Talmi Transformation for Unequal-Mass Particles and Related Formulas

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Three-dimensional polynomials which occur as coefficients of the exponential in the wavefunctions of the harmonic oscillator are used in nuclear physics and kinetic theory of gases. A generating function for these polynomials is used to simplify the calculation of several integrals. These include the integrals involving products of two and three polynomials and the coefficients of the Talmi transformation. Explicit formula in terms of recoupling coefficients of angular momentum theory are obtained.

IN the course of an investigation into the method of solving the Boltzmann equation by means of polynomial expansion of the distribution function we had occasion to derive the formulas referred to in the title. Our derivations are simpler than the ones published so far and do not involve the use of group theory. The results also have a simple structure and would perhaps be more suited for the formal study of the quantities involved. Because of its applications in nuclear theory and in kinetic theory of gases, the following may be of interest to mathematical physicists in general and is offered with little reference to any particular physical situation.

I. PRELIMINARIES

We are concerned with the orthogonal polynomials constructed from the components of three-dimensional vectors denoted by the usual bold-face symbols, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{r}$, $\mathbf{r}_1$, $\cdots$, etc. The polynomials are fully specified by a set of three indices n, l, m. The last two indices indicate the irreducible tensor character of the polynomial. For the algebra of irreducible tensors we follow the notation and phase conventions given in the book by Fano and Racah.[1] According to this book an irreducible tensor can be standard or contrastandard depending upon the way it transforms under rotations of the coordinate system. The standard tensors are characterized by a superscript in round brackets and the contrastandard ones by a superscript in square brackets. The prototypes of these tensors are the spherical harmonics which are defined in terms of the rotation matrix $\mathfrak{D}$ as follows

$$\mathfrak{Y}_m^{[l]}(\theta, \varphi) = i^l(2l + 1/4\pi)^{\frac{1}{2}}\mathfrak{D}_{0m}^{(l)}(\theta, \varphi) \tag{1.1}$$

This differs from the more common definition according to the convention of Condon and Shortley.[2] In the usual notation[3] the relationship is

$$\mathfrak{Y}_m^{[l]}(\theta, \varphi) = i^l Y_{lm}(\theta, \varphi). \tag{1.2}$$

We have

$$\mathfrak{Y}_m^{[l]*} = \mathfrak{Y}_m^{(l)} = (-)^{l-m}\mathfrak{Y}_{-m}^{[l]}. \tag{1.3}$$

The coupling rule for spherical harmonics consequently becomes (e.g., Ref. 3)

$$\mathfrak{Y}_{m_1}^{[l_1]}(\theta, \varphi)\mathfrak{Y}_{m_2}^{[l_2]}(\theta, \varphi) = \sum_l \sigma(l_1l_2l)(l_1m_1l_2m_2 \mid l\ m_1 + m_2)\mathfrak{Y}_{m_1+m_2}^{[l]}(\theta, \varphi) \tag{1.4}$$

where

$$\sigma(l_1l_2l) = i^{l_1+l_2-l}\left[\frac{(2l_1 + 1)(2l_2 + 1)}{4\pi(2l + 1)}\right]^{\frac{1}{2}}(l_10l_20 \mid l0). \tag{1.5}$$

The Clebsch–Gordan or Wigner coefficients denoted here by $(l_1m_1l_2m_2 \mid lm)$ or $(lm \mid l_1m_1l_2m_2)$ are the same as those of Condon and Shortley[2] and usually employed in angular momentum theory.[3]

The addition theorem of spherical harmonics for

$$\cos\theta = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2)$$

has the form

$$P_l(\cos\theta) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} \mathfrak{Y}_m^{[l]}(\theta_1, \varphi_1)\mathfrak{Y}_m^{(l)}(\theta_2, \varphi_2). \tag{1.6}$$

The following forms of the usual plane-wave expansion will be needed:

$$e^{2\mathbf{a}\cdot\mathbf{b}} = \sum_{l,m} 4\pi i^l j_l(-2iab)\mathfrak{Y}_m^{[l]}(\hat{\mathbf{a}})\mathfrak{Y}_m^{(l)}(\hat{\mathbf{b}}) = \sum \frac{2\pi^{\frac{1}{2}}}{\Gamma(n + 1)\Gamma(n + l + \frac{3}{2})}(ab)^{2n+l}\mathfrak{Y}_m^{[l]}(\hat{\mathbf{a}})\mathfrak{Y}_m^{(l)}(\hat{\mathbf{b}}). \tag{1.7}$$

[1] U. Fano and G. Racah, *Irreducible Tensorial Sets* (Academic Press Inc., New York, 1959).

[2] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, England, 1953).

[3] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

By $\hat{\mathbf{a}}$ we denote the angular variables associated with the vector $\mathbf{a}$. Differentiation with respect to a vector variable $\mathbf{a}$ will be written as $\partial/\partial\mathbf{a}$ or ∇_a. The polynomials constructed from the vector $\mathbf{r}$ will be denoted, e.g., by $\psi_m^{[nl]}(\mathbf{r})$ or $\psi_m^{(nl)}(\mathbf{r})$ depending upon their tensor character. For the sake of brevity, we sometimes write $\mathbf{n}$ for the set of indices n, l, m in which case, e.g.,

$$\psi_m^{[nl]} \equiv \psi^{[\mathbf{n}]}. \tag{1.8}$$

No confusion can arise since the positions in which the indices $\mathbf{n}$ occur are quite different from those in which the vectors $\mathbf{a}$, $\mathbf{r}$, $\cdots$, etc., occur.

We shall sometimes use the abbreviation

$$(2l+1)^{\frac{1}{2}} = \hat{l}. \tag{1.9}$$

The Talmi transformation[4-16] arises when one expresses the functions of the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ of two particles in terms of their center of mass and relative coordinates $\mathbf{R}$ and $\mathbf{r}$. For unequal masses the relations between the vectors are

$$\Gamma^2\mathbf{R} = \alpha_1^2\mathbf{r}_1 + \alpha_2^2\mathbf{r}_2, \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2; \tag{1.10a}$$

$$\mathbf{r}_1 = \mathbf{R} + (\alpha_2/\Gamma)^2\mathbf{r}, \qquad \mathbf{r}_2 = \mathbf{R} - (\alpha_1/\Gamma)^2\mathbf{r}; \tag{1.10b}$$

$$\Gamma^2 = \alpha_1^2 + \alpha_2^2, \qquad \gamma^{-2} = \alpha_1^{-2} + \alpha_2^{-2}. \tag{1.10c}$$

The quantities α_1^2 and α_2^2 are proportional to the masses of the particles, Γ^2 is proportional to the total mass, and γ^2 to the reduced mass. These relations hold also in kinetic theory work but the transformation is taken to apply to the velocities rather than to the positions.

