Tags: #logbook - Denison
Links:

Logbook_05_220128

A Numeric Project

B Analytic Project

Aims

- [ ] Find analytic conditions for A, B that match with our data

B.1 Notes

Although we are solving a DE, so there are constants, there is only one solution that satisfies our DE and is finite as $x \to \pm\infty$. This is the solution that Long found numerically, and we are trying to approximate with our values of A, B. Therefore, there should be a way of analytically finding what our values should be.

> **Martijn**: The coefficients A & B are the key parameters, and I cannot see how you can find them using our present approach—I don't think you will ever get a closed set of equations of which they are the solutions. In other words, you need a complementary set of information to find them. I recommend that you to the following. Substitute u^(0)+u^(1) into the DE and do a Taylor expansion around the origin. Since this is an even function you only get even orders; the DE needs to be satisfied at every order for the solution to be exact. That is too much to ask here since u^(0)+u^(1) is not an exact solution, but you can

> impose that it satisfies the DE at orders 0 and 2. This gives you two equations with two unknowns (namely A & B) that you can solve. Can you please try this? It should give a sensible result, i.e., a result that is close to Long's numerics, since we know that u^(0)+u^(1) is a decent approximation to the exact result. If it does then you can extend it to u^(0)+u^(1)+ u^(2)+u^(3) and see what you get.

B.2 Results

Using `04_taylor_220128.nb`:

1. Substitute in the $u^{(0)}+u^{(1)}$ into the differential equation
2. Taylor expand the result about $x=0$
3. Equate the x^0 and the x^2 coefficients (there will be no odd terms because the function is even) to 0 and use these expressions to solve for A, B

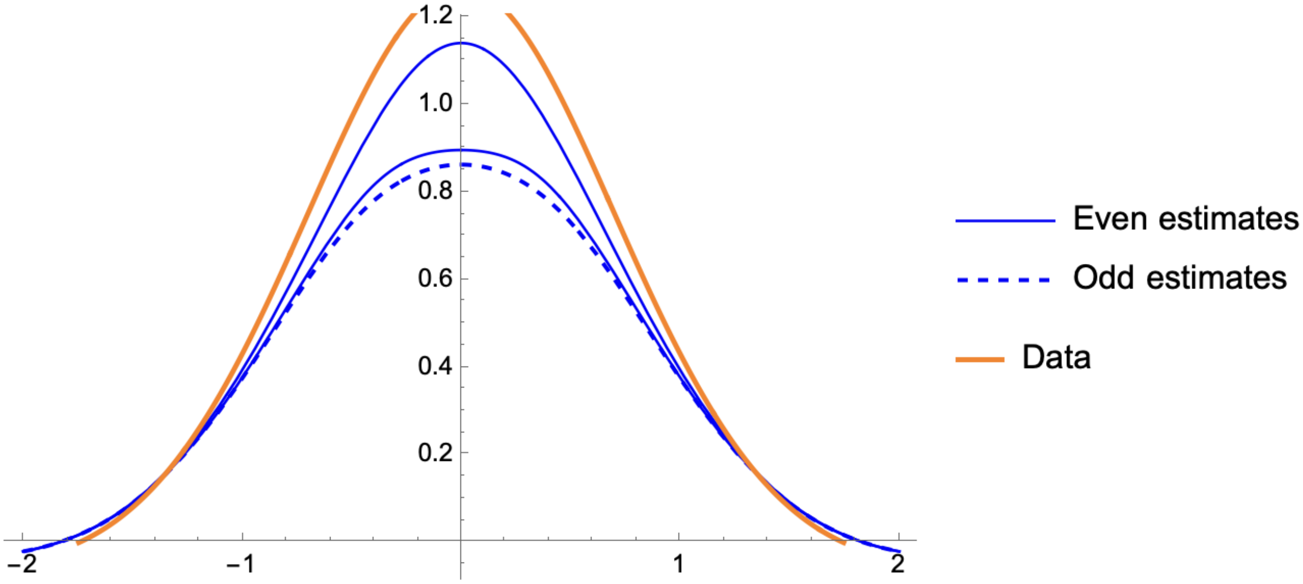
B.2.1 Solving with $u^{(1)}$

We get

$$A \to 0.242423, B \to 0.898534$$

which is not the same as what we found before $A=0.445408,\ B=1.02817$

- The ratio is also not the same
- Different amplitudes also



- Why does the amplitude not match?
  - We solved the DE so that to order x^2 it is 0, so should the solutions be matching at $x = 0$?

