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THE PENETRATION OF RADIATION WITH CONSTANT DRIVING TEMPERATURE

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THE PENETRATION OF RADIATION WITH CONSTANT DRIVING TEMPERATURE

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ABSTRACT

Exact and approximate solutions for the penetration of radiation with constant driving temperature are presented.

1. Scope. We investigate the diffusion equation for radiation under the following assumptions:

(a) no energy in the radiation field and c_v = constant

(b) constant density

(c) an opacity $\kappa = \kappa_0 \rho^{\alpha} T^{-n}$

(d) constant driving temperature

(e) plane geometry.

Many of these assumptions can be generalized or mitigated with some additional complication in the arguments. In only one place is there any real difficulty in generalizing (a) and (c), and this is in the homology transformation (see Section 5). It is possible to make (b) more realistic in some cases. And, as Marshak pointed out a number of years ago,* (d) can be generalized to cases where $T \propto t^p$ or $T \propto e^{pt}$. However, we shall restrict ourselves for the moment to discussion of the diffusion equation with limitations (a)-(e). We have a two-fold purpose in mind: to provide exact solutions for cases of different n and to elaborate a consistent set of analytical approximations to the solution of the diffusion equation. In our experience, assumptions (a)-(e) are remarkably

*R. E. Marshak, Los Alamos Scientific Laboratory Report LA-230, February 1945.

good for many classes of problems, and in any case, most radiation codes can be forced to obey these assumptions so that questions as to the accuracy of the code may be settled. With a view to satisfying the two purposes mentioned, we shall: first, make use of the well-known similarity transformation to reduce the partial differential equation to an ordinary differential equation; second, tabulate exact solutions; third, systematize the hierarchy of "constant flux" approximations; and finally, discuss the accuracy of the first few of these.

We do not claim originality for this work. On the contrary, most of what we are setting forth has been done many times; our aim is simply to make it readily available once and for all.

2. The similarity transformation. In general, the radiation diffusion equation is

$$\frac{\partial E}{\partial t} = \frac{ac}{3} \nabla \cdot \frac{1}{\kappa\rho} \nabla T^4, \tag{1}$$

where E is the internal energy per unit volume of the medium. We may note, moreover, that the flux, $\vec{F}$, is given by

$$\vec{F} = - \frac{ac}{3} \frac{1}{\kappa\rho} \nabla T^4 \tag{2}$$

and that the total internal energy of the wave is

$$E_{tot} = \int E dv, \tag{3}$$

where the range of integration is over the whole volume of the radiation wave.

Under our assumptions, the diffusion equation becomes

$$\rho c_v \frac{\partial T}{\partial t} = \frac{ac}{3} \frac{1}{\kappa_o \rho^{1+\alpha}} \frac{4}{n+4} \frac{\partial^2 T^{n+4}}{\partial x^2}, \tag{4}$$

where x is a geometrical variable. Thus

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T^{n+4}}{\partial x^2}, \tag{5}$$

where

$$K = \frac{ac}{3} \cdot \frac{1}{\kappa_o c_v \rho^{2+\alpha}} \cdot \frac{4}{n+4} . \tag{6}$$

The expressions for the flux and for the total energy in the radiation wave (expressed per unit normal area) become

$$F = -K\rho c_v \frac{\partial T^{4+n}}{\partial x} \tag{7}$$

and

$$E_{tot} = \rho c_v \int_0^{x_o} T dx, \tag{8}$$

where $x = 0$ is the point at which the constant driving temperature is applied, and $x = x_o$ is the head of the wave. It may be remarked in passing that the often-used quantity P, the radiation potential, is defined for a geometrical (as opposed to a mass) coordinate as

$$P = c_v \rho K T^{n+4} . \tag{9}$$

Now we define the similarity variable ξ for $t > 0$ as

$$\xi = A \frac{x}{t^{1/2}} . \tag{10}$$

It is easy to verify that

$$\left.\begin{aligned} \frac{\partial}{\partial t} &= - \frac{1}{2} \frac{\xi}{t} \frac{d}{d\xi} \\ \frac{\partial^2}{\partial x^2} &= \frac{A^2}{t} \frac{d^2}{d\xi^2} . \end{aligned}\right\} \tag{11}$$

At the same time, we find it convenient to express the temperature as a dimensionless variable, τ:

$$\tau = \frac{T(x,t)}{T_o} , \tag{12}$$

where T_o is the constant driving temperature. If we make use of Eqs. (11) and (12) in Eq. (5), we obtain

$$-\xi \frac{d\tau}{d\xi} = \frac{d^2 \tau^m}{d\xi^2} , \qquad t \neq 0 \tag{13}$$

where we have set

$$\left.\begin{aligned} m &= n + 4 \\ A^2 &= \frac{1}{2KT_o^{m-1}} \quad . \end{aligned}\right\} \tag{14}$$

In these new variables, Eqs. (7) and (8) for the flux and the total energy in the wave become

$$F = -c_v\rho\sqrt{\frac{KT_o^{m+1}}{2t}}\;\frac{d\tau^m}{d\xi} \tag{15}$$

and

$$E_{tot} = c_v\rho\sqrt{2KT_o^{m+1}t}\;\mathcal{E} \tag{16}$$

with

$$\mathcal{E} = \int_o^{\xi_o} \tau d\xi, \tag{17}$$

where ξ_o is that value of ξ which is obtained by substituting x_o and t in the definition (10) for ξ. The condition for energy conservation, namely, that

$$F(x = 0,t) = \frac{\partial E_{tot}}{\partial t}$$

is easily seen to be equivalent to

$$\mathcal{E} = - \frac{d\tau^m}{d\xi}\bigg|_{\xi=0}, \tag{18}$$

which can be obtained by differentiating Eq. (16) with respect to time and comparing the result with Eq. (15).

3. Boundary conditions. It seems physically obvious, it is borne out by experience, and it has been shown by Marshak that (except for the pathological -- because linear -- case,* $m = 1$) the boundary conditions which insure that a solution of Eq. (13) be a true radiation wave are

$$\text{(a) } \tau = 1 \text{ at } \xi = 0 \tag{19a}$$

$$\text{(b) at some point } \xi_0,\ \tau = \frac{d\tau^m}{d\xi} = 0. \tag{19b}$$

The first follows from the definition of τ, Eq. (12), the second is the condition which determines the front of the wave, where $T(\xi_0) = 0$ and the flux = 0, simultaneously.

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*The case $m = 1$ is the well-known heat conduction equation. Here the boundary condition (19b) is inapplicable and is replaced by one requiring that $\tau \to 0$ as $\xi \to \infty$. The relevant solution is

$$\tau = 1 - \sqrt{2}\,\Phi(\xi/\sqrt{2}),$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

The energy in the wave is still finite, and in the dimensionless variables is $2/\sqrt{\pi}$.

4. Formulation of the equation for numerical integration. Near $\xi = 0$, Eq. (13) looks like $(d^2\tau^m/d\xi^2) = 0$ and, therefore, τ^m is linear in ξ. Since τ is finite, all powers of τ are well-behaved near the origin. Near $\xi = \xi_o$, Eq. (13) is approximately

$$-\xi_o \frac{d\tau}{d\xi} = \frac{d^2\tau^m}{d\xi^2}, \tag{20}$$

and its asymptotic behavior may be found from the indicial equation. Let

$$\tau = \left[A(\xi_o - \xi)\right]^{1/p} + \ldots .$$

Then Eq. (20) gives, to lowest order,

$$-\xi_o \frac{1}{p} A^{1/p}(\xi_o - \xi)^{(1/p)-1} = \frac{m}{p}\left(\frac{m}{p} - 1\right)A^{m/p}(\xi_o - \xi)^{(m/p)-2}.$$

From this it follows that

$$p = m - 1.$$

This satisfies $(d\tau^m/d\xi) = 0$, which requires only $p < m$.

Since τ is now zero, powers of τ greater than the pth and the ξ axis osculate more and more closely, while powers smaller than the pth have a singular first derivative. A numerical integration will therefore proceed most conveniently after a change of variables. We put

$$\varphi = \tau^{m-1}. \tag{21}$$

Using this in Eq. (13), we obtain the following form, which we use for the numerical integrations:

$$\varphi'' = -\frac{\varphi'}{m\varphi}\left[\xi + \frac{m}{m-1}\varphi'\right]. \tag{22}$$

5. <u>Homology transformation.</u>* The search for ξ_0 will be facilitated by observing that Eq. (22) is invariant to the following transformation. Let us set

$$\left.\begin{aligned} \xi &= \alpha X \\ \varphi &= \beta \Phi \end{aligned}\right\} \tag{23}$$

in Eq. (22) and find the conditions on α and β which render it of the same form in X,Φ as it was in ξ,φ. Now

$$\varphi' = \frac{\beta}{\alpha}\Phi'$$

$$\varphi'' = \frac{\beta}{\alpha^2}\Phi''.$$

And so

*The arguments of this section were called to our attention by Peter Lax.

$$\Phi'' = \frac{\Phi'}{m\Phi}\left[\frac{\alpha^2}{\beta} X + \frac{m}{m - 1} \Phi'\right] . \tag{24}$$

Then if

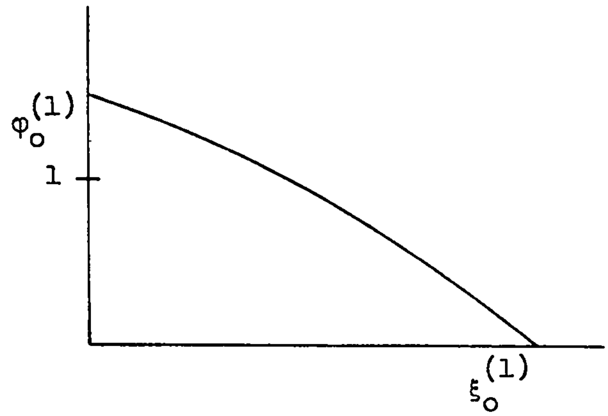
$$\alpha^2 = \beta, \tag{25}$$

Eq. (24) becomes

$$\Phi'' = - \frac{\Phi'}{m\Phi} \left[X + \frac{m}{m - 1} \Phi'\right] , \tag{26}$$

which is identical in form to Eq. (22).

This enables us to meet condition (19a) given an arbitrary solution. For, suppose that we have one integration which progresses backward from $\xi_o^{(1)}$ and crosses the φ-axis at $\varphi_o^{(1)}$, what should we use for $\xi_o^{(2)}$ to arrive at $\varphi_o^{(2)} = 1$?



In Eqs. (23), identify Φ, X with the first trial values of φ and ξ and the left-hand sides with the second:

$$\xi_o^{(2)} = \alpha\, \xi_o^{(1)}$$

$$\varphi_o^{(2)} = \beta\, \varphi_o^{(1)}.$$

The second of these gives

$$\beta = \frac{1}{\varphi_o^{(1)}} .$$

But from Eq. (25),

$$\alpha^2 = \beta, \qquad \alpha = \sqrt{\beta} = \frac{1}{\sqrt{\varphi_o^{(1)}}}$$

and

$$\xi_o^{(2)} = \frac{1}{\sqrt{\varphi_o^{(1)}}}\, \xi_o^{(1)}. \tag{27}$$

6. Numerical solutions. Equation (22) has been integrated by a fourth order Runge-Kutta method for

$$n = 0(0.5)5(1)10$$

$$m = 4(0.5)9(1)14.$$

For each m, two integrations were performed, the first from $\xi_o = 1$ backward to $\xi = 0$. After the first integration, the homology transformation (27) was used to find that ξ_o which would satisfy the boundary condition on φ at the origin.

