3.4 Stability analysis

3.4.1 Exercises

1. Let X^* be the set of distinguishable points, i.e. the set of equivalence classes of distance-zero points, in a pseudometric space X. Show that X^* is the terminal object in a (relatively small) category that also includes X as an object.

2. Define the radius $r(X) = \min\{D_X(x, X \setminus \{x\}) \mid x \in X\}$.

   a) Prove that $r : \mathsf{FinPMet} \to \mathbb{R}_{\geq 0}$ is continuous with respect to a suitable pseudometric on $\mathsf{FinPMet}$.

   b) Rigorously restate the assertion that $\max\{d_X(\operatorname{minmax}(X), x) \mid x \in X \setminus \{\operatorname{minmax}(X)\}\}$ varies continuously with perturbations in X. Prove that this rigorous statement is correct.

3. Let $\mathcal{D}$ be the set of finite decompositions of $[0, 1]$ into intervals of non-negative length, i.e. $0 = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n = 1$ for some $n \in \mathbb{N}$.

   a) Prove by construction that every $D \in \mathcal{D}$ is obtained as the (nonincreasing) sequence $m_i = \max\{\min\{d(x, \ell) \mid \ell \in L_i\} \mid x \in X^* \setminus L_i\}$ of a maxmin procedure on a finite pseudometric space X seeded with ℓ_0 so that $\max\{d(x, \ell_0) \mid x \in X^* \setminus \{\ell_0\}\} = 1$. Is it enough to consider only finite metric spaces?

   b) Define a reasonable metric or pseudometric on $\mathcal{D}$. Which is more appropriate?

   c) Prove that the map $(X, \ell_0) \mapsto D$ from part (a) is stable when $|X| \leq 3$.

4. Let $\operatorname{Par}(n)$ denote the set of *partitions* of n, $\operatorname{Par} = \bigsqcup_n \operatorname{Par}(n)$, and $\operatorname{Par}_{m \times n}$ denote the set of partitions whose Young diagrams are contained in the $m \times n$ rectangle; and let $\operatorname{Comp}(n)$ denote the set of *compositions* of n.

   a) Prove that, for any $x \in X \in \mathsf{FinPMet}$ with $N = |X|$, $Q^{\pm}(x, X) \in \operatorname{Par}_{N \times N}$.

   b) Furthermore prove that $Q^+(x, X) \in \operatorname{Comp}(N)$.

5. Define a *move* in $\operatorname{Comp}(n)$ to be of one of the forms

$$(\ldots, m, 1, n-1, \ldots) \leftrightarrow (\ldots, m, n, \ldots) \leftrightarrow (\ldots, m-1, 1, n, \ldots)$$

   Define the "rank-radius" $s(X) = \min\{Q^+(x, X) \mid x \in X\}$.

   a) Prove that a sufficiently small perturbation of any $x \in X \setminus \operatorname{fl}(X)$ produces at most one move difference in $Q^+(x, X)$, and that there exists such a perturbation for each such x.

   b) Suppose that X^* is in general position. Prove that then a sufficiently small perturbation of any $x \in X$ produces at most one move difference in $s(X)$, and that there exists such a perturbation for each such x.