Mathematical Analysis Homework 4: Solutions

Cory Nezin

September 20, 2017

1. Let $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^m$ be Cauchy with respect to $\| \|_{\infty}$. Prove that $\{a_{nk}\}_{n=1}^{\infty} \subseteq \mathbb{R}$ is Cauchy for each $k = 1, ..., m$ where $a_n = (a_{n1}, a_{n2}, ..., a_{nm})$
   Proof:
   $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^m$ is Cauchy $\rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n, p > N$, $\|a_n - a_p\|_{\infty} < \epsilon$.
   Since $a_n - a_p \in \mathbb{R}^m$ is finite,

$$
\begin{aligned}
\|a_n - a_p\|_{\infty} &= \sup\{|a_n - a_p|; n, p > N\} \\
&= max\{|a_n - a_p|; n, p > N\} \\
&= max\{|a_{n1} - a_{p1}|, |a_{n2} - a_{p2}|, ..., |a_{np} - a_{pm}|\} \\
&< \epsilon
\end{aligned}
$$

   Or, $|a_{nk} - a_{pk}| < \epsilon\ \forall k \in \{1, ..., m\};\ \forall n, m > N$
   So $\{a_{nk}\}_{n=1}^{\infty}$ is Cauchy. ■

2. Prove that $(\mathbb{R}^m, \| \|_{\infty})$ is a complete normed linear space assuming that $\mathbb{R}$ is Cauchy complete.
   Proof:
   From the previous problem, if $\{a_n\}_{n=1}^{\infty}$ is Cauchy, then $\{a_{nk}\}_{n=1}^{\infty} \in \mathbb{R}$ is a Cauchy sequence for fixed $k \in \{1, ..., m\}$
   By Cauchy completeness of $\mathbb{R}$, this sequence necessarily converges to some value $a_{\infty k} \in \mathbb{R}$. This implies $\forall \epsilon > 0, \exists N \in \mathbb{N}, a_{\infty k} \in \mathbb{R}$ such that $|a_{\infty k} - a_{nk}| < \epsilon; n > N$ Therefore $\epsilon > |a_{\infty k} - a_{nk}|\ \forall k \in \{1, ..., m\} \rightarrow \epsilon > max\{|a_{n1} - a_{p1}|, |a_{n2} - a_{p2}|, ..., |a_{np} - a_{pm}|\} = \|a_{\infty} - a_n\|_{\infty}$
   Since $a_{\infty} \in \mathbb{R}^m$, $(\mathbb{R}^m, \| \|_{\infty})$ is Cauchy complete. ■