Soc. **372**, 3867–3903 (2019)
6. A. Latorre, L. Ugarte, R. Villacampa, arXiv:2011.09916v1 [math.RA]
7. S. Rollenske, Proc. Lond. Math. Soc. **99**, 425–460 (2009)
8. S. Rollenske, J. Lond. Math. Soc. **79**, 346–362 (2009)
9. S. Rollenske, in *Complex and Differential Geometry*, ed. by W. Ebeling, K. Hulek, K. Smoczyk. Springer Proc. in Math., vol. 8 (Springer, Heidelberg, 2011), p. 369
10. S. Rollenske, A. Tomassini, X. Wang, Ann. Mat. Pura. Appl. **199**, 833–862 (2020)
11. Y. Sakane, Osaka J. Math. **13**, 187–212 (1976)
12. S.M. Salamon, J. Pure Appl. Algebra **157**, 311–333 (2001)
13. L. Ugarte, Transform. Groups **12**, 175–202 (2007)

The Dihedral Dunkl–Dirac Symmetry Algebra with Negative Clifford Signature

Alexis Langlois-Rémillard

Abstract The Dunkl–Dirac symmetry algebra is an associative subalgebra of the tensor product of a Clifford algebra and the faithful polynomial representation of a rational Cherednik algebra. In previous work, the finite-dimensional representations of the Dunkl–Dirac symmetry algebra in three dimensions linked with a dihedral group were given. We give here the necessary results to proceed to the same construction when the Clifford algebra in the tensor product has negative signature.

Keywords Dunkl operator · Dunkl–Dirac equation · Symmetry algebra · Total angular momentum algebra · Dihedral root systems

1 Introduction

Dunkl operators [3] generalise partial derivatives by introducing terms related to a reflection group $W \subset \mathscr{O}(N)$, its associated root system R, and a function $\kappa : R \to \mathbb{C}$ invariant on the W-orbits. Together with the multiplication operators and the group algebra $\mathbb{C}[W]$, they generate an associative algebra $\mathscr{A}_\kappa$ that is the faithful polynomial representation of a rational Cherednik algebra [4]. Given a Clifford algebra $Cl(N)$, there is an $\mathfrak{osp}(1|2)$-realisation inside the tensor product $\mathscr{A}_\kappa \otimes Cl(N)$ generated by the Dunkl–Dirac operator obtained by changing the partial derivatives by Dunkl operators and its dual symbol. The symmetry algebra $\mathfrak{SA}$ linked to a family of $\mathfrak{osp}(1|2)$-realisations containing the Dunkl realisation mentioned was characterised abstractly in [1], and it was shown in [6] that it is the full $\mathfrak{osp}(1|2)$-supercentraliser. The representation theory of these algebras is only known for a few specific cases.

In a recent article [2], we constructed the finite-dimensional representations of the dihedral Dunkl–Dirac symmetry algebra $\mathfrak{SA}_m \subset \mathscr{A}_\kappa \otimes Cl(3)$, that is, the symmetry algebra of the $\mathfrak{osp}(1|2)$-realisation linked to the group $W = \mathbb{Z}_2 \times D_{2m}$ acting on

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_50

the three-dimensional Euclidean space. A pair of ladder operators behaving nicely under the action of the double covering $\widetilde{W}$ of the group W was instrumental to this. As the construction was rather involved, only the case when the Clifford algebra had positive signature was considered, that is when the generators e_1, e_2, e_3 square to 1. The goal of this short contribution is to give the needed results to proceed to the same construction in the case when the Clifford algebra has negative signature, that is e_1, e_2, e_3 square to -1. To help compare, the sign introduced is given as $\varepsilon \in \{-1, +1\}$. We study thus here the algebra $\mathfrak{SA}_m^\varepsilon \subset \mathscr{A}_\kappa \otimes Cl^\varepsilon(3)$. The complete classification of the finite-dimensional representations is long and would greatly exceed the allowed space, we refer the readers to [2] for its details. We believe this contribution could be of help for interested readers who want to translate our results, since both conventions for Clifford algebras coexist and the two lead to non-isomorphic real Clifford algebras; multiplication of the generators by i gives the correspondence for complex Clifford algebras.

In Sect. 2 we present the general result needed for the construction. Proposition 1 gives the commutation relations respected by the algebra, where a small sign change appears. Proposition 2 compares the Casimir of the $\mathfrak{osp}(1|2)$ superalgebra with central elements of $\mathfrak{SA}_m^\varepsilon$, and two signs appear. As a consequence, the factorisation of the ladder operators changes slightly as shown in Proposition 5. The remaining steps of the construction of the finite-dimensional representations are then presented in Sect. 3.

2 The Dihedral Dunkl–Dirac Symmetry Algebra

In this section we present the necessary definitions and results on the dihedral Dunkl–Dirac symmetry algebra. We refer the readers to [2, Sects. 2 and 3] for more details, bearing in mind that $\varepsilon = +1$ there.

We consider the Euclidean space $\mathbb{R}^3$ with coordinate vectors ξ_1, ξ_2, ξ_3 and its canonical bilinear form $\langle -, - \rangle$. Let $W = \mathbb{Z}_2 \times D_{2m}$. Its root system R is

$$R = \{\alpha_0 := (0, 0, 1), -\alpha_0, \alpha_j := (\sin(j\pi/m), -\cos(j\pi/m), 0) \mid 1 \leq j \leq 2m\}. \tag{1}$$

The positive root system is $R_+ = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$ and the simple roots are given by α_0, α_1 and α_m. The related reflections $\sigma_\alpha(x) := x - 2\langle x, \alpha\rangle/\langle \alpha, \alpha\rangle$ are given in matrix form by

$$\sigma_0 := \sigma_{\alpha_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \sigma_j := \sigma_{\alpha_j} = \begin{pmatrix} \cos(2j\pi/m) & \sin(2j\pi/m) & 0 \\ -\sin(2j\pi/m) & \cos(2j\pi/m) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Let $\kappa : R \to \mathbb{C}$ be a function invariant on the W-orbits. The Dunkl operators are

$$\mathscr{D}_j f(x) := \partial_{x_j} f(x) + \sum_{\alpha \in R^+} \kappa(\alpha) \frac{f(x) - f(\sigma_\alpha(x))}{\langle x, \alpha \rangle} \left\langle \alpha, \xi_j \right\rangle. \tag{3}$$

The Dunkl operators, the group algebra $\mathbb{C}[W]$ and the multiplication operators generate a faithful representation denoted $\mathscr{A}_\kappa$ of a rational Cherednik algebra.

Let $\varepsilon \in \{-1, +1\}$ be a sign and $Cl^\varepsilon(3)$ be the Clifford algebra generated by the three anticommuting elements e_1, e_2, e_3 subject to

$$\{e_j, e_k\} = 2\varepsilon\delta_{ij}. \tag{4}$$

There is an $\mathfrak{osp}(1|2)$-realisation given by the Dunkl–Dirac operator $\underline{\mathscr{D}}$ and its dual symbol $\underline{x}$ in the tensor product $\mathscr{A}_\kappa \otimes Cl^\varepsilon(3)$:

$$\underline{\mathscr{D}} = \mathscr{D}_1 e_1 + \mathscr{D}_2 e_2 + \mathscr{D}_3 e_3, \qquad \underline{x} = x_1 e_1 + x_2 e_2 + x_3 e_3. \tag{5}$$

We are interested in the elements of $\mathscr{A}_\kappa \otimes Cl(3)$ supercommuting with the $\mathfrak{osp}(1|2)$-realisation, obtained in previous work [1]. First, the following elements in $W \otimes Cl^\varepsilon(3)$ anticommute with $\underline{\mathscr{D}}$ and $\underline{x}$:

$$\widetilde{\sigma}_\alpha = \sigma_\alpha \otimes \sum_{j=1}^{3} \langle \alpha, \xi_j \rangle e_j. \tag{6}$$

They generate a group that is isomorphic to either one of the two possible central extensions of W denoted by $\widetilde{W}^\varepsilon$ [5]. The simple roots become $\widetilde{\sigma}_0 := \widetilde{\sigma}_{\alpha_0}$, $\widetilde{\sigma}_1 := \widetilde{\sigma}_{\alpha_1}$, $\widetilde{\sigma}_m := \widetilde{\sigma}_{\alpha_1}$ and they respect the following relations depending on the parity of m and the value of ε:

$$\widetilde{\sigma}_j^2 = \varepsilon, \quad (\widetilde{\sigma}_0 \widetilde{\sigma}_m)^2 = -1, \quad (\widetilde{\sigma}_1 \widetilde{\sigma}_m)^m = (-1)^{m+1} \varepsilon^m. \tag{7}$$

The following linear combinations, called *one-index symmetries*, of $\mathbb{C}[W] \otimes Cl^\varepsilon(3)$ are distinguished:

$$O_j = \sum_{k=0}^{m} \kappa(\alpha_k) \langle \alpha_k, \xi_j \rangle \widetilde{\sigma}_{\alpha_k} = \frac{\varepsilon}{2}([\underline{\mathscr{D}}, x_i] - e_i) = \frac{\varepsilon}{2} \left(\sum_{k=1} e_k [\mathscr{D}_k, x_j] - e_j \right). \tag{8}$$

Defining $L_{ij} := x_i \mathscr{D}_j - x_j \mathscr{D}_i$, the following elements, named the *two-index symmetries*, commute with $\underline{\mathscr{D}}$ and $\underline{x}$

$$O_{ij} := L_{ij} + \frac{\varepsilon}{2}e_ie_j + O_ie_j - O_je_i, \tag{9}$$
$$= L_{ij} + \frac{\varepsilon}{2}e_ie_j + e_iO_j - e_jO_i. \tag{10}$$

The final symmetry is named *three-index symmetry* and is given by

$$O_{123} = -\frac{\varepsilon}{2}e_1e_2e_3 - O_1e_2e_3 - O_2e_3e_1 - O_3e_1e_2 + O_{12}e_3 + O_{31}e_2 + O_{23}e_1, \tag{11}$$
$$= -\frac{\varepsilon}{2}e_1e_2e_3 - e_2e_3O_1 - e_3e_1O_2 - e_1e_2O_3 + e_3O_{12} + e_2O_{31} + e_1O_{23}. \tag{12}$$

Definition 1 The dihedral *Dunkl–Dirac symmetry algebra* $\mathfrak{S}\mathfrak{A}_m^\varepsilon$ is the associative subalgebra of $\mathscr{A}_\kappa \otimes Cl^\varepsilon(3)$ generated by O_{12}, O_{31}, O_{23}, O_{123} and the group algebra $\mathbb{C}[\widetilde{W}^\varepsilon]$.