If we consider another example,

$$h(x) = 3\cos x + 1$$

which solves the DE

$$h'' - h + (1 + 6\cos x) = 0$$

and we use the ansatz

$$g(x) = A\cos x$$

Substituting it into the DE and expanding about $x = 0$ gives

$$(7 - 2A) + (A - 3)x^2 + \mathcal{O}(x^3)$$

Solving this to the 0th order gives $A \mapsto \frac{7}{2}$, so $h(0) \neq g(0)$ even though the DE is satisfied to this order.

B.2.2 Solving with $u^{(3)}$

Repeating this with $u^{(0)} + \cdots + u^{(3)}$, we get

$$A \to 0.334942, B \to 0.69456$$

B.2.3 Solving with $u^{(0)}$

With just $u^{(0)}$, we get

$$A \to 0.0333275, B \to 1.15336$$

B.2.4 Expanding only Fifth Order Terms

We have our function $u^{(1)}$ such that the sech terms are zero up to order sech^5. Therefore, perhaps including all the terms in our Taylor Expansion is too ambitious, and we should only consider the sech^5 terms.

Using `04_taylor_220131.nb`, we find that the Taylor expansion of the substituted DE has a x^4 leading term.

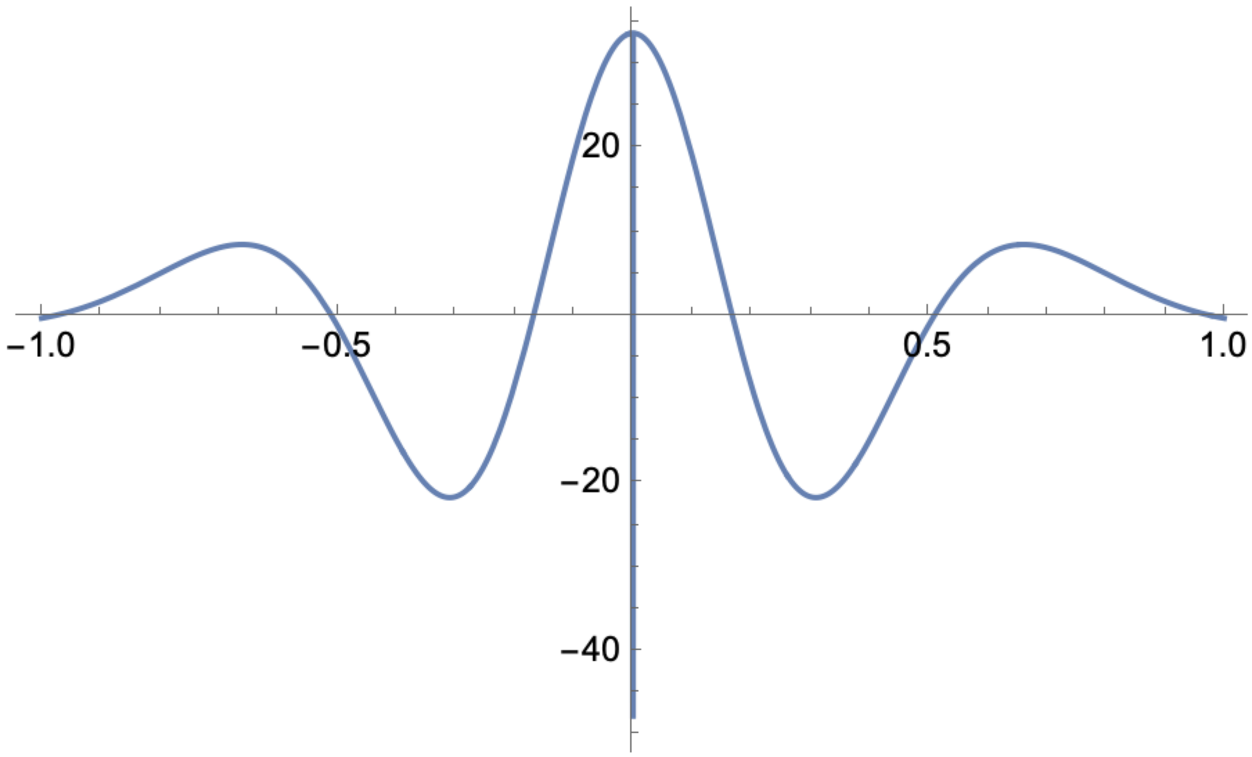
- If we add in sech^7, then we get x^2. sech^9 gives us x^0, and so on
- This is because higher order sech terms become negligible in the tails, but they are definitely a significant contribution when near $x = 0$
  - With higher order csch, we need equivalent $\sin$ terms in the numerator to "balance" them out and prevent divergence at $x = 0$.
- So I don't think we can leave out these higher order terms / ignore them when considering solving the DE around $x = 0$