The step interval for the integration was taken to be $\Delta\xi = -0.05$ except for the first step, which was $-\Delta\xi = \xi_o \bmod(.05)$. Integrations at $\Delta\xi = -0.1$ and $\Delta\xi = -.01$ showed no numerical differences to six significant figures.

Initial values for φ' and φ'' (this because of its indeterminacy at $\xi = \xi_o$) are needed to start the integration. These are obtained from the Taylor series near $\xi = \xi_o$:

$$\varphi = \varphi'(\xi_o - \xi) + \frac{1}{2}\varphi''(\xi_o - \xi)^2 + \dots .$$

Inserting this into Eq. (22), one obtains

$$\varphi'' \cong \frac{\varphi'}{m\varphi(\xi_o - \xi)} \left\{\xi_o - (\xi_o - \xi) - \frac{m}{m - 1}\left[\varphi' + \varphi''(\xi_o - \xi)\right]\right\} ;$$

therefore,

$$\varphi' = \xi_o \frac{m - 1}{m}$$

and

$$\varphi'' = -\frac{m - 1}{m^2} .$$

In Table I and the figures will be found the results of the integrations.

7. The constant flux approximation. It is often convenient to have a flexible, analytical expression for the temperature distribution in a radiation wave. Although our results here will be specifically for Eq. (13), which embodies the fundamental assumptions given on page 5, the method is of far wider applicability. We propose to exhibit a set of increasingly accurate (and complicated) approximate solutions to Eq. (13).

To this end, we shall convert the differential equation to a pair of integral equations. A first integration yields

$$\left.\frac{d\tau^m}{d\xi}\right|_{\xi_1}^{\xi_2} = -\int_{\xi_1}^{\xi_2} \xi \, \frac{d\tau}{d\xi} \cdot d\xi. \tag{28}$$

Particularizing with

$$\xi_1 = \xi, \qquad \xi_2 = \xi_0,$$

we find

$$\left.\frac{d\tau^m}{d\xi}\right|_{\xi} = \int_{\xi}^{\xi_0} \xi \, \frac{d\tau}{d\xi} \cdot d\xi. \tag{29}$$

It is a tautology to note that

$$\tau^m \Big|_{\xi}^{\xi_o} = \int_{\xi}^{\xi_o} \frac{d\tau^m}{d\xi}\, d\xi .$$

Hence, we have

$$\tau^m(\xi) = -\int_{\xi}^{\xi_o} \frac{d\tau^m}{d\xi} \cdot d\xi . \tag{30}$$

The scheme for iteration rests on Eqs. (29) and (30), applied successively. There is a certain amount of freedom allowable in setting up (and naming) a consistent approximation scheme. Our choice, which follows tradition and which has the advantage of automatic energy conservation, is the following:

(a) pick a temperature distribution which obeys the boundary conditions (19) but which is of arbitrary length, i.e., is a function of ξ/ξ_o with ξ_o as yet unspecified

(b) obtain $\frac{d\tau^m}{d\xi}$ (which is proportional to the flux) by inserting the above approximation in the right-hand side of Eq. (29)

(c) obtain $\tau^m(\xi)$ by inserting the above approximation to the derivative in the right-hand side of Eq. (30)

(d) determine ξ_o by the condition

$$\tau^m(0) = 1 \tag{31}$$

(e) to improve, use the mth root of the above-mentioned expression for $\tau^m(\xi)$ as the function referred to in (a).

The question is: does this iteration converge on the correct solution? We are in a position to answer this pragmatically, since we can compare it with the exact solution of Eq. (13), subject to the boundary conditions (19). In point of fact, only the first few iterations are of practical interest, since a) they turn out to be very good and b) the complexity of the higher iterates is sufficiently great that one may as well turn to the exact solution itself. We have neither the interest, the time, nor the erudition to satisfy prurient mathematical curiosity as to the rate and cause of convergence of the method sketched. It appears to work.

The zeroth approximation. We start by taking for our initial temperature distribution:

$$\left.\begin{aligned} \tau^{(0)} &= 1, \qquad \xi \leq \xi_0', \\ &= 0, \qquad \xi > \xi_0. \end{aligned}\right\} \tag{32}$$

Then if we approximate this step function by a continuous function whose limit is the above, substitute into Eq. (29) and pass to the limit after performing the integration, we find

$$\left[\frac{d\tau^m}{d\xi}\right]^{(0)} = -\xi_o\tau = -\xi_o, \qquad \xi \leq \xi_o. \tag{33}$$

Thus we see that the flux in this order of approximation is constant behind the head of the wave. Making use of Eq. (30) — step (c) — we have

$$\left[\tau^m(\xi)\right]^{(0)} = \xi_o(\xi_o - \xi)$$

$$= \xi_o^2(1 - \xi/\xi_o). \tag{34}$$

Then (step (d))

$$\left[\tau^m(0)\right]^{(0)} = \xi_o^2 ,$$

whence

$$\xi_o^{(0)} = 1. \tag{35}$$

In passing, we note that

$$-\left[\frac{d\tau^m}{d\xi}\right]^{(0)} = \mathcal{E}^{(0)} = 1. \tag{36}$$

Equations (32), (34), (35), and (36) completely specify the zeroth approximation.

Comments on the approximation scheme. To one coming upon the constant flux approximation for the first time and viewing the matter from a mathematical rather than a physical point of view, the construction of the zeroth approximation may look somewhat artificial. If we recall the form of the exact solution, we note first of all that $(d\tau^m/d\xi)$ is indeed surprisingly constant over a large portion of the wave. The temperature, on the other hand, remains quite close to the driving temperature for a good portion of the length of the wave. It may also seem inconsistent that

$$\tau^{(1)}(\xi) \neq \left\{\left[\tau^m(\xi)\right]^{(1)}\right\}^{1/m}.$$

However, the nature of an approximate solution is that it not be exact, i.e., that there be an inconsistency somewhere. The scheme which we present here has the characteristic that the flux, a derivative quantity, comes from a somewhat more sophisticated expression for $\tau^m(\xi)$ than the temperature distribution itself. This is in the spirit of a more balanced approximation as is evidenced by the fact that with this nomenclature, energy is identically conserved.

The one-halfth approximation. Before returning to our program, we present an approximation which is sometimes used. We shall call it the one-halfth approximation, since it turns out that it is intermediate in accuracy between our zeroth and our first approximation. It also illustrates another consistent set of energy-preserving approximations.

Using Eq. (34) and mindful of Eq. (35), we may write

$$\tau^m(\xi) = 1 - \xi/\xi_o . \tag{37}$$

Then

$$\frac{d\tau^m}{d\xi} = -\frac{1}{\xi_o} . \tag{38}$$

We evaluate $\mathcal{E}$ by taking τ as the mth root of Eq. (37):

$$\mathcal{E} = \int_0^{\xi_o} \tau d\xi = \int_0^{\xi_o} (1 - \xi/\xi_o)^{1/m} d\xi$$

$$\mathcal{E}^{(1/2)} = \xi_o \frac{m}{m + 1} . \tag{39}$$

Using Eqs. (38) and (39) in Eq. (18), we find that for energy conservation

$$\xi_o^{(1/2)} = \sqrt{\frac{m + 1}{m}} , \tag{40}$$

whence

$$\mathcal{E}^{(1/2)} = \frac{d\tau^m}{d\xi}\bigg|_{\xi=0}^{(1/2)} = \sqrt{\frac{m}{m + 1}} . \tag{41}$$

We see that the philosophy used here is to take a temperature distribution and use the <u>differential</u> rather than the <u>integral</u> representation for the flux, obtaining ξ_o by enforcing energy conservation.

We have tabulated the results of the one-halfth approximation in Tables II and III, but we have not carried out the three-halfth or higher approximations.

The first approximation. In accordance with our enunciated scheme, we take

$$\tau^{(1)}(\xi) = (1 - \xi/\xi_0)^{1/m}. \tag{42}$$

Before we continue with this approximation, we shall find it convenient to make some transformations. Thus Eq. (29) when integrated by parts gives

$$\frac{d\tau^m}{d\xi} = -\xi\tau - \int_\xi^{\xi_0} \tau d\xi. \qquad (43)^*$$

Another useful transformation comes about by introducing the variable

$$y = 1 - \xi/\xi_0. \tag{44}$$

Then any integral of the form

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*We now see how energy conservation is guaranteed. For

$$-\left.\frac{d\tau^m}{d\xi}\right|_{\xi=0} = 0 \cdot 1 + \int_0^{\xi_0} t d\xi = \mathcal{E},$$

and this is the condition, Eq. (18), that energy be conserved.

$$I = \int_{\xi}^{\xi_0} f(1 - \xi/\xi_0)d\xi$$

$$= -\xi_0 \int_{1-\xi/\xi_0}^{0} f(y)dy$$

or

$$I = \xi_0 \int_0^{1-\xi/\xi_0} f(y)dy. \tag{45}$$

Then, making use of Eqs. (29), (42), and (45),

$$\left(\frac{d\tau^m}{d\xi}\right)^{(1)} = -\xi_0 \left[y^{1/m} - \frac{1}{m+1} y^{(m+1)/m}\right]. \tag{46}$$

Integrating once more

$$\left[\tau^m(\xi)\right]^{(1)} = -\xi_0^2 \left[\int_0^{1-\xi/\xi_0} y^{1/m}dy - \frac{1}{m+1}\int_0^{1-\xi/\xi_0} y^{(m+1)/m}dy\right]$$

$$\left[\tau^m(\xi)\right]^{(1)} = \xi_0^2 \frac{m}{m+1} y^{(m+1)/m} \left(1 - \frac{1}{2m+1} y\right). \tag{47}$$

Then, putting $\xi = 0$ to determine ξ_0, we find

$$\xi_0^{(1)} = \frac{\sqrt{(m+1)(m+1/2)}}{m}. \tag{48}$$

Using this in Eq. (46) for y = 1, which corresponds to $\xi = 0$, we obtain

$$g^{(1)} = - \left. \frac{d\tau^m}{d\xi} \right|_{\xi=0}^{(1)} = \sqrt{\frac{m + 1/2}{m + 1}} \; . \tag{49}$$

Equations (42), (47), (48), and (49) define the first approximation.

