MAT-MEK4270: Mandatory Assignment 1

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Exercise 1.2.3

$u(t, x, y)$ is defined as

$$u(t, x, y) = e^{\imath(k_x x + k_y y - \omega t)} \tag{1}$$

, where $\imath = \sqrt{-1}$.

To show that Eq. 1 satisfies the wave equation, we insert it into the wave equation:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2 \nabla^2 u \\
\frac{\partial^2}{\partial t^2} e^{\imath(k_x x + k_y y - \omega t)} &= c^2 \nabla^2 e^{\imath(k_x x + k_y y - \omega t)} \\
\frac{\partial^2}{\partial t^2} e^{\imath(k_x x + k_y y - \omega t)} &= c^2 \left(\frac{\partial^2}{\partial x^2} e^{\imath(k_x x + k_y y - \omega t)} + \frac{\partial^2}{\partial y^2} e^{\imath(k_x x + k_y y - \omega t)} \right)
\end{aligned}$$

, and for each partial derivative, we get:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} e^{\imath(k_x x + k_y y - \omega t)} &= \frac{\partial}{\partial t} \imath\omega e^{\imath(k_x x + k_y y - \omega t)} \\
&= \imath^2 \omega^2 e^{\imath(k_x x + k_y y - \omega t)} \\
&= -\omega^2 e^{\imath(k_x x + k_y y - \omega t)}
\end{aligned}$$

,

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} e^{\imath(k_x x + k_y y - \omega t)} &= \frac{\partial}{\partial x} \imath k_x e^{\imath(k_x x + k_y y - \omega t)} \\
&= \imath^2 k_x^2 e^{\imath(k_x x + k_y y - \omega t)} \\
&= -k_x^2 e^{\imath(k_x x + k_y y - \omega t)}
\end{aligned}$$

, and

$$\begin{aligned}
\frac{\partial^2}{\partial y^2} e^{\imath(k_x x + k_y y - \omega t)} &= \frac{\partial}{\partial y} \imath k_y e^{\imath(k_x x + k_y y - \omega t)} \\
&= \imath^2 k_y^2 e^{\imath(k_x x + k_y y - \omega t)} \\
&= -k_y^2 e^{\imath(k_x x + k_y y - \omega t)}
\end{aligned}$$

. What we end up with is

$$
\begin{aligned}
-\omega^2 e^{\imath(k_x x+k_y y-\omega t)} &= c^2\left(-k_x^2 e^{\imath(k_x x+k_y y-\omega t)} - k_y^2 e^{\imath(k_x x+k_y y-\omega t)}\right)\\
-\omega^2 e^{\imath(k_x x+k_y y-\omega t)} &= -c^2\left(k_x^2 e^{\imath(k_x x+k_y y-\omega t)} + k_y^2 e^{\imath(k_x x+k_y y-\omega t)}\right)\\
\omega^2 e^{\imath(k_x x+k_y y-\omega t)} &= c^2\left(k_x^2 e^{\imath(k_x x+k_y y-\omega t)} + k_y^2 e^{\imath(k_x x+k_y y-\omega t)}\right)\\
\omega^2 e^{\imath(k_x x+k_y y-\omega t)} &= c^2 k_x^2 e^{\imath(k_x x+k_y y-\omega t)} + c^2 k_y^2 e^{\imath(k_x x+k_y y-\omega t)}\\
\omega^2 &= c^2 k_x^2 + c^2 k_y^2\\
\omega^2 &= c^2\left(k_x^2 + k_y^2\right)\\
\omega &= \pm c\sqrt{k_x^2 + k_y^2}
\end{aligned}
$$

. This means that as long as the last line of the equation above is satisfied, then u satisfies the wave equation.

1 1.2.4

A discrete version of Eq. 1 for when $m_x = m_y \rightarrow k_x = k_y = k$ is given by

$$u_{ij}^n = e^{\imath(kh(i+j)-\tilde{\omega} n\Delta t)} \tag{2}$$

, and the discretized wave equation is given by

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = c^2\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2}\right) \tag{3}$$

.

