Functional Analysis Homework 2: Solutions

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1. Prove that $\|\vec{z}\|_\infty = max\{|z_k|, 1 \leq k \leq n\}$ (denoted $max|z_k|$) is a norm on $\mathbb{C}^n$
   Proof:
   Since $|z_k| \geq 0 \; \forall z_k \in \mathbb{C}, max|z_k| \geq 0 \rightarrow \|z\|_\infty \geq 0$ (1)
   $\|z\|_\infty = 0 \rightarrow |z_k| \leq 0 \; \forall k$. Since $|z_k| \geq 0$, then by trichotemy, $|z_k| = 0 \; \forall k \rightarrow z_k = 0 \; \forall k$ (2)
   First we show that multiplication of positive scalars by a positive scalar commutes with max:
   $A = max\{a_k\} \Leftrightarrow A \geq a_k \Leftrightarrow \alpha A \geq \alpha a_k \Leftrightarrow \alpha A = max\{\alpha a_k\}$ Now,
   $\|\alpha z\|_\infty = max\{|\alpha z_k|\} = max\{|\alpha||z_k|\} = |\alpha| max\{|z_k|\} = |\alpha| \, \|z_k\|_\infty$ (3)
   By the triangle inequality, $|x_k + y_k| \leq |x_k| + |y_k| \rightarrow$
   $\|\vec{x} + \vec{y}\|_\infty = max\{|x_k + y_k|\} \leq max\{|x_k| + |y_k|\}$
   Let $C = \{|x_i| + |y_j|; x_i, y_j \in \vec{x}, \vec{y}; i, j \leq n\}$
   Let $c = \{|x_i| + |y_i|; x_i, y_i \in \vec{x}, \vec{y}; i \leq n\}$
   since $c \subseteq C$ and both sets are finite, then by Homework 3 (Math Analysis) $\sup c \leq \sup C \rightarrow max\{c\} \leq max\{C\}$, or
   $max\{c\} = max\{|x_k| + |y_k|\} \leq max\{C\} = max\{|x_k|\} + max\{|y_k|\} = \|x\|_\infty + \|y\|_\infty$ (4) ■

2. $f \in l^\infty(\mathbb{Z}) = \{f : \mathbb{Z} \rightarrow \mathbb{C}$ such that $f(\mathbb{Z})$ is a bounded subset of $\mathbb{C}\}$
   $\|f\|_\infty = \sup\{|f(k)| : k \in \mathbb{Z}\}$. Prove that $\|.\|_\infty$ is a norm on $l^\infty(\mathbb{Z})$.
   Proof:
   Since $|f(k)| \geq 0, \sup\{|f(k)|\} \geq 0 \rightarrow \|f\|_\infty \geq 0$ (1)
   $\|f\|_\infty = 0 \rightarrow \sup\{|f(k)|\} = 0 \rightarrow |f(k)| \leq 0$. Since $|f(k)| \geq 0$, by trichotemy we have $|f(k)| = 0$ (2)
   $\|\alpha f\|_\infty = \sup\{|\alpha f(k)|\} = \sup\{|\alpha||f(k)|\} = |\alpha| \sup\{|f(k)|\} = \alpha \, \|f\|_\infty$ (3)

By the triangle ineuqality we have
$\|f+g\|_\infty = \sup\{|f(k)+g(k)|\} \leq \sup\{|f(k)|+|g(k)|\}$
Let $C = \{|f(i)|+|g(j)|; i,j \in \mathbb{Z}\}$
Let $c = \{|f(i)|+|g(i)|; i \in \mathbb{Z}\}$
Since $c \subseteq C$ and both sets are bounded, then by Homework 3 (Math Analysis) $\sup c \leq \sup C \rightarrow \|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ (4) ■