<u>The second approximation</u>. According to our scheme,

$$\tau^{(2)}(\xi) = \left\{ \left[\tau^m(\xi) \right]^{(1)} \right\}^{1/m} \tag{50}$$

or

$$\tau^{(2)}(\xi) = \left(\frac{m + 1/2}{m} \right)^{1/m} y^{(m+1)/m^2} \left(1 - \frac{1}{2m + 1} y \right)^{1/m} . \tag{51}$$

We find Eq. (51), the integrand necessary for proceeding with our second approximation, intractable in closed form. However, the quantity $y/(2m+1)$ has an upper bound of 1/9 for all cases of physical interest. Hence, a binomial expansion of the third factor of Eq. (51) appears attractive; carrying out the expansion, we find

$$\left. \begin{aligned} \tau^{(2)} &= \left(\frac{m + 1/2}{m} \right)^{1/m} \sum_{\nu=0}^{\infty} \eta_\nu y^{(m+1)/m^2 + \nu} \\ \eta_0 &= 1 \\ \eta_\nu &= \frac{1}{(2m + 1)^\nu \nu!} \prod_{\mu=0}^{\nu-1} \left(\mu - \frac{1}{m}\right), \qquad \nu \neq 0. \end{aligned} \right\} \tag{52}$$

As a computational aid, we notice that

$$\eta_{\nu+1} = \frac{\nu - (1/m)}{(2m + 1)(\nu + 1)} \eta_\nu. \tag{53}$$

Substituting Eq. (51) into Eq. (29) and making use of Eq. (45), we find

$$\frac{d\tau^m}{d\xi} = -\xi\tau^{(2)} - \xi_o \int_0^{1-\xi/\xi_o} \tau^{(2)} dy.$$

This becomes, after some reductions,

$$\left(\frac{d\tau^m}{d\xi}\right)^{(2)} = -\xi_o \left(\frac{m + 1/2}{m}\right)^{1/m} \sum_{\nu=0}^{\infty} \left[\frac{1 - (\beta + \nu)}{\beta + \nu} \eta_{\nu-1} + \eta_\nu\right] y^{\beta+\nu}, \tag{54}$$

where we have set

$$\left.\begin{aligned} \eta_{-1} &= 0 \\ \beta &= \frac{m + 1}{m^2} . \end{aligned}\right\} \tag{55}$$

The final integration gives us

$$\left[\tau^m(\xi)\right]^{(2)} = \xi_o^2 \left(\frac{m + 1/2}{m}\right)^{1/m} \sum_{\nu=0}^{\infty} \frac{1}{\beta + \nu + 1}$$

$$\cdot \left[\frac{1 - (\beta + \nu)}{\beta + \nu} \eta_{\nu-1} + \eta_\nu\right] y^{\beta+\nu+1}. \tag{56}$$

It is necessary to evaluate this expression and its derivative numerically for $\xi = 0$ to find the values of $\xi_o^{(2)}$ and $\varepsilon^{(2)}$. Actually, the series converges rapidly even in the least favorable case of $m = 4$.*
Equations (51), (54), and (56), together with

$$\left[\tau^m(0)\right]^{(2)} = 1$$

and

$$\varepsilon^{(2)} = \left.\frac{d\tau^m}{d\xi}\right|_{\xi=0} ,$$

define the second approximation.

8. Accuracy of the approximations. The results of the approximations are given in Tables II and III.

Except for the one-halfth approximation, the relative errors $\xi_o^{(i)}/\xi_o - 1$ are roughly proportional to m^{-i-1} as are the relative errors in $\varepsilon_o^{(i)}$. This illustrates that the scheme furnishes in some sense an expansion in powers of $1/m$. At $n = 0$, the errors in the $\xi_o^{(i)}$'s are 20, 4, and 0.8%, respectively, and in the $\varepsilon_o^{(i)}$'s, 6, 0.8, and 0.2%, respectively.

It is no surprise that the one-halfth approximation does not give errors proportional to $m^{-3/2}$. In fact, the error for $\xi_o^{(1/2)}$ is about proportional to $m^{-1.2}$ and that for $\varepsilon_o^{(1/2)}$ to $m^{-0.85}$. However, for $0 < n < 10$, the errors of the one-halfth approximation always lie, in absolute value, between those for the zeroth and first.

*Five terms give five significant figures in $\xi_o^{(2)}$.

The functions τ and τ^m for n = 0, 3, and 10 are given in the figures, together with such of the approximations as are sufficiently distinct from them to be drawn. It can be seen that τ approaches $\tau^{(0)}$ and the flux becomes more constant as n grows.

The successive approximations approach τ monotonely in the sense that $\xi_o^{(i)}$ approaches ξ_o from below while for values of ξ sufficiently less than ξ_o, $\tau^{(i)}$ approaches τ from above.

TABLE I

$n = 0.00 \quad m = 4.00 \quad \xi_0 = 1.231173 \quad \mathcal{E} = 0.940689$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.940689	1.000000	-0.705517
0.05	0.988029	0.952971	-0.940388	0.964516	-0.713836
0.10	0.975615	0.905972	-0.939455	0.928616	-0.722202
0.15	0.962726	0.859036	-0.937842	0.892295	-0.730614
0.20	0.949327	0.812200	-0.935494	0.855554	-0.739072
0.25	0.935376	0.765500	-0.932353	0.818387	-0.747576
0.30	0.920829	0.718979	-0.928350	0.780795	-0.756126
0.35	0.905633	0.672681	-0.923409	0.742774	-0.764721
0.40	0.889728	0.626655	-0.917441	0.704322	-0.773361
0.45	0.873043	0.580956	-0.910347	0.665437	-0.782046
0.50	0.855497	0.535641	-0.902008	0.626117	-0.790776
0.55	0.836992	0.490778	-0.892289	0.586359	-0.799550
0.60	0.817411	0.446438	-0.881025	0.546161	-0.808368
0.65	0.796611	0.402704	-0.868020	0.505521	-0.817230
0.70	0.774419	0.359669	-0.853034	0.464437	-0.826136
0.75	0.750611	0.317439	-0.835766	0.422907	-0.835086
0.80	0.724905	0.276137	-0.815835	0.380928	-0.844078
0.85	0.696924	0.235908	-0.792741	0.338499	-0.853113
0.90	0.666156	0.196927	-0.765806	0.295616	-0.862192
0.95	0.631869	0.159407	-0.734073	0.252279	-0.871312
1.00	0.592959	0.123623	-0.696114	0.208484	-0.880475
1.05	0.547627	0.089937	-0.649616	0.164231	-0.889679
1.10	0.492578	0.058871	-0.590388	0.119516	-0.898926
1.15	0.420470	0.031257	-0.509169	0.074337	-0.908214
1.20	0.306146	0.008784	-0.374535	0.028694	-0.917543
	0.000000	0.000000	0.000000	0.000000	-0.923380

$n = 0.50 \quad m = 4.50 \quad \xi_0 = 1.200171 \quad \mathcal{E} = 0.946970$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.946970	1.000000	-0.736532
0.05	0.989280	0.952656	-0.946700	0.962979	-0.744301
0.10	0.978143	0.905339	-0.945863	0.925569	-0.752110
0.15	0.966559	0.858080	-0.944413	0.887768	-0.759957
0.20	0.954490	0.810909	-0.942299	0.849573	-0.767843
0.25	0.941897	0.763862	-0.939463	0.810983	-0.775768
0.30	0.928731	0.716976	-0.935840	0.771995	-0.783731
0.35	0.914939	0.670292	-0.931355	0.732609	-0.791733
0.40	0.900458	0.623856	-0.925921	0.692821	-0.799772
0.45	0.885213	0.577718	-0.919439	0.652631	-0.807850
0.50	0.869119	0.531932	-0.911790	0.612036	-0.815965
0.55	0.852069	0.486560	-0.902835	0.571034	-0.824118
0.60	0.833938	0.441673	-0.892405	0.529624	-0.832308
0.65	0.814567	0.397348	-0.880293	0.487803	-0.840535
0.70	0.793762	0.353676	-0.866242	0.445569	-0.848799
0.75	0.771269	0.310761	-0.849928	0.402922	-0.857100
0.80	0.746759	0.268728	-0.830923	0.359859	-0.865438
0.85	0.719783	0.227723	-0.808657	0.316378	-0.873812
0.90	0.689709	0.187930	-0.782327	0.272477	-0.882222
0.95	0.655597	0.149578	-0.750754	0.228155	-0.890669
1.00	0.615955	0.112972	-0.712076	0.183410	-0.899152
1.05	0.568154	0.078541	-0.663038	0.138239	-0.907670
1.10	0.506760	0.046947	-0.596964	0.092642	-0.916223
1.15	0.416468	0.019414	-0.495199	0.046616	-0.924813
1.20	0.082325	0.000013	-0.098801	0.000160	-0.933437
	0.000000	0.000000	0.000000	0.000000	-0.933467

TABLE I (cont.)

$n = 1.00 \quad m = 5.00 \quad \xi_0 = 1.176525 \quad \mathcal{E} = 0.952049$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.952049	1.000000	-0.761639
0.05	0.990294	0.952402	-0.951805	0.961736	-0.768907
0.10	0.980197	0.904828	-0.951046	0.923109	-0.776208
0.15	0.969677	0.857306	-0.949729	0.884115	-0.783543
0.20	0.958700	0.809865	-0.947806	0.844754	-0.790910
0.25	0.947223	0.762537	-0.945222	0.805024	-0.798310
0.30	0.935200	0.715355	-0.941913	0.764922	-0.805743
0.35	0.922575	0.668359	-0.937807	0.724449	-0.813208
0.40	0.909286	0.621589	-0.932821	0.683601	-0.820706
0.45	0.895257	0.575093	-0.926855	0.642378	-0.828236
0.50	0.880396	0.528922	-0.919793	0.600777	-0.835799
0.55	0.864597	0.483134	-0.911494	0.558797	-0.843393
0.60	0.847724	0.437796	-0.901787	0.516437	-0.851020
0.65	0.829611	0.392982	-0.890461	0.473695	-0.858678
0.70	0.810048	0.348781	-0.877249	0.430569	-0.866368
0.75	0.788758	0.305295	-0.861806	0.387057	-0.874090
0.80	0.765374	0.262645	-0.843674	0.343159	-0.881842
0.85	0.739386	0.220982	-0.822222	0.298873	-0.889627
0.90	0.710055	0.180493	-0.796542	0.254196	-0.897442
0.95	0.676243	0.141421	-0.765244	0.209128	-0.905288
1.00	0.636048	0.104100	-0.726021	0.163667	-0.913165
1.05	0.585863	0.069021	-0.674529	0.117811	-0.921073
1.10	0.517209	0.037011	-0.600616	0.071559	-0.929011
1.15	0.397274	0.009896	-0.465297	0.024909	-0.936980
	0.000000	0.000000	0.000000	0.000000	-0.941220

$n = 1.50 \quad m = 5.50 \quad \xi_0 = 1.157887 \quad \mathcal{E} = 0.956241$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.956241	1.000000	-0.782379
0.05	0.991132	0.952192	-0.956018	0.960711	-0.789195
0.10	0.981897	0.904406	-0.955324	0.921080	-0.796039
0.15	0.972264	0.856668	-0.954118	0.881106	-0.802912
0.20	0.962196	0.809004	-0.952355	0.840788	-0.809813
0.25	0.951655	0.761442	-0.949981	0.800125	-0.816742
0.30	0.940593	0.714017	-0.946936	0.759114	-0.823700
0.35	0.928955	0.666761	-0.943151	0.717754	-0.830686
0.40	0.916678	0.619715	-0.938545	0.676045	-0.837699
0.45	0.903686	0.572922	-0.933020	0.633984	-0.844741
0.50	0.889887	0.526430	-0.926462	0.591570	-0.851810
0.55	0.875170	0.480295	-0.918731	0.548802	-0.858907
0.60	0.859398	0.434580	-0.909658	0.505679	-0.866031
0.65	0.842398	0.389355	-0.899028	0.462199	-0.873183
0.70	0.823949	0.344707	-0.886567	0.418360	-0.880362
0.75	0.803756	0.300735	-0.871920	0.374162	-0.887569
0.80	0.781424	0.257560	-0.854603	0.329603	-0.894802
0.85	0.756391	0.215330	-0.833938	0.284681	-0.902063
0.90	0.727823	0.174238	-0.808924	0.239396	-0.909351
0.95	0.694396	0.134536	-0.777979	0.193746	-0.916665
1.00	0.653788	0.096584	-0.738350	0.147729	-0.924006
1.05	0.601266	0.060935	-0.684449	0.101345	-0.931374
1.10	0.524036	0.028608	-0.601270	0.054591	-0.938769
1.15	0.336800	0.002515	-0.389493	0.007467	-0.946189
	0.000000	0.000000	0.000000	0.000000	-0.947362

TABLE I (cont.)

