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ASYMPTOTIC ANALYSIS OF NONLINEAR MARSHAK WAVES*

E. W. LARSEN† AND G. C. POMRANING‡

Abstract. The classic Marshak wave equation (an equilibrium diffusion radiative transfer description) is obtained as the lowest order approximation in an asymptotic analysis of a system of time dependent nonequilibrium radiative transfer equations. The next approximation leads to a more general equilibrium diffusion approximation, which contains the radiative energy in the description. We derive an asymptotic solution of this higher order equilibrium diffusion approximation by including the smallness parameter in both the independent time variable and the dependent variable of the problem. The solution obtained is applicable over a longer time interval than the solution of the Marshak equation. Its main qualitative feature is that the predicted position of the wave front lags behind the Marshak prediction.

1. Introduction. The nonlinear equilibrium diffusion equation for radiative energy transfer, in the absence of conduction and convection and with the neglect of radiative energy content, was derived by Marshak [1] by introducing several physically motivated approximations into a system of nonequilibrium radiative transfer equations. Marshak treated this equation as an isolated approximation without regard to a systematic procedure of obtaining higher order approximations. It is given by

$$\frac{\partial}{\partial t} E_m(\theta) = \vec{\nabla} \cdot \left[\frac{ac}{3} \lambda(\theta) \vec{\nabla} \theta^4\right] + Q, \tag{1}$$

where $\theta(\vec{r}, t)$ is the material temperature, $E_m(\theta)$ is the material energy density, $\lambda(\theta)$ is the Rosseland mean free path of radiation, $Q(\vec{r}, t)$ is the external source of heat, c is the speed of light, and a is the radiation constant ($a = 4\sigma/c$, where σ is the Stefan–Boltzmann constant). Experience has shown that (1) is a reasonably accurate physical description for many radiative transfer problems.

If one introduces into (1) the realistic idealizations that both $E_m(\theta)$ and $\lambda(\theta)$ are simple powers of θ, then for certain problems the resulting equation is susceptible to analytic methods [2]. In particular, if the source Q is a delta function in both space and time, this nonlinear partial differential equation admits a closed form similarity solution in any one-dimensional geometry. This solution exhibits the well known "Marshak wave" [1], [2], which is identically zero ahead of a wave front whose position varies as a fractional power of time. Far behind the front, the solution varies slowly in space and time.

Equation (1) is a simplified version of the more general equilibrium diffusion equation which includes the radiative energy content of the problem. While it is generally true for problems of interest that the radiative energy density is small compared to the material energy density, it may not be small enough to neglect completely. With the inclusion of this effect, (1) is replaced by ([3])

$$\frac{\partial}{\partial t}[a\theta^4 + E_m(\theta)] = \vec{\nabla} \cdot \left[\frac{ac}{3} \lambda(\theta) \vec{\nabla} \theta^4\right] + Q. \tag{2}$$

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† Theoretical Division, University of California, Los Alamos Scientific Laboratory, Los Alamos, New Mexico, 87545.

‡ School of Engineering and Applied Science, University of California, Los Angeles, California, 90024.

Unfortunately, the addition of the $a\theta^4$ term destroys the ability to generate a closed form solution for the delta function source problem, or any other problem of interest. Partly for this reason, there has appeared no quantitative analysis of (2) in the literature.

The purpose of this paper is two-fold. First, we show that (1) and (2) correspond to zeroth and first-order results derived from a systematic asymptotic analysis of a more detailed radiative transfer description. Secondly, we generate (using asymptotics) a correction to the previously mentioned Marshak solution which accounts for the small but nonnegligible radiative energy density term contained in (2).

The principal idea in our asymptotic analysis is the use of a single time variable η which contains the elements of both the short- and long-time behavior of the Marshak wave. We define η implicitly by

$$\eta = \tau - \varepsilon f(\eta), \qquad f(0) = 0, \quad \varepsilon \ll 1, \tag{3}$$

where τ is the unperturbed time variable and f is a function to be determined. This ansatz, which is a slight variation of the Poincaré method [4], corresponds to introducing a small shift in the wave position and speed.

