Lecture Notes IX: The Universe From Here On Out

Cory Schillaci

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Yesterday we discussed the history of the Big Bang up to structure formation. For now, let's just gloss over the formation of large scale structure. Today we will talk about observations of the present day universe, and possible futures. Then there will be a twist at the end.

For reference purposes, recall that the FRW metric is

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \tag{1}$$

where the scale factor $a(t)$ is dimensionless. We will explore the curvature k more today.

1 The Present Day Universe

To understand the our universe, we need to know more about two things. First, what does the curvature k mean and what is its observed value? We'll also need to know more about the evolution of $a(t)$.

1.1 Curvature

For the simplest case, when the curvature is zero, we get a universe that is spatially Euclidean. This means that the metric can be written as

$$ds^2 = -dt^2 + a(t) \left(dx^2 + dy^2 + dz^2 \right) \tag{2}$$

It should surprise you that the only interesting thing left is the scale factor. When the curvature is zero, the resulting geometry is *spatially flat*. Often, laziness leads us to omit the qualifier and just call the metric flat, but because the scale factor is time dependent it might not be true that the four-dimensional space-time is truly flat. If $a(t)$ were also constant, we would get a totally flat space-time equivalent to the

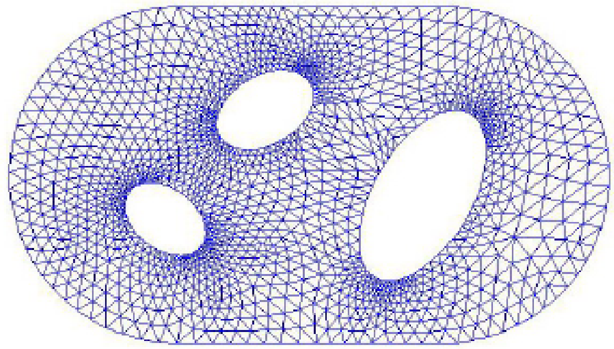

Figure 1: An example of a multiply connected space. The "handles" prevent the possibility of continuously deforming a closed path into any other closed path.

Minkowski spacetime of special relativity. In any case, the universe is also spatially infinite.

For non-zero curvature, let's consider the possibilities of positive or negative curvature separately. If $k > 0$, the metric has a singularity at the radius $r = 1/\sqrt{k}$. The singularity is actually an artifact of our choice of coordinates[1]. If we make the change of coordinates $r = k^{-1/2} \sin \chi$ the metric becomes

$$ds^2 = -dt^2 + \frac{a^2(t)}{k} \left(d\chi^2 + \sin^2 \chi d\theta^2 + \sin^2 \chi \sin^2 \theta d\phi^2 \right) \tag{3}$$

In these coordinates, there is no singularity. The spatial part of this metric is the natural measure of distance on a Euclidean three-sphere (the surface of a four-dimensional ball). This implies that the universe is finite and *closed*, in the sense that if I go far enough space is periodic in all directions.

The only remaining possibility is negative curvature. This case corresponds to a geometry which looks like a three dimensional hyperbolic surface embedded in a four-dimensional space. Although we call FRW universes with negative curvature *open* to suggest that they are infinite, there is a possibility that the space is *nonsimply-connected* and finite. An example of a nonsimply connected space is shown in Figure 1.

1.2 The scale factor

Current evidence suggests that the universe is expanding at an accelerating rate. This is due to the specific balance of matter and dark energy we observe. Naturally, the expansion is encoded in the scale factor's time dependence. Let's explore that now.

[1]General relativity allows us a freedom to change coordinates using a smooth transformation called a *diffeomorphism*. Unfortunately some choices will obscure the physics, which require a different coordinate system to get a better understanding.

In addition to the metric (1), we also know how to determine the evolution of the scale factor a using the Friedmann equations. The rate at which the scale factor changes is usually notated as the Hubble parameter H,

$$H \equiv \frac{\dot{a}}{a} \tag{4}$$

where $\dot{a}$ is the rate at which the scale factor changes with respect to time[2].

How can we determine H? Well, I'll just tell you:

$$H^2 = \frac{8\pi G}{3} \left(\rho_{\text{matter}} + \rho_{\text{radiation}} + \rho_{\text{vacuum}} - \rho_{\text{curvature}}\right) \tag{5}$$

These ρ's are energy densities due to different types of stuff. Matter means non-relativistic particles, and radiation means relativistic particles (including photons). In our present universe, $\rho_{\text{radiation}} \ll \rho_{\text{matter}}$ so that the second term can be neglected. The other two contributions involve more explanation.

