Adaptive Filters Homework 1

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1. Let $\{u_i\}_{i=1}^r$ be r linearly independent $M \times 1$ vectors, and $\{v_i\}_{i=1}^r$ be r linearly independent $N \times 1$ vectors. Show that the $M \times N$ matrix A given by:

   $$A = \sum_{i=1}^{r} u_i v_i^H$$

   Proof:
   Let $y \in \mathcal{R}_A$, then for some $x \in M^{N\times 1}$
   $y = Ax = \left(\sum_{i=1}^r u_i v_i^H\right) x$. Since matrix multiplication is distributive,
   $y = Ax = \sum_{i=1}^r u_i v_i^H x$. Since matrix multiplication is associative,
   $y = Ax = \sum_{i=1}^r u_i (v_i^H x)$. Since $v_i^H x$ is scalar,
   $y = Ax = \sum_{i=1}^r (v_i^H x) u_i$
   Let V be the matrix obtained by concatenating all v_i vertically. Since all rows are lienarly independent, the system is consistent. Therefore if we let $Vx = c$, we can choose an x producing any vector and we have: $y = \sum_{i=1}^r c_i u_i$, so $\mathcal{R}_{\mathcal{A}}$ is spanned by $\{u_i\}_{i=1}^r$.
   Since $\{u_i\}_{i=1}^r$ is linearly independent, it is a basis for $\mathcal{R}_A$ and
   $rank\ A = dim\ \mathcal{R}_{\mathcal{A}} = r$ ■

2. Done

3. The matrix inversion lemma is:

   $$(B^{-1} + CD^{-1}C^H)^{-1} = B - BC(D + C^H BC)^{-1} C^H B$$

   (a) Given $R = \delta I + \alpha u u^H$, use the matrix inversion lemma to find a simplified formula for R^{-1}.
   Let $D = 1/\alpha$, $B = 1/\delta \times I$, and $C = u$ then we have:

$R = B + D^{-1}CC^H$. Since D is scalar, $R = B + CD^{-1}C^H$.
Leveraging the lemma we have
$R^{-1} = 1/\delta \times I - 1/\delta \times u(1/\alpha + 1/\delta u^H u)^{-1} u^H I/\delta$
Since $(1/\alpha + 1/\delta u^H u)$ is a scalar,
$R^{-1} = I/\delta - \frac{1}{\delta^2(1/\alpha+(1/\delta)u^H u)} uu^H$

(b) Give a specific condition for α under which R^{-1} would not exist.
Setting the denominator to zero we have
$\frac{1}{\alpha} = \frac{1}{\delta} uu^H \implies \alpha = \frac{\delta}{uu^H}$

4. Consider a signal u formed as a superposition of two sinewaves at respective frequencies w_1, w_2, and additive white noise. Specifically, with:

$$s_i = \frac{1}{\sqrt{M}} \begin{bmatrix} 1 \\ e^{-j\omega_i} \\ \vdots \\ e^{-j(M-1)\omega_i} \end{bmatrix}$$

$$u = \alpha_1 s_1 + \alpha_2 s_2 + v$$

where $\alpha_i, i = 1, 2$ are uncorrelated random complex amplitudes (actually they represent amplitude and phase), and $v(n)$ is $0 - mean$ white noise vector (the components of v are uncorrelated with each other and the α_i values). Assume $E(\alpha_i) = 0$, which would be consistent for example by assuming each has a phase uniform from 0 to 2π, and:

$$E(|\alpha_i|^2) = \sigma_i^2, i = 1, 2$$
$$E(vv^H) = \sigma_v I$$

Also assume $\sigma_1^2 > \sigma_2^2$, and that $|\omega_1 - \omega_2| = \frac{2\pi}{M} k$ for some integer k not a multiple of M

(a) Check that s_1, s_2 are unit orthogonal vectors.

$$\langle s_1, s_2 \rangle = \sum_{n=0}^{M-1} \left(\frac{1}{\sqrt{M}} e^{-jn\omega_1} \frac{1}{\sqrt{M}} e^{jn\omega_2} \right) = \frac{1}{M} \sum_{n=1}^{M-1} e^{jn(\omega_2 - \omega_1)}$$
$$= \frac{1 - e^{jM(\omega_2-\omega_1)}}{1 - e^{j(\omega_2-\omega_1)}} = \frac{1 - e^{j2\pi(k_2-k_1)}}{1 - e^{j(\omega_2-\omega_1)}}$$