It is the full centraliser of the $\mathfrak{osp}(1|2)$-realisation [6].

Proposition 1 *The element O_{123} commutes with every element of $\mathfrak{S}\mathfrak{A}_m^\varepsilon$; the two-index symmetries respect*

$$\begin{aligned}
[O_{12}, O_{31}] &= O_{23} + 2O_1O_{123} + \varepsilon[O_2, O_3],\\
[O_{23}, O_{12}] &= O_{31} + 2O_2O_{123} + \varepsilon[O_3, O_1],\\
[O_{31}, O_{23}] &= O_{12} + 2O_3O_{123} + \varepsilon[O_1, O_2],
\end{aligned} \tag{13}$$

and the elements of $\widetilde{W}^\varepsilon$ interact as

$$\begin{aligned}
\widetilde{\sigma}_0O_{12} &= O_{12}\widetilde{\sigma}_0, & \widetilde{\sigma}_jO_{12} &= -O_{12}\widetilde{\sigma}_j,\\
\widetilde{\sigma}_0O_{31} &= -O_{31}\widetilde{\sigma}_0, & \widetilde{\sigma}_jO_{31} &= (\cos(2j\pi/m)O_{31} + \sin(2j\pi/m)O_{23})\widetilde{\sigma}_j,\\
\widetilde{\sigma}_0O_{23} &= -O_{23}\widetilde{\sigma}_0, & \widetilde{\sigma}_jO_{23} &= (-\cos(2j\pi/m)O_{31} + \sin(2j\pi/m)O_{23})\widetilde{\sigma}_j.
\end{aligned} \tag{14}$$

Proof The relations (13) come from [1, Theorem 3.12]. For (14), remark that it is equivalent to consider σ_kL_{ij} by the definition (9) of O_{ij} and that (6) of $\widetilde{\sigma}_j$. Then only $\widetilde{\sigma}_jO_{12}$ is not direct, and we get:

$$\sigma_jL_{12} = \sigma_j(x_1\mathscr{D}_2 - x_2\mathscr{D}_1) = (\sin^2(2j\pi/m) + \cos^2(2j\pi/m))L_{21}\sigma_j = -L_{12}\sigma_j.$$

Working out the remaining terms of (9) gives the rest. □

We are interested in the representation theory of $\mathfrak{S}\mathfrak{A}_m^\varepsilon$. The construction uses ladder operators, and their factorisations in turn follow from the next proposition.

Proposition 2 *The three-index symmetry squares to*

$$O_{123}^2 = -\frac{\varepsilon}{4} + O_1^2 + O_2^2 + O_3^2 + \varepsilon(O_{12}^2 + O_{31}^2 + O_{23}^2). \tag{15}$$

Proof Express O_{123}^2 as the product of the two expressions (11) and (12)

$$\begin{aligned} O_{123}^2 &= \Big(-\frac{\varepsilon}{2}e_1e_2e_3 - O_1e_2e_3 - \underline{O_2e_3e_1} - O_3e_1e_2 + O_{12}e_3 + O_{31}e_2 + O_{23}e_1\Big) \\ &\quad\times\Bigg(-\frac{\varepsilon}{2}e_1e_2e_3 - e_2e_3O_1 - e_3e_1O_2 - e_1e_2O_3 + \underline{e_3O_{12}} + e_2O_{31} + e_1O_{23} \\ &= -\frac{\varepsilon}{4} - O_1^2 - O_2^2 - O_3^2 + \varepsilon(O_{12}^2 + O_{31}^2 + O_{23}^2\Bigg) + Q, \end{aligned} \tag{16}$$

where Q expresses the 42 remaining "cross terms". We show now that $Q = 2(O_1^2 + O_2^2 + O_3^2)$. Replace in Q all instances of O_{ij} on the left with (9), and all instances on the right by (10). For example, the terms below produce $2(O_1^2 + O_2^2 + O_3^2)$ (the underlined term comes from the two underlined terms in the product)

$$\begin{aligned} A &= \varepsilon((\underline{O_2e_1O_{12}} - O_{12}e_1O_2) + (O_1e_3O_{31} - O_{31}e_3O_1) + (O_3e_2O_{23} - O_{23}e_2O_3)) \\ &= \varepsilon\Big(\underline{O_2e_1L_{12} + \tfrac{1}{2}O_2e_2 + \varepsilon O_2^2 - O_2e_1e_2O_1} - L_{12}e_1O_2 + \tfrac{1}{2}e_2O_2 + O_1e_1e_2O_2 + \varepsilon O_2^2 \\ &\quad + O_1e_3L_{31} + \tfrac{1}{2}O_1e_1 + \varepsilon O_1^2 + O_1e_1e_3O_3 - L_{31}e_3O_1 + \tfrac{1}{2}e_1O_1 + O_3e_3e_1O_1 + \varepsilon O_1^2 \\ &\quad + O_3e_2L_{23} + \tfrac{1}{2}O_2e_3 + \varepsilon O_3^2 + O_3e_3e_2O_2 - L_{23}e_3O_3 + \tfrac{1}{2}e_3O_3 + O_2e_2e_3O_3 + \varepsilon O_3^2\Big) \\ &= 2\varepsilon^2(O_1^2 + O_2^2 + O_3^2) + B, \qquad \text{with } B \text{ the remaining part.} \end{aligned}$$

After doing this procedure for all terms, and further simplifications, one reaches

$$\begin{aligned} Q &= 2(O_1^2 + O_2^2 + O_3^2) \\ &\quad + \frac{\varepsilon}{2}\begin{pmatrix} L_{12}e_1e_2(\varepsilon - e_3e_1L_{31} - e_2e_3L_{23} + 2\varepsilon e_3O_3) \\ +L_{31}e_3e_1(\varepsilon - e_1e_2L_{12} - e_2e_3L_{23} + 2\varepsilon e_2O_2) \\ +L_{23}e_2e_3(\varepsilon - e_1e_2L_{12} - e_3e_1L_{31} + 2\varepsilon e_1O_1) \end{pmatrix} + \frac{\varepsilon}{2}\begin{pmatrix} (\varepsilon - L_{31}e_3e_1 - L_{23}e_2e_3 + 2\varepsilon O_3e_3)e_1e_2L_{12} \\ +(\varepsilon - L_{12}e_1e_2 - L_{23}e_2e_3 + 2\varepsilon O_2e_2)e_3e_1L_{31} \\ +(\varepsilon - L_{12}e_1e_2 - L_{31}e_3e_1 + 2\varepsilon O_1e_1)e_2e_3L_{23} \end{pmatrix}. \end{aligned}$$

The last line is zero. To prove this, replace the O_j by their last definition (8) in terms of commutators $C_{kj} := [\mathscr{D}_k, x_j]$ and apply the following identity [1, Thm 2.5]

$$L_{ij}L_{kl} + L_{ki}L_{jl} + L_{jk}L_{il} = L_{ij}C_{kl} + L_{ki}C_{jl} + L_{jk}C_{il}, \tag{17}$$

keeping in mind that $L_{ii} = 0$, $L_{ij} = -L_{ji}$ and $C_{ij} = C_{ji}$. □

This proposition yields in fact a correspondence between the Casimir of the Lie algebra $\mathfrak{osp}(1|2)$ and a central element in the symmetry algebra. Similar statements hold for any reflection group in any dimension, see [6].

The finite-dimensional representations are constructed via ladder operators. In the classical non-Dunkl case, the ladder operators for the $\mathfrak{so}(3)$ algebra are given by the following linear combinations of the two-index symmetries:

$$O_0 := -iO_{12}, \quad O_+ := iO_{31} + O_{23}, \quad O_- := iO_{31} - O_{23}. \tag{18}$$

For ease of notation, denote the following combination of one-index symmetries (note that they vanish when $\kappa = 0$):

$$T_0 := i O_3, \quad T_+ := O_1 + i O_2, \quad T_- := O_1 - i O_2. \tag{19}$$

Proposition 3 *The commutation relations respected by O_0, O_+ and O_- are*

$$\begin{aligned}
[O_0, O_+] &= +O_+ + \{O_{123}, T_+\} + \varepsilon[T_0, T_+],\\
[O_0, O_-] &= -O_- + \{O_{123}, T_-\} - \varepsilon[T_0, T_-],\\
[O_0, O_+] &= \ 2O_0 - \{O_{123}, T_0\} + \varepsilon[T_+, T_-],
\end{aligned} \tag{20}$$

and those with T_0, T_+ and T_- are

$$\begin{aligned}
T_0 O_0 &= O_0 T_0, & T_0 O_+ &= -O_+ T_0, & T_0 O_- &= -O_- T_0,\\
T_+ O_0 &= -O_0 T_+, & T_+ O_- &= -O_+ T_-, & T_- O_+ &= O_- T_+,\\
T_- O_0 &= -O_0 T_-, & T_- T_0 &= -T_0 T_-, & T_+ T_0 &= -T_0 T_+.
\end{aligned} \tag{21}$$

Proof Use the commutation relations of Proposition 1. □

In this new basis, the following expressions hold.