$n = 2.00 \quad m = 6.00 \quad \xi_o = 1.142817 \quad \mathcal{E} = 0.959760$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.959760	1.000000	-0.799800
0.05	0.991838	0.952015	-0.959555	0.959850	-0.806209
0.10	0.983329	0.904052	-0.958915	0.919379	-0.812643
0.15	0.974444	0.856132	-0.957803	0.878585	-0.819102
0.20	0.965147	0.808280	-0.956174	0.837468	-0.825586
0.25	0.955401	0.760524	-0.953979	0.796026	-0.832094
0.30	0.945157	0.712893	-0.951160	0.754258	-0.838626
0.35	0.934364	0.665419	-0.947650	0.712163	-0.845183
0.40	0.922956	0.618140	-0.943369	0.669740	-0.851764
0.45	0.910860	0.571097	-0.938225	0.626986	-0.858370
0.50	0.897983	0.524334	-0.932105	0.583902	-0.864999
0.55	0.884213	0.477905	-0.924872	0.540486	-0.871653
0.60	0.869411	0.431868	-0.916356	0.496737	-0.878330
0.65	0.853401	0.386294	-0.906344	0.452653	-0.885032
0.70	0.835953	0.341263	-0.894560	0.408233	-0.891758
0.75	0.816761	0.296874	-0.880639	0.363476	-0.898507
0.80	0.795407	0.253243	-0.864079	0.318382	-0.905280
0.85	0.771287	0.210521	-0.844167	0.272948	-0.912076
0.90	0.743484	0.168900	-0.819821	0.227174	-0.918896
0.95	0.710499	0.128642	-0.789284	0.181058	-0.925739
1.00	0.669590	0.090126	-0.749356	0.134600	-0.932606
1.05	0.614746	0.053973	-0.693062	0.087797	-0.939496
1.10	0.527001	0.021422	-0.598510	0.040650	-0.946409
	0.000000	0.000000	0.000000	0.000000	-0.952348

$n = 2.50 \quad m = 6.50 \quad \xi_o = 1.130378 \quad \mathcal{E} = 0.962756$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.962756	1.000000	-0.814639
0.05	0.992439	0.951865	-0.962565	0.959117	-0.820683
0.10	0.984551	0.903750	-0.961972	0.917931	-0.826749
0.15	0.976306	0.855675	-0.960940	0.876442	-0.832836
0.20	0.967671	0.807664	-0.959427	0.834647	-0.838945
0.25	0.958608	0.759742	-0.957386	0.792547	-0.845076
0.30	0.949071	0.711935	-0.954761	0.750139	-0.851227
0.35	0.939007	0.664276	-0.951488	0.707424	-0.857401
0.40	0.928356	0.616799	-0.947491	0.664399	-0.863595
0.45	0.917040	0.569541	-0.942679	0.621064	-0.869811
0.50	0.904971	0.522546	-0.936943	0.577418	-0.876048
0.55	0.892035	0.475864	-0.930148	0.533459	-0.882306
0.60	0.878094	0.429552	-0.922127	0.489187	-0.888586
0.65	0.862967	0.383675	-0.912668	0.444600	-0.894886
0.70	0.846423	0.338313	-0.901494	0.399698	-0.901207
0.75	0.828147	0.293560	-0.888236	0.354479	-0.907549
0.80	0.807701	0.249533	-0.872380	0.308942	-0.913912
0.85	0.784447	0.206378	-0.853182	0.263087	-0.920296
0.90	0.757398	0.164289	-0.829496	0.216912	-0.926700
0.95	0.724895	0.123534	-0.799403	0.170417	-0.933125
1.00	0.683773	0.084514	-0.759263	0.123600	-0.939570
1.05	0.626596	0.047909	-0.700561	0.076459	-0.946036
1.10	0.525320	0.015232	-0.591357	0.028996	-0.952523
	0.000000	0.000000	0.000000	0.000000	-0.956474

TABLE I (cont.)

$n = 3.00 \quad m = 7.00 \quad \xi_o = 1.119935 \quad \mathcal{E} = 0.965336$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.965336	1.000000	-0.827431
0.05	0.992958	0.951736	-0.965159	0.958486	-0.833146
0.10	0.985606	0.903490	-0.964606	0.916685	-0.838880
0.15	0.977916	0.855282	-0.963643	0.874597	-0.844633
0.20	0.969854	0.807134	-0.962231	0.832221	-0.850405
0.25	0.961385	0.759067	-0.960323	0.789556	-0.856197
0.30	0.952463	0.711110	-0.957868	0.746601	-0.862007
0.35	0.943038	0.663291	-0.954803	0.703355	-0.867836
0.40	0.933048	0.615641	-0.951054	0.659818	-0.873684
0.45	0.922420	0.568198	-0.946534	0.615987	-0.879551
0.50	0.911063	0.521003	-0.941137	0.571862	-0.885436
0.55	0.898868	0.474101	-0.934730	0.527443	-0.891340
0.60	0.885694	0.427549	-0.927151	0.482728	-0.897263
0.65	0.871362	0.381409	-0.918188	0.437716	-0.903205
0.70	0.855637	0.335758	-0.907567	0.392407	-0.909165
0.75	0.838198	0.290686	-0.894916	0.346799	-0.915143
0.80	0.818594	0.246309	-0.879713	0.300892	-0.921140
0.85	0.796160	0.202770	-0.861191	0.254685	-0.927155
0.90	0.769849	0.160264	-0.838150	0.208177	-0.933189
0.95	0.737851	0.119064	-0.808523	0.161366	-0.939240
1.00	0.696590	0.079587	-0.768243	0.114252	-0.945310
1.05	0.637040	0.042576	-0.707092	0.066835	-0.951398
1.10	0.517073	0.009882	-0.577616	0.019112	-0.957505
	0.000000	0.000000	0.000000	0.000000	-0.959944

$n = 3.50 \quad m = 7.50 \quad \xi_o = 1.111043 \quad \mathcal{E} = 0.967582$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.967582	1.000000	-0.838571
0.05	0.993410	0.951624	-0.967416	0.957936	-0.843989
0.10	0.986526	0.903264	-0.966899	0.915601	-0.849424
0.15	0.979320	0.854940	-0.965997	0.872993	-0.854876
0.20	0.971761	0.806672	-0.964672	0.830113	-0.860344
0.25	0.963813	0.758481	-0.962882	0.786959	-0.865830
0.30	0.955432	0.710392	-0.960576	0.743530	-0.871332
0.35	0.946569	0.662433	-0.957693	0.699825	-0.876852
0.40	0.937164	0.614633	-0.954164	0.655844	-0.882388
0.45	0.927144	0.567029	-0.949903	0.611586	-0.887941
0.50	0.916422	0.519657	-0.944807	0.567050	-0.893510
0.55	0.904888	0.472564	-0.938747	0.522235	-0.899096
0.60	0.892403	0.425801	-0.931564	0.477140	-0.904699
0.65	0.878788	0.379430	-0.923050	0.431765	-0.910318
0.70	0.863807	0.333523	-0.912931	0.386108	-0.915954
0.75	0.847136	0.288169	-0.900837	0.340169	-0.921606
0.80	0.828314	0.243481	-0.886240	0.293947	-0.927274
0.85	0.806655	0.199600	-0.868358	0.247441	-0.932959
0.90	0.781058	0.156720	-0.845940	0.200651	-0.938661
0.95	0.749581	0.115117	-0.816794	0.153575	-0.944378
1.00	0.708241	0.075224	-0.776433	0.106213	-0.950112
1.05	0.646255	0.037847	-0.712766	0.058563	-0.955862
1.10	0.497012	0.005281	-0.551470	0.010626	-0.961628
	0.000000	0.000000	0.000000	0.000000	-0.962904

TABLE I (cont.)

$n = 4.00 \quad m = 8.00 \quad \xi_o = 1.103380 \quad \mathcal{E} = 0.969555$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.969555	1.000000	-0.848361
0.05	0.993808	0.951525	-0.969399	0.957453	-0.853509
0.10	0.987336	0.903066	-0.968913	0.914649	-0.858673
0.15	0.980557	0.854640	-0.968064	0.871586	-0.863852
0.20	0.973441	0.806266	-0.966817	0.828263	-0.869046
0.25	0.965953	0.757965	-0.965131	0.784681	-0.874255
0.30	0.958052	0.709761	-0.962956	0.740838	-0.879479
0.35	0.949688	0.661679	-0.960236	0.696733	-0.884719
0.40	0.940803	0.613747	-0.956902	0.652365	-0.889973
0.45	0.931327	0.566000	-0.952872	0.607735	-0.895242
0.50	0.921172	0.518474	-0.948045	0.562841	-0.900526
0.55	0.910232	0.471211	-0.942298	0.517682	-0.905825
0.60	0.898368	0.424262	-0.935472	0.472258	-0.911139
0.65	0.885403	0.377685	-0.927364	0.426568	-0.916468
0.70	0.871102	0.331551	-0.917704	0.380611	-0.921811
0.75	0.855137	0.285946	-0.906122	0.334387	-0.927170
0.80	0.837043	0.240980	-0.892089	0.287894	-0.932543
0.85	0.816114	0.196791	-0.874809	0.241132	-0.937930
0.90	0.791206	0.153574	-0.852995	0.194101	-0.943333
0.95	0.760257	0.111605	-0.824335	0.146799	-0.948750
1.00	0.718887	0.071332	-0.783941	0.099226	-0.954181
1.05	0.654377	0.033622	-0.717667	0.051380	-0.959627
1.10	0.441361	0.001440	-0.486803	0.003263	-0.965088
	0.000000	0.000000	0.000000	0.000000	-0.965457

