Mathematical Analysis Homework 2: Solutions

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1. Let $\emptyset \neq A \subseteq \mathbb{R}$. Suppose $\alpha = \sup A$ exists in $\mathbb{R}$. Prove that if $\alpha \in A$, then $\alpha = max\{A\}$. Moveover prove that α is unique.

   Proof:
   By the supremum principle, $a \leq \alpha \ \forall a \in A$.
   By the hypothesis, $\alpha \in A$.
   Thus $a = max\{A\}$. $\blacksquare$
   Moreover, suppose $b = max\{a\}$. Then $b \geq a \ \forall a \in A$.
   Since $\alpha \in A$, $b \geq \alpha$. Since $\alpha = \sup A$, $\alpha \geq b$.
   By trichotemy, if $b > \alpha$ then $\alpha \ngeq b$ thus $\alpha = b$. Therefore $b = \alpha$ and α is unique. $\blacksquare$

2. Prove that $a = \sup\{r \in Q : r < a\}$ for $a \in \mathbb{R}$. Proof:
   First we prove a useful lemma:
   For $a, b \in \mathbb{R}, \exists r \in \mathbb{Q}$ such that $b < r < a$.
   Proof:
   By corollary 1 to the Archimedian Principle: $\exists n \in \mathbb{N}$ such that $x > \frac{1}{n} \ \forall x > 0 \in \mathbb{R} \to (a - b) > \frac{1}{n} \to n(a - b) > 1$.
   By corollary 2 to the Archimedian Principle: $\exists m \in \mathbb{N}$ such that $nb < m < na \to b < \frac{m}{n} < a \to \exists \frac{m}{n} \in \mathbb{Q}$ such that $b < r < a$ $\blacksquare$

   Now since by definition of the set, a is an upper bound, we must only prove that is the smallest possible upper bound. Suppose $b < a$. By the Lemma, $\exists r \in \mathbb{Q}$ such that $b < r < a$. So b is not an upper bound and a is the least upper bound. $\blacksquare$