Proposition 4 *The square of the three-index symmetry becomes*

$$O_{123}^2 = -\frac{\varepsilon}{4} + T_+T_- - T_0^2 - \varepsilon(O_0^2 - O_0 + O_+O_- + 2O_{123}T_0), \tag{22}$$

$$= -\frac{\varepsilon}{4} + T_-T_+ - T_0^2 - \varepsilon(O_0^2 + O_0 - O_-O_+ - 2O_{123}T_0). \tag{23}$$

Furthermore, the following equations hold

$$O_+O_- = \varepsilon T_+T_- - (O_0 - 1/2)^2 - \varepsilon(\varepsilon O_{123} + T_0)^2, \tag{24}$$

$$O_-O_+ = \varepsilon T_-T_+ - (O_0 + 1/2)^2 - \varepsilon(\varepsilon O_{123} - T_0)^2. \tag{25}$$

Proof We prove (22) by directly rewriting from the definitions (18) and (19):

$$\begin{aligned}
O_{12}^2 &= -O_0^2, & O_{31}^2 + O_{23}^2 &= -O_+O_- + O_0 - 2O_{123}T_0 + \frac{\varepsilon}{2}[T_+, T_-],\\
O_3^2 &= T_0^2, & O_1^2 + O_2^2 &= T_+T_- - \frac{1}{2}[T_+, T_-].
\end{aligned}$$

Equation (23) is similar, and the expressions (24) and (25) follow directly. □

Proposition 5 *The following operators*

$$L_+ := \frac{1}{2}\{O_0, O_+\} \quad \textit{and} \quad L_- := \frac{1}{2}\{O_0, O_-\} \tag{26}$$

are ladder operators with respect to O_0 in the sense that

$$[O_0, L_+] = +L_+, \qquad [O_0, L_-] = -L_-, \tag{27}$$

and the products of two of them admit the following factorisations

$$L_+L_- = -((O_0 - 1/2)^2 + \varepsilon(\varepsilon O_{123} + T_0)^2)((O_0 - 1/2)^2 - \varepsilon T_+T_-), \tag{28}$$
$$L_-L_+ = -((O_0 + 1/2)^2 + \varepsilon(\varepsilon O_{123} - T_0)^2)((O_0 + 1/2)^2 - \varepsilon T_-T_+). \tag{29}$$

Proof That L_+ and L_- are ladder operators comes from Proposition 3

$$\begin{aligned}2[O_0, L_\pm] &= [O_0, \{O_0, O_\pm\}] = \{O_0, [O_0, O_\pm]\}\\ &= \{O_0, \pm O_\pm + \{O_{123}, T_\pm\} \pm \varepsilon[T_0, T_\pm]\} = \pm\{O_0, O_\pm\} = 2L_\pm,\end{aligned}$$

where Eq. (14) was used in the second line. The proof of the factorisation is the same as [2, Proposition 3.8] using the ε variants of the commutation relations. □

3 Sketch of the Finite-Dimensional Representations Construction

Everything needed for the construction of the finite-dimensional representations is in place. Doing it would, however, greatly exceed the scope of this note. We give below a sketch of the steps needed and refer the readers to [2] for the details.

1. Any finite-dimensional $\mathfrak{S}\mathfrak{A}^\varepsilon_m$-representation decomposes as a $\widetilde{W}^\varepsilon$-representation into a direct sum of spin irreducible $\widetilde{W}^\varepsilon$-representations by Maschke's Theorem (the irreducible representations for these groups can be found in [2, Thm A.5]). Let $\widetilde{W}^\varepsilon_0$ be the subgroup of $\widetilde{W}^\varepsilon$ generated by elements commuting with O_0. The associative subalgebra of $\mathfrak{S}\mathfrak{A}^\varepsilon_m$ generated by O_0, L_+, L_-, O_{123} and $\widetilde{W}^\varepsilon_0$ has a triangular decomposition. Use this triangular decomposition and the ladder operators to give a basis of O_0- and O_{123}-eigenvectors for any irreducible $\mathfrak{S}\mathfrak{A}^\varepsilon_m$-representation. (See [2, Lem. 4.3].)
2. Thus start from a general O_0- and O_{123}-eigenbasis. The elements v_j^+ and v_j^- of this basis are obtained from multiple applications of the ladder operators on a first pair v_0^+, v_0^-. Use the two factorisations (28) and (29) to create equations $L_+v_j^- = A(j)v_{j+1}^-$ and $L_-v_j^+ = A(j)v_{j+1}^+$. The terms $A(j)$ will depend on the first $\widetilde{W}^\varepsilon$-representation, and on the eigenvalues of O_{123} and O_0. Then irreducibility and the finite-dimension give conditions on $A(j)$. (See [2, (4.21)–(4.23)].)
3. Solve the system obtained for the values of the O_{123}- and O_0-eigenvalues keeping track of the conditions on κ. (See [2, (4.28)].)
4. Furthermore, the unitarity of the representations can be studied in the same fashion by looking at positivity constraints in the $A(j)$. (See [2, Sect. 3.3 and Lem. 4.4].)

Acknowledgements We wish to thank Hendrik De Bie, Roy Oste and Joris Van der Jeugt for helpful discussions and support. ALR holds scholarships from the Fonds de recherche Nature et technologies (Québec) 270527 and the Excellence of Science Research Project 30889451. This support is gratefully acknowledged.

References

1. H. De Bie, R. Oste, J. Van der Jeugt, Lett. Math. Phys. **108**, 1905–1953 (2018). https://doi.org/10.1007/s11005-018-1065-0
2. H. De Bie, A. Langlois-Rémillard, R. Oste, J. Van der Jeugt, J. Algebra. **591**, 170–216 (2022). https://doi.org/10.1016/j.jalgebra.2021.09.025
3. C.F. Dunkl, Trans. Am. Math. Soc. **311**, 167–183 (1989). https://doi.org/10.2307/2001022
4. P. Etingof, V. Ginzburg, Invent. math. **147**, 243–348 (2002). https://doi.org/10.1007/s002220100171
5. A.O. Morris, Proc. Lond. Math. Soc. **s3-32**, 403–420 (1976). https://doi.org/10.1112/plms/s3-32.3.403
6. R. Oste, Supercentralizers for deformations of the Pin osp dual pair. arXiv:2110.15337

Lie Structure on Hopf Algebra Cohomology

Tekin Karadağ

Abstract We calculate the Gerstenhaber bracket (graded Lie bracket) on Hopf algebra and Hochschild cohomologies of the Taft algebra T_n for any integer $n > 2$ which is a nonquasi-triangular Hopf algebra. We show that the bracket is indeed zero on Hopf algebra cohomology of T_n, as in all known quasi-triangular Hopf algebras. This example is the first known bracket computation for a nonquasi-triangular Hopf algebra. We also explore a general formula for the bracket on Hopf algebra cohomology of any Hopf algebra with bijective antipode on the bar resolution that is reminiscent of Gerstenhaber's original formula for Hochschild cohomology [8, Sect. 5]. In order to find the expression, we use the composition of various isomorphisms and an embedding from Hopf algebra cohomology into Hochschild cohomology.

Keywords Hochschild cohomology · Hopf algebra cohomology · Gerstenhaber bracket · Taft algebra

1 Introduction

In 1945, Hochschild introduced homology and cohomology of any associative algebra [7]. Almost two decades later, Gerstenhaber showed that a Hochschild cohomology ring is an algebra with an associative product (cup product) and is a Lie algebra with nonassociative Lie bracket (Gerstenhaber bracket) [2]. The Gerstenhaber bracket is originally defined on bar complex which makes the bracket impossible to calculate by the definition.

The Gerstenhaber bracket was originally defined on Hochschild cohomology by Gerstenhaber himself [2, Sect. 1.1] which makes the Hochschild cohomology a G-algebra together with the cup product. In 1992, M. Gerstenhaber and D. Schack conjectured that Hopf algebra cohomology has a G-structure as well [3]. Farinati and Solotar showed that for any Hopf algebra A, Hopf algebra cohomology is a G-

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_51

algebra [1]. In the same year, R. Taillefer used a different approach and solve the same conjecture [13]. In 2016, Hermann [6, Theorem 6.3.12, Corollary 6.3.15] proved that the Gerstenhaber bracket on the Hopf algebra cohomology of a quasi-triangular Hopf algebra is trivial. We use the technique introduced by Negron and Witherspoon [10] and find the bracket structure on Hochschild cohomology of a Taft algebra which is a nonquasi-triangular Hopf algebra [8, Sect. 4]. Moreover, it is known that for any Hopf algebra with bijective antipode, the Hopf algebra cohomology can be embedded into the Hochschild cohomology [4]. A Taft algebra also has a bijective antipode. Then, we consider the Hochschild and Hopf algebra cohomologies of a Taft algebra that were done by Nguyen [12], find the embedding from the Hopf algebra cohomology into the Hochschild cohomology, and finally find the corresponding bracket on the Hopf algebra cohomology of a Taft algebra [8, Sect. 4]. As a result, we show that the bracket is indeed zero, as in all known quasi-triangular Hopf algebras. This example is the first known bracket computation on Hopf algebra cohomology of a nonquasi-triangular Hopf algebra. Moreover, it is known that any non-semisimple Hopf algebra of dimension p^2 for a prime p over an algebraically closed field is isomorphic to a Taft algebra [11, Theorem 6.5]. Hence, on Hopf algebra cohomology of such a Hopf algebra, the Lie structure is Abelian.

2 Gerstenhaber Bracket on Cohomologies of a Taft Algebra

A Hopf algebra is an algebra over a field k with a coalgebra structure, i.e. it has a comultiplication, counit and antipode which satisfy coassociativity, counit property and antipode property. Group algebras, universal enveloping algebras over a Lie algebra, quantum enveloping algebras, and quantum elementary Abelian groups are a few examples of Hopf algebras. For $n > 2$, a Taft algebra is defined as the k-algebra generated by g and x satisfying the relations : $g^n = 1$, $x^n = 0$ and $xg = \omega gx$ where ω is a primitive nth root of unity. A Taft algebra has also a Hopf algebra structure with the maps:

- Comultiplication: $\Delta(g) = g \otimes g$, $\Delta(x) = 1 \otimes x + x \otimes g$
- Counit: $\varepsilon(g) = 1$, $\varepsilon(x) = 0$
- Antipode: $S(g) = g^{-1}$, $S(x) = -xg^{-1}$.

