Question 5

Use the definition of Θ in order to show the following:

Definition:

Let f and g be two functions Z^+ to Z^+
$f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$.

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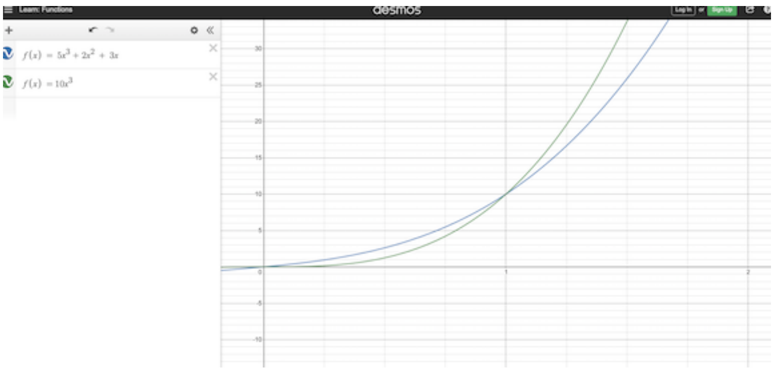
a. $5n^3 + 2n^2 + 3n = \Theta(n^3)$

Proof that f is $O(g)$

$f(n) = 5n^3 + 2n^2 + 3n$
$g(n) = n^3$

Claim $f = O(g)$:

1. Select $c = 10$ and $n_0 = 1$. We will show that for any $n \geq 1$, $f(n) \leq 10g(n)$.
2. Explicitly, $5n^3 + 2n^2 + 3n \leq 10n^3$.



Facts:
1. Since $n \geq 1$, $n \leq n^2 \leq n^3$.

Substitutions:

$f(n) = 5n^3 + 2n^2 + 3n$	...now given fact 1, substitute n^3 for both n^2 and n...
$f(n) \leq 5n^3 + 2n^3 + 3n^3$	...now adding factors of n^3 we obtain...
$f(n) \leq 10n^3$	...add observing $g(n) = n^3$ we substitute and obtain...
$f(n) \leq 10g(n)\blacksquare$	

Proof that f is $\Omega(g)$

$f(n) = 5n^3 + 2n^2 + 3n$
$g(n) = n^3$

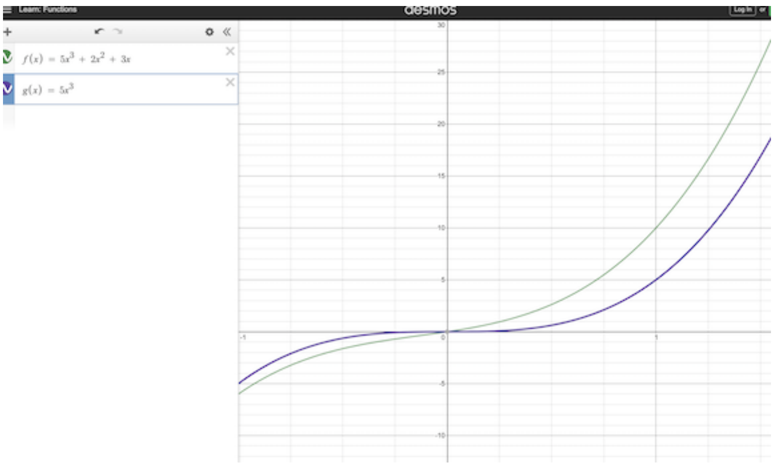
Noting:

If $f(n)$ is a polynomial of degree k, then $f = \Omega(n^k)$ only if the coefficient of the n^k term in f (call it a_k) is positive. Here are combinations for c and n_0 that suffice as a witness to show that $f = \Omega(n^k)$.

If f has no negative coefficients, then $c = a_k$ and $n_0 = 1$ suffice. If f has negative coefficients (but $a_k > 0$), then let A be the sum of the absolute values of the negative coefficients in $f(n)$. The choices $c = a_k/2$ and $n_0 = max\{1, 2A/(a_k)\}$ are sufficient.

Claim $f = \Omega(g)$:

1. Noting the polynomial has all positive coefficients, $c = a_k$ and $n_0 = 1$ will suffice.
2. Select $c = 5$ and $n_0 = 1$. We will show that for any $n \geq 1$, $f(n) \geq 5g(n)$.
2. Explicitly, $5n^3 + 2n^2 + 3n \geq 5n^3$.



Facts:
1. Since $n \geq 1$, $3n \geq 0$.
2. Since $n \geq 1$, $2n^2 \geq 0$.
3. Since $n \geq 1$, $5n^3 \geq 0$.

Building the inequality:
Starting with fact 1...

$3n \geq 0$	...now using fact 2 add the inequalities to obtain...
$2n^2 + 3n \geq 0$	...now using fact 3, we can add $5n^3$ to both sides of the inequality...
$5n^3 + 3n^2 + 3n \geq 5n^3$	...fact 3 also ensures sign of inequality does not change...