II. GENERATING FUNCTION FOR THREE-DIMENSIONAL POLYNOMIALS

Consider the polynomials $\xi_m^{[nl]}(\mathbf{r})$ defined through the generating function

$$\begin{aligned} G(\mathbf{a},\mathbf{r}) &\equiv e^{-a^2+2\mathbf{a}\cdot\mathbf{r}} \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{(-)^n}{n!}\, a^{2n+l}\mathcal{Y}_m^{(l)}(\hat{\mathbf{a}})\xi_m^{[nl]}(\mathbf{r}). \end{aligned} \tag{2.1}$$

[4] I. Talmi, Helv. Phys. Acta **25**, 185 (1952).
[5] R. Thieberger, Nucl. Phys. **2**, 533 (1956–1957).
[6] K. W. Ford and E. J. Konopinski, Nucl. Phys. **9**, 218 (1957–1958).
[7] M. Moshinsky, Nucl. Phys. **13**, 104 (1959).
[8] T. A. Brody, Rev. Mex. Fis. **8**, 139 (1959).
[9] R. D. Lawson and M. Goeppert-Mayer, Phys. Rev. **117**, 174 (1960).
[10] V. V. Balashov and V. A. Eltekov, Nucl. Phys. **16**, 423 (1960).
[11] A. Arima and T. Terasawa, Progr. Theoret. Phys. (Kyoto) **23**, 115 (1960).
[12] T. A. Brody and M. Moshinsky, *Tables of Transformation Brackets* (Universidad de Mexico, Mexico City, 1960).
[13] M. Moshinsky and T. A. Brody, Rev. Mex. Fis. **9**, 181 (1960).
[14] T. A. Brody, G. Jacob, and M. Moshinsky, Nucl. Phys. **17**, 16 (1960).
[15] B. Kaufman and C. Noak, J. Math. Phys. **6**, 142 (1965).
[16] Yu. F. Smirnov, Nucl. Phys. **27**, 177 (1961).

Since the generating function is a scalar, $\xi_m^{[nl]}$ must transform like a spherical harmonic. The parity property, $\xi_m^{[nl]}(-\mathbf{r}) = (-)^l\xi_m^{[nl]}(\mathbf{r})$, is appropriately reflected in the relation

$$G(\mathbf{a}, -\mathbf{r}) = G(-\mathbf{a}, \mathbf{r}). \tag{2.2}$$

Using the identity

$$\begin{aligned} a(\partial/\partial a)G(\mathbf{a},\mathbf{r}) &\equiv \mathbf{a}\cdot(\partial/\partial\mathbf{a})G(\mathbf{a},\mathbf{r}) \\ &= -2\mathbf{a}\cdot(\mathbf{a}-\mathbf{r})G(\mathbf{a},\mathbf{r}), \end{aligned} \tag{2.3}$$

one can show that

$$\nabla_r^2\{G(\mathbf{a},\mathbf{r})e^{-\frac{1}{2}r^2}\} = \left(r^2 - 2a\frac{\partial}{\partial a} - 3\right)G(\mathbf{a},r)e^{-\frac{1}{2}r^2}, \tag{2.4}$$

from which, on using (2.1), it follows that

$$(-\nabla^2 + r^2)(e^{-\frac{1}{2}r^2}\xi_m^{[nl]}) = [2(2n+l)+3]e^{-\frac{1}{2}r^2}\xi_m^{[nl]}. \tag{2.5}$$

This shows that, apart from the normalization, $\exp(-\frac{1}{2}r^2)\xi_m^{[nl]}$ are the wavefunctions of the 3-dimensional harmonic oscillator. We shall write

$$\begin{aligned} \psi_m^{[nl]}(\alpha\mathbf{r}) &= N_{nl}^{-1}\left(\frac{\alpha^2}{\pi}\right)^{\frac{3}{4}} e^{-\frac{1}{2}\alpha^2r^2}\xi_m^{[nl]}(\alpha\mathbf{r}) \\ &\equiv \mathcal{R}_{nl}(\alpha r)\mathcal{Y}_m^{[l]}(\theta,\varphi). \end{aligned} \tag{2.6}$$

The function $\mathcal{R}_{nl}(\alpha r)$ then agrees with the usual definition of the radial function if the constant N_{nl} is chosen such that

$$\int \psi_m^{(nl)}(\alpha\mathbf{r})\psi_{m'}^{[n'l']}(\alpha\mathbf{r})\, d\mathbf{r} = \delta_{nn'}\,\delta_{ll'}\,\delta_{mm'}. \tag{2.7}$$

The explicit form of ξ is obtained by using on the left-hand side of (2.1), Eq. (1.7) and the relation (Ref. 17, p. 189)

$$\begin{aligned} &e^{-a^2}j_{l+\frac{1}{2}}(-2iar) \\ &= (-)^{l+\frac{1}{2}}(iar)^{l+\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(-a^2)^n}{\Gamma(n+l+\frac{3}{2})}L_n^{l+\frac{1}{2}}(r^2). \end{aligned} \tag{2.8}$$

Accordingly,

$$\xi_m^{[nl]}(\mathbf{r}) = \frac{2\pi^{\frac{3}{2}}\Gamma(n+1)}{\Gamma(n+l+\frac{3}{2})}L_n^{l+\frac{1}{2}}(r^2)r^l\mathcal{Y}_m^{[l]}(\hat{\mathbf{r}}). \tag{2.9}$$

The Laguerre polynomials L_n^α occurring here have been defined in Ref. 17, p. 188. This definition is the same as that of the Sonine polynomials $S_\alpha^{(n)}$

[17] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II.

used in kinetic theory work, e.g., in the book of Chapman and Cowling.[18] The use of this function provides the link between the two subjects.

If $A(\mathbf{r})$ is any function of $\mathbf{r}$, possibly containing differential operators, then the integral

$$\int e^{-r^2}\xi_m^{(nl)}(\mathbf{r})A(\mathbf{r})\xi_{m'}^{[n'l']}(\mathbf{r})\,d\mathbf{r} \tag{2.10}$$

can be obtained by evaluating the coefficient of $\chi_m^{[nl]}(\mathbf{a})\chi_{m'}^{(n'l')}(\mathbf{b})$ in the integral

$$\int e^{-r^2}G(\mathbf{a},\mathbf{r})A(\mathbf{r})G(\mathbf{b},\mathbf{r})\,d\mathbf{r}, \tag{2.11}$$

where

$$\chi_m^{(nl)}(\mathbf{a}) = \frac{(-)^n}{n!}\,a^{2n+l}\mathfrak{Y}_m^{(l)}(\hat{\mathbf{a}}). \tag{2.12}$$

The integral (2.10) is related to the matrix element of the operator $A(\mathbf{r})$ with respect to harmonic oscillator wavefunctions. It is often more convenient to evaluate the integral (2.11).