As well as Hochschild cohomology, we also define Hopf algebra cohomology of a Hopf algebra. It is proved that Hopf algebra cohomology has also G-algebra structure [1, 13], i.e. it has a cup product and a graded Lie bracket which are compatible. We know that the Lie bracket is trivial when the Hopf algebra is quasi-triangular [1, 6, 13]. To see the Lie structure on nonquasi-triangular case, we compute the Gerstenhaber bracket on the Hopf algebra cohomology of a Taft algebra over a field k of characteristic 0 which is a nonquasi-triangular Hopf algebra [5].

A finite group G acts by automorphisms on a k-algebra A. The *skew group algebra* $A \rtimes G$ is $A \otimes kG$ as a vector space, with the multiplication

$$(a_1 \otimes g_1)(a_2 \otimes g_2) = a_1({}^{g_1}a_2) \otimes g_1 g_2$$

where ${}^{g}a$ represent the action of g on a. A Taft algebra T_n can be seen as a skew group algebra $A \rtimes G$ for $A = k[x]/(x^n)$ and $G = \mathbb{Z}/n\mathbb{Z}$ [5].

2.1 Lie Structure on Hochschild Cohomology of $A = k[x]/(x^n)$

In order to calculate the bracket on the Hopf algebra cohomology of a Taft algebra T_n, first, we calculate the bracket on Hochschild cohomology of A. Computing the bracket on the bar resolution is not an ideal method. Instead, we consider the following resolution of A:

$$\mathbb{A} : \cdots \xrightarrow{v.} A^e \xrightarrow{u.} A^e \xrightarrow{v.} A^e \xrightarrow{u.} A^e \xrightarrow{\pi} A \longrightarrow 0 \tag{1}$$

where $A^e = A \otimes A^{op}$, $u = x \otimes 1 - 1 \otimes x$, $v = x^{n-1} \otimes 1 + x^{n-2} \otimes x + \cdots + 1 \otimes x^{n-1}$, and π is the multiplication. The following theorem which is the combination of [10, Theorem 3.2.5] and [10, Lemma 3.4.1] allows us to use the resolution (1) for the bracket calculation.

Theorem 1 *Suppose $\mathbb{A} \xrightarrow{\mu} A$ is a projective A-bimodule resolution of A that satisfies some hypotheses [10, hypotheses 3.1]. Let $\phi : \mathbb{A} \otimes_A \mathbb{A} \to \mathbb{A}$ be any contracting homotopy for the chain map $F_{\mathbb{A}} : \mathbb{A} \otimes_A \mathbb{A} \to \mathbb{A}$ defined by $F_{\mathbb{A}} := (\mu \otimes_A id_{\mathbb{A}} - id_{\mathbb{A}} \otimes_A \mu)$, i.e.*

$$d(\phi) := d_{\mathbb{A}}\phi + \phi d_{\mathbb{A}\otimes_A\mathbb{A}} = F_{\mathbb{A}}. \tag{2}$$

Then for cocycles f and g in $Hom_{A^e}(\mathbb{A}, A)$, the bracket given by

$$[f, g]_\phi = f \circ_\phi g - (-1)^{(|f|-1)(|g|-1)} g \circ_\phi f \tag{3}$$

where the circle product is

$$f \circ_\phi g = f\phi(id_{\mathbb{A}} \otimes_A g \otimes_A id_{\mathbb{A}})\Delta^{(2)} \tag{4}$$

agrees with the Gerstenhaber bracket on cohomology.

We find the needed maps ϕ and Δ in the formula (4) for the resolution (1) [8, (3.3), (3.4)] and find the bracket structure on Hochschild cohomology of A [8, Sect. 3].

2.2 Lie Structure on Hochschild Cohomology of a Taft Algebra

Now, we consider $\mathcal{D} = A^e \rtimes G$ which is isomorphic to a subalgebra of T_n^e. By using the resolution $\mathbb{A}$ in (1), we derive the resolution $T_n \otimes_{\mathcal{D}} \mathbb{A}$ of $T_n \otimes_{\mathcal{D}} A$ with $T_n \otimes_{\mathcal{D}} A^e$ in each degree. It is known that, $T_n \cong T_n^e \otimes_{\mathcal{D}} A$ as T_n-bimodules [14, Sect. 3.5], which also implies $A \otimes T_n \cong T_n^e \otimes_{\mathcal{D}} A^e$ [8, (4.2)]. Then, the resolution $T_n \otimes_{\mathcal{D}} \mathbb{A}$ of $T_n \otimes_{\mathcal{D}} A$ turns into the following resolution of T_n:

$$\tilde{\mathbb{A}} : \cdots \xrightarrow{\tilde{u}\cdot} A \otimes T_n \xrightarrow{\tilde{v}\cdot} A \otimes T_n \xrightarrow{\tilde{u}\cdot} A \otimes T_n \xrightarrow{\tilde{\pi}\cdot} T_n \longrightarrow 0 \tag{5}$$

where $\tilde{v} = v \otimes id_{kG}$, $\tilde{u} = u \otimes id_{kG}$, and $\tilde{\pi} = \pi \otimes id_{kG}$.

Next, we find the needed maps $\tilde{\phi}$ and $\tilde{\Delta}$ [8, Lemmas 4.6, 4.10], use the formula (3) and calculate the bracket on Hochschild cohomology of T_n for degree 1 and degree 2 elements ($\tilde{f}_{xg^i}$ and $\tilde{f}_{g^j}$, respectively) as follows [8, Sect. 4]:

$$[\tilde{f}_{xg^i}, \tilde{f}_{xg^j}] = 0, [\tilde{f}_{xg^i}, \tilde{f}_{g^j}] = \begin{cases} -(n-2)g^j, & i = 0 \\ (\omega^{-i}+1)g^{i+j}, & i \neq 0 \end{cases}, [\tilde{f}_{g^i}, \tilde{f}_{g^j}] = 0. \tag{6}$$

By the relation between the bracket and the cup product, brackets in higher degrees can be determined, since the Hochschild cohomology is generated as an algebra under cup product in degrees 1 and 2.

2.3 Lie Structure on Hopf Algebra Cohomology of a Taft Algebra

It is known that the Hopf algebra cohomology of a Hopf algebra with a bijective antipode can be embedded into Hochschild cohomology of the algebra [4]. Since all finite dimensional Hopf algebras and many known infinite dimensional Hopf algebras have a bijective antipode, the condition on the Hopf algebras is not too restrictive.

A Taft algebra is a Hopf algebra with a bijective antipode as it is finite dimensional. Hence there is an embedding from the Hopf algebra cohomology to Hochschild cohomology of T_n. The Hopf algebra and Hochschild cohomologies of T_n were calculated in [12, Sect. 8] as follows:

$$\mathrm{H}^n(T_n, k) = \begin{cases} k & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad \mathrm{HH}^n(T_n) = \begin{cases} k & \text{if } n \text{ is even,} \\ Span_k\{x\} & \text{if } n \text{ is odd.} \end{cases} \tag{7}$$

We summarize all the work in this section with the following theorem:

Theorem 2 *The graded Lie algebra structure on Hopf algebra cohomology of a Taft algebra is Abelian.*

Proof Since a Taft algebra is finite dimensional, it has a bijective antipode. Then, the Hopf algebra cohomology of T_n can be embedded into Hochschild cohomology of T_n [4]. The Hochschild and Hopf algebra cohomologies of T_n in (7) implies that the embedding is identity in even degrees and zero map in odd degrees. The even degree elements in $\mathrm{HH}^*(T_n)$ is in form f_{g_i} and the bracket for degree 2 elements is $[\tilde{f}_{g^i}, \tilde{f}_{g^j}] = 0$ by (6). This implies the bracket for any two even degree elements is also 0 as the bracket of higher even degree elements is determined by bracket of degree 2 elements. Therefore, the corresponding bracket on Hopf algebra cohomology of T_n is 0.

This example is the first known bracket computation for a nonquasi-triangular algebra. Later, the author and S. Witherspoon obtains the same result by using a homotopy lifting technique [9, Sect. 5].

The following is a direct result of Theorem 2 and [11, Theorem 6.5]:

Corollary 1 *Let A be a non-semisimple Hopf algebra of dimension p^2 for a prime p over an algebraically closed field. Then the bracket structure on the Hopf algebra cohomology of A is Abelian.*

3 Gerstenhaber Bracket for Hopf Algebras

As it is mentioned before, the Gerstenhaber bracket is defined on Hochschild cohomology via the bar resolution. However, for a Hopf algebra with a bijective antipode, we know that there is an embedding from the Hopf algebra cohomology to Hochschild cohomology. To explore a bracket formula on the Hopf algebra cohomology, we take the following resolution which is the bar resolution of k [8, Lemma 5.1]:

$$P_\bullet : \cdots \xrightarrow{d_3} A^{\otimes 3} \xrightarrow{d_2} A^{\otimes 2} \xrightarrow{d_1} A \xrightarrow{\varepsilon} k \longrightarrow 0.$$

Then, we find an equivalent formula for the bracket on Hochschild cohomology. Then, we use the isomorphism in the proof of the Eckmann-Shapiro lemma [8, Lemma 5.3] with the embedding in [14, Theorem 9.4.5] and give a new bracket formula for the Hopf algebra cohomology [8, Sect. 5]:

$$[f, g] = \varepsilon_*(\eta_*(f) \circ \eta_*(g)) - (-1)^{(m-1)(n-1)} \varepsilon_*(\eta_*(g) \circ \eta_*(f)) \tag{8}$$

where $f \in \mathrm{Hom}_A(P_m, k)$, $g \in \mathrm{Hom}_A(P_n, k)$, the unit map $\eta_* : \mathrm{Hom}_A(P_\bullet, k) \to \mathrm{Hom}_A(P_\bullet, A^{ad})$, the counit map $\varepsilon_* : \mathrm{Hom}_A(P_\bullet, A) \to \mathrm{Hom}_A(P_\bullet, k)$, and

$$\varepsilon_*((\eta_*(f)\circ\eta_*(g))(1\otimes c^1\otimes c^2\otimes\cdots\otimes c^{m+n-1})) = \varepsilon\Big(\sum\sum_{i=1}^{m}(-1)^{(n-1)(i-1)}$$
$$\eta(f(1\otimes c_1^1\otimes c_1^2\otimes\cdots\otimes c_1^{i-1}\otimes c_1^*\otimes c_1^{i+n}\otimes\cdots\otimes c_1^{m+n-1}))$$
$$c_2^1c_2^2\cdots c_2^{i-1}c_2^*c_2^{i+n}\cdots c_2^{m+n-1}S(c_3^1c_3^2\cdots c_3^{m+n-1})\Big)$$

with

$$\Delta(c^*) = \sum c_1^*\otimes c_2^* \text{ and}$$
$$c^* = \sum \eta(g(1\otimes c_1^i\otimes c_1^{i+1}\otimes\cdots\otimes c_1^{i+n-1}))c_2^ic_2^{i+1}\cdots c_2^{i+n-1}.$$

Therefore, the formula in (8) is a general expression of the Gerstenhaber bracket on a Hopf algebra cohomology which is indeed inherited from the formula of the bracket on Hochschild cohomology.

