Functional Analysis Homework 1: Solutions

Cory Nezin

September 9, 2017

1. (a) Prove that

$$z_1\overline{z_2} = \sum_{k=0}^{3} \frac{j^k}{4}|z_1 + j^k z_2|^2 \text{ for } z_1, z_2 \in \mathbb{C}$$

Proof:
First note that

$$\sum_{k=0}^{3} j^k z = z\sum_{k=0}^{3} j^k = z(1 + j - 1 - j) = 0 \text{ for } z \in \mathbb{C}$$

and

$$\sum_{k=0}^{3} j^{2k} z = z\sum_{k=0}^{3} j^{2k} = z(1 - 1 + 1 - 1) = 0 \text{ for } z \in \mathbb{C}$$

$$\begin{aligned}
\sum_{k=0}^{3} \frac{j^k}{4}|z_1 + j^k z_2|^2 &= \frac{1}{4}\sum_{k=0}^{3} j^k(z_1 + j^k z_2)\overline{(z_1 + j^k z_2)} \\
&= \frac{1}{4}\sum_{k=0}^{3} j^k[z_1\overline{z_1} + j^{-k}z_1\overline{z_2} + j^k z_2\overline{z_1} + z_2\overline{z_2}] \\
&= 0 + \frac{1}{4}\sum_{k=0}^{3} z_1\overline{z_2} + 0 + 0 \\
&= z_1\overline{z_2}
\end{aligned}$$

■

(b) Let $(V, \langle \cdot, \cdot \rangle)$ be a complex inner product space. Prove that

$$\langle \vec{w}, \vec{v} \rangle = \sum_{k=0}^{3} \frac{j^k}{4} \|\vec{w} + j^k \vec{v}\|^2$$

Similar to the last proof:

$$\begin{aligned}
\sum_{k=0}^{3} \frac{j^k}{4} \|\vec{w} + j^k \vec{v}\|^2 &= \frac{1}{4} \sum_{k=0}^{3} j^k \langle \vec{w} + j^k \vec{v}, \vec{w} + j^k \vec{v} \rangle \\
&= \frac{1}{4} \sum_{k=0}^{3} j^k [\langle \vec{w}, \vec{w} \rangle + \langle \vec{w}, j^k \vec{v} \rangle + \langle j^k \vec{v}, \vec{w} \rangle + \langle \vec{v}, \vec{v} \rangle] \\
&= \frac{1}{4} \sum_{k=0}^{3} j^k [\langle \vec{w}, \vec{w} \rangle + j^{-k} \langle \vec{w}, \vec{v} \rangle + j^k \langle \vec{v}, \vec{w} \rangle + \langle \vec{v}, \vec{v} \rangle] \\
&= 0 + \frac{1}{4} \sum_{k=0}^{3} \langle \vec{w}, \vec{v} \rangle + 0 + 0 \\
&= \langle \vec{w}, \vec{v} \rangle
\end{aligned}$$

■