When $A \equiv 1$, (2.11) becomes

$$\pi^{\frac{3}{2}}\exp(2\mathbf{a}\cdot\mathbf{b}), \tag{2.13}$$

from which, on using (1.7), it follows that

$$\int e^{-r^2}\xi_m^{(nl)}(\mathbf{r})\xi_{m'}^{[n'l']}(\mathbf{r})\,d\mathbf{r} = \frac{2\pi^3\Gamma(n+1)}{\Gamma(n+l+\frac{3}{2})}\,\delta_{nn'}\,\delta_{ll'}\,\delta_{mm'}. \tag{2.14}$$

Hence in (2.6)

$$N_{nl}^2 = \frac{2\pi^{\frac{3}{2}}\Gamma(n+1)}{\Gamma(n+l+\frac{3}{2})}. \tag{2.15}$$

The main advantage of the generating function arises from the separability of its arguments, because of which many integrals can be evaluated in the form (2.11). We list these properties

$$G(\mathbf{a},\mathbf{r})G(\mathbf{b},\mathbf{r}) = G(\mathbf{a}+\mathbf{b},\mathbf{r})e^{2\mathbf{a}\cdot\mathbf{b}}, \tag{2.16a}$$

$$G(\mathbf{a},\mathbf{r})G(\mathbf{b},\mathbf{r})G(\mathbf{c},\mathbf{r}) = G(\mathbf{a}+\mathbf{b}+\mathbf{c},\mathbf{r})e^{2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})}. \tag{2.16b}$$

From (1.10),

$$G(\mathbf{a},\Gamma\mathbf{R}) = G\left(\frac{\alpha_1}{\Gamma}\,\mathbf{a},\,\alpha_1\mathbf{r}_1\right)G\left(\frac{\alpha_2}{\Gamma}\,\mathbf{a},\,\alpha_2\mathbf{r}_2\right), \tag{2.17}$$

$$G(\mathbf{a},\gamma\mathbf{r}) = G\left(\frac{\alpha_2}{\Gamma}\,\mathbf{a},\,\alpha_1\mathbf{r}_1\right)G\left(\frac{\alpha_1}{\Gamma}\,\mathbf{a},\,-\alpha_2\mathbf{r}_2\right), \tag{2.18}$$

$$G(\mathbf{a},\alpha_1\mathbf{r}_1) = G\left(\frac{\alpha_1}{\Gamma}\,\mathbf{a},\,\Gamma\mathbf{R}\right)G\left(\frac{\alpha_2}{\Gamma}\,\mathbf{a},\,\gamma\mathbf{r}\right), \tag{2.19}$$

$$G(\mathbf{a},\alpha_2\mathbf{r}_2) = G\left(\frac{\alpha_2}{\Gamma}\,\mathbf{a},\,\Gamma\mathbf{R}\right)G\left(\frac{\alpha_1}{\Gamma}\,\mathbf{a},\,-\gamma\mathbf{r}\right), \tag{2.20}$$

$$\int e^{-r^2}G(\mathbf{a},\mathbf{r})\,d\mathbf{r} = \int e^{-(\mathbf{a}-\mathbf{r})^2}\,d\mathbf{r} = \pi^{\frac{3}{2}}. \tag{2.21}$$

It is convenient to introduce the weight function

$$w(\alpha,r) = (\alpha^2/\pi)^{\frac{3}{2}}\exp(-\alpha^2r^2);\quad w(r)\equiv w(1,r) \tag{2.22}$$

with the aid of which an arbitrary function of $\mathbf{r}$ may be expanded as

$$f(\mathbf{r}) = w(\alpha,r)\sum_{\mathbf{n}} f^{(\mathbf{n})}(\alpha)\xi^{[\mathbf{n}]}(\alpha\mathbf{r}), \tag{2.23a}$$

$$f^{(\mathbf{n})}(\alpha) = N_{nl}^{-2}\int d\mathbf{r}\,f(\mathbf{r})\xi^{(\mathbf{n})}(\alpha\mathbf{r}). \tag{2.23b}$$

The quantities $f^{(\mathbf{n})}(\alpha)$ are linear combinations of a finite number of moments of the function $f(\mathbf{r})$ and occur in kinetic theory of gases when functions of velocity are expanded near the local equilibrium. This expansion is quite different from the expansion in terms of harmonic oscillator wavefunctions which may be used in shell model theory,

$$f(\mathbf{r}) = \sum f_{\mathrm{ho}}^{(\mathbf{n})}(\alpha)\psi^{[\mathbf{n}]}(\alpha\mathbf{r}), \tag{2.24a}$$

$$f_{\mathrm{ho}}^{(\mathbf{n})}(\alpha) = \int d\mathbf{r}\,f(\mathbf{r})\psi^{(\mathbf{n})}(\alpha\mathbf{r}). \tag{2.24b}$$

There is a third possibility, also utilized in shell theory[4]:

$$f(\mathbf{r}) = \sum_{\mathbf{n}} f_t^{(\mathbf{n})}\xi^{[\mathbf{n}]}(\mathbf{r}), \tag{2.25a}$$

$$f_t^{(\mathbf{n})} = N_{nl}^{-2}\int w(r)f(\mathbf{r})\xi^{(\mathbf{n})}(\mathbf{r})\,d\mathbf{r}. \tag{2.25b}$$

For the special case $f(\mathbf{r}) \equiv f(r)\mathfrak{Y}_{m'}^{[l']}(\hat{\mathbf{r}})$, we have

$$f_t^{(\mathbf{n})} = \delta_{ll'}\,\delta_{mm'}\int w(r)f(r)r^lL_n^{l+\frac{1}{2}}(r^2)r^2\,dr \tag{2.26}$$

which may be further expressed in terms of the Talmi integrals[4] $I_p(f)$ by using the power series for $L_n^{l+\frac{1}{2}}(r^2)$,

$$f_t^{(\mathbf{n})} = \delta_{ll'}\,\delta_{mm'}\sum_{p=\frac{1}{2}l}^{2n+\frac{1}{2}l} A(nl,p)I_p(f), \tag{2.27}$$

$$I_p(f) = 2[\Gamma(p+\tfrac{3}{2})]^{-1}\int_0^\infty r^{2p}\,e^{-r^2}\,f(r)r^2\,dr, \tag{2.28}$$

$$A(nl;p) = \frac{(-)^{p-l/2}}{2\pi^{\frac{3}{2}}}\binom{n+l+\frac{1}{2}}{n-p+\frac{1}{2}l}\frac{\Gamma(p+\frac{3}{2})}{\Gamma(p+1-\frac{1}{2}l)}, \tag{2.29}$$

$$\binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}. \tag{2.30}$$

[18] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-uniform Gases* (Cambridge University Press, Cambridge, England, 1939), also 2nd ed. (1952).