Actually, the governing equation of general relativity looks like this:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{6}$$

The $g^{\mu\nu}$ is the metric, in our case the FRW metric. For a general spacetime the curvature is measured by the different R symbols, in our case this is encoded in k. Matter, which we think of as the sources of gravity, shows up in the so-called *stress energy tensor* on the right hand side of the equation. Finally the cosmological constant Λ appears.

Roughly speaking, I can just move the curvature and cosmological constant to the other side of the equality and think of them as a sources for gravitation. This is a traditional move made by cosmologists, and one consequence is that they rename the cosmological constant *dark energy*, suggesting that it somehow represents a form of energy or mass.

When Einstein formulated general relativity, he noticed that a constant had to be added to the equations in order to make a stationary universe. Furthermore, there was no structural reason *not* to add this constant, the most general version of his equations allowed for one[3]. This original motivation was abandoned when Hubble made his famous diagram showing that the universe is expanding. but quantum field theory actually predicts that the vacuum is not empty. Instead, it may have some energy density. The vacuum energy density is ρ_{vac}.

The curvature energy density is not really an energy density at all either. Again it's just a way of rewriting part of the Friedmann equation to make everything look like an energy density. In the present day this term is negligible, so we can ignore it when

[2] If you know calculus, the overdot in physics typically indicates a time derivative. For example, $\dot{x} = \frac{dx}{dt} = v$.

[3] In hindsight, we shouldn't expect any possible term to be zero unless forced to be by symmetry!

determining the evolution of H. We can, however, use the equations to find the sign of the curvature.

If I consider only the true matter density ρ, the Friedmann equations tell us that there is a critical energy density,

$$\rho_{\text{crit}} = \frac{3H^2}{8\pi G} \tag{7}$$

and that

$$\frac{\rho}{\rho_{\text{crit}}} - 1 = \frac{\kappa}{H^2 a^2} \tag{8}$$

In this way, the matter content of the universe determines the sign of the curvature. For example, if $\rho > \rho_{\text{crit}}$ the universe will be closed. Note that the critical density is not constant, because the Hubble parameter is changing in time.

Before people took dark energy seriously, the amount of stuff in the universe was the only input which determined the ultimate fate of things. If there was enough stuff, the universe would have to be closed. If there wasn't, the universe would be open. Experimentally, $\Omega = 0.3 \pm 0.1$ seemed to indicate that the latter case described what we see.

Modern observations suggest that not all of the energy density is accounted for by matter, radiation, and curvature. We currently call the excess dark energy, because we don't have a good theory to predict its magnitude or provide an understanding of its origin. It could truly be just a constant in the equations, or it might be a dynamical effect from quantum theories. Unfortunately, the prediction from QFT for ρ_{vac} is around the Planck scale, which is too big by thirty-one orders of magnitude! So we might need a new understanding of the vacuum energy's magnitude, or we might be observing something entirely new.

One common misconception, perpetuated from a time in the not-so-distant past when dark energy wasn't taken seriously, was that open and flat universe expand forever while closed ones recollapse. If we include dark energy, the condition that the universe expands forever is actually

$$\frac{\rho_\Lambda}{\rho_{\text{crit}}} \geq \begin{cases} 0 & 0 \leq \Omega \leq 1 \\ 4\Omega \cos^3\left[\frac{1}{3}\cos^{-1}\left(\frac{1-\Omega}{\Omega}\right) + \frac{4\pi}{3}\right] & \Omega > 1 \end{cases} \tag{9}$$

2 The Future of the Universe

Presently, our understanding of dark energy and the shape of the universe is insufficiently precise to distinguish between the possible endings of the universe. The most likely scenarios are probably either the big rip or the big freeze.

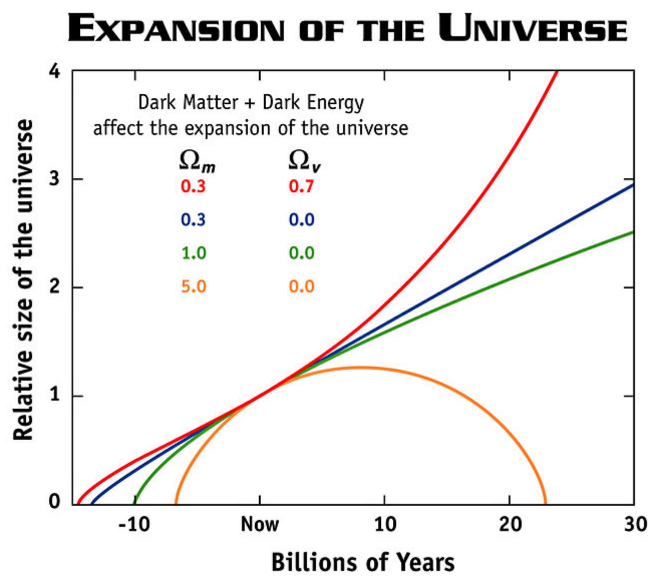

Figure 2: Historically, we thought that the possible outcomes would be either a crunch or eternal inflation. These two possibilities, determined by the balance of the vacuum and matter energy densities, are illustrated here. The Ω's quoted are simply the energy densities divided by the critical density. From http://map.gsfc.nasa.gov/universe/uni_fate.html