Now observing $g(n) = n^3$ and $f(n) = 5n^3 + 2n^2 + 3n$ we substitute and obtain...
$f(n) \geq 5g(n)$■

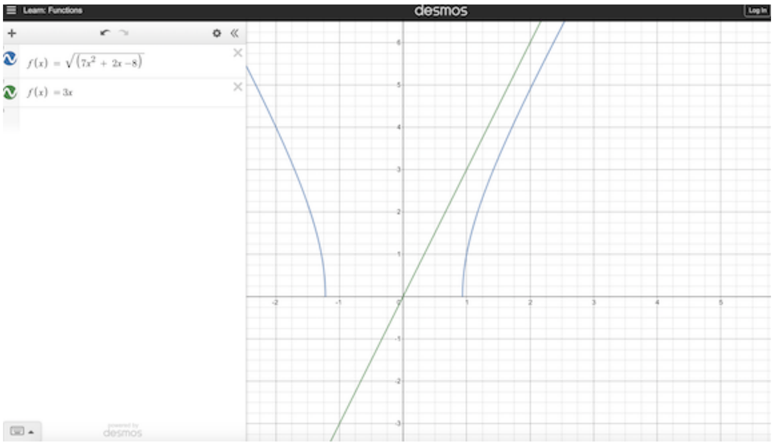
b. $\sqrt{7n^2 + 2n - 8} = O(n)$

Proof that f is $O(g)$

$f(n) = \sqrt{7n^2 + 2n - 8}$
$g(n) = n$

Claim $f = O(g)$:

1. Select $c = 3$ and $n_0 = 1$. We will show that for any $n \geq 1$, $f(n) \leq 3g(n)$.
2. Explicitly, $\sqrt{7n^2 + 2n - 8} \leq 3n$.



Facts:
1. Since $n \geq 1$, $n \leq n^2$.
2. $0 \geq -8$
3. Since $n^2 \geq 1$, $9n^2 - 8 \geq 1$

Substitutions:

$f(n) = \sqrt{7n^2 + 2n - 8}$	...now given fact 1, substitute n^2 for n...
$f(n) \leq \sqrt{7n^2 + 2n^2 - 8}$	...now consolidating terms we obtain...
$f(n) \leq \sqrt{9n^2 - 8}$	...now given fact 2 we obtain...
$f(n) \leq \sqrt{9n^2}$	...and given fact 3, we can take square root to obtain...
$f(n) \leq 3n$	...add observing $g(n) = 3n$ we substitute and obtain...
$f(n) \leq 3g(n)\blacksquare$	

Proof that f is $\Omega(g)$

$f(n) = \sqrt{7n^2 + 2n - 8}$
$g(n) = n$

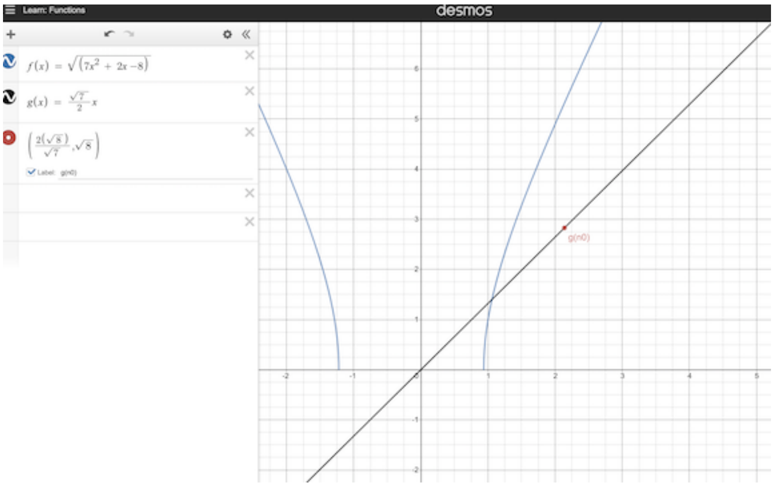
Noting:

If $f(n)$ is a polynomial of degree k, then $f = \Omega(n^k)$ only if the coefficient of the n^k term in f (call it a_k) is positive. Here are combinations for c and n_0 that suffice as a witness to show that $f = \Omega(n^k)$.

If f has no negative coefficients, then $c = a_k$ and $n_0 = 1$ suffice. If f has negative coefficients (but $a_k > 0$), then let A be the sum of the absolute values of the negative coefficients in $f(n)$. The choices $c = a_k/2$ and $n_0 = max\{1, 2A/(a_k)\}$ are sufficient.

Claim $f = \Omega(g)$:

1. Given the presence of negative coefficients (but $a_k > 0$), we calculate $A = \sqrt{|-8|}$.
2. Additionally, $a_k = \sqrt{7}$.
3. As such, we select $c = \frac{\sqrt{7}}{2}$ and $n_0 = max\{1, \frac{2(\sqrt{8})}{\sqrt{7}}\}$ which leaves us with $n_0 = \frac{2(\sqrt{8})}{\sqrt{7}}$.
4. We will show that for any $n \geq \frac{2(\sqrt{8})}{\sqrt{7}}$, $f(n) \geq \frac{\sqrt{7}}{2} g(n)$.
5. Explicitly, $\sqrt{7n^2 + 2n - 8} \geq \frac{\sqrt{7}}{2} n$.



Building the inequality:

$f(n) = \sqrt{7n^2 + 2n - 8}$ ...now factoring out n^2 we obtain...

$f(n) = \sqrt{n^2(7 + \frac{2}{n} - \frac{8}{n^2})}$ ...now we can take the n^2 out of root to obtain...

$f(n) = n\sqrt{7 + \frac{2}{n} - \frac{8}{n^2}}$ ...knowing the limit of a sum is the sum of the limits...

$f(n) \geq \sqrt{7}n$ ...as it will never obtain this value (asymptotic)...

$f(n) \geq \frac{\sqrt{7}n}{2}$ ...and diving by a positive 2 maintains the inequality...

Now observing $g(n) = n$ we substitute and obtain...
$f(n) \geq \frac{\sqrt{7}}{2} g(n) \blacksquare$