The principal mathematical difficulty occurs in the determination of $f(\eta)$ in (3). In a "standard" asymptotic expansion, f, which is contained in the $O(\varepsilon)$ equation, would be determined by a solvability condition which is derived in a straightforward way from the (nonlinear) $O(1)$ equation. This appears not to be possible here for the following reasons.

First, the homogeneous $O(1)$ equation has an infinite number of solutions corresponding to different initial data. We choose the special (similarity) solution corresponding to a delta-function initial condition. This solution is weak; its derivatives are unbounded at the wave front. Next, the $O(\varepsilon)$ equation also has an infinite number of solutions (corresponding to different initial data), one of which is a similarity solution which has the same general form as the $O(1)$ solution. (See (42), (43), and (50), (51).) However, the exponent of η differs in these two forms, and this leads to basically different equations for the two functions of the remaining similarity variable ξ.

To obtain our result, we adopt the following straightforward but rather inelegant procedure. We formally solve the $O(\varepsilon)$ equation by means of a power series and then require this series to have the proper differentiability properties; this requirement uniquely determines f.

In § 2 we introduce the radiative transfer equations which are the basis for our analysis, cast them into dimensionless form, discuss the ordering of the four resulting dimensionless parameters, and obtain (1) and (2) as zeroth and first-order approximations. Then in § 3 we derive an asymptotic solution of an initial value problem for (2), and we compare the differences between this solution and the corresponding Marshak solution. Our analysis indicates that the asymptotic wave front lags behind the Marshak front. We discuss this in detail and provide a physical explanation.

2. The radiative transfer equations. For concreteness, we consider slab geometry with a constant (independent of temperature) heat capacity and a radiation mean-free path which varies as the cube of the temperature. Other geometries and powers of temperature are handled with equal ease. If we take the nonequilibrium (also called the two-temperature) diffusion equation as the fundamental description of the radiative transfer process and consider an infinite homogeneous medium with a slab, pulsed source at $z = 0$ and $t = 0$, we have [5]

$$\frac{\partial E}{\partial t} - \frac{\partial}{\partial z}\left[\frac{c}{3} l \left(\frac{\theta}{\theta_r}\right)^3 \frac{\partial E}{\partial z}\right] = \frac{c}{l}\left(\frac{\theta_r}{\theta}\right)^3 (a\theta^4 - E), \tag{4}$$

$$c_v \frac{\partial \theta}{\partial t} = \frac{c}{l}\left(\frac{\theta_r}{\theta}\right)^3 (E - a\theta^4), \tag{5}$$

with initial and boundary conditions

$$\begin{aligned} &E(-\infty, t) = E(\infty, t) = 0, \\ &c_v\theta(z, 0) + E(z, 0) = Q\delta(z). \end{aligned} \tag{6}$$

Here $E(z, t)$ is the radiation energy density, $\theta(z, t)$ is the material temperature, c_v is the material heat capacity per unit volume, θ_r is a reference temperature at which the radiation mean-free path is l, and Q is the source strength per unit area. The constants l and θ_r are not known individually, but the ratio l/θ_r^3 is assumed to be given.

We expect a wave-like solution to the above equations. In front of the wave, E and θ are zero, or nearly so. Far behind the wave, the solution as a function of space is nearly a constant, with this constant plateau value decreasing in time as the wave moves outward. In the vicinity of the wave front, we identify a characteristic distance L and a characteristic time T, and we introduce the dimensionless variables

$$z' \equiv z/L, \qquad t' \equiv t/T, \tag{7}$$

$$u(z', t') = \frac{E(z, t)}{a\theta_r^4}, \tag{8}$$

$$v(z', t') = \frac{\theta(z, t)}{\theta_r}, \tag{9}$$

where θ_r characterizes the temperatures of interest. We conceptually take L to be the distance over which θ and E drop from their plateau values (roughly θ_r and $a\theta_r^4$) to zero, and T to be the time for the wave front to move a distance L. Then the differential operators $\partial/\partial z'$ and $\partial/\partial t'$ are of order one. That is, in a distance $\Delta z' = 1$ ($\Delta z = L$), u and v go from their plateau values [which are $O(1)$] to zero; and in a time $\Delta t' = 1$ ($\Delta t = T$), u and v go from zero to their plateau values.

