Mathematical Analysis Homework 1: Solutions

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Let

$$S = \{\gamma \in \mathbb{Q} : \gamma \leq 0 \text{ or } \gamma > 0 \text{ and } \gamma^2 < 2\}$$

$$T = \{\gamma \in \mathbb{Q} : \gamma > 0 \text{ and } \gamma^2 > 2\}$$

1. Prove that $s < t$ for $s \in S$ and $t \in T$
   If $s \leq 0$ then since $t > 0$, $t > s$
   If $s \geq 0$ then since $t^2 > 2$ and $s^2 < 2$, $t^2 > 2 > s^2 \rightarrow t^2 > s^2 \rightarrow t^2 - s^2 > 0 \rightarrow (t-s)(t+s) > 0$
   Since $s \geq 0$ and $t > 0$, $t + s > 0$, and therefore $t - s > 0$ or, $t > s$
   $\blacksquare$

2. Prove that if $t \in T$ then $\exists s \in T$ such that $s < t$
   Proof: Suppose
   $$s = t - \frac{t^2 - 2}{t + 2} = \frac{2(t+1)}{t+2}$$
   Let
   $$t = \frac{p}{q}; p, q \in \mathbb{Z}$$
   Then
   $$s = \frac{2(p/q + 1)}{p/q + 2} = \frac{2p + 2q}{p + 2q}$$
   which is rational.
   We have
   $$\left(\frac{2(t+1)}{t+2}\right)^2 = 4\left(\frac{t^2 + 2t + 1}{t^2 + 4t + 4}\right)$$

We wish to evaluate whether this is greater than 2 to see if the result is in T. Suppose

$$4\left(\frac{t^2+2t+1}{t^2+4t+4}\right) > 2$$

iff

$$\left(\frac{t^2+2t+1}{t^2+4t+4}\right) > 1/2$$

The numerator and denominator are both squares and therefore positive, so

$$t^2+2t+1 > t^2/2+2t+2$$
$$t^2 > 2$$

This is guaranteed by definition of T so s is in T. $t^2 > 2$, so $t^2-2 > 0$, and

$$\frac{(t^2-2)}{t+2} > 0 \rightarrow -\frac{(t^2-2)}{t+2} < 0$$

Therefore

$$t-\frac{t^2-2}{t+2} < t \rightarrow s > t \ \forall t \in T$$

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3. Prove that if $t \in S$ then $\exists s \in S$ such that $s > t$ The proof follows similarly: Suppose

$$s = t-\frac{t^2-2}{t+2} = \frac{2(t+1)}{t+2}$$

Let

$$t = \frac{p}{q}; p, q \in \mathbb{Z}$$

Then

$$s = \frac{2(p/q+1)}{p/q+2} = \frac{2p+2q}{p+2q}$$

which is rational.
We also have that

$$\left(\frac{2(t+1)}{t+2}\right)^2 = 4\left(\frac{t^2+2t+1}{t^2+4t+4}\right)$$

We wish to evaluate whether this is less than 2 to see if the result is in S. Suppose

$$4\left(\frac{t^2+2t+1}{t^2+4t+4}\right) \leq 2$$

iff

$$\left(\frac{t^2+2t+1}{t^2+4t+4}\right) \leq 1/2$$

$$t^2+2t+1 \leq t^2/2+2t+2$$

$$t^2 < 2$$

Since this condition is guaranteed, s is in S.
Since $t > 0$, then $t+2 > 0$, so

$$\frac{(t^2-2)}{t+2} \leq 0 \to -\frac{(t^2-2)}{t+2} \geq 0$$

for $t^2 < 2$ and $t > 0 \to t+2 > 0$

$$t - \frac{t^2-2}{t+2} > t \to s > t \;\forall t \in S$$

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