III. INTEGRALS INVOLVING TRIPLE PRODUCTS OF ξ's

In view of (2.25), the evaluation of (2.10) often involves an integration of products of three ξ's. From the Wigner–Eckart theorem or Eq. (1.4) we can write

$$\int w(r)\xi_{m_3}^{(n_3 l_3)}(\mathbf{r})\xi_{m_2}^{[n_2 l_2]}(\mathbf{r})\xi_{m_1}^{[n_1 l_1]}(\mathbf{r})\, d\mathbf{r}$$

$$\equiv K(\mathbf{n}_3;\mathbf{n}_2\mathbf{n}_1) = (l_1 m,\ l_2 m_2 \mid l_3 m_3)K(n_3 l_3; n_2 l_2 n_1 l_1). \tag{3.1}$$

This may be considered a definition of the symbols K, to evaluate which we have to pick out the coefficient of $\chi^{[\mathbf{n}_3]}(\mathbf{c})\chi^{(\mathbf{n}_2)}(\mathbf{b})\chi^{(\mathbf{n}_1)}(\mathbf{a})$ in the integral [Eqs. (2.16b) and (2.21)]

$$\int w(r)G(\mathbf{c},\mathbf{r})G(\mathbf{b},\mathbf{r})G(\mathbf{a},\mathbf{r})\, d\mathbf{r}$$

$$= \exp\left[2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a})\right]. \tag{3.2}$$

In expanding the right-hand side according to (1.7), the angular parts can be reexpressed by means of the following formula:

$$\sum_{m_1',m_2',m_3'} (-)^{l_3'-m_3'}\mathfrak{Y}_{m_1'}^{[l_1']}(\hat{\mathbf{c}})\mathfrak{Y}_{m_2'}^{[l_2']}(\hat{\mathbf{c}})\mathfrak{Y}_{-m_3'}^{(l_3')}(\hat{\mathbf{b}})$$

$$\times \mathfrak{Y}_{m_1'}^{(l_1')}(\hat{\mathbf{b}})\mathfrak{Y}_{m_2'}^{(l_2')}(\hat{\mathbf{a}})\mathfrak{Y}_{m_3'}^{(l_3')}(\hat{\mathbf{a}})$$

$$= \sum_{l_1,l_2,l_3}\ \sum_{m_1,m_2,m_3} (l_1 m_1\ l_2 m_2 \mid l_3 m_3)\bar{W}\begin{pmatrix} l_1 & l_2 & l_3 \\ l_1' & l_2' & l_3' \end{pmatrix}$$

$$\times \hat{l}_1\hat{l}_2\sigma(l_1'\ l_2'\ l_3)\sigma(l_2'\ l_3'\ l_1)\sigma(l_3'\ l_1'\ l_2)$$

$$\times \mathfrak{Y}_{m_3}^{[l_3]}(\hat{\mathbf{c}})\mathfrak{Y}_{m_2}^{(l_2)}(\hat{\mathbf{b}})\mathfrak{Y}_{m_1}^{(l_1)}(\hat{\mathbf{a}}). \tag{3.3}$$

Both sides are evidently scalar and the formula represents the effect of a recoupling as evidenced by the appearance of the coefficient $\bar{W}$ which is the same as Wigner's 6-j symbol [Fano and Racah,[1] Eq. (11.7)].

Finally,

$$K(n_3 l_3; n_2 l_2 n_1 l_1) = (-)^{n_1+n_2+n_3} n_1!\, n_2!\, n_3!\, \hat{l}_1\hat{l}_2$$

$$\times \sum \sigma(l_1'\ l_2'\ l_3)\sigma(l_2'\ l_3'\ l_1)\sigma(l_3'\ l_1'\ l_2)$$

$$\times \bar{W}\begin{pmatrix} l_1 & l_2 & l_3 \\ l_1' & l_2' & l_3' \end{pmatrix}\left[\frac{N_{n_1'l_1'}N_{n_2'l_2'}N_{n_3'l_3'}}{n_1'!\, n_2'!\, n_3'!}\right]^2. \tag{3.4}$$

The summation variables are l_1', l_2', and l_3' whose values are restricted by the functions occurring in the sum. The values of n_i' are given by the three relations obtained by equating the powers of a, b, and c,

$$2n_1 + l_1 = 2n_2' + l_2' + 2n_3' + l_3',$$

$$2n_2 + l_2 = 2n_3' + l_3' + 2n_1' + l_1',$$

$$2n_3 + l_3 = 2n_1' + l_1' + 2n_2' + l_2'.$$

It follows that for given n_i, l_i, the sum $2n_i' + l_i' = p_i'$ has a fixed value which depends only on the former numbers,

$$2p_i' = p_j + p_k - p_i, \qquad p_i = 2n_i + l_i, \tag{3.5}$$

(i, j, k) cyclic permutation of 1, 2, and 3.

Thus, the quantities in the square brackets in (3.4) actually are

$$[N_{n_1'l_1'}(n_1'!)^{-1}]^2 = 2\pi^{\frac{3}{2}}\{\Gamma[\tfrac{1}{2}(p_1' - l_1' + 2)]$$

$$\times \Gamma[\tfrac{1}{2}(p_1' + l_1' + 3)]\}^{-1}. \tag{3.6}$$

The restrictions above establish also a relation between the p_i (hence n_i) values. Since $p_1 + p_2 = p_3 + 2p_3'$ and $p_3' \geq 0$, it follows that the integral vanishes unless

$$p_i + p_j \geq p_k, \quad (i, j, k) \text{ cyclic permutation of 1, 2, and 3.} \tag{3.7}$$

This is a symmetric relation between the free indices of independent polynomials on the left-hand side of (3.1). This may be called a scalar triangular relation corresponding to the fact that the sum of two sides of a triangle is always greater than the third. This is to be contrasted with the *vector* triangular relations which hold between the numbers l_i in virtue of the Wigner coefficient.

IV. RELATIONS INVOLVING MOSHINSKY'S COEFFICIENT $B(nl,n'l',p)$

This coefficient is defined by means of the formula[7,12,14]

$$\int_0^\infty \mathcal{R}_{nl}(r)V(r)\mathcal{R}_{n'l'}(r)r^2\, dr = \sum_p B(nl, n'l', p)I_p(V). \tag{4.1}$$

The explicit form of B has been obtained using the explicit power series for $\mathcal{R}$ functions. This coefficient occurs in calculations of nuclear shell theory and has been well tabulated.[12–14] It is related to the triple product integral of the last section—a relation which also shows the role of the $\bar{W}$-coefficient in this coefficient.