We lastly need to point that, the definition in (8) on Hopf algebra cohomology of a Hopf algebra can be considered as a similar definition of M. Gerstenhaber's original bracket formula on Hochschild cohomology of an algebra. Unfortunately, we cannot use Gerstenhaber's definition to calculate the bracket structure on Hochschild cohomology of an algebra. Similarly, we cannot use the formula in (8) to calculate the bracket structure on Hopf algebra cohomology of a Hopf algebra. However, (8) gives us a general expression of the bracket on the Hopf algebra cohomology.

References

1. M.A. Farinati, A. Solotar, G-structure on the cohomology of Hopf algebras. Proc. Am. Math. Soc. **132**(10) (2004)
2. M. Gerstenhaber, The cohomology structure of an associative ring. Ann. Math. (2) **78**, 267–288 (1963)
3. M. Gerstenhaber, S.D. Schack, Algebras, bialgebras, quantum groups, and algebraic deformations. Contemp. Math. **134**, 51–92 (1992)
4. V. Ginzburg, S. Kumar, Cohomology of quantum groups at roots of unity. Duke Math. J. **69**, 179–198 (1993)
5. E. Gunnlaugsdóttir, Monoidal structure of the category of u_q^+-modules, Special issue on linear algebra methods in representation theory. Linear Algebra Appl. **365**, 183–199 (2003)
6. R. Hermann, *Monoidal Categories and the Gerstenhaber Bracket in Hochschild Cohomology* (American Mathematical Society, 2016)
7. G. Hochschild, On the cohomology groups of an associative algebra. Ann. Math. **2**, 58–67 (1945)
8. T. Karadağ, Gerstenhaber bracket on Hopf algebra and Hochschild cohomologies, to appear in J. Pure Appl. Algebra (2022). https://doi.org/10.1016/j.jpaa.2021.106903
9. T. Karadağ, S. Witherspoon, Lie brackets on Hopf algebra cohomology, to appear in Pac. J. Math. https://doi.org/10.2140/pjm.2022.316.395
10. C. Negron, S. Witherspoon, An alternate approach to the Lie bracket on Hochschild cohomology. Homol. Homotopy Appl. **18**(1) (2016)
11. S. Ng, Non-semisimple Hopf algebras of dimension p^2. J. Algebra **255**(1), 182–197 (2002). MR 1935042 (2003h:16067)

12. V.C. Nguyen, Tate and Tate-Hochschild cohomology for finite dimensional Hopf algebras. J. Pure Appl. Algebra **217**, 1967–1979 (2013)
13. R. Taillefer, Injective Hopf bimodules, cohomologies of infinite dimensional Hopf algebras and graded-commutativity of the Yoneda product. J. Algebra. **276**(1), 259–279 (2004)
14. S. Witherspoon, *Hochschild Cohomology for Algebras*, vol. 204 (American Mathematical Society, 2019)

Filtration Associated to an Abelian Inner Ideal and the Speciality of the Subquotient of a Lie Algebra

Esther García, Miguel Gómez Lozano, and Rubén Muñoz Alcázar

Abstract For any abelian inner ideal B of a Lie algebra L such that $[B, \mathrm{Ker}_L B]^n \subseteq B$ for some $n \in \mathbb{N}$ we build a bounded filtration $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ whose first nonzero term $\mathcal{F}_{-n}$ is B, $\mathcal{F}_{n-1} = \mathrm{Ker}_L B$ and $\mathcal{F}_n = L$. The extremes of the induced $\mathbb{Z}$-graded Lie algebra $\hat{L} = \mathcal{F}_{-n} \oplus \mathcal{F}_{-n+1}/\mathcal{F}_{-n} \oplus ... \oplus \mathcal{F}_n/\mathcal{F}_{n-1}$ are the Jordan pair $V = (\mathcal{F}_{-n},\ \mathcal{F}_n/\mathcal{F}_{n-1})$ and coincide with the subquotient $(B, L/\mathrm{Ker}_L B)$. Thanks to this filtration, we can prove that when a Lie algebra L is strongly prime and $\mathrm{Ker}_L B$ is not a subalgebra of L, then subquotient $(B, L/\mathrm{Ker}_L B)$ is a special strongly prime Jordan pair.

Keywords Lie algebra · Inner ideal · Filtration · Jordan pair · Subquotient · Speciality

1 Preliminaries

Throughout this chapter we are going to introduce definitions and results which are necessary for the development of subsequent sections.

By a ring of scalars Φ we understand an associative, commutative and unitary ring. We will be deal with Lie algebras L, associative algebras R and linear Jordan pairs V over a ring of scalars Φ containing $\frac{1}{2}$ and $\frac{1}{3}$. As usual, $[x, y]$ will denote the Lie bracket of two elements x, y of L, and the product of elements of R will be written by juxtaposition. Any associative algebra R gives rise to a Lie algebra $R^{(-)}$ with Lie bracket $[x, y] := xy - yx$, for all $x, y \in R$. If R has an involution $*$ we will consider the Lie subalgebra $\mathrm{Skew}(R, *) = \{x \in R \mid x^* = -x\}$ of $R^{(-)}$. Jordan

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_52

triple products of a Jordan pair $V = (V^+, V^-)$ will be written by $\{x, y, z\}$ for any $x, z \in V^\sigma, y \in V^{-\sigma}, \sigma = \pm$. The reader is referred to [1, 2] and [4] for basic results, notation and terminology on Lie algebras and Jordan pairs.

Definition 1 A Φ-module B of a Lie algebra L is called an abelian inner ideal if $[B, B] = 0$ and $[B, [B, L]] \subseteq B$. The kernel of an abelian inner ideal is

$$\mathrm{Ker}_L B = \{x \in L \mid [B, [B, x]] = 0\}$$

Associated to an abelian inner ideal B of L we can consider the subquotient $(B, L/\mathrm{Ker}_L B)$, which is a linear Jordan pair with products

$$\{b_1, \bar{x}, b_2\} = [[b_1, x], b_2] \qquad \{\bar{x}, b_1, \bar{y}\} = \overline{[[x, b_1], y]}$$

for every $b_1, b_2 \in B$ and every $\bar{x}, \bar{y} \in L/\mathrm{Ker}_L B$, see [7, 3.2].

Definition 2 Let Φ be a ring of scalars and let L be a Lie algebra over Φ. For each $x \in L$, we define the linear map $\mathrm{ad}_x : L \longrightarrow L$ as $\mathrm{ad}_x(y) := [x, y]$ for every $y \in L$. We will say that L is a nondegenerate Lie algebra if, for every $x \in L$ such that $\mathrm{ad}_x^2(L) = 0$, then $x = 0$. An element x in a Lie algebra L is called a Jordan element of L if $\mathrm{ad}_x^3(L) = 0$.

Definition 3 Let L be a Lie algebra. A finite $\mathbb{Z}$-grading is a non-trivial $\mathbb{Z}$-grading of L such that the support $\mathrm{supp}\, L = \{m \in \mathbb{Z} \mid L_m \neq 0\}$ is finite. In this case $L = L_{-n} \oplus L_{-(n-1)} \oplus \ldots \oplus L_0 \oplus \ldots \oplus L_{n-1} \oplus L_n$ for some positive integer n. If $L_{-n} + L_n \neq 0$, we will call such a grading a $(2n+1)$-grading. Note that if L is nondegenerate then both L_{-n} and L_n are non-zero. If $L = L_{-n} \oplus \ldots \oplus L_n$ is a $(2n+1)$-graded Lie algebra, then $V = (L_{-n}, L_n)$ is a Jordan pair with products $\{x, y, z\} = [[x, y], z]$ and $\{y, x, t\} = [[y, x], t]$ for every $x, z \in L_{-n}$ and every $y, t \in L_n$, which is called the associated Jordan pair of L.

Definition 4 Let L be a Lie algebra over Φ. A $\mathbb{Z}$-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ is a chain of submodules of L $\ldots \subset \mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ such that $[\mathcal{F}_i, \mathcal{F}_j] \subset \mathcal{F}_{i+j}$ for every $i, j \in \mathbb{Z}$. A $\mathbb{Z}$-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ is bounded if there exist $n, m \in \mathbb{Z}$, with $n < m$, such that $\mathcal{F}_i = 0$ for every $i \leq n$ and $\mathcal{F}_j = L$ for every $j \geq m$. If $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ is a $\mathbb{Z}$-filtration of a Lie algebra L over Φ, we can consider the Φ-module

$$\hat{L} = \ldots \oplus \underbrace{\mathcal{F}_{i-1}/\mathcal{F}_{i-2}}_{\hat{L}_{i-1}} \oplus \underbrace{\mathcal{F}_i/\mathcal{F}_{i-1}}_{\hat{L}_i} \oplus \underbrace{\mathcal{F}_{i+1}/\mathcal{F}_i}_{\hat{L}_{i+1}} \oplus \ldots \qquad (\star)$$

with product $[\bar{x}, \bar{y}] = \overline{[x, y]} \in \mathcal{F}_{i+j}/\mathcal{F}_{i+j-1}$ for every $\bar{x} = x + \mathcal{F}_{i-1} \in \mathcal{F}_i/\mathcal{F}_{i-1}$ and every $\bar{y} = y + \mathcal{F}_{j-1} \in \mathcal{F}_j/\mathcal{F}_{j-1}$. Thereby $\hat{L}$ has structure of $\mathbb{Z}$-graded Lie algebra and it is called the induced graded Lie algebra (see [5, p. 351]).

