Question 3a - Exercise 4.1.3

b. $f(x) = 1/(x^2 - 4)$ Not a function. If x is -2 or 2, 1/0 is not defined.

c. $f(x) = \sqrt{x^2}$ It is a function and is well-defined.

Question 3b - Exercise 4.1.5

b. {4, 9, 16, 25}

d. {0, 1, 2, 3, 4, 5}

h. {(1,1), (2,1), (3,1), (1,2), (2,2), (3,2), (1,3), (2,3), (3,3)}

i. {(1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,2), (3,3), (3,4)}

l. { $\emptyset$, {2}, {3}, {2, 3} }

Question 4 I. a - Exercise 4.2.2

c. $h : \mathbf{Z} \to \mathbf{Z}$. $h(x) = x^3$
one-to-one
but not onto because for all x in $\mathbf{Z}$ there exists no $h(x) = 2$

g. $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$, $f(x, y) = (x + 1, 2y)$
one-to-one
but not onto because for all (x, y) in $\mathbf{Z} \times \mathbf{Z}$ there exists no $f(x, y) = (1, 1)$

k. $f : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$, $f(x, y) = 2^x + y$.
not one-to-one because $f(0, 2) = f(1, 1) = 3$
and not onto because $f(x, y) = 0$ does not exist for $\mathbf{Z}^+ \times \mathbf{Z}^+$

Question 4 I. b - Exercise 4.2.4

b. $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.
not one-to-one because $f(000) = f(100) = 100$
and not onto because for all strings x in $\{0, 1\}^3$ there is no $f(x) = 000$.

c. $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example $f(011) = 110$.
both one-to-one and onto

d. $f : \{0, 1\}^3 \to \{0, 1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example,$f(100) = 1001$.
one-to-one
but not onto because for all strings x in $\{0, 1\}^3$ there is no $f(x) = 1000$.

g. Let A be defined to be the set {1, 2, 3, 4, 5, 6, 7, 8} and let B = {1}. $f : P(A) \to P(A)$. For $X \subseteq A, f(X) = X - B$. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.
not one-to-one because for X = {2, 3, 4} and X = {1, 2, 3, 4}, f(X) = {2, 3, 4}
and not onto because for any $X \subseteq A$, there is no solution to $F(X) = \{1, 2, 3, 4\}$

Question 4 II.

Give an example of a function on the set of integers to the set of positive integers ($f : \mathbf{Z} \to \mathbf{Z}^+$) that is:

a. one-to-one, but not onto
$$f(x) = \begin{cases} \text{-2x} & \text{for } x < 0 \\ \text{2x + 3} & \text{for } x \geq 0 \end{cases}$$

b. onto, but not one-to-one
$f(x) = |x| + 1$

c. one-to-one and onto
$$f(x) = \begin{cases} (\,|x| \;\; \times 2) - 1 & \text{for } x < 0 \\ (\text{x } \times 2) + 1 & \text{for } x \geq 0 \end{cases}$$

d. neither one-to-one nor onto
$f(x) = x^2 + 2$ will map any integer (positive or negative) to the positive set of integers, but not one-to-one as x = -1 and x = 1 both map to 3, nor onto as not all positive integers are the offsets of perfect squares

Question 5a - Exercise 4.3.2

c. $f : \mathbf{R} \to \mathbf{R}$. $f(x) = 2x + 3$
$f^{-1}(x) = \frac{x-3}{2}$

d. Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. $f : P(A) \to \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. For $X \subseteq A, f(X) = |X|$. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.
The function f is not one-to-one, so f^{-1} is not well-defined.

g. $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$.
$f^{-1}(x)$ is obtained by taking the input string and reversing the bits.

i. $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$, $f(x, y) = (x + 5, y - 2)$
$f^{-1}(x, y) = (x - 5, y + 2)$

Question 5b - Exercise 4.4.8

The domain and target set of functions f, g, and h are Z. The functions are defined as:

- $f(x) = 2x + 3$
- $g(x) = 5x + 7$
- $h(x) = x^2 + 1$

c. $f \circ h$
$f \circ h(x) = 5(x^2 + 1) + 7 = 5x^2 + 5 + 7 = 5x^2 + 12$

d. $h \circ f$
$h \circ f(x) = (2x + 3)^2 + 1 = 4x^2 + 12x + 10$

Question 5c - Exercise 4.4.2

Consider three functions f, g, and h, whose domain and target are **Z**. Let

$f(x) = x^2$ $\qquad g(x) = 2^x$ $\qquad h(x) = \lceil \frac{x}{5} \rceil$

b. Evaluate $f \circ h(52) = 121$

c. Evaluate $g \circ h \circ f(4) = 16$

d. Give a mathematical expression for $h \circ f = \left\lceil \frac{x^2}{5} \right\rceil$.

Question 5d - Exercise 4.4.6

Define the following functions f, g, and h:

- $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.
- $g : \{0, 1\}^3 \to \{0, 1\}^3$. The output of g is obtained by taking the input string and reversing the bits. For example, $g(011) = 110$.
- $h : \{0, 1\}^3 \to \{0, 1\}^3$. The output of h is obtained by taking the input string x, and replacing the last bit with a copy of the first bit. For example, $h(011) = 010$.

c. What is $h \circ f(010)$?
(111)

d. What is the range of $h \circ f$?
{(101), (111)}

e. What is the range of $g \circ f$?
$\{(001), (101), (111)\}$

Question 5e - Exercise 4.4.4 (Extra Credit)

Let $f : X \to Y$ and $g : Y \to Z$ be two functions.

c. Is it possible that f is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

No. Given that there exists 2 or more x such that $f(x_1) = f(x_2)$, we conclude there exists 2 or more cases such that $g(f(x_1)) = g(f(x_2))$, which precludes a one-to-one mapping.

d. Is it possible that g is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

Yes. As demonstrated in question (a) g may not be one-to-one, but $g \circ f$ may be.