Combining (4)–(9), we obtain

$$\left(\frac{l}{cT}\right)\frac{\partial u}{\partial t'} - \frac{1}{3}\left(\frac{l}{L}\right)^2 \frac{\partial}{\partial z'}\left(v^3 \frac{\partial u}{\partial z'}\right) = \frac{1}{v^3}(v^4 - u), \tag{10}$$

$$\left(\frac{c_v l}{acT\theta_r^3}\right)\frac{\partial v}{\partial t'} = \frac{1}{v^3}(u - v^4), \tag{11}$$

$$\left(\frac{c_v l}{acT\theta_r^3}\right) v(z', 0) + \left(\frac{l}{cT}\right) u(z', 0) = \left(\frac{lQ}{TLca\theta_r^4}\right)\delta(z'). \tag{12}$$

Now we consider the dimensionless coefficients in these equations. We note that the ratio of radiative energy density to material energy density is given by

$$\frac{E}{c_v\theta} = \frac{a\theta_r^4 u}{c_v\theta_r v} = \left(\frac{a\theta_r^3}{c_v}\right)\left(\frac{u}{v}\right). \tag{13}$$

For the problem of interest, this ratio is small when the wave is fully developed. Therefore, since u/v is $O(1)$, we shall in the ordering process assume

$$\frac{a\theta_r^4}{c_v\theta_r} = \frac{a\theta_r^3}{c_v} \equiv \varepsilon, \tag{14}$$

when

$$(c_v\theta_r + a\theta_r^4) \approx c_v\theta_r = \frac{Q}{L}. \tag{15}$$

Equation (15) defines quantitatively what is meant by the wave being fully developed. Namely, the wave is fully developed when it has penetrated the characteristic distance L. We take (15) as defining θ_r, i.e.,

$$\theta_r \equiv \frac{Q}{c_v L}. \tag{16}$$

This definition implies that we are interested in a time interval which starts in the vicinity of the time when the wave is first fully developed. With these definitions, we find that two of the four dimensionless coefficients in (10)–(12) can be written

$$\frac{c_v l}{acT\theta_r^3} = \frac{lQ}{TLca\theta_r^4} = \left(\frac{1}{\varepsilon}\right)\left(\frac{l}{cT}\right). \tag{17}$$

The parameter l/cT in (17) is the ratio of the time required for a photon moving with the speed of light to traverse a mean-free path to the time T required for the wave to move a distance L. We expect this to be very small, in particular smaller than ε. We write

$$\frac{l}{cT} = O(\varepsilon^{n+1}), \tag{18}$$

where $n > 0$ is at present unspecified. We return to this point shortly. Thus the first and third coefficients in (12) are of order ε^n, and (11) then shows that $u - v^4$ is of order ε^n. From (10), we conclude that

$$\left(\frac{l}{L}\right)^2 = O(\varepsilon^n). \tag{19}$$

The ordering process completed, we define parameters α and β, both of order one, by the equations

$$\left(\frac{l}{cT}\right) = \alpha\varepsilon^{n+1}, \tag{20}$$

$$\frac{1}{3}\left(\frac{l}{L}\right)^2 = \beta\varepsilon^n, \tag{21}$$

with ε given by (14). If we change independent variables once again according to

$$\tau \equiv t'/\alpha, \qquad x \equiv z'/\sqrt{\beta}, \tag{22}$$

and define

$$S \equiv 1/\sqrt{\beta}, \tag{23}$$

then (10) through (12) take the form

$$\varepsilon^{n+1}\frac{\partial u}{\partial \tau} - \varepsilon^n \frac{\partial}{\partial x}\left(v^3 \frac{\partial u}{\partial x}\right) = \frac{1}{v^3}(v^4 - u), \tag{24}$$

$$\varepsilon^n \frac{\partial v}{\partial \tau} = \frac{1}{v^3}(u - v^4), \tag{25}$$

$$\varepsilon u(x,0)+v(x,0)=S\delta(x). \tag{26}$$

The overall energy balance for the problem follows from the addition of (24) and (25), i.e.,

$$\frac{\partial}{\partial\tau}(\varepsilon u+v)=\frac{\partial}{\partial x}\left(v^3\frac{\partial u}{\partial x}\right). \tag{27}$$

Integration of (27) over space and time gives, using (26),

$$\int_{-\infty}^{\infty}[\varepsilon u(x,t)+v(x,t)]\,dx=S, \tag{28}$$

which just states that the original energy put into the medium must be conserved.