$n = 4.50 \quad m = 8.50 \quad \xi_o = 1.096707 \quad \mathcal{E} = 0.971302$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.971302	1.000000	-0.857031
0.05	0.994160	0.951437	-0.971155	0.957026	-0.861935
0.10	0.988054	0.902890	-0.970696	0.913806	-0.866852
0.15	0.981654	0.854373	-0.969895	0.870340	-0.871783
0.20	0.974933	0.805907	-0.968717	0.826628	-0.876727
0.25	0.967855	0.757509	-0.967123	0.782668	-0.881686
0.30	0.960381	0.709202	-0.965066	0.738459	-0.886657
0.35	0.952463	0.661011	-0.962490	0.694002	-0.891642
0.40	0.944044	0.612963	-0.959331	0.649295	-0.896641
0.45	0.935055	0.565089	-0.955508	0.604337	-0.901653
0.50	0.925412	0.517424	-0.950925	0.559129	-0.906679
0.55	0.915007	0.470011	-0.945459	0.513669	-0.911718
0.60	0.903707	0.422896	-0.938957	0.467957	-0.916771
0.65	0.891334	0.376136	-0.931219	0.421992	-0.921836
0.70	0.877654	0.329799	-0.921979	0.375773	-0.926915
0.75	0.862341	0.283969	-0.910870	0.329300	-0.932008
0.80	0.844923	0.238752	-0.897361	0.282572	-0.937114
0.85	0.824683	0.194286	-0.880649	0.235588	-0.942233
0.90	0.800439	0.150762	-0.859416	0.188349	-0.947365
0.95	0.770019	0.108459	-0.831245	0.140852	-0.952510
1.00	0.728660	0.067836	-0.790857	0.093097	-0.957668
1.05	0.661511	0.029824	-0.721853	0.045085	-0.962840
	0.000000	0.000000	0.000000	0.000000	-0.967683

TABLE I (cont.)

$n = 5.00 \quad m = 9.00 \quad \xi_o = 1.090844 \quad \varepsilon = 0.972859$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.972859	1.000000	-0.864764
0.05	0.994475	0.951359	-0.972720	0.956645	-0.869444
0.10	0.988695	0.902733	-0.972286	0.913055	-0.874136
0.15	0.982634	0.854136	-0.971527	0.869231	-0.878841
0.20	0.976266	0.805586	-0.970411	0.825171	-0.883558
0.25	0.969556	0.757102	-0.968900	0.780875	-0.888288
0.30	0.962465	0.708704	-0.966948	0.736342	-0.893029
0.35	0.954948	0.660415	-0.964503	0.691572	-0.897783
0.40	0.946949	0.612263	-0.961501	0.646564	-0.902549
0.45	0.938400	0.564276	-0.957866	0.601317	-0.907328
0.50	0.929218	0.516488	-0.953502	0.555831	-0.912118
0.55	0.919300	0.468939	-0.948291	0.510105	-0.916921
0.60	0.908513	0.421676	-0.942084	0.464138	-0.921736
0.65	0.896681	0.374751	-0.934685	0.417931	-0.926562
0.70	0.883573	0.328231	-0.925831	0.371482	-0.931401
0.75	0.868861	0.282198	-0.915158	0.324791	-0.936252
0.80	0.852075	0.236754	-0.902138	0.277857	-0.941115
0.85	0.832484	0.192037	-0.885962	0.230679	-0.945990
0.90	0.808877	0.148233	-0.865286	0.183257	-0.950877
0.95	0.778985	0.105623	-0.837602	0.135591	-0.955776
1.00	0.737670	0.064679	-0.797254	0.087680	-0.960687
1.05	0.667736	0.026390	-0.725369	0.039522	-0.965609
	0.000000	0.000000	0.000000	0.000000	-0.969640

$n = 6.00 \quad m = 10.00 \quad \xi_o = 1.081022 \quad \varepsilon = 0.975517$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.975517	1.000000	-0.877965
0.05	0.995012	0.951226	-0.975391	0.955995	-0.882252
0.10	0.989790	0.902465	-0.974998	0.911775	-0.886550
0.15	0.984311	0.853731	-0.974312	0.867339	-0.890858
0.20	0.978547	0.805040	-0.973302	0.822689	-0.895176
0.25	0.972469	0.756407	-0.971933	0.777822	-0.899505
0.30	0.966038	0.707853	-0.970163	0.732738	-0.903843
0.35	0.959212	0.659398	-0.967943	0.687437	-0.908192
0.40	0.951938	0.611067	-0.965213	0.641919	-0.912551
0.45	0.944152	0.562887	-0.961902	0.596182	-0.916920
0.50	0.935775	0.514888	-0.957920	0.550227	-0.921299
0.55	0.926705	0.467108	-0.953156	0.504052	-0.925688
0.60	0.916816	0.419588	-0.947466	0.457658	-0.930087
0.65	0.905938	0.372380	-0.940662	0.411043	-0.934496
0.70	0.893843	0.325544	-0.932492	0.364208	-0.938916
0.75	0.880208	0.279159	-0.922600	0.317151	-0.943345
0.80	0.864561	0.233322	-0.910463	0.269873	-0.947784
0.85	0.846162	0.188163	-0.895271	0.222373	-0.952234
0.90	0.823752	0.143868	-0.875642	0.174650	-0.956693
0.95	0.794895	0.100716	-0.848914	0.126703	-0.961162
1.00	0.753751	0.059194	-0.808726	0.078533	-0.965641
1.05	0.677664	0.020424	-0.730469	0.030139	-0.970130
	0.000000	0.000000	0.000000	0.000000	-0.972920

TABLE I (cont.)

$n = 7.00 \quad m = 11.00 \quad \xi_o = 1.073118 \quad \mathcal{E} = 0.977700$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.977700	1.000000	-0.888818
0.05	0.995454	0.951117	-0.977585	0.955460	-0.892772
0.10	0.990692	0.902246	-0.977227	0.910723	-0.896735
0.15	0.985692	0.853399	-0.976601	0.865787	-0.900707
0.20	0.980428	0.804590	-0.975679	0.820652	-0.904687
0.25	0.974873	0.755836	-0.974428	0.775318	-0.908676
0.30	0.968990	0.707154	-0.972809	0.729784	-0.912673
0.35	0.962740	0.658562	-0.970775	0.684050	-0.916679
0.40	0.956071	0.610084	-0.968273	0.638116	-0.920693
0.45	0.948923	0.561744	-0.965233	0.591981	-0.924716
0.50	0.941220	0.513571	-0.961571	0.545644	-0.928748
0.55	0.932866	0.465599	-0.957183	0.499106	-0.932788
0.60	0.923738	0.417867	-0.951931	0.452365	-0.936836
0.65	0.913673	0.370423	-0.945636	0.405422	-0.940893
0.70	0.902447	0.323325	-0.938053	0.358276	-0.944959
0.75	0.889745	0.276645	-0.928837	0.310926	-0.949032
0.80	0.875099	0.230477	-0.917476	0.263372	-0.953115
0.85	0.857764	0.184946	-0.903162	0.215614	-0.957205
0.90	0.836452	0.140233	-0.884493	0.167652	-0.961305
0.95	0.808595	0.096614	-0.858690	0.119484	-0.965412
1.00	0.767703	0.054592	-0.818741	0.071110	-0.969528
1.05	0.684350	0.015419	-0.732950	0.022531	-0.973652
	0.000000	0.000000	0.000000	0.000000	-0.975562

$n = 8.00 \quad m = 12.00 \quad \xi_o = 1.066619 \quad \mathcal{E} = 0.979526$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.979526	1.000000	-0.897899
0.05	0.995824	0.951025	-0.979421	0.955013	-0.901567
0.10	0.991447	0.902062	-0.979092	0.909843	-0.905243
0.15	0.986849	0.853120	-0.978516	0.864489	-0.908926
0.20	0.982006	0.804215	-0.977667	0.818950	-0.912616
0.25	0.976891	0.755359	-0.976515	0.773227	-0.916314
0.30	0.971470	0.706569	-0.975023	0.727319	-0.920019
0.35	0.965705	0.657863	-0.973148	0.681225	-0.923731
0.40	0.959549	0.609261	-0.970837	0.634946	-0.927451
0.45	0.952942	0.560787	-0.968028	0.588480	-0.931178
0.50	0.945814	0.512468	-0.964639	0.541828	-0.934912
0.55	0.938071	0.464335	-0.960572	0.494989	-0.938654
0.60	0.929597	0.416424	-0.955696	0.447962	-0.942402
0.65	0.920232	0.368782	-0.949839	0.400748	-0.946158
0.70	0.909761	0.321461	-0.942765	0.353346	-0.949921
0.75	0.897875	0.274531	-0.934141	0.305756	-0.953692
0.80	0.884112	0.228081	-0.923466	0.257977	-0.957470
0.85	0.867731	0.182231	-0.909939	0.210009	-0.961254
0.90	0.847426	0.137157	-0.892151	0.161852	-0.965046
0.95	0.820525	0.093133	-0.867232	0.113504	-0.968846
1.00	0.779943	0.050670	-0.827578	0.064967	-0.972652
1.05	0.687580	0.011166	-0.732435	0.016239	-0.976465
	0.000000	0.000000	0.000000	0.000000	-0.977735

TABLE I (cont.)

$n = 9.00 \quad m = 13.00 \quad \xi_0 = 1.061182 \quad \varepsilon = 0.981075$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.981075	1.000000	-0.905608
0.05	0.996139	0.950948	-0.980978	0.954634	-0.909028
0.10	0.992090	0.901906	-0.980674	0.909097	-0.912455
0.15	0.987834	0.852884	-0.980141	0.863388	-0.915888
0.20	0.983349	0.803896	-0.979355	0.817508	-0.919328
0.25	0.978609	0.754953	-0.978287	0.771456	-0.922773
0.30	0.973583	0.706072	-0.976904	0.725231	-0.926225
0.35	0.968234	0.657269	-0.975164	0.678833	-0.929684
0.40	0.962516	0.608563	-0.973018	0.632262	-0.933149
0.45	0.956375	0.559975	-0.970406	0.585518	-0.936620
0.50	0.949742	0.511531	-0.967253	0.538600	-0.940097
0.55	0.942528	0.463260	-0.963463	0.491508	-0.943580
0.60	0.934620	0.415197	-0.958913	0.444242	-0.947070
0.65	0.925865	0.367384	-0.953437	0.396801	-0.950566
0.70	0.916055	0.319873	-0.946810	0.349185	-0.954069
0.75	0.904887	0.272728	-0.938708	0.301394	-0.957577
0.80	0.891910	0.226035	-0.928642	0.253427	-0.961092
0.85	0.876388	0.179909	-0.915823	0.205285	-0.964613
0.90	0.857006	0.134521	-0.898844	0.156966	-0.968140
0.95	0.831016	0.090141	-0.874766	0.108471	-0.971674
1.00	0.790783	0.047288	-0.835448	0.059799	-0.975213
1.05	0.686460	0.007516	-0.727869	0.010949	-0.978759
	0.000000	0.000000	0.000000	0.000000	-0.979553