Definition 5 An associative algebra R is semiprime if, for every nonzero ideal I of R, $I^2 \neq 0$, and it is prime if $IJ \neq 0$ for every pair of nonzero ideals I, J of R. If R

is an associative algebra with involution $*$, we say that and ideal I of R is an $*$-ideal if $y^* \in I$ for every $y \in I$, and we say that R is $*$-prime if $IJ \neq 0$ for every nonzero $*$-ideals I, I of R.

The extended centroid of R will be denoted by $C(R)$ (see [3, Sect. 2.3] for its definition and main properties). When R is semiprime, $C(R)$ is von Neumann regular, and when R is prime, $C(R)$ is a field. The central closure of R is $\hat{R} = C(R) + C(R)R$ and R is centrally closed if it coincides with its central closure. When R has an involution $*$, this involution extends to $C(R)$ and to $\hat{R}$. If R is a prime associative algebra with involution $*$, the involution is of the first kind when every element in $C(R)$ is symmetric with respect to $*$, and it is of the second kind if there are nonzero skew-symmetric elements in $C(R)$.

A Lie algebra L is said to be prime if $[I, J] \neq 0$ for every nonzero ideals I, J of L. If L is prime and nondegenerate, we say that L is a strongly prime Lie algebra. By [8, Theorem 1.6], we know that L is a strongly prime Lie algebra if and only if for every $x, y \in L$ such that $[x, [y, L]] = 0$, we have that $x = 0$ or $y = 0$.

Definition 6 An associative pair over a ring of scalars Φ is a pair $A = (A^+, A^-)$ of Φ-modules with a triple product such that $uv(xyz) = u(vxy)z = (uvx)yz$ for every $x, z, u \in A^\sigma$ and every $y, v \in A^{-\sigma}$, where $\sigma = \pm$.

Example 1 If X and Y are two Φ-modules over a ring of scalars Φ, then the pair $V = (\mathrm{Hom}_\Phi(X, Y), \mathrm{Hom}_\Phi(Y, X))$ with triple product $f_1 g_1 f_2$ and $g_1 f_1 g_2$ for every $f_1, f_2 \in \mathrm{Hom}_\Phi(X, Y)$ and for every $g_1, g_2 \in \mathrm{Hom}_\Phi(Y, X)$, is an associative pair.

If $A = (A^+, A^-)$ is an associative pair over a ring of scalars Φ, then the pair of Φ-modules (A^+, A^-) with products $\{x, y, z\} = xyz + zyx$ and $\{y, x, t\} = yxt + txy$ for every $x, z \in A^\sigma$ and for every $y, t \in A^{-\sigma}$, $\sigma = \pm$, is a Jordan pair denoted by $(A^+, A^-)^{(+)}$.

Definition 7 Let Φ be a ring of scalars an let $V = (V^+, V^-)$ be a Jordan pair over Φ. We say that V is special if is a subpair of the Jordan pair $A = (A^+, A^-)^{(+)}$ for some associative pair $A = (A^+, A^-)$ over Φ.

Example 2 Let X and Y be two modules over a ring of scalars Φ. Then the Jordan pair $V = (\mathrm{Hom}_\Phi(X, Y), \mathrm{Hom}_\Phi(Y, X))^{(+)}$ is special.

2 Filtration Associated to an Abelian Inner Ideal

In this section we will construct a filtration associated to an abelian inner ideal. If B is an abelian inner ideal of a Lie algebra L such that $[B, \mathrm{Ker}_L B]^n \subset B$ for some $n \in \mathbb{N}$, we will show that B induces a bounded filtration of L starting on B and whose second last submodule coincides with $\mathrm{Ker}_L B$.

In [9, Theorem 1.2] a bounded filtration from -2 to 2 associated to a Jordan element was built. Generalizing this idea, given an abelian inner ideal B of a Lie algebra L, we are going to construct a bounded filtration of L associated to B.

Theorem 1 ([6, Theorem 3.1]) *Let L be a Lie algebra and let B be an abelian inner ideal of L. Let us suppose that there exists $n \in \mathbb{N}$ such that $[B, Ker_L B]^n \subseteq B$. Then the chain*

$$\cdots \subset \mathcal{F}_{-n} \subset \mathcal{F}_{-n+1} \subset \cdots \subset \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n \subset \ldots,$$

where $\mathcal{F}_{-m} := \{0\}$ and $\mathcal{F}_m := L$ for every $m \geq n$, and

$$\begin{aligned} \mathcal{F}_{-n} &:= B,\ \mathcal{F}_{-k} := [B, \mathrm{Ker}_L B]^k + B \text{ for } k = 1, \ldots, n-1, \\ \mathcal{F}_0 &:= \{x \in L \mid [x, B] \subseteq B\} \\ \mathcal{F}_s &:= \mathrm{ad}^{n-s-1}_{[B, \mathrm{Ker}_L B]}(Ker_L B) + \mathcal{F}_0 \text{ for } s = 1, \ldots, n-1,\ \mathcal{F}_n := L \end{aligned}$$

is a bounded filtration of L.

Remark 1 The graded Lie algebra induced by this filtration $\hat{L} = \mathcal{F}_{-n} \oplus \mathcal{F}_{-n+1}/\mathcal{F}_{-n} \oplus \cdots \oplus \mathcal{F}_n/\mathcal{F}_{n-1}$ has associated Jordan pair $V = (\mathcal{F}_{-n}, \mathcal{F}_n/\mathcal{F}_{n-1})$ equal to the subquotient $(B, L/\mathrm{Ker}_L B)$.

In the following results we will show that the hypothesis $[B, \mathrm{Ker}_L B]^n = 0$ for some $n \in \mathbb{N}$ is quite natural for large families of Lie algebras.

Remark 2 In [10, Proposition 3.5(d)] it was shown that for any abelian inner ideal B of a centrally closed prime associative algebra R, $[B, \mathrm{Ker}_{R^{(-)}} B]$ is nilpotent of index k with $k \leq 3$. This result easily extends to semiprime associative algebras. Indeed, given an abelian inner ideal of a semiprime associative algebra R, since R is a subdirect product of prime associative algebras R_i, B decomposes into a subdirect product of abelian inner ideals B_i of R_i. For each i, let us consider the central closure $\hat{R}_i$, and let us extend B_i to an abelian inner ideal $\hat{B}_i = C(R_i)B_i$ of $\hat{R}_i$. Then $[\hat{B}_i, \mathrm{Ker}_{\hat{R}_i} \hat{B}_i]$ is nilpotent of index ≤ 3 for each i, and therefore $[B, \mathrm{Ker}_{R^{(-)}} B]$ is nilpotent of index ≤ 3. In particular, every abelian inner ideal B of a semiprime associative algebra R satisfies $[B, \mathrm{Ker}_{R^{(-)}} B]^k = 0 \subseteq B$ for some $k \leq 3$.

Proposition 1 ([6, Proposition 4.2]) *Let R be a centrally closed associative algebra with involution $*$. Suppose that R is $*$-prime and that $\frac{1}{2}, \frac{1}{3}, \frac{1}{5} \in \Phi$. Let $L = Skew(R, *)$ and let B be an abelian inner ideal of L. Then $[B, Ker_L B]$ is a nilpotent subalgebra of L of index n, where we have the following possibilities for n:*

(a) If R is $*$-prime not prime or if it is prime and the involution is of the second kind, $n \leq 3$.
(b) If R is prime and the involution is of the first kind, then $n \leq 4$. In particular, if $[L, L] = 0$, $n = 1$; otherwise $b^3 = 0$ for every $b \in B$ and either

- *there exists $b \in B$ such that $b^3 = 0$, $b^2 \neq 0$. In this case $n = 2$ and L admits a 3-grading $L = L_{-1} \oplus L_0 \oplus L_1$ with $B = L_{-1}$ and $KerB = L_{-1} \oplus L_0$, or*
- *$B^2 = 0$ and $n \leq 4$. If, moreover, $B(KerB)B = 0$, then $n \leq 3$.*

Since every semiprime associative algebra R with involution $*$ is a subdirect product of $*$-prime associative algebras R_i, given an abelian inner ideal B of $L = \mathrm{Skew}(R, *)$ we can consider the projections B_i of B onto $L_i = \mathrm{Skew}(R_i, *)$. Let $\hat{R}_i$ be the central closure of each R_i and let $\hat{B}_i = H(C(R_i), *)B_i$ be the abelian inner ideal generated by B_i in $\hat{L}_i = \mathrm{Skew}(\hat{R}_i, *)$. By Proposition 1, $[\hat{B}_i, \mathrm{Ker}_{\hat{L}_i} \hat{B}_i]$ is nilpotent of index ≤ 4, so $[B_i, \mathrm{Ker}_{L_i} B_i]$ is also nilpotent of index ≤ 4 for each i. Thus $[B, \mathrm{Ker}_L B]$ is nilpotent of index ≤ 4.

Remark 3 Let L be nondegenerate. Then for every nonzero abelian inner ideal B of finite length of L there exists a finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ such that $B = L_n$ (this is always the case when L is nondegenerate finite dimensional), see [7, Corollary 6.2]. With respect to this grading, $\mathrm{Ker}_L B = L_{-(n-1)} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$, so $[B, \mathrm{Ker}_L B] \subset L_1 \oplus \cdots \oplus L_n$ implies that $[B, \mathrm{Ker}_L B]^n \subseteq B$.

3 The Speciality of the Subquotient

In this last section we will apply the filtration associated to an abelian inner ideal to give a sufficient condition for the speciality of the subquotient associated to such an abelian inner ideal.

We will use the following lemma, which has appeared several times in the literature and can be found for example in [1, Theorem 11.34].