If we envision an expansion in ε, then to lowest order both (24) and (25) give

$$u=v^4+O(\varepsilon^n), \tag{29}$$

and using this in (27) gives

$$\frac{\partial v}{\partial\tau}=\frac{\partial}{\partial x}\left(v^3\frac{\partial v^4}{\partial x}\right)+O(\varepsilon), \tag{30}$$

which is just the Marshak description (1), for this problem. This lowest order description is independent of n, the exponent introduced in (18). To obtain (2), the Marshak-like equation with radiative energy included, we must assume $n>1$. Then (27) and (29) give

$$\frac{\partial}{\partial\tau}(\varepsilon v^4+v)=\frac{\partial}{\partial x}\left(v^3\frac{\partial v^4}{\partial x}\right)+O(\varepsilon^n). \tag{31}$$

Since for problems of interest (2) contains the most important correction to (1) [2], [3], we conclude that for such problems $n>1$. We restrict ourselves to this case for the remainder of this paper. Then (29) gives $u=v^4+\cdots$, where the dots refer to terms of order ε^n (at most), which will not play a role in our analysis.

Physically, $u=v^4$ implies that at any point in space and time, the radiation field is in thermodynamic equilibrium with the material temperature. The resulting diffusion description, (31) or equivalently (2), is accordingly referred to as the "equilibrium" diffusion approximation. We take (31) as the basis for our asymptotic analysis in the next section. The corresponding initial and boundary conditions (at $x=\pm\infty$) are:

$$\varepsilon v^4(x,0)+v(x,0)=S\delta(x), \tag{32}$$

$$v(-\infty,t)=v(+\infty,t)=0. \tag{33}$$

3. Asymptotic analysis. In this section we derive the leading terms of an asymptotic solution of (31). Our ansatz is

$$v(x,\tau)=V_0(x,\eta)+\varepsilon V_1(x,\eta)+O(\varepsilon^2), \tag{34}$$

where the time variable η is defined implicitly by

$$\eta=\tau-\varepsilon f(\eta)+O(\varepsilon^2),\qquad f(0)=0, \tag{35}$$

and f is a function to be determined. Since

$$\frac{\partial}{\partial\tau}=[1-\varepsilon f'(\eta)]\frac{\partial}{\partial\eta}+O(\varepsilon^2), \tag{36}$$

(31)–(36) lead to the following zeroth and first-order equations:

$$\frac{\partial}{\partial \eta} V_0 = \frac{4}{7} \frac{\partial^2}{\partial x^2} V_0^7, \tag{37}$$

$$\frac{\partial}{\partial \eta} (V_1 + V_0^4) - f'(\eta) \frac{\partial}{\partial \eta} V_0 = 4 \frac{\partial^2}{\partial x^2} (V_0^6 V_1). \tag{38}$$

The corresponding initial and boundary conditions are

$$V_0(x, 0) = 0, \quad x \neq 0; \qquad \lim_{\eta \to 0} \int_{-\infty}^{+\infty} V_0(x, \eta)\, dx = S; \tag{39}$$

$$V_1(x, 0) + V_0^4(x, 0) = 0, \quad x \neq 0; \tag{40}$$

$$\lim_{\eta \to 0} \int_{-\infty}^{+\infty} [V_1(x, \eta) + V_0^4(x, \eta)]\, dx = 0;$$

$$V_0(\pm\infty, \eta) = V_1(\pm\infty, \eta) = 0. \tag{41}$$

First we consider the problem for V_0. Since the conditions on the problem are such that the solution behaves the same for small η as for large x, namely v vanishes, a similarity solution suggests itself. In our particular problem, the similarity variable ξ is given by