$n = 10.00 \quad m = 14.00 \quad \xi_0 = 1.056566 \quad \varepsilon = 0.982407$

ξ	τ	τ^m	$(\tau^m)'$	τ^{m-1}	$(\tau^{m-1})'$
0.00	1.000000	1.000000	-0.982407	1.000000	-0.912235
0.05	0.996409	0.950881	-0.982317	0.954308	-0.915439
0.10	0.992642	0.901772	-0.982033	0.908456	-0.918647
0.15	0.988681	0.852681	-0.981537	0.862443	-0.921862
0.20	0.984505	0.803622	-0.980806	0.816270	-0.925082
0.25	0.980089	0.754605	-0.979811	0.769935	-0.928308
0.30	0.975405	0.705646	-0.978521	0.723439	-0.931539
0.35	0.970415	0.656759	-0.976899	0.676781	-0.934775
0.40	0.965078	0.607962	-0.974896	0.629961	-0.938017
0.45	0.959342	0.559276	-0.972456	0.582979	-0.941265
0.50	0.953139	0.510725	-0.969507	0.535835	-0.944518
0.55	0.946386	0.462336	-0.965960	0.488527	-0.947776
0.60	0.938974	0.414141	-0.961695	0.441057	-0.951041
0.65	0.930755	0.366181	-0.956554	0.393423	-0.954310
0.70	0.921527	0.318504	-0.950321	0.345626	-0.957585
0.75	0.910998	0.271172	-0.942681	0.297665	-0.960865
0.80	0.898723	0.224267	-0.933159	0.249539	-0.964151
0.85	0.883977	0.177900	-0.920982	0.201249	-0.967442
0.90	0.865444	0.132236	-0.904745	0.152795	-0.970739
0.95	0.840318	0.087540	-0.881466	0.104175	-0.974041
1.00	0.800463	0.044338	-0.842511	0.055391	-0.977349
1.05	0.678353	0.004369	-0.716406	0.006441	-0.980662
	0.000000	0.000000	0.000000	0.000000	-0.981097

TABLE II

n	ξ_o	$\xi_o^{(0)}$	$\epsilon^{(0)}$	$\xi_o^{(1/2)}$	$\epsilon^{(1/2)}$	$\xi_o^{(1)}$	$\epsilon^{(1)}$	$\xi_o^{(2)}$	$\epsilon^{(2)}$
0.0	1.231173	1.000	$1.9 \cdot 10^{-1}$	1.118	$9.2 \cdot 10^{-2}$	1.1859	$3.7 \cdot 10^{-2}$	1.220790	$8.4 \cdot 10^{-3}$
0.5	1.200171	1.000	$1.7 \cdot 10^{-1}$	1.106	$7.9 \cdot 10^{-2}$	1.1653	$2.9 \cdot 10^{-2}$	1.193075	$5.9 \cdot 10^{-3}$
1.0	1.176525	1.000	$1.5 \cdot 10^{-1}$	1.095	$6.9 \cdot 10^{-2}$	1.1489	$2.3 \cdot 10^{-2}$	1.171459	$4.3 \cdot 10^{-3}$
1.5	1.157887	1.000	$1.4 \cdot 10^{-1}$	1.087	$6.1 \cdot 10^{-2}$	1.1355	$1.9 \cdot 10^{-2}$	1.154145	$3.2 \cdot 10^{-3}$
2.0	1.142817	1.000	$1.2 \cdot 10^{-1}$	1.080	$5.5 \cdot 10^{-2}$	1.1242	$1.6 \cdot 10^{-2}$	1.139974	$2.5 \cdot 10^{-3}$
2.5	1.130378	1.000	$1.2 \cdot 10^{-1}$	1.074	$5.0 \cdot 10^{-2}$	1.1147	$1.4 \cdot 10^{-2}$	1.128167	$2.0 \cdot 10^{-3}$
3.0	1.119935	1.000	$1.1 \cdot 10^{-1}$	1.069	$4.5 \cdot 10^{-2}$	1.1066	$1.2 \cdot 10^{-2}$	1.118182	$1.6 \cdot 10^{-3}$
3.5	1.111043	1.000	$1.0 \cdot 10^{-1}$	1.065	$4.2 \cdot 10^{-2}$	1.0995	$1.0 \cdot 10^{-2}$	1.109630	$1.3 \cdot 10^{-3}$
4.0	1.103380	1.000	$9.4 \cdot 10^{-2}$	1.061	$3.9 \cdot 10^{-2}$	1.0933	$9.1 \cdot 10^{-3}$	1.102224	$1.0 \cdot 10^{-3}$
4.5	1.096707	1.000	$8.8 \cdot 10^{-2}$	1.057	$3.6 \cdot 10^{-2}$	1.0878	$8.1 \cdot 10^{-3}$	1.095749	$8.7 \cdot 10^{-4}$
5.0	1.090844	1.000	$8.3 \cdot 10^{-2}$	1.054	$3.4 \cdot 10^{-2}$	1.0830	$7.2 \cdot 10^{-3}$	1.090042	$7.4 \cdot 10^{-4}$
6.0	1.081022	1.000	$7.5 \cdot 10^{-2}$	1.049	$3.0 \cdot 10^{-2}$	1.0747	$5.8 \cdot 10^{-3}$	1.080443	$5.4 \cdot 10^{-4}$
7.0	1.073118	1.000	$6.8 \cdot 10^{-2}$	1.044	$2.7 \cdot 10^{-2}$	1.0679	$4.8 \cdot 10^{-3}$	1.072686	$4.0 \cdot 10^{-4}$
8.0	1.066619	1.000	$6.2 \cdot 10^{-2}$	1.041	$2.4 \cdot 10^{-2}$	1.0623	$4.1 \cdot 10^{-3}$	1.066289	$3.1 \cdot 10^{-4}$
9.0	1.061182	1.000	$5.8 \cdot 10^{-2}$	1.038	$2.2 \cdot 10^{-2}$	1.0575	$3.5 \cdot 10^{-3}$	1.060924	$2.4 \cdot 10^{-4}$
10.0	1.056566	1.000	$5.4 \cdot 10^{-2}$	1.035	$2.0 \cdot 10^{-2}$	1.0534	$3.0 \cdot 10^{-3}$	1.056360	$2.0 \cdot 10^{-4}$

$$\epsilon^{(n)} = \frac{\xi_o - \xi_o^{(n)}}{\xi_o}$$

TABLE III

n	$\mathcal{E}$	$\mathcal{E}^{(0)}$	$\epsilon^{(0)}$	$\mathcal{E}^{(1/2)}$	$\epsilon^{(1/2)}$	$\mathcal{E}^{(1)}$	$\epsilon^{(1)}$	$\mathcal{E}^{(2)}$	$\epsilon^{(2)}$
0.0	0.940689	1.000	$-6.3 \cdot 10^{-2}$	0.894	$4.9 \cdot 10^{-2}$	0.9487	$-8.5 \cdot 10^{-3}$	0.942356	$-1.8 \cdot 10^{-3}$
0.5	0.946970	1.000	$-5.6 \cdot 10^{-2}$	0.905	$4.5 \cdot 10^{-2}$	0.9535	$-6.9 \cdot 10^{-3}$	0.948184	$-1.3 \cdot 10^{-3}$
1.0	0.952049	1.000	$-5.0 \cdot 10^{-2}$	0.913	$4.1 \cdot 10^{-2}$	0.9574	$-5.6 \cdot 10^{-3}$	0.952960	$-9.6 \cdot 10^{-4}$
1.5	0.956242	1.000	$-4.6 \cdot 10^{-2}$	0.920	$3.8 \cdot 10^{-2}$	0.9608	$-4.7 \cdot 10^{-3}$	0.956942	$-7.3 \cdot 10^{-4}$
2.0	0.959760	1.000	$-4.2 \cdot 10^{-2}$	0.926	$3.5 \cdot 10^{-2}$	0.9636	$-4.0 \cdot 10^{-3}$	0.960311	$-5.7 \cdot 10^{-4}$
2.5	0.962756	1.000	$-3.9 \cdot 10^{-2}$	0.931	$3.3 \cdot 10^{-2}$	0.9661	$-3.5 \cdot 10^{-3}$	0.963195	$-4.6 \cdot 10^{-4}$
3.0	0.965336	1.000	$-3.6 \cdot 10^{-2}$	0.935	$3.1 \cdot 10^{-2}$	0.9682	$-3.0 \cdot 10^{-3}$	0.965693	$-3.7 \cdot 10^{-4}$
3.5	0.967582	1.000	$-3.4 \cdot 10^{-2}$	0.939	$2.9 \cdot 10^{-2}$	0.9701	$-2.6 \cdot 10^{-3}$	0.967876	$-3.0 \cdot 10^{-4}$
4.0	0.969555	1.000	$-3.1 \cdot 10^{-2}$	0.943	$2.8 \cdot 10^{-2}$	0.9718	$-2.3 \cdot 10^{-3}$	0.969800	$-2.5 \cdot 10^{-4}$
4.5	0.971302	1.000	$-3.0 \cdot 10^{-2}$	0.946	$2.6 \cdot 10^{-2}$	0.9733	$-2.1 \cdot 10^{-3}$	0.971508	$-2.1 \cdot 10^{-4}$
5.0	0.972859	1.000	$-2.8 \cdot 10^{-2}$	0.949	$2.5 \cdot 10^{-2}$	0.9747	$-1.9 \cdot 10^{-3}$	0.973034	$-1.8 \cdot 10^{-4}$
6.0	0.975517	1.000	$-2.5 \cdot 10^{-2}$	0.953	$2.3 \cdot 10^{-2}$	0.9770	$-1.5 \cdot 10^{-3}$	0.975646	$-1.3 \cdot 10^{-4}$
7.0	0.977700	1.000	$-2.3 \cdot 10^{-2}$	0.957	$2.1 \cdot 10^{-2}$	0.9789	$-1.3 \cdot 10^{-3}$	0.977798	$-1.0 \cdot 10^{-4}$
8.0	0.979526	1.000	$-2.1 \cdot 10^{-2}$	0.961	$1.9 \cdot 10^{-2}$	0.9806	$-1.1 \cdot 10^{-3}$	0.979602	$-7.8 \cdot 10^{-5}$
9.0	0.981075	1.000	$-1.9 \cdot 10^{-2}$	0.964	$1.8 \cdot 10^{-2}$	0.9820	$-9.2 \cdot 10^{-4}$	0.981136	$-6.2 \cdot 10^{-5}$
10.0	0.982407	1.000	$-1.8 \cdot 10^{-2}$	0.966	$1.7 \cdot 10^{-2}$	0.9832	$-8.0 \cdot 10^{-4}$	0.982455	$-4.9 \cdot 10^{-5}$