Since $\omega_2 - \omega_1$ is not a multiple of 2π the denominator is not 0 and since $k_2 - k_1$ is an integer, the numerator is $1 - 1 = 0$, so the vectors are orthogonal.
Similarly, $\langle s_i, s_i \rangle =$
$frac1M \sum_{n=1}^{M-1} e^{jn(\omega_i - \omega_i)} = \frac{1}{M} \sum_{n=1}^{M-1} 1 = 1$
■

(b) Verify the following formula $R = E(u(n)u^H(n))$:

$$R = \sigma_1^2 s_1 s_1^H + \sigma_2^2 s_2 s_2^H + \sigma_v I$$

$$\begin{aligned} R &= E(u(n)u^H(n)) \\ &= E((\alpha_1 s_1 + \alpha_2 s_2 + v)(\alpha_1 s_1 + \alpha_2 s_2 + v)^H) \\ &= E((\alpha_1 s_1 + \alpha_2 s_2 + v)(\alpha_1^* s_1^H + \alpha_2^* s_2 + v^H)) \end{aligned}$$

Note that any cross terms (v multiplied by a_i or a_i multiplied by $a_j, j \neq i$) have an expected value of 0 since they are uncorrelated and have mean 0. Since expectation is linear, we can seperate expressions containing these terms into their own expectation and move the multiplicative constant outside thus completely eliminating the term leaving:

$$\begin{aligned} R &= E(u(n)u^H(n)) \\ &= E(|\alpha_1|^2 s_1 s_1^H + |\alpha_2|^2 s_2 s_2^H + vv^H) \\ &= E(|\alpha_1|^2 s_1 s_1^H) + E(|\alpha_2|^2 s_2 s_2^H) + E(vv^H) \\ &= s_1 s_1^H E(|\alpha_1|^2) + s_2 s_2^H E(|\alpha_2|^2) + E(vv^H) \\ &= \sigma_1^2 s_1 s_1^H + \sigma_2^2 s_2 s_2^H + \sigma_v^2 I \end{aligned}$$

(c) Show that s_1, s_2 are each eigenvectors of R, and specify the corresponding eigenvalues

$$
\begin{aligned}
Rs_1 &= (\sigma_1^2 s_1 s_1^H + \sigma_2^2 s_2 s_2^H + \sigma_v^2 I)s_1 \\
&= \sigma_1^2 s_1 s_1^H s_1 + \sigma_2^2 s_2 s_2^H s_1 + \sigma_v^2 s_1 \\
&= \sigma_1^2 s_1 + \sigma_2^2 s_2 s_2^H s_1 + \sigma_v^2 s_1 \text{ (by normality)} \\
&= \sigma_1^2 s_1 + 0 + \sigma_v^2 s_1 \text{ (by orthogonality)} \\
&= (\sigma_1^2 + \sigma_v^2) s_1
\end{aligned}
$$

Thus s_1 is an eigenvector for R and $\sigma_1^2 + \sigma_v^2$ is the corresponding eigenvalue.
Similarly, S_2 is an eigenvector for R and $\sigma_2^2 + \sigma_v^2$ is the corresponding eigenvalue.
■

(d) If q is a unit vector such that $q \perp s_i, i = 1, 2$, show that it is an eigenvector of R and specify the corresponding eigenvalue.

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\begin{aligned}
Rq &= (\sigma_1^2 s_1 s_1^H + \sigma_2^2 s_2 s_2^H + \sigma_v^2 I)q \\
&= \sigma_1^2 s_1 s_1^H q + \sigma_2^2 s_2 s_2^H q + \sigma_v^2 q \\
&= 0 + 0 + \sigma_v^2 q \text{ (by orthogonality)} \\
&= (\sigma_v^2) q
\end{aligned}
$$

So q is an eigenvector of R and the corresponding eigenvalue is σ_v^2
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(e) Consider the orthonormal vectors $\{s_1, s_2, q_3, \ldots, q_M\}$. That is, we take vectors $q_i, 3 \le i \le M$, which are unit length, mutually orthogonal, and orthogonal to $s_i, i = 1, 2$. Examine this set to list all distinct eigenvalues of R, with multiplicity, and identify a basis for the corresponding eigenspaces.

Since R is an $M \times M$ matrix, it has M eigenvalues. As we showed, two of those eigenvalues are given by $\sigma_1^2 + \sigma_v^2$ and $\sigma_2 + \sigma_v^2$. For a given q_i, it must be an eigenvector with eigenvalue σ_v^2. Since each q_i are mutually orthogonal, they are linearly independent and therefore they each correspond to one eigenvalue in the spectrum. Therefore σ_v^2 has multiplicity $M - 2$ which accounts for all eigenvalues.