$$\xi = \frac{x}{\eta^{1/8}}, \tag{42}$$

and $V_0(x, \eta)$ is written

$$V_0(x, \eta) = \frac{\psi(\xi)}{\eta^{1/8}}. \tag{43}$$

Equation (37) then reduces to

$$\frac{d^2 \psi^7}{d\xi^2} + \frac{7}{32} \frac{d}{d\xi} (\xi \psi) = 0, \tag{44}$$

with boundary conditions

$$\psi(-\infty) = \psi(+\infty) = 0, \tag{45}$$

and the energy conservation condition

$$\int_{-\infty}^{\infty} \psi(\xi)\, d\xi = S, \tag{46}$$

which follows from (39). The solution of (44)–(46) is easily found to be

$$\psi(\xi) = \begin{cases} [\frac{3}{32}(\xi_0^2 - \xi^2)]^{1/6}, & \xi^2 < \xi_0^2, \\ 0, & \xi^2 \geqq \xi_0^2, \end{cases} \tag{47}$$

where ξ_0 is given by

$$\xi_0 = \left[\left(\frac{4}{\sqrt{\pi}} \right) \left(\frac{32}{3} \right)^{1/6} \frac{\Gamma(2/3)}{\Gamma(1/6)} \right]^{3/4} S^{3/4} = (0.8574) S^{3/4}, \tag{48}$$

and where $\Gamma(z)$ is the usual gamma function (See [6, pp. 253–254] for a result leading to

(48).) The interesting aspect of this solution is that the position of the wave front, $x_0(\eta)$, is given by

(49) $$x_0(\eta)=\xi_0\eta^{1/8},$$

with the solution vanishing identically for $|x|\geqq x_0(\eta)$. The corresponding solutions in the other one-dimensional geometries and for other assumed power dependences of c_v and λ on θ are given in [7].

Since the time variable η is defined in terms of ε, it follows that ε is implicitly contained in V_0. However, if we set $\varepsilon=0$ in (35), the time variable becomes unperturbed and V_0 reduces exactly to the classic Marshak solution. We shall see that the inclusion of the ε term in the definition of η makes it possible to eliminate a secularity which arises in the problem for V_1.

This problem (for V_1) is given by (38), (40), and (41). Since V_0 is identically zero for $|x|\geqq x_0(\eta)$, it follows from these equations that V_1 must be identically zero in this region also. For $|x|<x_0(\eta)$, we seek a similarity solution of the same general form as V_0. Thus

(50) $$V_1(x,\eta)=\frac{1}{\eta^\gamma}\phi(\xi),$$

with $\xi=x\eta^{-1/8}$ and $\phi(\xi)\equiv 0$ for $|\xi|\geqq\xi_0$. Introducing (50) and V_0 into (38), we find that for each term to have a common power of η we must choose

(51) $$\gamma=\tfrac{1}{2}\quad\text{and}\quad f'(\eta)=k\eta^{-3/8},$$

where k is a constant to be determined. The resulting equation for $\phi(\xi)$ on $|\xi|<\xi_0$ is:

(52) $$\begin{aligned}3\frac{d^2}{d\xi^2}(\xi_0^2-\xi^2)\phi+\xi\frac{d}{d\xi}\phi+4\phi&\\ =\frac{4}{3}\left(\frac{3}{32}\right)^{2/3}&[\xi_0^2(\xi_0^2-\xi^2)^{-1/3}-4(\xi_0^2-\xi^2)^{2/3}]\\ -\frac{k}{3}\left(\frac{3}{32}\right)^{1/6}&[\xi_0^2(\xi_0^2-\xi^2)^{-5/6}-4(\xi_0^2-\xi^2)^{1/6}].\end{aligned}$$

In order that a delta function not be generated by H at $\xi=0$ and at the wave fronts $\xi=\pm\xi_0$, it is necessary that $\phi'(\xi)$ be continuous at $\xi=0$ and