We will insert the discrete equation into the discretized wave equation and simplify. We will do this part by part, starting with the left–hand side of the equation:

$$
\begin{aligned}
&\frac{e^{\imath(kh(i+j)-\tilde{\omega}(n+1)\Delta t^2)} - 2e^{\imath(kh(i+j)-\tilde{\omega}(n)\Delta t^2)} + e^{\imath(kh(i+j)-\tilde{\omega}(n-1)\Delta t^2)}}{\Delta t^2}\\
=&\frac{e^{\imath(kh(i+j)-\tilde{\omega} n\Delta t^2)}\left(e^{\imath(-\tilde{\omega}\Delta t^2)} - 2 + e^{\imath(\tilde{\omega}\Delta t^2)}\right)}{\Delta t^2}
\end{aligned}
$$

, and now the first part of the right–hand side of the equation:

$$\frac{e^{\imath(kh(i+1+j)-\tilde{\omega}n\Delta t^2)} - 2e^{\imath(kh(i+j)-\tilde{\omega}n\Delta t^2)} + e^{\imath(kh(i-1+j)-\tilde{\omega}n\Delta t^2)}}{h^2}$$
$$= \frac{e^{\imath(kh(i+j)-\tilde{\omega}n\Delta t^2)}(e^{\imath kh} - 2 + e^{\imath(-kh)})}{h^2}$$

. Now we can put them together and get:

$$\frac{e^{\imath(kh(i+j)-\tilde{\omega}n\Delta t^2)}(e^{\imath(-\tilde{\omega}\Delta t^2)} - 2 + e^{\imath(\tilde{\omega}\Delta t^2)})}{\Delta t^2} =$$
$$c^2\left(\frac{e^{\imath(kh(i+j)-\tilde{\omega}n\Delta t^2)}(e^{\imath kh} - 2 + e^{\imath(-kh)})}{h^2} + \frac{e^{\imath(kh(i+j)-\tilde{\omega}n\Delta t^2)}(e^{\imath kh} - 2 + e^{\imath(-kh)})}{h^2}\right)$$
$$\frac{e^{\imath(-\tilde{\omega}\Delta t^2)} - 2 + e^{\imath(\tilde{\omega}\Delta t^2)}}{\Delta t^2} = c^2\left(\frac{e^{\imath kh} - 2 + e^{\imath(-kh)}}{h^2} + \frac{e^{\imath kh} - 2 + e^{\imath(-kh)}}{h^2}\right)$$
$$\frac{2\cos(\tilde{\omega}\Delta t^2) - 2}{\Delta t^2} = c^2\left(\frac{2\cos(kh) - 2}{h^2} + \frac{2\cos(kh) - 2}{h^2}\right)$$
$$\frac{2\cos(\tilde{\omega}\Delta t^2) - 2}{\Delta t^2} = c^2\frac{4\cos(kh) - 4}{h^2}$$
$$(2\cos(\tilde{\omega}\Delta t^2) - 2) = \frac{\Delta t^2 c^2}{h^2}(4\cos(kh) - 4)$$
$$2(\cos(\tilde{\omega}\Delta t^2) - 1) = 4C^2(\cos(kh) - 1)$$
$$\cos(\tilde{\omega}\Delta t^2) - 1 = 2C^2(\cos(kh) - 1)$$
$$\cos(\tilde{\omega}\Delta t^2) - 1 = 2\left(\frac{1}{\sqrt{2}}\right)^2(\cos(kh) - 1)$$
$$\cos(\tilde{\omega}\Delta t^2) - 1 = 2\frac{1}{2}(\cos(kh) - 1)$$
$$\cos(\tilde{\omega}\Delta t^2) - 1 = \cos(kh) - 1$$
$$\tilde{\omega}\Delta t^2 = kh$$
$$\tilde{\omega} = \frac{kh}{\Delta t^2}$$
$$\tilde{\omega} = \frac{kh\sqrt{2}}{\Delta t^2\sqrt{2}}$$
$$\tilde{\omega} = ck\sqrt{2}$$
$$\tilde{\omega} = c\sqrt{2k^2}$$
$$\tilde{\omega} = \omega,\ \forall k_x = k_y = k$$

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