$$\epsilon^{(n)} = \frac{\mathcal{E} - \mathcal{E}^{(n)}}{\mathcal{E}}$$

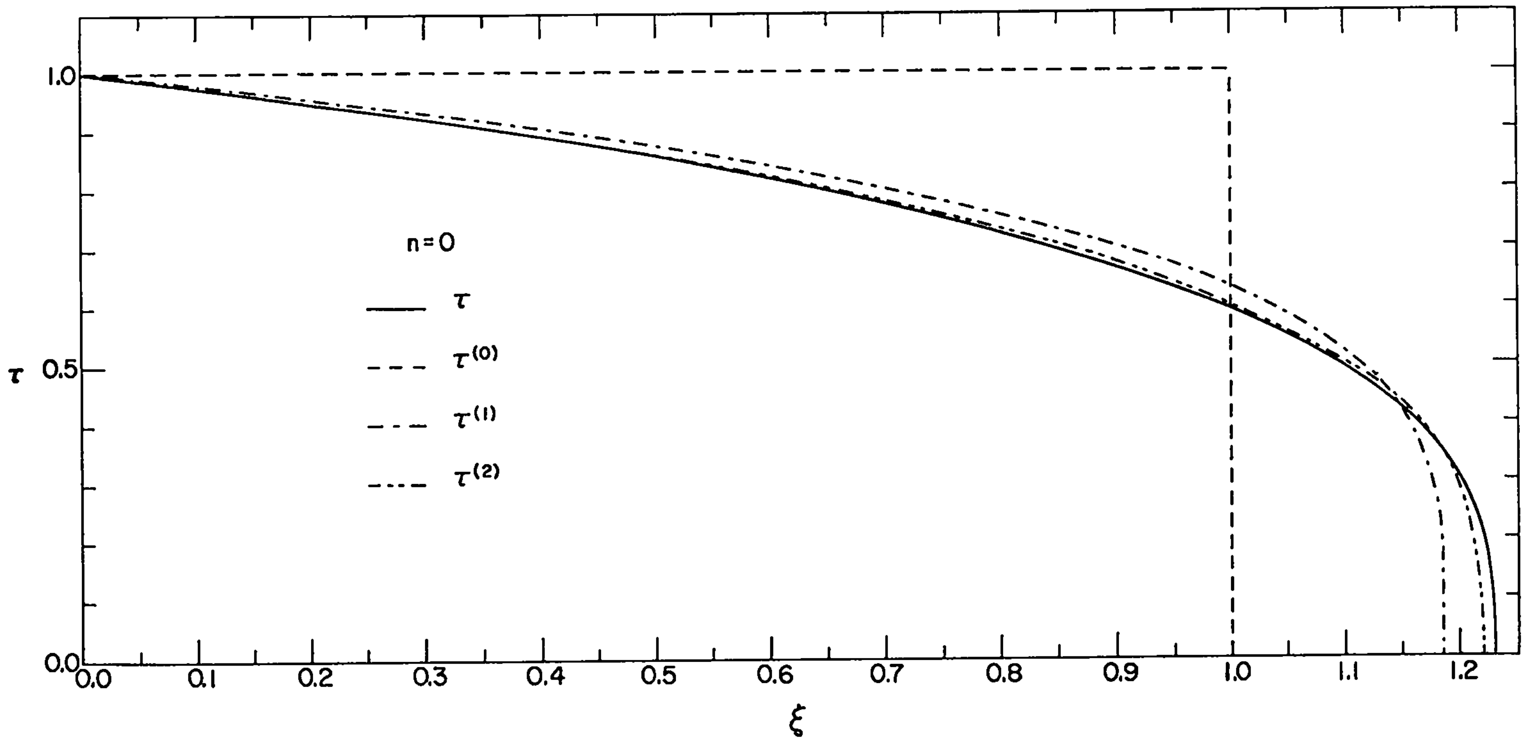
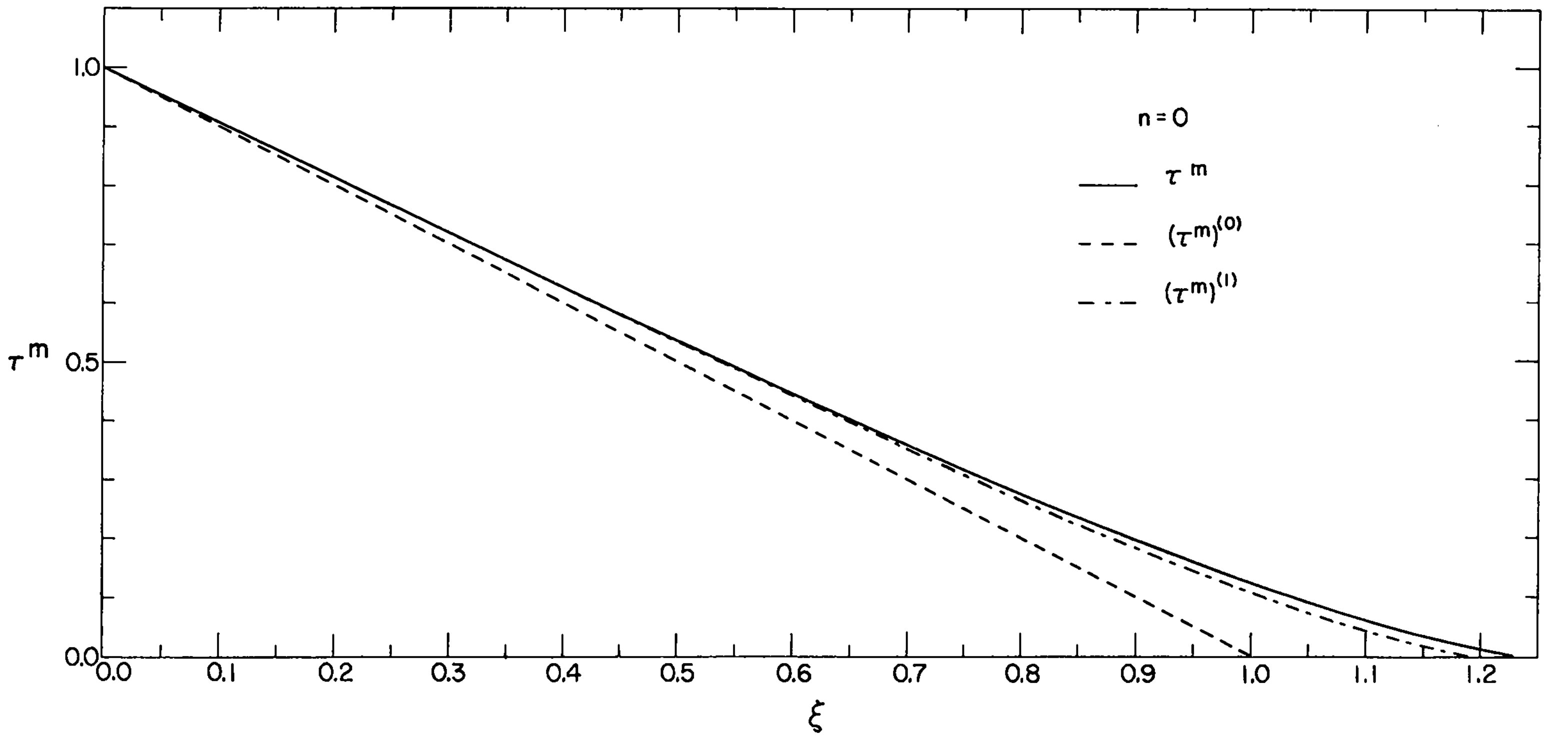


Figure 1



n = 0
τ^m
$(\tau^m)^{(0)}$
$(\tau^m)^{(1)}$
τ^m
1.0
0.5
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
ξ