(53) $$0=\phi(\xi)=\frac{d}{d\xi}(\xi_0^2-\xi^2)\phi(\xi),\qquad \xi=\pm\xi_0.$$

If for $|\xi|<\xi_0$ we express ϕ as the sum of two functions ϕ_1 and ϕ_2, with ϕ_1 corresponding to the first term on the right side of (52) and ϕ_2 corresponding to the second, then ϕ_1 and ϕ_2 can be represented as

(54) $$\phi_i(\xi)=(\xi_0^2-\xi^2)^{m_i}\sum_{n=0}^{\infty}a_{in}(\xi_0^2-\xi^2)^n,$$

with (clearly)

(55) $$m_1=\tfrac{2}{3},\qquad m_2=\tfrac{1}{6}.$$

These exponents are consistent with (53).

Formally introducing the series (54) into (52) and equating the coefficients of $(\xi_0^2-\xi^2)^n$ gives

$$\phi(\xi)=\frac{1}{9}\left(\frac{3}{32}\right)^{2/3}(\xi_0^2-\xi^2)^{2/3}\left[1-\sum_{n=1}^{\infty}b_n\left(1-\frac{\xi^2}{\xi_0^2}\right)^n\right] \tag{56}$$
$$-\frac{k}{6}\left(\frac{3}{32}\right)^{1/6}(\xi_0^2-\xi^2)^{1/6}\left[1-\sum_{n=1}^{\infty}c_n\left(1-\frac{\xi^2}{\xi_0^2}\right)^n\right],$$

where

$$b_1=\tfrac{3}{5}, \tag{57}$$

$$b_n=\left[\frac{6n^2+4n-1}{(3n+2)(2n+3)}\right]b_{n-1},\qquad n\geqq 2, \tag{58}$$

and

$$c_1=\tfrac{3}{28}, \tag{59}$$

$$c_n=\left[\frac{12n^2-4n-3}{2(6n+1)(n+1)}\right]c_{n-1},\qquad n\geqq 2. \tag{60}$$

For large n, we have

$$b_n=\left[1-\frac{3}{2n}+O\left(\frac{1}{n^2}\right)\right]b_{n-1} \tag{61}$$

and

$$c_n=\left[1-\frac{3}{2n}+O\left(\frac{1}{n^2}\right)\right]c_{n-1}, \tag{62}$$

from which it follows that

$$b_n=\frac{B}{n^{3/2}}+O\left(\frac{1}{n^{5/2}}\right) \tag{63}$$

and

$$c_n=\frac{C}{n^{3/2}}+O\left(\frac{1}{n^{5/2}}\right). \tag{64}$$

Therefore the two series in (56) converge for $|\xi|\leqq\xi_0$. However, unless k is properly chosen, $\phi'(\xi)$ is not continuous at $\xi=0$. To show this, we write (near $\xi=0$)

$$\phi(\xi)=[1+O(\xi^2)]\sum_{n=1}^{\infty}\left[\frac{a}{n^{3/2}}+O\left(\frac{1}{n^{5/2}}\right)\right]\left(1-\frac{\xi^2}{\xi_0^2}\right)^n, \tag{65}$$

where

$$a=\frac{k}{6}\left(\frac{3}{32}\right)^{1/6}\xi_0^{1/3}C-\frac{1}{9}\left(\frac{3}{32}\right)^{2/3}\xi_0^{4/3}B. \tag{66}$$

Hence,

$$\phi'(\xi)=-\frac{2\xi}{\xi_0^2}\sum_{n=1}^{\infty}\left[\frac{a}{n^{1/2}}+O\left(\frac{1}{n^{3/2}}\right)\right]\left(1-\frac{\xi^2}{\xi_0^2}\right)^n+\cdots. \tag{67}$$

Making use of the result [6, p. 280]

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}\left(1-\frac{\xi^2}{\xi_0^2}\right)^n \approx \frac{\xi_0\sqrt{\pi}}{|\xi|} \quad \text{for} \quad \xi \approx 0, \tag{68}$$

equation (67) gives

$$\phi'(\xi) = -\frac{2a\sqrt{\pi}}{\xi_0}\frac{\xi}{|\xi|}+\cdots, \tag{69}$$

where the dots refer to terms which vanish as $\xi \to 0$.