2.1 Big Rip

For some kinds of dark energy, the contribution of the dark energy density will actually cause the scale factor to diverge at a finite time. The important property is the ratio

$$w = \frac{p}{\rho} \tag{10}$$

of the pressure to energy density. For $w < -1$, a big rip will occur at

$$t - t_0 \approx \frac{2}{3|1-w|H_0\sqrt{1-\rho_{\rm mat}/\rho_{\rm crit}}} \tag{11}$$

Using realistic values of the Hubble constant and matter energy density along with the arbitrary choice $w = -1.5$, this formula predicts the big rip in about 22 billion years. Note that a cosmological constant has $w = -1$.

2.2 Big Freeze

If dark energy is due to the cosmological constant then $w = -1$ and the universe will eventually be vacuum energy dominated. When the other terms in (5) become negligible, the scale factor becomes exponential in time:

$$a(t)^2 = e^{Ht} \tag{12}$$

As $t \to \infty$, the universe cools. The CMB and CνB are redshifted to arbitrarily low energies. The density of matter becomes too small to sustain stellar genesis and the stars eventually all burn away. In GUT theories, the protons eventually decay and stellar remnants dissipate. Only those stars which evolved into black holes leave lasting marks as the universe becomes black hole dominated. Even black holes will eventually evaporate as the universe becomes uniform and matter densities become arbitrarily small.

2.3 Big Crunch

In a big crunch scenario, the gravitational attraction of matter causes the universe to eventually collapse back in on itself. The end state of the universe would then be something like the initial state in which $a \to 0$. Current cosmological data suggests that the universe is not just expanding, but that its expansion is accelerating. This makes a big crunch unlikely.

2.4 Big Bounce

The big bounce is a cyclic model of the universe. If the universe eventually experiences a big crunch, but quantum effects prevent a complete collapse back to $a = 0$, a big bang could follow. Then another crunch, another bang, ad infinitum. Since it doesn't look good for the crunch, this scenario is also unlikely.

3 The Cosmological Constant Problem, the Multiverse and the Anthropic Principle

Supposing that some unknown physics corrects the QFT prediction for cosmological constant. In order to reproduce the correct value, the correction has to agree with the QFT vacuum energy up to the 120^{th} decimal place. This is called fine tuning, and it's fine tuning to a degree far more severe than some other areas of physics (such as the Higgs or neutrino masses).

One of the predictions from inflation is that there should be causally disconnected regions of spacetime, i.e. other universes. The collection of these is called the multiverse. At the same time, string theory has a "problem" in that the vacuum of string theory appears to be non-unique. If we combine these ideas, we can imagine that each universe might have a different set of physical constants covering all of the possibilities. But how likely is it that we find ourselves in the universe we see? Why not in one of the many, many other possibilities?

Many physicists are now beginning to argue that an additional selection principal is at work. The very fact that we are here to learn about physics requires that we are in a special universe that supports intelligent life. Thus we select only from the possible universes in which we could exist. this is called the anthropic principal. It is currently our best explanation for the small value of the cosmological constant, and could also explain other fine tuning problems of physics. Unfortunately, it seems to be ultimately untestable.

4 The Measure Problem

In the last section we asked an innocent question about probabilities. It turns out that probabilities are a bit hard to define in infinite systems.

Quick Question: What do you think the problems are? Can you think of ways around them?

The main problem is that any event with a finite probability per volume will occur an infinite number of times. Infinities are tricky animals. As an example, let's compare the number of positive integers to the number of integers. Naively, it seems like there are about two integers for every positive integer. But I can place the negative and positive integers in one-to-one correspondence[4]. This means that the sets are the same size. Hmmm...

Quick Question: Which is bigger, the set of all integers or the set of all rational numbers?

Of course I could also map two positive integers to each integer, which makes it look like there are more of them. Even very rare events will happen an infinite number of times in the multiverse, so there is no notion of relative probabilities that can be formed by taking ratios. Normally, we can define a probability as

$$P(A) = \frac{\text{number of occurences of } A}{\text{number of chances for } A \text{ to occur}}$$

Of course in the multiverse this gives $\frac{\infty}{\infty}$ which is indeterminate. This is roughly what physicists call the *measure problem*.

[4] In other words, there exists a bijection from $\mathbb{Z}$ to $\mathbb{Z}^+$.