Figure 2

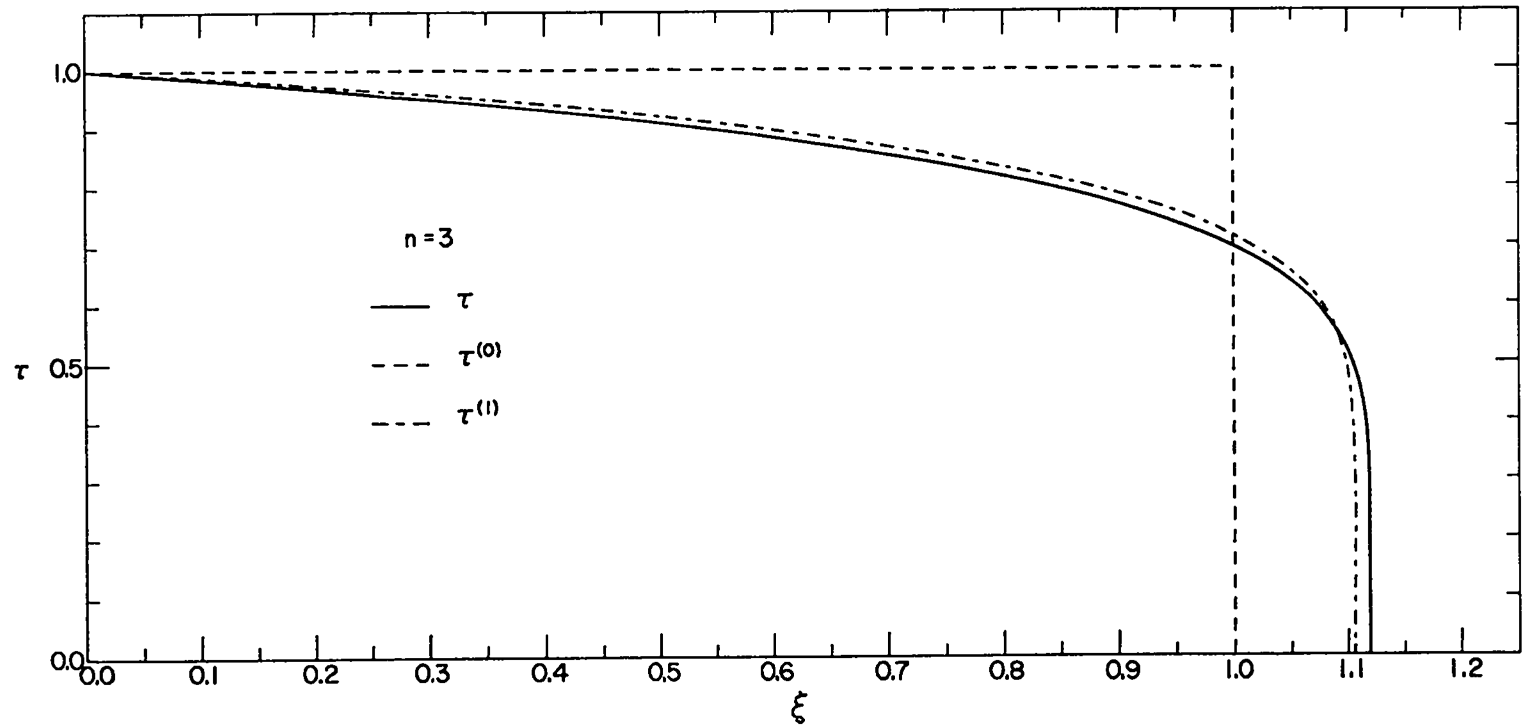


Figure 3

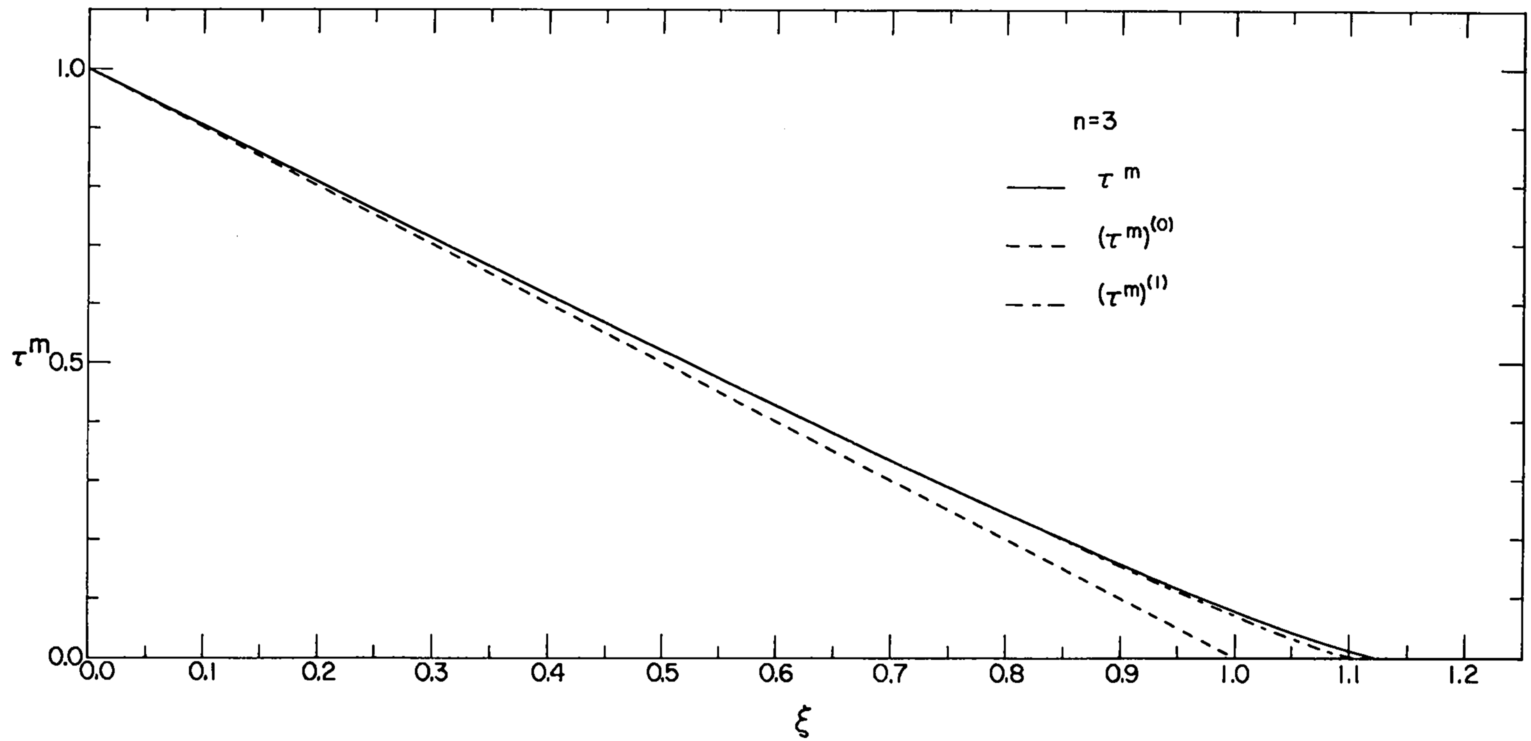

n=3
τ^m
$(\tau^m)^{(0)}$
$(\tau^m)^{(1)}$
τ^m
1.0
0.5
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
ξ

Figure 4

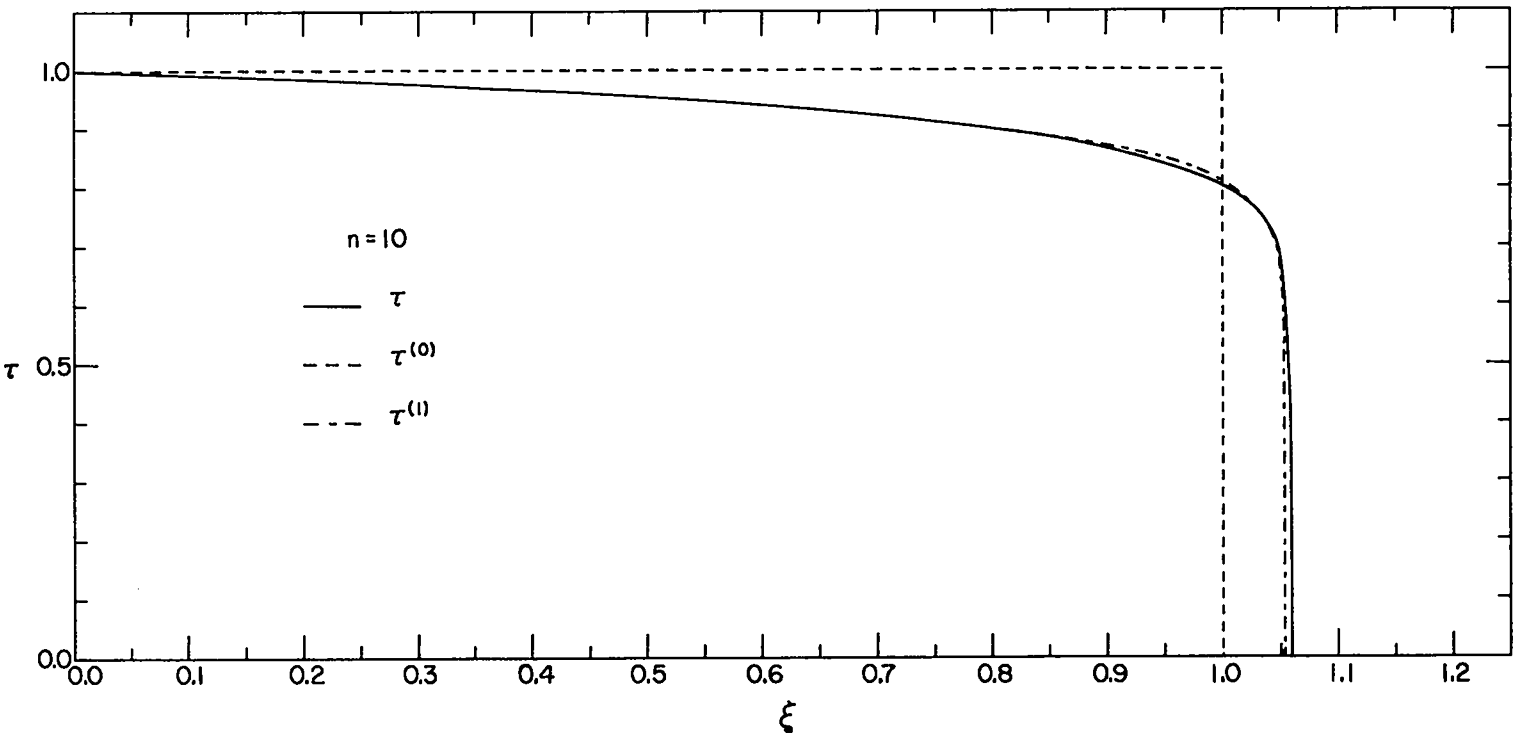


Figure 5

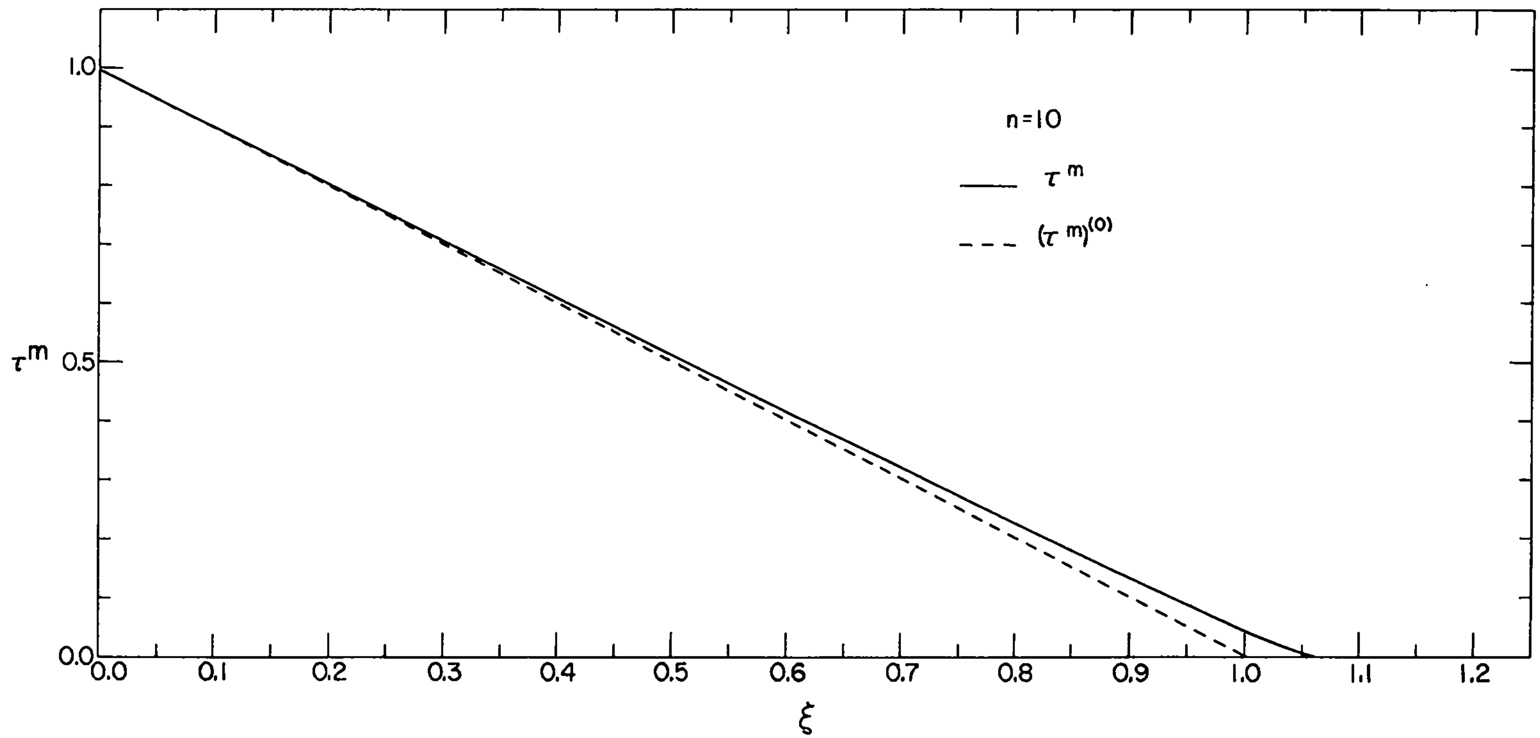


Figure 6