Thus $\phi'(\xi)$ is continuous (and, in fact, vanishes) at $\xi = 0$ if and only if $a = 0$. Imposing this condition, we find from (66) that

$$k = \frac{\xi_0}{6}\sqrt{\frac{3}{2}\frac{B}{C}}, \tag{70}$$

where, by (61) and (62),

$$\frac{B}{C} = \lim_{n\to\infty}\frac{b_n}{c_n} = \frac{25}{9}\prod_{n=1}^{\infty}\frac{2(6n^2+4n-1)(6n+1)(n+1)}{(12n^2-4n-3)(3n+2)(2n+3)}. \tag{71}$$

This infinite product is of the form

$$\frac{B}{C} = \frac{25}{9}\prod_{n=1}^{\infty}\left[1+O\left(\frac{1}{n^2}\right)\right], \tag{72}$$

and hence it exists. In fact, it can be expressed as a finite product of gamma functions [6, pp. 238–239]:

$$\frac{B}{C} = \frac{25\sqrt{\pi}}{3}\frac{\Gamma(\frac{2}{3})\Gamma((5+\sqrt{10})/6)\Gamma((5-\sqrt{10})/6)}{\Gamma(\frac{1}{6})\Gamma((8+\sqrt{10})/6)\Gamma((8-\sqrt{10})/6)} = 8.527. \tag{73}$$

Equation (70) now gives

$$k = 1.741\xi_0 = 1.493\, S^{3/4}. \tag{74}$$

It remains to show that V_1 satisfies the initial condition (40). First we note that by construction, $V_1(x, 0) = V_0(x, 0) = 0$ for $x \neq 0$, and so

$$V_1(x, 0) + V_0^4(x, 0) = 0, \qquad x \neq 0. \tag{75}$$

Next, since for any η, V_0 and V_1 vanish for $|x| \geqq \xi_0\, \eta^{1/8}$, (37), (38), (43), (50), and (51) give

$$I = \int_{-\infty}^{+\infty} [V_1(x, \eta) + V_0^4(x, \eta)]\, dx = \frac{1}{\eta^{3/8}}\int_{-\xi_0}^{\xi_0} [\phi(\xi) + \psi^4(\xi)]\, d\xi, \tag{76}$$

where I is independent of η. Since the last integral in (76) is independent of η, it follows that I must be zero, and hence

$$\lim_{n\to 0}\int_{-\infty}^{+\infty} [V_1(x, \eta) + V_0^4(x, \eta)]\, dx = 0. \tag{77}$$

The $O(1)$ and $O(\varepsilon)$ terms of our asymptotic solution are now fully determined, and have the form $V_0 = \eta^{-1/8}\, \psi(\xi)$, $V_1 = \eta^{-1/2}\phi(\xi)$. In order for the ordering process to be consistent with this solution, V_1 cannot be large compared to V_0. This is true for intermediate and large times, but not for very short times ($0 \leqq \eta \leqq O(\varepsilon)$) because here the wave is not fully developed in accordance with the scaling assumptions outlined in § 2. Thus, within this initial layer there exists a transient solution not accounted for by the above asymptotic solution.

In order to assign initial conditions to the asymptotic solution, we have assumed that the transient layer is initially zero and that the initial condition for the original problem is appropriate to the asymptotic solution. This is the simplest such assumption which leads to a tractable asymptotic analysis for which the total energy is conserved. Similar types of assumptions are made in fluid models of gas-dynamic problems, wherein initial conditions for the fluid equations are taken to conserve, at each spatial point, the mass, momentum, and the energy of the initial condition of the gas-dynamic problem. Ideally, one would *derive* the appropriate initial conditions by constructing an initial layer and performing a matched asymptotic expansion. However, the nonlinearities in both the present and the gas-dynamical problems make such a derivation difficult to implement[1].