Let the Talmi integral of the function $N_{nl}r^l L_n^{l+\frac{1}{2}}(r^2)$ be denoted by $I_p(nl)$. By (2.28)

$$I_p(nl) = 2[\Gamma(p + \tfrac{3}{2})]^{-1}N_{nl}\int_0^\infty r^{2p}\, e^{-r^2} r^l L_n^{l+\frac{1}{2}}(r^2)r^2\, dr. \tag{4.2}$$

Performing the angular integrals in (3.1) and converting the radial parts to the form (4.1) by using (2.6) we obtain the relations

$$K(n_3 l_3; n_2 l_2\, n_1 l_1) = N_{n_1 l_1}N_{n_2 l_2}N_{n_3 l_3}\sigma(l_1\ l_2\ l_3)$$

$$\times \sum B(n_2 l_2, n_3 l_3, p)I_p(n_1 l_1). \tag{4.3}$$

Also

$$\sum_p B(n_2l_2, n_3l_3, p)I_p(n_1l_1)$$
$$= \sum_p B(n_3l_3, n_1l_1, p)I_p(n_2l_2)$$
$$= \sum_p B(n_1l_1, n_2l_2, p)I_p(n_3l_3). \tag{4.4}$$

These relations may be used to calculate the coefficients K, since B's have already been tabulated[12-14] and $I_p(nl)$ can be calculated easily.

On the other hand, the coefficient B may also be expressed in terms of K's. The integral

$$\int w(r)\xi^{(n_3)}f(\mathbf{r})\xi^{[n_1]}\,d\mathbf{r}$$

may be evaluated in two ways for the special case when $f(\mathbf{r}) \equiv f(r)\mathcal{Y}_{m_2}^{[l_2]}$. Using (2.25), (2.27), and (3.1) one gets

$$(l_1m_1\ l_2m_2 \mid l_3m_3) \sum_{n_2,p} K(n_3l_3; n_2l_2\ n_1l_1)A(n_2l_2, p)I_p(f).$$

Performing the angular integrals first and then using (4.1) on the radial part, one gets for the same quantity

$$(l_1m_1\ l_2m_2 \mid l_3m_3)\sigma(l_1\ l_2\ l_3)N_{n_1l_1}N_{n_3l_3}$$
$$\times \sum_p B(n_1l_1\ n_3l_3, p)I_p(f);$$

hence the relation

$$B(n_1l_1, n_3l_3, p) = [N_{n_1l_1}N_{n_3l_3}\sigma(l_1\ l_2\ l_3)]^{-1}$$
$$\times \sum_{n_2} K(n_3l_3; n_2l_2\ n_1l_1)A(n_2l_2; p). \tag{4.5}$$

With (3.4) it shows the role of the $\bar{W}$-coefficient in the formation of B. Of course, l_2 must satisfy the triangular restrictions, $l_1 + l_3 < l_2 < |l_1 - l_3|$, if this relation is to hold. The sum over n_2 is restricted among other things by l_2 and p. It serves eventually to eliminate l_2 on the right-hand side. A symmetric expression in l_1 and l_3 is obtained by setting $l_2 = l_1 + l_3$.

V. TALMI COEFFICIENTS FOR UNEQUAL MASSES

In nuclear physics where the harmonic oscillator functions can be applied, the particles in general have the same mass. Hence that is the case most often studied. However, Smirnov[16] has drawn attention to the problem of separating the center-of-mass motion of several nucleons in which harmonic oscillator functions of different masses may be used. In kinetic theory the case of unequal masses arises in the study of transport properties of gas mixtures.

Talmi coefficients[4-16] are defined by the following relation:

$$\psi_{m_1}^{[n_1l_1]}(\alpha_1\mathbf{r}_1)\psi_{m_2}^{[n_2l_2]}(\alpha_2\mathbf{r}_2)$$
$$= \sum_{NLM,nlm} T\begin{Bmatrix}(\Gamma)NLM & (\alpha_1)n_1l_1m_1\\ (\gamma)nlm & (\alpha_2)n_2l_2m_2\end{Bmatrix}$$
$$\times \psi_M^{[NL]}(\Gamma\mathbf{R})\psi_m^{[nl]}(\gamma\mathbf{r}). \tag{5.1}$$

Various notations have been used for this quantity. We use the one which is most descriptive. We shall also use the abbreviated form $T(\Gamma\mathbf{N}, \gamma\mathbf{n} \mid \alpha_1\mathbf{n}_1, \alpha_2\mathbf{n}_2)$. Sometimes the scale parameters on one or both sides will also be dropped,

$$T(\Gamma\mathbf{N}, \gamma\mathbf{n} \mid \alpha_1\mathbf{n}_1, \alpha_2\mathbf{n}_2)$$
$$= \int \psi^{(\mathbf{N})}(\Gamma\mathbf{R})\psi^{(\mathbf{n})}(\gamma\mathbf{r})\psi^{[\mathbf{n}_1]}(\alpha_1\mathbf{r}_1)\psi^{[\mathbf{n}_2]}(\alpha_2\mathbf{r}_2)\,d\mathbf{R}\,d\mathbf{r}. \tag{5.2}$$

Using (2.6) and (1.10) we may write

$$T(\Gamma\mathbf{N}, \gamma\mathbf{n} \mid \alpha_1\mathbf{n}_1, \alpha_2\mathbf{n}_2) = \frac{\bar{T}(\Gamma\mathbf{N}, \gamma\mathbf{n} \mid \alpha_1\mathbf{n}_1, \alpha_2\mathbf{n}_2)}{N_{NL}N_{nl}N_{n_1l_1}N_{n_2l_2}}, \tag{5.3}$$

$$\bar{T}(\Gamma\mathbf{N}, \gamma\mathbf{n} \mid \alpha_1\mathbf{n}_1, \alpha_2\mathbf{n}_2)$$
$$= \int w(\Gamma, R)w(\gamma, r)\xi^{(\mathbf{N})}(\Gamma\mathbf{R})\xi^{(\mathbf{n})}(\gamma\mathbf{r})$$
$$\times \xi^{[\mathbf{n}_1]}(\alpha_1\mathbf{r}_1)\xi^{[\mathbf{n}_2]}(\alpha_2\mathbf{r}_2)\,d\mathbf{R}\,d\mathbf{r}. \tag{5.4}$$

