Homework II: Vectors, Matrices and Complex Numbers

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There were a lot of definitions to learn today, and we covered them fairly quickly in class. The best way to internalize it all is to practice, practice, practice with everything. That's the goal of this homework: to get familiar with all the operations now so that you can understand the physics later. Good luck!

At the end of of class, read the article "The Unreasonable Effectiveness of Mathematics in the Natural Sciences." You should also read chapter 6 of Weinberg up to page 144, stopping at the paragraph which begins "The symmetry underlying the electroweak theory..." The footnote on pages 138 and 139 is partially a discussion of natural units that may help illuminate some ideas from the first day.

Problem 1 *Dreams of a Final Theory*

Today's reading, which you should do at the end of class, is about beauty in science. Have you ever felt that a physical principle or idea was beautiful? Can you explain why to someone?

Problem 2 Linear Operations on Vectors

Let $\vec{U} = (1, 1)$ and $\vec{V} = \hat{x} + 2\hat{y}$. Write each of the following vectors in both the column vector and basis vector notations:

a) $\vec{A} = \vec{U} - \vec{V}$

b) $\vec{B} = \vec{U} + 3\vec{V}$

c) $\vec{C} = \frac{1}{\sqrt{3}}\vec{V}$

d) $\vec{D} = 2(\vec{A} + \frac{1}{2}\vec{U})$

Write the results of (a) and (b) in polar coordinates.

Problem 3 Visualizing Vectors

Sketch each of the vectors from Problem 2 on a two dimensional coordinate system.

Problem 4 Multiplying Vectors (The Dot or Inner Product)

Rewrite each of the following in row times column format, then find the dot product.

1. $\vec{U} \cdot \vec{U} = U^2 = |\vec{U}|^2$
2. $\vec{U} \cdot \vec{V}$
3. $\vec{A} \cdot \vec{B}$

For the following problems, use these definitions:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Problem 5 Operations on Matrices

Write out the results of the following matrix operations:

a) M^T (Transpose of M)

b) SM (S times M)

c) MS (M times S)

d) $\det S = |S|$

Problem 6 The Determinant

Show that $\begin{vmatrix} 2 & 4 & 8 \\ 3 & 5 & 9 \\ 4 & 6 & 0 \end{vmatrix} = \det \begin{pmatrix} 2 & 4 & 8 \\ 3 & 5 & 9 \\ 4 & 6 & 0 \end{pmatrix} = 20$

Problem 7 Properties of Matrices

Define each of the following properties and find which of the matrices S, M or N satisfies the property.

a) Diagonal

b) Symmetric

c) Unitary

Problem 8 Combined Matrix and Vector Operations

Give the following quantities. Before computing anything, write down whether the answer will be a matrix, a vector or a scalar.

a) IM

b) $S\vec{V}$

c) $\vec{U} \cdot (M\vec{V})$

d) $M^T M\vec{V}$

Problem 9 Converting Complex Numbers

Write $2 - 3i$ in polar coordinates. Now write $6e^{i\pi/8}$ in cartesian coordinates.

Problem 10 Working with Complex Numbers

Define three complex numbers: $a = 1$, $b = 1 + i$ and $c = 2 - 3i$. Give the results of the following operations:

a) $a + b$

b) ac

c) $bc - a$

d) $\mathrm{Re}(b)$

e) $\mathrm{Im}(c)$

f) a/c

g) $|c|$

h) **Bonus:** $\sin b$.

Problem 11 $\mathbf{n^{th}}$ roots of unity

Find $z = \sqrt[3]{1}$. Note that there should be more than one answer! What are all the the solutions to $z = \sqrt[4]{1}$? How many solutions are there to $z = \sqrt[n]{1}$ for arbitrary n? These are called the n^{th} roots of unity.

Consider now the equation $z = \sqrt[n]{x}$. How many solutions are there if both z and x must be real numbers? What if z is complex and x is real? Does it make a difference if I let x be complex as well?

Problem 12 Matrix Representation of Complex Numbers (Bonus)

There is a way to represent complex numbers as 2x2 matrices. Can you figure out what it is? Explain the proper way to add, multiply, and find the norm using the matrices which exactly reproduces the normal behavior of complex numbers.

Problem 13 Quaternions (Bonus)

In the 19th century, William Hamilton invented *quaternions*. In some sense, these are an extension of the complex numbers. The four quaternions satisfy:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1}$$

It is not safe to assume that multiplication of quaternions is commutative, in fact you can show that $ij = -k = -ji$!

1. Find expressions for the products jk, kj, ik, and ki.
2. Count the roots of unity as in problem 11, but now allow the solutions to be quaternions.
3. Describe how to divide one quaternion by another.