Lecture Notes IV: More Quantum Mechanics

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1 More on Stern-Gerlach measurements

We will start today off with a more in depth discussion of the SG measuring apparatus. Throughout, we will continue to think of column vectors in the basis where

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

1.1 Spin causes the deflection

We have been successful at reproducing the outcome of the SG experiment simply by using a two dimensional vector space for our state vector. We know that one eigenstate of the SG operator is always deflected up, while the other eigenstate is always deflected down. We have not asked whether this state has any other physical meaning. In fact, the property that the SG device measures is something called intrinsic angular momentum or *spin*.

In classical mechanics, angular momentum is related to the speed at which an object rotates and to its inertia (a property describing how hard it is to make it rotate faster or slower). Angular momentum is a vector quantity $\vec{L}$ with components in each direction of three dimensional space. The three components of the angular momentum vector can be any real number.

In quantum mechanics, fundamental particles are point like objects. They have no spatial extent, and therefore the concept of rotation has no meaning. Nevertheless, all fundamental particles are found to possess an angular momentum. This is a purely quantum mechanical effect. Even if the particles were just tiny and not exactly, their surfaces would have to spin faster than the speed of light to have the observed angular momenta!

Angular momentum is a dimensionful quantity, so we will need to modify our expected eigenvalues to reflect this. The SG experiment is capable of measuring the component of the angular momentum on a single axis. Let's say the one we have been dealing with measures along $\hat{z}$. Careful analysis of the experiment shows that the component is aways found to be $\vec{L} \cdot \hat{z} = \pm\hbar/2$. We thus modify our Hermitian operator for the observable to reflect this:

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

This is our first glimpse of $\hbar$ in action. Particles with a maximum component of intrinsic angular momentum $\hbar s$ along a given axis are said to have *spin* s. For a particular with spin s, the possible values of any component of the intrinsic angular momentum are $s, s-1, s-2, \ldots, 2-s, 1-s, -s$.

Quick Question: What is the spin of the silver atom in the SG experiment?

1.2 Rotating the Stern-Gerlach device

Based on the discussion in the previous section, it seems reasonable that we could measure the other components of the angular momentum L by rotating the SG apparatus. For example, rotating by $\pi/2$ would allow us to measure the project along the x-axis. Measuring the spin along the other two axes of the coordinate system is done with the operators:

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

To find the projection of the angular momentum along any unit vector $\hat{v}$, we can use the operator

$$S_v = (\hat{v} \cdot \hat{x})S_x + (\hat{v} \cdot \hat{y})S_y + (\hat{v} \cdot \hat{z})S_z \tag{4}$$

2 Uncertainty Principles

In the SG apparatus, we found that the eigenstates of the measurement were different if we rotated the apparatus. Fundamentally, this is due to the fact that matrix multiplication is not commutative. Two matrices A and B are said to commute when $AB = BA$. A quantity called the commutator,

$$[A, B] \equiv AB - BA \tag{5}$$

Quick Question: Is $[A, B]$ equal to $[B, A]$?

Quick Question: What is $[A, A]$?

In what circumstance can we first do measurement A and collapse the wave function, then do measurement B, then do A again and get the same result as the first time? Well, we would have to have some eigenvector of A that is also an eigenvector of B so that the second measurement doesn't have to change to state. We can only find states which are eigenvectors of both A and B simultaneously if the commutator vanishes. Suppose such a state exists and

$$A|\psi\rangle = a|\psi\rangle \tag{6}$$

$$B|\psi\rangle = b|\psi\rangle \tag{7}$$

Then we would also have

$$BA|\psi\rangle = Ba|\psi\rangle = ab|\psi\rangle \tag{8}$$

$$AB|\psi\rangle = Ab|\psi\rangle = ab|\psi\rangle \tag{9}$$

Subtracting these gives the relation

$$(AB - BA)|\psi\rangle = 0 \tag{10}$$

This is generally only true if $[A, B] = 0$. When the commutator is not zero, the second measurement (B) will collapse the state to some state which is not an eigenstate of A, meaning that the state again has no definite value of A! This means that there is no way to simultaneously assign a definite value for both observables to the system!

In statistics, the width of a distribution is described by a quantity called the standard deviation. Using the notation $\langle A\rangle$ to mean the average of the variable A, the standard deviation is defined as

$$\Delta A \equiv \sqrt{\langle A^2\rangle - \langle A\rangle^2} = \sqrt{\langle (A - \langle A\rangle)^2\rangle} \tag{11}$$

If a probability distribution is infinitely narrow, that means that outcome is always the same. The wider it is, the more likely it is to be far from the average.