It follows that

$$\xi^{[\mathbf{n}_1]}(\alpha_1\mathbf{r}_1)\xi^{[\mathbf{n}_2]}(\alpha_2\mathbf{r}_2)$$
$$= \sum_{NLM,nlm} \bar{T}(\Gamma\mathbf{N},\gamma\mathbf{n} \mid \alpha_1\mathbf{n}_1, \alpha_2\mathbf{n}_2)\xi^{[\mathbf{N}]}(\Gamma\mathbf{R})\xi^{[\mathbf{n}]}(\gamma\mathbf{r}). \tag{5.5}$$

The dependence on the numbers m can be separated in two Wigner coefficients and the T-coefficient can then be expressed in terms of the transformation brackets of Moshinsky,[7,8,11-14] which has been often investigated for the equal-mass case and for which tables have been prepared.

$$T(\mathbf{N}\,\mathbf{n} \mid \mathbf{n}_1\mathbf{n}_2) = \sum_\lambda (lm\ LM \mid \lambda\mu)(\lambda\mu \mid l_1m_1\ l_2m_2)$$
$$\times i^{l_1+l_2-l-L}\langle nl, NL, \lambda \mid n_1l_1, n_2l_2, \lambda\rangle. \tag{5.6}$$

To obtain explicit formulas for $\bar{T}$ note that from (2.19) and (2.20)

$$G(\mathbf{a}_1, \alpha_1\mathbf{r}_1)G(\mathbf{a}_2, \alpha_2\mathbf{r}_2) = G\left(\frac{\alpha_1}{\Gamma}\,\mathbf{a}_1, \Gamma\mathbf{R}\right)G\left(\frac{\alpha_2}{\Gamma}\,\mathbf{a}_1, \gamma\mathbf{r}\right)$$
$$\times G\left(\frac{\alpha_2}{\Gamma}\,\mathbf{a}_2, \Gamma\mathbf{R}\right)G\left(\frac{\alpha_1}{\Gamma}\,\mathbf{a}_2, -\gamma\mathbf{r}\right). \tag{5.7}$$

Then from (2.16b), (2.22), and (3.2)

$$\int w(\Gamma, R)w(\gamma, r)G(\mathbf{a}_1, \alpha_1\mathbf{r}_1)G(\mathbf{a}_2, \alpha_2\mathbf{r}_2) \times G(\mathbf{A}, \Gamma\mathbf{R})G(\mathbf{a}, \gamma\mathbf{r})\, d\mathbf{R}\, d\mathbf{r} \equiv \exp\left[\frac{2}{\Gamma}(\alpha_1\mathbf{a}_1\cdot\mathbf{A} + \alpha_2\mathbf{a}_2\cdot\mathbf{A} + \alpha_2\mathbf{a}_1\cdot\mathbf{a} - \alpha_1\mathbf{a}_2\cdot\mathbf{a})\right]. \tag{5.8}$$

In virtue of (5.4), then, $\bar{T}$ is obtained by picking the coefficient of $\chi^{(n_1)}(\mathbf{a}_1)\chi^{(n_2)}(\mathbf{a}_2)\chi^{[N]}(\mathbf{A})\chi^{[n]}(\mathbf{a})$ in the right-hand side. This procedure is similar to the one described for the case of triple product integrals. The right-hand sides of (5.8) and (3.8) are thus the generating functions for $\bar{T}$ and K coefficients. Multiplying by appropriate spherical harmonics and integrating over all angles, one obtains, because of (1.4), a product over four Wigner coefficients which is to be summed over the m numbers of four l's. This gives rise to an X-coefficient

$$\sum_{m',m'',M'M''} (l'm'\ L'M' \mid l_1m_1)(l''m''\ L''M'' \mid l_2m_2) \times (l'm'\ l''m'' \mid lm)(L'M'\ L''M'' \mid LM) = \hat{l}_1\hat{l}_2\hat{l}\hat{L} \times \sum_\lambda (lm\ LM \mid \lambda\mu)(\lambda\mu \mid l_1m_1\ l_2m_2)X\begin{bmatrix} l' & l'' & l \\ L' & L'' & L \\ l_1 & l_2 & \lambda \end{bmatrix}. \tag{5.9}$$

From (1.7) and (2.15) the nonangular factors are

$$\left[\frac{N_{N'L'}}{N'!}\frac{N_{N''L''}}{N''!}\frac{N_{n'l'}}{n'!}\frac{N_{n''l''}}{n''!}\right]^2 \times (\alpha_1a_1A)^{P'}(\alpha_2a_2A)^{P''}(\alpha_2a_1a)^{p'}(-\alpha_1a_2a)^{p''}$$

where $P = 2N + L$, etc.

Comparing the powers of a_1, a_2, A, and a, respectively we have

$$\begin{aligned} P' + p' &= p_1, \qquad P'' + p'' = p_2; \\ P' + P'' &= P, \qquad p' + p'' = p, \end{aligned} \tag{5.10}$$

so that $P + p = p_1 + p_2$ or

$$2n_1 + l_1 + 2n_2 + l_2 = 2N + L + 2n + l. \tag{5.11}$$

This is referred to as the equation for conservation of energy in nuclear theory literature and also follows from (5.1) by using the differential equation (2.5).

Collecting all the terms we get finally

$$\bar{T}\begin{bmatrix} (\Gamma)NLM & (\alpha_1)n_1l_1m_1 \\ (\gamma)nlm & (\alpha_2)n_2l_2m_2 \end{bmatrix} = (-)^{n_1+n_2+N+n} n_1!\, n_2!\, N!\, n!\ \hat{l}_1\hat{l}_2\hat{L}\hat{l} \times \sum (-)^{l''}(\alpha_1/\Gamma)^{2N'+L'+2n''+l''}(\alpha_2/\Gamma)^{2N''+L''+2n'+l'} \times \left[\frac{N_{N'L'}}{N'!}\frac{N_{N''L''}}{N''!}\frac{N_{n'l'}}{n'!}\frac{N_{n''l''}}{n''!}\right]^2 \times \sigma(l'L'l_1)\,\sigma(l''L''l_2)\,\sigma(l'l''l)\,\sigma(L'L''L) \times X\begin{bmatrix} l' & l'' & l \\ L' & L'' & L \\ l_1 & l_2 & \lambda \end{bmatrix}(lmLM \mid \lambda\mu)(\lambda\mu \mid l_1m_1\, l_2m_2). \tag{5.12}$$

The sum is over all the primed variables and λ. The restrictions on the sum are those arising from the functions occurring in the expression and the conditions (5.10) and (5.11). Because of the latter, (5.10) fixes only three out of the four variables N', N'', n', n'' for a given set of l-values. A sum over all allowable values of the one independent one must be performed. This expression is fully symmetric under the indices of $\bar{T}$-coefficient and involves only standard functions. The derivation given here and the formula (5.12) may be compared with those given previously in the literature,[4-16] even those for the equal-mass case. Many of the results of previous workers have been expressed in terms of the transformation bracket defined in (5.6). An expression for it is obtained by comparing (5.6) and (5.12).