Lemma 1 *Let $L = L_{-n} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_n$ be a $(2n+1)-\mathbb{Z}$-graded Lie algebra. Then the pair $V := (L_{-n}, L_n)$ with product $\{x, y, z\} := [x, [y, z]]$ for $x, z \in L_{\sigma n}$ and $y \in L_{-\sigma n}$, $\sigma = \pm$ is a Jordan pair. Moreover, for any $i \in 1, 2, \ldots, n-1$ the pair of linear maps (Ψ_i, Ψ_{-i})*

$$\Psi_i : L_{-n} \to Hom(L_i, L_{i-n}) \qquad \Psi_{-i} : L_n \to Hom(L_{i-n}, L_i)$$

defined by $\Psi_{\sigma i}(x)(y) = ad_x y$ for any $x \in L_{-\sigma n}$ and any $y \in L_i$ if $\sigma = +$ or $y \in L_{i-n}$ if $\sigma = -$ is a homomorphism of Jordan pairs between V and the special Jordan pair $(Hom(L_i, L_{i-n}), Hom(L_{i-n}, L_i))^{(+)}$.

In the following theorem we will give sufficient conditions to assure that the pair of homomorphisms (Ψ_{n-1}, Ψ_{1-n}) is a monomorphism and therefore the subquotient $(B, L/\mathrm{Ker}_L B)$ is a special Jordan pair.

Theorem 2 ([10, Corollary 3.4]) *Let L be a strongly prime Lie algebra over a ring of scalars Φ. Let B be an abelian inner ideal of L and consider* $\mathrm{Ker}_L B$. *Suppose that there exists $n \in \mathbb{N}$ such that $[B, \mathrm{Ker}_L B]^n \subseteq B$. If* $\mathrm{Ker}_L B$ *is not a subalgebra of L, then the subquotient $(B, L/\mathrm{Ker}_L B)$ is a special Jordan pair.*

Acknowledgements The three authors were partially supported by Junta de Andalucía FQM-264 and by B4: Proyecto Puente Universidad de Málaga *Sistemas de Jordan, álgebras de Lie y estructuras relacionadas*.

References

1. A. Fernández López, *Jordan Structures in Lie Algebras* (American Mathematical Society, Providence, RI, 2019)
2. O. Loos, *Jordan Pairs* (Springer, Berlin-New York, 1975)
3. K.I. Beidar, W.S. Martindale III., A.V. Mikhalev, *Rings with Generalized Identities* (Marcel Dekker Inc, New York, 1996)
4. N. Jacobson, *Lie Algebras* (Dover Publications Inc, New York, 1979)
5. E.I. Zelmanov, Mat. Sb. (N.S.) **124(166)**, 33–37 (1984)
6. E. García, M. Gómez Lozano, R. Muñoz Alcázar, *A filtration associated to an abelian inner ideal of a Lie algebra (submitted)*
7. A. Fernández López, E. García, M. Gómez Lozano, E. Neher, https://doi.org/10.1093/imrn/rnm051
8. E. García, M. Gómez Lozano, https://doi.org/10.1016/j.jalgebra.2006.11.003
9. E. García, M. Gómez Lozano, https://doi.org/10.1080/00927872.2011.590955
10. E. García, M. Gómez Lozano, R. Muñoz Alcázar, https://doi.org/10.1016/j.jalgebra.2020.07.013

Nilpotent Inner Derivations in Prime Superalgebras

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Abstract In this contribution, we summarize an in-depth analysis of the nilpotency index of nilpotent homogeneous inner superderivations in associative prime superalgebras with and without superinvolution. We also present examples of all the different cases that our analysis exhibits for the nilpotency indices of the inner superderivations [5]. This work fits with Herstein's branch of the theory that studies nilpotent inner derivations in algebras.

Keywords Associative superalgebra · Lie superalgebra · Inner superderivation · Superinvolution · Skew-symmetric element

1 Preliminaries

This work can be found with all the details in [5]. General notions about superalgebras can be found in [4, 6–8].

In this first section, we will only present the definitions necessary to understand the theorems in Sect. 2.

Definition 1 Throughout this work, Φ will denote a scalar ring with $\frac{1}{2}$. We say that a non-necessarily associative algebra R over Φ is a superalgebra if $R = R_0 \oplus R_1$ and $R_i R_j \subset R_{i+j}$ with i, j indices modulo 2. For any superalgebra $R = R_0 \oplus R_1$ we can consider the automorphism $\sigma : R \to R$ such that $R_0 = \{x \in R \mid \sigma(x) = x\}$ and $R_1 = \{x \in R \mid \sigma(x) = -x\}$.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 396,
https://doi.org/10.1007/978-981-19-4751-3_53

Recall that $R = R_0 \oplus R_1$ is an associative superalgebra if it is associative as an algebra. In such R we can define a new product, called the super-bracket, as follows:

$$[x, y] := xy - (-1)^{|x||y|} yx \tag{1}$$

for any homogeneous $x, y \in R$. Then R with the superbracket is a Lie superalgebra.

Given an associative superalgebra, the adjoint operator at $a \in R_0 \cup R_1$ is defined by $\mathrm{ad}_a(x) := [a, x]$ for any homogeneous $x \in R$ and we say that a is ad-nilpotent of index n if $\mathrm{ad}_a^n(x) = 0$ for every $x \in R$ and there exists $y \in R$ such that $\mathrm{ad}_a^{n-1}(y) \neq 0$.

Definition 2 A semiprime associative superalgebra R is a superalgebra without nonzero nilpotent graded ideals. We remark that a semiprime associative superalgebra is just an associative superalgebra which is semiprime as an algebra (for every nonzero ideal I of R, $I^2 \neq 0$). A prime associative superalgebra R is an associative superalgebra without nonzero orthogonal graded ideals (for every nonzero graded ideals I, J of R, $IJ \neq 0$).

Definition 3 Given an associative superalgebra R with superinvolution $*$, that is, $*$ is a 0-degree linear map such that for every homogeneous $x, y \in R$, $(x^*)^* = x$ and $(xy)^* = (-1)^{|x||y|} y^* x^*$, the set of skew-symmetric elements $K := \mathrm{Skew}(R, *) = \{a \in R \mid a^* = -a\}$ and the set of symmetric elements $H := \mathrm{Sym}(R, *) = \{a \in R \mid a^* = a\}$ are Lie subsuperalgebras and Jordan subsuperalgebras of R respectively. Since $\frac{1}{2} \in \Phi$, $R = H \oplus K$. We will denote $H_i = H \cap R_i$ and $K_i = K \cap R_i$, $i = 0, 1$.

Definition 4 Let R be a semiprime associative superalgebra. Since R is semiprime as an algebra, we can consider the extended centroid $C(R)$ of R (see [1, Sect. 2.3] for further information). Let $\hat{R} = RC(R) + C(R)$ be the central closure of R. Let $\sigma : R \to R$ be the automorphism associated to the $\mathbb{Z}_2$-grading of R ($\sigma^2 = id$). This automorphism can be extended to $\hat{R}$ and we denote this extension by $\hat{\sigma}$. Since $\hat{\sigma}^2 = id$, $\hat{R}$ is again a superalgebra and $\hat{\sigma}(C(R)) = C(R)$, i.e., $C(R) = C(R)_0 + C(R)_1$ where $C(R)_0 = \{\lambda + \hat{\sigma}(\lambda) \mid \lambda \in C(R)\}$ and $C(R)_1 = \{\lambda - \hat{\sigma}(\lambda) \mid \lambda \in C(R)\}$. We will say that R is centrally closed if $R = \hat{R}$, i.e., if R is centrally closed as an algebra.

2 Ad-nilpotent Elements of R and $\mathrm{Skew}(R, *)$

In this section we present the description, depending on the index of ad-nilpotency, of ad-nilpotent elements in a prime associative superalgebra R and of ad-nilpotent elements in K when a superinvolution is considered.

First, let us state the description of homogeneous ad-nilpotent elements in a prime associative superalgebra R.

Theorem 1 ([5, Theorem 3.2]) *Let us consider a prime associative superalgebra $R = R_0 \oplus R_1$, let $\hat{R}$ denote the central closure of R, and let $a \in R_0 \cup R_1$ be a homo-*

geneous ad-nilpotent element of index n. If R is free of $\binom{n}{s}$-torsion and free of s-torsion, for $s = [\frac{n+1}{2}]$, then:

1. *If $a \in R_0$, n is odd and exists $\lambda \in C(R)_0$ such that $a - \lambda \in \hat{R}$ is nilpotent of index $\frac{n+1}{2}$.*
2. *If $a \in R_1$, then*

   a. *if $n \equiv_4 1$ and R is free of $\binom{\frac{n-1}{2}}{\frac{s-1}{2}}$-torsion, then a is nilpotent of index $\frac{n+1}{2}$.*
   b. *if $n \equiv_4 2$ then there is $\lambda \in C(R)_0$ such that $(a^2 - \lambda) \in \hat{R}$ is nilpotent of index $\frac{n+2}{4}$.*
   c. *the cases $n \equiv_4 0$ and $n \equiv_4 3$ do not occur.*

Let R be a prime associative superalgebra with superinvolution. In the following two results we will describe the homogeneous ad-nilpotent elements of $K = \mathrm{Skew}(R, *)$.

