Mathematical Analysis Homework 6: Solutions

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1. Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(a, b) = |a - b|$
   Prove that:

   (a) $d(a, b) \geq 0$

   (b) $d(a, b) = 0 \implies a = b$

   (c) $d(a, b) = d(b, a)$

   (d) $d(a, b) \leq d(a, c) + d(c, b)$

   Proof (a):
   $d(a, b) = |a - b|$. Since $|x| \geq 0 \; \forall x \in \mathbb{R}, |a - b| > 0$. $\blacksquare$
   Proof (b, $\implies$):
   $0 = d(a, b) = |a - b| \implies a - b = 0 \implies a = b$
   Proof (b, $\impliedby$):
   $d(a, b) = |a - b| = |b - b| = |0| = 0$. $\blacksquare$
   Proof (c):
   $d(a, b) = |a - b| = |-1||a - b| = |(-1)(a - b)| = |b - a| = d(b, a)$. $\blacksquare$
   Proof (d):
   $d(a, b) = |a - b| = |a - c + (c - b)| \leq |a - c| + |c - b| = d(a, c) + d(c, b)$.
   $\blacksquare$

2. (ii) Let $1 < a \in \mathbb{R}$. Prove that $\{a^n\}_{n=1}^{\infty}$ is not bounded above.
   Proof:
   Suppose the sequence is bounded, then by the monotone convergence theorem, the sequence converges.
   So let $\lim_{n\to\infty} a^n = L$
   Then $\exists N \in \mathbb{N}$ such that $|a^n - L| < \epsilon \; \forall n \geq N \implies$
   $|a^{N+M+1} - L| < \epsilon$ for $M \geq 0 \implies$

$|a \times a^{N+M} - L| < \epsilon \implies$
$|a||a^{N+M} - \frac{L}{a}| < \epsilon (\text{since } a \neq 0) \implies$
$|a^{N+M} - \frac{L}{a}| < \frac{\epsilon}{|a|} < \epsilon$ (since $|a| > 1$) $\implies$
For any $m = N + M \geq N$, $|a^m - \frac{L}{|a|}| < \frac{\epsilon}{|a|} < \epsilon$
$\lim_{M\to\infty} a_M = \frac{L}{|a|}$
Since $a \neq 1$, this is a contradiction, so the sequence must not be bounded. ■
(i) Prove that $\{10^n\}_{n=1}^{\infty}$ is not bounded above.
Proof:
Since $\mathbb{R} \ni 10 > 1$, this is proved by 2 (ii). ■