Homework III: Quantum Mechanics

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Unless otherwise noted, in this homework you should use notation in which

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

In this basis, we define the operators:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

Problem 1 Eigensystems

Find the eigenvalues and eigenvectors of σ_x.

Problem 2 Vector spaces (Optional)

Show that the complex numbers form a vector space when associated with the real numbers as scalars. Also find a prescription for the inner product that satisfies the definition from class.

Problem 3 Classical versus quantum probability

Suppose that someone has put a spinning top into a closed box. They haven't told you which way they initially spun it, and they have no physical or mental prejudice towards picking one orientation over the other. What is the probability that, when you open the box, the top is spinning clockwise when viewed from above? How about counterclockwise? When you open the box, what happens to the top?

Let's now pretend that the top is a quantum mechanical system, and that the only property we care about is the direction it spins. Call the state where it is spinning clockwise $|+\rangle$ and the counterclockwise spin $|-\rangle$. Suppose someone prepares a state

$|\psi\rangle = (|+\rangle + |-\rangle)/\sqrt{2}$ in the box. When you look into the box, what is the chance that you will see the top spinning clockwise? If you do see the top spinning clockwise, what is its wave function? Did something physically change when you looked at the top? What happens if you close the box and then open it again?

Comment on the difference between the meaning of probability in the first and second cases. Also contrast the effect of observing the top in each case.

Problem 4 Cat States

This problem is motivated by some work I did when I was an undergraduate physics major, so if you want to know more just ask! Physicists in the last decade have become very good and making and controlling things called *Bose-Einstein condensates* (BECs), which we will talk about in more detail later. For now, all you need to know is that these are systems of many atoms cooled to extremely low temperatures (a billion times colder than empty space). One remarkable property of BECs is that they demonstrate strongly quantum mechanical behavior even though they can be millimeters across. That may sound pretty small, but it's much bigger than an atom!

Researchers have used BECs to make an analog of Schrödinger's cat. If we allow atoms to be in one of two places (let's call them left and right, although the spatial orientation is irrelevant), and we really just want to know how many are in each, we could describe a system of N particles by a single number indicating how many particles are on the "left". For example, $|13\rangle$ would be the state with 13 atoms in the "left" and $N - 13$ on the right.

Scientists have created extreme states, $|\psi\rangle = (|0\rangle + |N\rangle)/\sqrt{2}$, which are often referred to as *cat states* (although several physics journals have banned any reference to cats because of the popularity of this term). Describe what happens when someone measures the state by answering the following questions:

a) What is the observable?

b) What are the possible outcomes of a measurement?

c) What is the state after measurement for each of the possible outcomes?

d) Can you tell the difference between the cat state and the state $|N\rangle$ in a single experiment?

e) After many measurements, can you tell whether the prepared state was a cat state or if each time the scientist had simply created either $|0\rangle$ or $|N\rangle$ when setting up the experiment?

Problem 5 Basic Properties of Quantum States

Consider the state $|\psi\rangle = a\,|+\rangle + b\,|-\rangle$ where $|a|^2 + |b|^2 = 1$.

- What is the probability of measuring the state in $|-\rangle$?
- Construct a state $|\phi\rangle$ which is orthogonal to $|\psi\rangle$, i.e. $\langle\phi|\psi\rangle = 0$.
- If I measure the state many times with σ_z, what is the average value of my measurements?

Problem 6 Changing Basis

A basis in linear algebra is essentially a way of explicitly writing an abstract vector. Choosing a basis in quantum mechanics is exactly the same thing as choosing which way the x- and y-axes point relative to physical vector in the plane. Let's see an example of how this is totally arbitrary, and how to switch between arbitrary choices.

Suppose that instead of using the definitions in (1), we wanted to describe states using the definitions

$$|+\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3}$$

How would the notation for the state $|\psi\rangle$ of Problem 5 change? Assume that the column vectors have the same meaning, only the ket notation has changed. Note that the physical meaning is identical, but our way of describing it looks different!

Optional Suppose instead we wanted to write

$$|+\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} c \\ d \end{pmatrix} \tag{4}$$

Find values for a, b, c and d such that none are zero, $\langle 0|0\rangle = \langle 1|1\rangle = 1$ (the states are normalized) and $\langle 0|1\rangle = 0$ (the states are perpendicular or orthogonal).

Problem 7 Completeness

Show that, for any

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

we always have

$$|\psi\rangle = |+\rangle\langle +|\psi\rangle + |-\rangle\langle -|\psi\rangle\,.$$

This relation implies that any state can be built out of $|+\rangle$ and $|-\rangle$.

Equivalently, $|+\rangle\langle+| + |-\rangle\langle-| = I$. Show this using the appropriate matrix multiplication.

Generically, the sum over the normalized eigenvectors of a Hermitian matrix gives the identity: $\sum_\lambda |\lambda\rangle\langle\lambda| = I$.