Theorem 2 ([5, Theorem 4.3]) *Let $R = R_0 \oplus R_1$ be a prime associative superalgebra of characteristic $p > n$ with superinvolution $*$, let $\hat{R}$ be its central closure, let $a \in K_0 := \mathrm{Skew}(R, *)_0$ be an ad-nilpotent element of K of index $n > 1$ and let $s = [\frac{n+1}{2}]$. Then*

1. *If $n \equiv_4 0$ then a is nilpotent of index $s + 1$, ad-nilpotent of R and of R_0 of index $n + 1$ and satisfies $a^s K a^s = 0$. Moreover, the index of ad-nilpotence of a in K_0 can be $n - 1$ or n.*
2. *If $n \equiv_4 1$ then there exists $\lambda \in \mathrm{Skew}(C(R), *)_0$ such that $a - \lambda \in \hat{R}$ is nilpotent of index s and a is ad-nilpotent of R, of R_0 and of K_0 of index n.*
3. *The case $n \equiv_4 2$ is not possible.*
4. *If $n \equiv_4 3$ then either:*

   a. *there exists $\lambda \in \mathrm{Skew}(C(R), *)_0$ such that $a - \lambda \in \hat{R}$ is nilpotent of index s and a is ad-nilpotent of R, of R_0 and of K_0 of index n, or*
   b. *a is nilpotent of index $s + 1$, ad-nilpotent of K_0 of index n, ad-nilpotent of R and of R_0 of index $n + 2$ and satisfies $a^s k a^{s-1} - a^{s-1} k a^s = 0$ for every $k \in K$. In particular R satisfies $a^s K a^s = 0$.*

Theorem 3 ([5, Theorem 4.4]) *Let $R = R_0 \oplus R_1$ be a prime associative superalgebra of characteristic $p > n$ with superinvolution $*$, let $\hat{R}$ be its central closure, let $a \in K_1 := \mathrm{Skew}(R, *)_1$ be an ad-nilpotent element of K of index $n > 1$ and let $s = [\frac{n+1}{2}]$.*

1. *If $n \equiv_8 0$ then a is nilpotent of index $s + 1$, ad-nilpotent of R of index $n + 1$ and $a^s K a^s = 0$ (so $a^s R a^s$ is a commutative trivial local superalgebra).*
2. *If $n \equiv_8 1$ then $a^{s-1} \in H_0$, and a is nilpotent of index s and ad-nilpotent of R of index n.*
3. *If $n \equiv_8 2$ then there exists $\lambda \in \mathrm{Skew}(C(R), *)_0$ such that $a^2 - \lambda \in \hat{R}$ is nilpotent of index $\frac{s+1}{2}$ and a is ad-nilpotent of R of index n.*
4. *If $n \equiv_8 5$ then $a^{s-1} \in K_0$, and a is nilpotent of index s and ad-nilpotent of R of index n.*

5. *If* $n \equiv_8 6$ *then there exists* $\lambda \in \mathrm{Skew}(C(R), *)_0$ *such that* $a^2 - \lambda \in \hat{R}$ *is nilpotent of index* $\frac{s+1}{2}$ *and a is ad-nilpotent of* R *of index* n.
6. *If* $n \equiv_8 7$ *then* a *is nilpotent of index* $s+1$*, ad-nilpotent of* R *of index* $n+2$ *and* $a^s k a^{s-1} + (-1)^{|k|} a^{s-1} k a^s = 0$ *for every homogeneous* $k \in K$ *(so* $a^s R a^s$ *is a commutative trivial local superalgebra).*
7. *The cases* $n \equiv_8 3$ *and* $n \equiv_8 4$ *do not occur.*

Remark 1 We highlight that some of the types appearing in these last three theorems are new models than only appear in the supersetting (see, [2, Theorems 4.4 and 5.6]).

3 Examples

In this section we are going to construct examples of all types of homogeneous ad-nilpotent elements appearing in Theorem 1, and in Theorems 2 and 3. The examples of even ad-nilpotent elements of R and of K are based on the examples of ad-nilpotent elements in the non-super setting, see [3].

Definition 5 Let Φ be a ring of scalars and let r, s be natural numbers. The matrix algebra $\mathcal{M}_{r+s}(\Phi)$ with

$$\mathcal{M}(r|s)_0 := \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} : A \in \mathcal{M}_r(\Phi), D \in \mathcal{M}_s(\Phi) \right\} \text{ and}$$

$$\mathcal{M}(r|s)_1 := \left\{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} : B \in \mathcal{M}_{r,s}(\Phi), C \in \mathcal{M}_{s,r}(\Phi) \right\}$$

becomes an $\mathbb{Z}_2$-graded associative algebra. It will be denoted $\mathcal{M}(r|s) = \mathcal{M}(r|s)_0 + \mathcal{M}(r|s)_1$. We will use the notation $\mathcal{M}(r) = \mathcal{M}(r|r)$.

Definition 6 Let r and s be two natural numbers with odd $r > 1$ and even s, let F be a field with involution denoted by $\overline{\alpha}$ for any $\alpha \in F$, and let R be the superalgebra $\mathcal{M}(r|s)$ over F. Let $\{e_{i,j}\}$ denote the matrix units, and define

$$\begin{array}{l} H = \sum_{i=1}^{r} (-1)^i e_{i,r+1-i} \in \mathcal{M}_r(F) \ \ (\text{notice } H = H^t = H^{-1}) \\ J = \sum_{i=1}^{s} (-1)^i e_{i,s+1-i} \in \mathcal{M}_s(F) \ \ (\text{notice } J^t = -J = J^{-1}). \end{array}$$

The map $* : R \to R$ given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} H & 0 \\ 0 & J \end{bmatrix}^{-1} \overline{\begin{bmatrix} A & -B \\ C & D \end{bmatrix}}^t \begin{bmatrix} H & 0 \\ 0 & J \end{bmatrix}$$

defines a superinvolution in R. In particular

$$e_{i,j}^* = (-1)^{j-i} e_{r-j+1,r-i+1} \text{ for every } i, j \in \{1, \dots, r\},$$

$$e_{r+i,r+j}^* = (-1)^{j-i} e_{r+s-j+1,r+s-i+1} \text{ for every } i, j \in \{1, \dots, s\} \text{ and}$$

$$e_{i,r+j}^* = (-1)^{i-j+1} e_{r+s+1-j,r+1-i} \text{ for every } i \in \{1, ..., r\} \text{ and } j \in \{1, ..., s\}.$$

The associative superalgebra R is a simple superalgebra with superinvolution, and its extended centroid $C(R)$, which coincides with $Z(R)$, is isomorphic to F. Moreover, the extension of the superinvolution $*$ to $C(R)$ is isomorphic to the involution $-$ of F.

3.1 Examples of Even Ad-nilpotent Elements of R and of **Skew($R, *$)**

Let F be a field with involution $-$ and characteristic zero (or big enough). Let k be an even number($k \geq 2$), let $r = 3k + 3$ and $s = 2k$, and let us consider the associative superalgebra $R = \mathcal{M}(r|s)$ over F with the superinvolution defined in Definition 6. Let us denote by K the skew-symmetric elements of R with respect to $*$. Consider the following nilpotent matrices:

$$T := \sum_{i=k+2}^{2k+1} e_{i,i+1} \in R_0 \text{ (nilpotent of index } k+1) \tag{2}$$

$$S := \sum_{i=1}^{k-1} (e_{i,i+1} + e_{r-i,r-i+1}) \in R_0 \text{ (nilpotent of index } k) \tag{3}$$

$$U := \sum_{i=1}^{k-1} e_{r+i,r+i+1} + \sum_{i=k+1}^{2k-1} e_{r+i,r+i+1} \in R_0 \text{ (nilpotent of index } k). \tag{4}$$

T is ad-nilpotent of R and of R_0 of index $2k + 1$, and S and U are ad-nilpotent elements of R and of R_0 of index $2k - 1$. Notice that $T^* = -T$, $S^* = -S$ and $U^* = -U$ so $T, S, U \in K_0$. Then, for any $\lambda \in \text{Skew}(F, -)$,

1. $T + \lambda I$ is an example of case (1) of Theorem 1 and of case (2) of Theorem 2,
2. $S + \lambda I$ is an example of case (1) of Theorem 1 and of case (4.a) of Theorem 2,

If $\mathrm{Skew}(F, -) = 0$:

3. T is an example of case (4.b) of Theorem 2,
4. $T + S$ is an example of case (1) of Theorem 2,
5. $T + U$ is an example of case (1) of Theorem 2 such that the index in K_0 decreases.

3.2 *Examples of Odd Ad-nilpotent Elements of R and of* $\mathbf{Skew}(R, *)$

Let F be a field of characteristic zero (or big enough) and with identity involution, let $r > 1$ be an odd number, let $s = r - 1$, and consider the superalgebra $R = \mathcal{M}(r|s)$ with the superinvolution given in Definition 6. Again, let us denote by K the skew-symmetric elements of R with respect to $*$. Let us consider $T := \sum_{i=1}^{r-1} e_{i,r+i} \in R_1$ and

$$A = T - T^* = \sum_{i=1}^{r-1} e_{i,r+i} + \sum_{i=2}^{r} e_{r+i-1,i} \in K_1 \text{ (nilpotent of index } 2r-1). \qquad (5)$$

1. For $r = 10t + 1$, A^5 is an example of case (1) of Theorem 3,
2. For $r = 10t + 3$, A^5 is an example of case (2) of Theorem 3,
3. For $r = 10t + 5$, A^5 is an example of case (3) of Theorem 3,
4. For $r = 10t + 7$, A^5 is an example of case (4) of Theorem 3,
5. For $r = 10t + 9$, A^5 is an example of case (5) of Theorem 3,
6. A is an example of case (6) of Theorem 3.

Acknowledgements The three authors were partially supported by FQM-264 and by B4: Proyecto Puente Universidad de Málaga *Sistemas de Jordan, álgebras de Lie y estructuras relacionadas*.

References

1. K.I. Beidar, W.S. Martindale III., A.V. Mikhalev, *Rings with Generalized Identities* (Marcel Dekker, New York, 1996), pp. 68–82
2. J. Brox, E. García, M. Gómez Lozano, R. Muñoz Alcázar, G. Vera de Salas, Bull. Malays. Math. Sci. Soc, https://doi.org/10.1007/s40840-020-01064-w
3. J. Brox, E. García, M. Gómez Lozano, R. Muñoz Alcázar, G. Vera de Salas, Bull. Malays. Math. Sci. Soc, https://doi.org/10.1007/s40840-021-01206-8
4. M. Fošner, Commun. Algebra **32**(2), 689–705 (2004)
5. E. García, M. Gómez Lozano, G. Vera de Salas, Linear Multilinear Algebra, https://doi.org/10.1080/03081087.2021.1919594
6. F. Montaner, Commun. Algebra **26**(7), 2337–2349 (1998)
7. S. Montgomery, J. Algebra **195**(2), 558–579 (1997)
8. M. Scheunert, *The Theory of Lie Superalgebras* (Springer, Berlin, 1979)