Functional Analysis Homework 3: Solutions

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Let $(E, \| \ \|) be a normed linear space.$

1. Porve that $| \, \|x\| - \|y\| \, | \leq \|x - y\|$
   Proof:
   By the triangle inequality, $\|x + y\| \leq \|x\| + \|y\|$
   So, $\|(x - y) + y\| \leq \|x - y\| + \|y\|$
   And $\|x - y\| \geq \|x\| - \|y\|$
   Similarly, $\|y - x\| \geq \|y\| - \|x\| \rightarrow \|x - y\| \geq -(\|x\| - \|y\|)$ Thus $\|x - y\| \geq | \, \|x\| - \|y\| \, |$ ■

2. If $\lim_{n\to\infty} x_n = x, x_n, x \in E \ \forall n$, then show that $\lim_{n\to\infty} \|x_n\| = \|x\|$.
   Proof:
   $\forall \epsilon > 0, \exists N_\epsilon$ such that $\forall n > N_\epsilon, \|x - x_n\| < \epsilon$
   By the last problem, this implies that $| \, \|x\| - \|x_n\| \, | \leq \epsilon$ under the same conditions. So $\lim_{n\to\infty} \|x_n\| = \|x\|$ ■

3. $\lim_{n\to\infty} \alpha_n = \alpha, \lim_{n\to\infty} x_n = x$ where $\alpha_n, \alpha \in \mathbb{C}$. Prove that $\lim_{n\to\infty} \alpha_n x_n = \alpha x$
   Proof:
   By definition of limits, $\forall \sqrt{\epsilon} > 0, \exists N_\epsilon$ such that $\forall n > N_{\sqrt{\epsilon}}$,
   $|\alpha_n - \alpha| < \sqrt{\epsilon}$
   $\|x_n - x\| < \sqrt{\epsilon}$
   And $|\alpha_n - \alpha| \, \|x_n - x\| < \epsilon$
   By properties of norms, this expression is equal to $\|(\alpha_n - \alpha)(x_n - x)\|$
   So $\lim_{n\to\infty}((\alpha_n - \alpha)(x_n - x)) = 0$
   Expanding the expression we have $\lim_{n\to\infty}(\alpha_n x_n - \alpha_n x - \alpha x_n + \alpha x) = 0$
   Which is equal to
   $\lim_{n\to\infty} \alpha_n x_n - \lim_{n\to\infty} \alpha_n x - \lim_{n\to\infty} \alpha x_n + \lim_{n\to\infty} \alpha x = 0 \rightarrow$

$\lim_{n\to\infty} \alpha_n x_n = x \lim_{n\to\infty} \alpha_n + \alpha \lim_{n\to\infty} x_n - \alpha x = 2\alpha x - \alpha x = \alpha x$ ■

4. State the converse of (2).
   If $\lim_{n\to\infty} \|x_n\| = \|x\|$, $\lim_{n\to\infty} x_n = x, x_n, x \in E\ \forall n$
   This is clearly not true. Take the case $x_n = (-1)^n$. The limit of the norms is 1 (the norms are always 1) but the sequence does not converge.
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