Let us now examine the leading term of the asymptotic solution. In terms of the original variables of the problem, it is:

$$\theta(z,t)=\begin{cases}\left\{\dfrac{1}{y^{1/8}}\left[\lambda_1\left(\lambda_0^2-\dfrac{z^2}{y^{1/4}}\right)\right]\right\}^{1/6}, & |z|<\lambda_0 y^{1/8},\\ 0, & |z|\geqq\lambda_0 y^{1/8},\end{cases}\tag{78}$$

where y is defined implicitly in terms of t by

$$y+\lambda_2 y^{5/8}=t,\tag{79}$$

and the constants λ_0, λ_1, and λ_2 are given by

$$\lambda_0=(0.8574)\left(\frac{Q}{c_v}\right)^{3/4}\left(\frac{ac}{3c_v}\frac{l}{\theta_r^3}\right)^{1/8},\tag{80}$$

$$\lambda_1=\left(\frac{9c_v}{32ac}\right)\left(\frac{\theta_r^3}{l}\right),\tag{81}$$

$$\lambda_2=(2.388)\left(\frac{a}{c_v}\right)^{5/8}\left(\frac{3Q^2}{cc_v^2}\frac{\theta_r^3}{l}\right)^{3/8}.\tag{82}$$

These constants are expressed in terms of quantities explicitly given in the original problem (4)–(6).

The classic Marshak solution is obtained by setting $\lambda_2=0$ in (78) and (79), giving $y=t$. In our solution, λ_2 can be written as

$$\lambda_2=(2.388)\left(\frac{T\alpha}{\beta}\right)^{3/8}\varepsilon,\tag{83}$$

and hence $\lambda_2 y^{5/8}$ is small (of order ε) compared to y for $y\geqq T$.

For $\lambda_2>0$, (79) implies $y\leqq t$, with equality holding only at $t=0$. For intermediate and large times, (79) and (83) give, correct to order ε,

$$y=t-\lambda_2 t^{5/8}.\tag{84}$$

Therefore, for such times, the solution (78) can be interpreted as a classic Marshak wave

[1] The method of matched asymptotic expansions has been applied to linearized gas-dynamic problems, yielding initial conditions for the fluid variables which do in fact conserve the pointwise mass, momentum, and energy of the initial condition for the gas-dynamical problem. See [8].

which began to propagate at time $\lambda_2 t^{5/8}$ instead of time zero. Qualitatively, the solution (78) lags behind the Marshak solution.

The following remarks pertain to intermediate and large times t, for which (84) is valid.

The position of the wave front moving to the right is

$$z_0(t)=\lambda_0 y^{1/8}=\lambda_0(t-\lambda_2 t^{5/8})^{1/8}, \tag{85}$$

whereas the Marshak prediction is

$$z_1(t)=\lambda_0 t^{1/8}. \tag{86}$$

The difference of these positions is

$$z_0(t)-z_1(t)\approx -\frac{\lambda_0\lambda_2}{8}\frac{1}{t^{1/4}}, \tag{87}$$

which tends to zero as $t\to\infty$.

However, the time at which the wave front moving to the right passes the point z is

$$t_0(z)=\left(\frac{z}{\lambda_0}\right)^8+\lambda_2\left(\frac{z}{\lambda_0}\right)^5, \tag{88}$$

whereas the Marshak prediction is

$$t_1(z)=\left(\frac{z}{\lambda_0}\right)^8. \tag{89}$$

The difference of these times is

$$t_0(z)-t_1(z)=\lambda_2\left(\frac{z}{\lambda_0}\right)^5, \tag{90}$$

which tends to infinity as $z\to\infty$.

Therefore the two solutions coalesce in the sense that for large t, the difference in the position of the wave fronts tends to zero. On the other hand the two solutions diverge in the sense that for large z, the difference in the times required for the two waves to pass through z tends to infinity.

Another way to describe this is that if one is interested in the position of the wave front as a function of time, the two solutions agree (for large t), whereas if one is interested in the time at which the wave front passes a certain position, then the two solutions do not agree (for large z).

Physically, the fact that the solution (78) lags behind the Marshak wave can be understood by interpreting the term εv^4 in (31) as corresponding to a nonlinear addition to the heat capacity given by $4\varepsilon v^3$. Then, since the background material heats up to a lower temperature for a given energy input, the radiative wave front feels a smaller driving potential, and hence must move slower.

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