Other less symmetric formulas can be derived for the $\bar{T}$-coefficient. In one form $\bar{T}$ is expressed as a sum over to K-functions and one X-function.

The following special case is useful when it is required to transform the functions of one vector variable:

$$\bar{T}\begin{bmatrix} \Gamma\ \mathbf{N} & \alpha_1\ \mathbf{n}_1 \\ \gamma\ \mathbf{n} & \alpha_2\ \mathbf{0} \end{bmatrix} = (4\pi)^{\frac{1}{2}}(-)^{N+n+n_1}\frac{n_1!\, N_{NL}^2N_{nl}^2}{N!\,n!} \times \left(\frac{\alpha_1}{\Gamma}\right)^{2N+L}\left(\frac{\alpha_2}{\Gamma}\right)^{2n+l}\sigma(l\,L\,l_1)(lm\,LM \mid l_1m_1). \tag{5.13}$$

This is most easily derived by considering the generating function integral (5.8) without the term depending on $\mathbf{r}_2$, i.e., put $\mathbf{a}_2 = 0$. Of course, it follows also from (5.12). For the case $\alpha_1 = \alpha_2 = 1$, Moshinsky has given a formula [Ref. 7, Eq. (60)] for the quantity $\langle nl, NL, \lambda \mid 0l_1, 0l_2, \lambda\rangle$ which may also be derived from (5.6) and (5.12). Since all the numbers n and l are either positive or zero; the requirement $|L' - l'| \le l_1 \le L' + l'$ together with Eq. (5.10) and $n_1 = 0$ gives $N' = n' = 0$ and $l_1 = L' + l'$. Similarly for $n_2 = 0$, $N'' = n'' = 0$, and $l_2 = L'' + l''$. The rest is straightforward.

VI. SYMMETRY RELATIONS, SUM RULES, AND RECURSION FORMULAS

By putting $\mathbf{a}_1 = 0$ in (5.8) instead of $\mathbf{a}_2$ and using (5.13) one finds

$$\bar{T}(\Gamma\mathbf{N}, \gamma\mathbf{n} \mid \alpha_1\mathbf{n}_1, \alpha_2\mathbf{0}) = (-)^l(\alpha_1/\alpha_2)^{2N+L-2n-l}\bar{T}(\Gamma\mathbf{N}, \gamma\mathbf{n} \mid \alpha_1\mathbf{0}, \alpha_2\mathbf{n}_2) = (\alpha_1/\alpha_2)^{2N+L-2n-l}\bar{T}(\Gamma\mathbf{n}, \gamma\mathbf{N} \mid \alpha_1\mathbf{n}_1, \alpha_2\mathbf{0}). \quad (6.1)$$

Similarly,

$$T(\Gamma\mathbf{N}, \gamma\mathbf{0} \mid \alpha_1\mathbf{n}_1, \alpha_2\mathbf{n}_2) = (-)^{l_2}(\alpha_1/\alpha_2)^{2n_1+l_1-2n_2-l_2}T(\Gamma\mathbf{0}, \gamma\mathbf{N} \mid \alpha_1\mathbf{n}_1, \alpha_2\mathbf{n}_2). \quad (6.2)$$

There does not seem to be any such simple relation when all indices are nonvanishing.

From the integral (5.4) it is seen that the change of scale $(\alpha_1, \alpha_2) \to (\beta\alpha_1, \beta\alpha_2)$ can be compensated by a corresponding change $(\mathbf{r}_1, \mathbf{r}_2) \to (\beta^{-1}\mathbf{r}_1, \beta^{-1}\mathbf{r}_2)$; since the integral remains unaltered by the latter we have

$$T(\Gamma\mathbf{N}\ \gamma\mathbf{n} \mid \alpha_1\mathbf{n}_1\ \alpha_2\mathbf{n}_2) = T((\beta\Gamma)\mathbf{N}, (\beta\gamma)\mathbf{n} \mid (\beta\alpha_1)\mathbf{n}_1, (\beta\alpha_2)\mathbf{n}_2). \quad (6.3)$$

In particular for the equal-mass transformation, $\alpha_1 = \alpha_2 = \alpha$,

$$T((2^{\frac{1}{2}}\alpha)\mathbf{N}, (2^{-\frac{1}{2}}\alpha)\mathbf{n} \mid \alpha\mathbf{n}_1, \alpha\mathbf{n}_2) = T((2^{\frac{1}{2}})\mathbf{N}, (2^{-\frac{1}{2}})\mathbf{n} \mid 1\mathbf{n}_1, 1\mathbf{n}_2). \quad (6.4)$$

The right-hand side is the standard form in which equal-mass transformation is calculated. To establish a relation between the equal-mass and unequal-mass case we note that from the identity

$$G(\mathbf{a}, \alpha\mathbf{r}) = \exp[(\alpha^2 - 1)a^2]G(\alpha\mathbf{a}, \mathbf{r}) \quad (6.5)$$

it follows that

$$\xi_m^{(nl)}(\alpha\mathbf{r}) = \sum_{n'=0}^{n} \epsilon(\alpha, n'nl)\xi_m^{(n'l)}(\mathbf{r}), \quad (6.6)$$

$$\epsilon(\alpha, n'nl) = \binom{n}{n'}(1 - \alpha^2)^{n-n'}\alpha^{2n'+l}. \quad (6.7)$$

Then from (5.5)

$$\sum_{N=N'}^{\infty}\sum_{n=n'}^{\infty} \epsilon(2^{-\frac{1}{2}}\Gamma, N'NL)\,\epsilon(2^{\frac{1}{2}}\gamma, n'nl) \times \bar{T}(\Gamma\mathbf{N}, \gamma\mathbf{n} \mid \alpha_1\mathbf{n}_1, \alpha_2\mathbf{n}_2) = \sum_{n_1'=0}^{n_1}\sum_{n_2'=0}^{\infty} \epsilon(\alpha_1, n_1'n_1l_1)\,\epsilon(\alpha_2, n_2'n_2l_2) \times \bar{T}(2^{\frac{1}{2}}N'LM, 2^{-\frac{1}{2}}n'lm \mid 1n_1'l_1m_1, 1n_2l_2m_2). \quad (6.8)$$

The simplest form of sum rules are, of course, the orthogonality relations in which a product of two T-coefficients is summed over common indices $(\mathbf{N}, \mathbf{n})$ or $(\mathbf{n}_1\mathbf{n}_2)$. These follow from the completeness of the $\psi^{(\mathbf{n})}$ functions. A variety of other relations may be obtained by the use of the generating functions. For example, in Eq. (5.8) one may set $\mathbf{A} = \mathbf{a} = \mathbf{a}_1 = \mathbf{a}_2$; then the right-hand side becomes a scalar. By comparing the coefficients of a power of A on the two sides one has the result that product of a $\bar{T}$-coefficient with $\bar{W}$-coefficient and two Wigner coefficients and summed over an appropriate number of indices is equal to a constant. It is now clear that other sum rules result by setting a different set of vectors equal in expression (5.8). Similar sum rules for the K-coefficients can be obtained.

