Question 5

Use the definition of $\Theta$ in order to show the following:

**Definition:**

Let f and g be two functions $Z^+$ to $Z^+$
$f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$.

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a. $5n^3 + 2n^2 + 3n = \Theta(n^3)$

**Proof that $f$ is $O(g)$**

$f(n) = 5n^3 + 2n^2 + 3n$
$g(n) = n^3$

**Claim $f = O(g)$:**

1. Select $c = 10$ and $n_0 = 1$. We will show that for any $n \geq 1$, $f(n) \leq 10g(n)$.
2. Explicitly, $5n^3 + 2n^2 + 3n \leq 10n^3$.

**Facts:**
1. Since $n \geq 1$, $n \leq n^2 \leq n^3$.

**Substitutions:**

| | |
|---|---|
| $f(n) = 5n^3 + 2n^2 + 3n$ | ...now given fact 1, substitute $n^3$ for both $n^2$ and $n$... |
| $f(n) \leq 5n^3 + 2n^3 + 3n^3$ | ...now adding factors of $n^3$ we obtain... |
| $f(n) \leq 10n^3$ | ...add observing $g(n) = n^3$ we substitute and obtain... |
| $f(n) \leq 10g(n)\blacksquare$ | |

**Proof that $f$ is $\Omega(g)$**

$f(n) = 5n^3 + 2n^2 + 3n$
$g(n) = n^3$

**Noting:**

If $f(n)$ is a polynomial of degree $k$, then $f = \Omega(n^k)$ only if the coefficient of the $n^k$ term in $f$ (call it $a_k$) is positive. Here are combinations for $c$ and $n_0$ that suffice as a witness to show that $f = \Omega(n^k)$.

If $f$ has no negative coefficients, then $c = a_k$ and $n_0 = 1$ suffice. If $f$ has negative coefficients (but $a_k > 0$), then let $A$ be the sum of the absolute values of the negative coefficients in $f(n)$. The choices $c = a_k/2$ and $n_0 = max\{1, 2A/(a_k)\}$ are sufficient.

**Claim $f = \Omega(g)$:**

1. Noting the polynomial has all positive coefficients, $c = a_k$ and $n_0 = 1$ will suffice.
2. Select $c = 5$ and $n_0 = 1$. We will show that for any $n \geq 1$, $f(n) \geq 5g(n)$.
2. Explicitly, $5n^3 + 2n^2 + 3n \geq 5n^3$.

**Facts:**
1. Since $n \geq 1$, $3n \geq 0$.
2. Since $n \geq 1$, $2n^2 \geq 0$.
3. Since $n \geq 1$, $5n^3 \geq 0$.

**Building the inequality:**
Starting with fact 1...

| | |
|---|---|
| $3n \geq 0$ | ...now using fact 2 add the inequalities to obtain... |
| $2n^2 + 3n \geq 0$ | ...now using fact 3, we can add $5n^3$ to both sides of the inequality... |
| $5n^3 + 3n^2 + 3n \geq 5n^3$ | ...fact 3 also ensures sign of inequality does not change... |

Now observing $g(n) = n^3$ and $f(n) = 5n^3 + 2n^2 + 3n$ we substitute and obtain...
$f(n) \geq 5g(n)$■

b. $\sqrt{7n^2 + 2n - 8} = O(n)$

**Proof that $f$ is $O(g)$**

$f(n) = \sqrt{7n^2 + 2n - 8}$
$g(n) = n$

**Claim $f = O(g)$:**

1. Select $c = 3$ and $n_0 = 1$. We will show that for any $n \geq 1$, $f(n) \leq 3g(n)$.
2. Explicitly, $\sqrt{7n^2 + 2n - 8} \leq 3n$.

**Facts:**
1. Since $n \geq 1$, $n \leq n^2$.
2. $0 \geq -8$
3. Since $n^2 \geq 1$, $9n^2 - 8 \geq 1$

**Substitutions:**

| | |
|---|---|
| $f(n) = \sqrt{7n^2 + 2n - 8}$ | ...now given fact 1, substitute $n^2$ for $n$... |
| $f(n) \leq \sqrt{7n^2 + 2n^2 - 8}$ | ...now consolidating terms we obtain... |
| $f(n) \leq \sqrt{9n^2 - 8}$ | ...now given fact 2 we obtain... |
| $f(n) \leq \sqrt{9n^2}$ | ...and given fact 3, we can take square root to obtain... |
| $f(n) \leq 3n$ | ...add observing $g(n) = 3n$ we substitute and obtain... |
| $f(n) \leq 3g(n)\blacksquare$ | |

**Proof that $f$ is $\Omega(g)$**

$f(n) = \sqrt{7n^2 + 2n - 8}$
$g(n) = n$

**Noting:**

If $f(n)$ is a polynomial of degree $k$, then $f = \Omega(n^k)$ only if the coefficient of the $n^k$ term in $f$ (call it $a_k$) is positive. Here are combinations for $c$ and $n_0$ that suffice as a witness to show that $f = \Omega(n^k)$.

If $f$ has no negative coefficients, then $c = a_k$ and $n_0 = 1$ suffice. If $f$ has negative coefficients (but $a_k > 0$), then let $A$ be the sum of the absolute values of the negative coefficients in $f(n)$. The choices $c = a_k/2$ and $n_0 = max\{1, 2A/(a_k)\}$ are sufficient.

**Claim $f = \Omega(g)$:**

1. Given the presence of negative coefficients (but $a_k > 0$), we calculate $A = \sqrt{|-8|}$.
2. Additionally, $a_k = \sqrt{7}$.
3. As such, we select $c = \frac{\sqrt{7}}{2}$ and $n_0 = max\{1, \frac{2(\sqrt{8})}{\sqrt{7}}\}$ which leaves us with $n_0 = \frac{2(\sqrt{8})}{\sqrt{7}}$ .
4. We will show that for any $n \geq \frac{2(\sqrt{8})}{\sqrt{7}}$, $f(n) \geq \frac{\sqrt{7}}{2} g(n)$.
5. Explicitly, $\sqrt{7n^2 + 2n - 8} \geq \frac{\sqrt{7}}{2} n$.

**Building the inequality:**

| | |
|---|---|
| $f(n) = \sqrt{7n^2 + 2n - 8}$ | ...now factoring out $n^2$ we obtain... |
| $f(n) = \sqrt{n^2(7 + \frac{2}{n} - \frac{8}{n^2})}$ | ...now we can take the $n^2$ out of root to obtain... |
| $f(n) = n\sqrt{7 + \frac{2}{n} - \frac{8}{n^2}}$ | ...knowing the limit of a sum is the sum of the limits... |
| $f(n) \geq \sqrt{7}n$ | ...as it will never obtain this value (asymptotic)... |
| $f(n) \geq \frac{\sqrt{7}n}{2}$ | ...and diving by a positive 2 maintains the inequality... |

Now observing $g(n) = n$ we substitute and obtain...
$f(n) \geq \frac{\sqrt{7}}{2} g(n) \blacksquare$