Mathematical Analysis Homework 3: Solutions

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1. Assume $\mathbb{R}$ is an ordered field satisfying least upper bound or supremum principle. Prove that $\mathbb{R}$ has the completeness property.

   Proof:
   Let $\emptyset \neq A \subseteq \mathbb{R}$ be bounded above and not below. Let $T = \{b \in \mathbb{R}; x \leq b \ \forall x \in A\}$
   By the supremum principle, A has a least upper bound in $\mathbb{R}$. That is, $\exists \alpha \in \mathbb{R}$ such that $\alpha \leq b \ \forall b \in T$.

   For any $x \in \mathbb{R}$ if $x \notin A$, $x > a \ \forall a \in A$ (since A is not bounded below) thus $x \in T$ Therefore A and T form a partition over $\mathbb{R}$, so either $\alpha \in S$ or $\alpha \in T$.

   If $\alpha \in S$ then since it the supremum of A, by Homework 2 it is also the maximum of S.
   If $\alpha \in T$ then since $\alpha \leq b \ \forall b \in T$ (by the supreum principle) α is the minimum of T.
   Thus $\mathbb{R}$ has the completeness property. ■

2. Let $\emptyset \neq A \subseteq B \subseteq \mathbb{R}$ and given that B is bounded, prove that A is bounded and

   (a) $\sup A \leq \sup B$

   (b) $\inf B \leq \inf A$

   Proof (A is bounded):
   B is bounded $\rightarrow \ \forall b \in B, |b| \leq M \in \mathbb{R}$.

$A \subseteq B \to \ \forall a \in A, a \in B \to |a| \leq M$. So A is bounded. ■

Proof (a):
$\forall b \in B, b \leq \sup B \to a \leq \sup B \ \forall a \in A \to \sup B$ is an upper bound for A. So $\sup A \leq \sup B$ ■
Proof (b):
$\forall b \in B, b \geq \inf B \to a \geq \inf B \ \forall a \in A \to \inf B$ is a lower bound for A. So $\inf A \geq \inf B$ ■