In numerical evaluation of these complicated expressions, especially for constructing tables of values, it is often more convenient to work with recursion relations. For Talmi coefficients such recursion relations have been derived using the corresponding ones for Laguerre polynomials and spherical harmonics.[11,13,14] Since we now have generating function for composite polynomials the recursion relations for ξ or ψ may be obtained more directly. The normalization of ξ-function is especially useful for this purpose as it avoids many square rooted coefficients in such formulas. We give some brief examples. Let the contrastandard components of a vector $\mathbf{a}$ be denoted by

$$a_{\pm1}^{[1]} = (\mp ia_x + a_y)/\sqrt{2}, \qquad a_0^{[1]} = ia_z. \quad (6.9)$$

Then taking account of the phase conventions and using (1.4) we have

$$a_\nu^{[1]}\chi_m^{(nl)}(\mathbf{a}) = -(n+1) \times \left(\frac{l}{2l+1}\right)^{\frac{1}{2}}(l-1\ m-\nu\ 1\nu \mid lm)\chi_{m-\nu}^{(n+1,l-1)}(\mathbf{a}) + \left(\frac{l+1}{2l+1}\right)^{\frac{1}{2}}(l+1\ m-\nu\ 1\nu \mid lm)\chi_{m-\nu}^{(n,l+1)}(\mathbf{a}), \quad (6.10)$$

and using the gradient formula (e.g., Ref. 3, p. 124 with appropriate phase changes),

$$\nabla_{a\,\nu}^{[1]}\chi_m^{(nl)}(\mathbf{a}) = (2n + 2l + 1) \times \left(\frac{l}{2l+1}\right)^{\frac{1}{2}}(l-1\ m+\nu\ 1\nu \mid lm)\chi_{m-\nu}^{(n,l-1)}(\mathbf{a}) - 2\left(\frac{l+1}{2l+1}\right)^{\frac{1}{2}}(l+1\ m-\nu\ 1\nu \mid lm)\chi_{m-\nu}^{(n-1,l+1)}(\mathbf{a}). \quad (6.11)$$

The generating function yields

$$\nabla_{a\,\nu}^{[1]}G(\mathbf{a}, \mathbf{r}) = (2r_\nu^{[1]} - 2a_\nu^{[1]})G(\mathbf{a}, \mathbf{r}), \quad (6.12)$$

$$\nabla_{r\,\nu}^{[1]}G(\mathbf{a}, \mathbf{r}) = 2a_\nu^{[1]}G(\mathbf{a}, \mathbf{r}). \quad (6.13)$$

Hence comparing the coefficients with the help of (6.10) and (6.11) we immediately get

$$\nabla_\nu^{[1]}\xi_m^{[nl]} = 2\left(\frac{l}{2l-1}\right)^{\frac{1}{2}}(lm\ 1\nu \mid l-1\ m+\nu)\xi_{m+\nu}^{[n,l-1]} - 2n\left(\frac{l+1}{2l+3}\right)^{\frac{1}{2}}(lm\ 1\nu \mid l+1\ m+\nu)\xi_{m+\nu}^{[n-1,l+1]}, \quad (6.14)$$

$$r_\nu^{[1]}\xi_m^{[nl]} = \left(\frac{l}{2l-1}\right)^{\frac{1}{2}}(lm\ 1\nu \mid l-1\ m+\nu) \times [\xi_{m+\nu}^{[n,l-1]} - \xi_{m+\nu}^{[n+1,l-1]}] + \left(\frac{l+1}{2l+3}\right)^{\frac{1}{2}}(lm\ 1\nu \mid l+1\ m+\nu) \times [-n\xi_{m+\nu}^{[n-1,l+1]} + (n+l+\tfrac{3}{2})\xi_{m+\nu}^{[n,l+1]}]. \quad (6.15)$$

Of course, these expressions can be calculated by using the definition (2.9) and corresponding formulas for L_n^α, etc., but the calculations would be rather long. The calculation of relations involving scalar operators such as $r\partial/\partial r$ or r^2 are even shorter as will be seen from the following:

$$\begin{aligned} 4r^2G(\mathbf{a},\mathbf{r}) &= \left(2\mathbf{a}+\frac{\partial}{\partial\mathbf{a}}\right)\cdot\left(2\mathbf{a}+\frac{\partial}{\partial\mathbf{a}}\right)G(\mathbf{a},\mathbf{r}) \\ &= \left(4a^2+4a\frac{\partial}{\partial a}+6+\nabla_a^2\right)G(a,r); \end{aligned} \quad (6.16)$$

$$\begin{aligned} r\frac{\partial}{\partial r}G \equiv \mathbf{r}\cdot\frac{\partial}{\partial\mathbf{r}}G &= \mathbf{a}\cdot\left(2\mathbf{a}+\frac{\partial}{\partial\mathbf{a}}\right)G \\ &= \left(2a^2+a\frac{\partial}{\partial a}\right)G; \end{aligned} \quad (6.17)$$

$$a^2\chi_m^{(nl)} = -(n+1)\chi_m^{(n+1,l)}; \quad (6.18)$$

$$a\frac{\partial}{\partial a}\chi_m^{(nl)} = (2n+l)\chi_m^{(nl)}; \quad (6.19)$$

$$\nabla_a^2\chi_m^{(nl)} = -\frac{1}{n}[(2n+l)(2n+l+1)-l(l+1)] \times \chi_m^{(n-1,l)}. \quad (6.20)$$

The generating function (2.1) may also be expanded in Cartesian coordinates in which case it generates products of three Hermite polynomials in x, y, and z (e.g., Ref. 17, p. 194). Such a product can be expressed as a linear combination of $\xi_m^{[nl]}(r, \theta, \varphi)$. The coefficients of this expansion are related to the transformation brackets $(n_xn_yn_z \mid nlm)$ which have been discussed in connection with the nuclear shell model.[19,20] The use of generating function may be expected to simplify these calculations also.

[19] Z. Pluhař and J. Tolar, Czech. J. Phys. **B14**, 287 (1964).
[20] E. Chacón and M. de Llano, Rev. Mex. Fis. **12**, 57 (1963).