Uncertainty relations in quantum mechanics give the mimimum product of the standard deviations, i.e. $\Delta A \Delta B$, in the probability of two observables for any possible state. Let's try to compute this for S_z and S_x. First of all, what exactly is the average of A and how do we compute it?

$$\langle A\rangle = \sum_\lambda P(\lambda)\lambda \tag{12}$$

The average is just what we would get if we perform the experiment a large number of times, sum the outcomes and divide by the number of experiments. But notice that

$$\langle\psi| A |\psi\rangle = \langle\psi| \left(\sum_\lambda \lambda |\lambda\rangle \langle\lambda|\right) |\psi\rangle \tag{13}$$

$$= \sum_\lambda \lambda \langle\psi|\lambda\rangle \langle\lambda|\psi\rangle \tag{14}$$

$$= \sum_\lambda \lambda \langle\lambda|\psi\rangle^* \langle\lambda|\psi\rangle \tag{15}$$

$$= \sum_\lambda \lambda| \langle\lambda|\psi\rangle | \tag{16}$$

$$= \sum_\lambda \lambda P(\lambda) = \langle A\rangle \tag{17}$$

Let $|\psi\rangle$ be any state vector. Then, by definition,

$$(\Delta A \Delta B)^2 = \langle\psi|(A - \langle A\rangle)^2|\psi\rangle \langle\psi|(B - \langle B\rangle)^2|\psi\rangle \tag{18}$$
$$= \langle\psi_A|\psi_A\rangle \langle\psi_B|\psi_B\rangle \tag{19}$$

where $(A - \langle A\rangle)|\psi\rangle = |\psi_A\rangle$.

You will prove this afternoon that $|\langle\alpha|\beta\rangle|^2 \le \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$. Therefore

$$(\Delta A \Delta B)^2 \ge |\langle\psi_A|\psi_B\rangle|^2 \tag{20}$$

Some algebra shows that

$$\langle\psi_A|\psi_B\rangle = \frac{1}{2}\langle\psi|([A,B] + AB + BA)|\psi\rangle \tag{21}$$

The commutator is not Hermitian but may be written as $[A,B] = iC$ with C Hermitian. The combination $AB+BA$, is Hermitian. We see that (21) can be written

$$(\Delta A \Delta B)^2 \ge \frac{1}{4}\langle AB + BA\rangle^2 + \frac{1}{4}\langle C\rangle^2 \tag{22}$$

Since the first time on the right side is the square of a real number, it is always greater than or equal to zero. Thus the inequality

$$(\Delta A \Delta B)^2 \ge \frac{1}{4}\langle C\rangle^2 \tag{23}$$

also holds. Finally, we take the square root to find that

$$\Delta A \Delta B \ge \sqrt{\frac{1}{4}\langle C\rangle^2} = \sqrt{\frac{1}{4}|i\langle C\rangle|^2} = \frac{1}{2}|\langle[A,B]\rangle| \tag{24}$$

I will repeat the result since it is so important:

$$\Delta A \Delta B \ge \frac{1}{2}|\langle[A,B]\rangle| \tag{25}$$

3 Multi-particle states

When we discuss quantum computers, we will need to be able to combine the quantum states of two systems into a single vector. Say we have a particle A in the state $|\uparrow\rangle_A$ and a particle B in the state $|\downarrow\rangle_B$. If the system we wish to describe is the system containing both particles, we can notate it in a couple of different ways:

$$|\psi\rangle_{A+B} = |\uparrow\rangle_A |\downarrow\rangle_B = |\uparrow\downarrow\rangle \tag{26}$$

Quick Question: What are the possible combinations of single particle basis states which make a basis for the two particle states?

In column notation, this would correspond to

$$|\psi\rangle_{A+B} = \begin{pmatrix}1\\0\end{pmatrix}_A \begin{pmatrix}0\\1\end{pmatrix}_B = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \tag{27}$$

For the last equality we chose a basis in which

$$|\uparrow\uparrow\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |\uparrow\downarrow\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad |\downarrow\uparrow\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |\downarrow\downarrow\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{28}$$

We can choose to make a measurement on just one of these particles. For example, the matrix which observes the z component of spin for particle A is

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \tag{29}$$

3.1 Spin statistics

The postulates for constructing single particle quantum mechanics require that every vector in the vector space corresponded to a physical system. For multiple particle systems this is not the case! This calls for a new postulate that applies only to these systems:

Postulate 5 (spin statistics): Particles of the same type are called *indistinguishable*. If they are bosons (fermions), i.e. particles with an integer (half integer) spin quantum number, only vectors which are symmetric (antisymmetric) under exchange of any two particles are physical.

It's not always obvious when two particles are indistinguishable, but a pretty good rule of thumb is to treat them as such if they are doing anything together and are both the same kind of particle. Examples of indistinguishable particles include two electrons, or two silver atoms, or two photons. A photon is distinguishable from an electron or a positron.

To understand the concept of particle exchange, let's work with a definite system of two particles described by

$$|\psi\rangle = |\uparrow\downarrow\rangle \tag{30}$$

Particle exchange simply takes two particles, A and B, from a system and swaps their labels. For the state in (31) this has the effect

$$|\uparrow\downarrow\rangle \to |\downarrow\uparrow\rangle \tag{31}$$

A state is symmetric under exchange if the exchange gives exactly the same state back. The state

$$|\uparrow\uparrow\rangle \to |\uparrow\uparrow\rangle \tag{32}$$

is an example of this. If exchange gives the same vector back but with a minus sign, for example

$$|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \to |\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle = -(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{33}$$

then we call it antisymmetric. Some states, such as the choice in (31), do not have definite symmetry under exchange. These last states are unphysical unless the particles involved are distinguishable.

One consequence of Postulate 5 is that indistinguishability is reflected in the state vectors of multi-particle systems. The state vectors must treat the two particles on the same footing.