> I see. The various csch orders can cancel the divergence because they have different factors of sin in the numerator, so they are of the same order at the origin but not at infinity (this was just a rephrasing of your statement in my own words). So it is all tangled up.

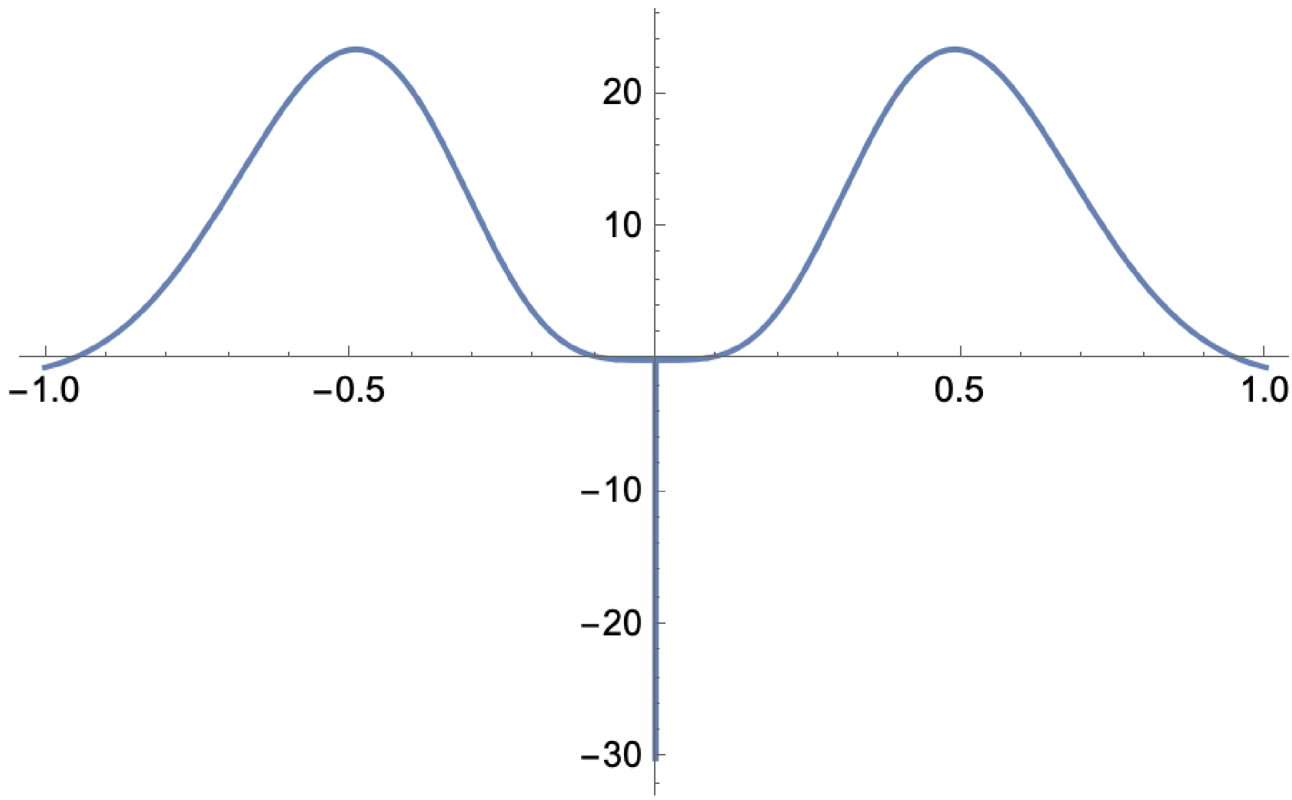
B.2.5 Plotting DE Remainder

Should we even be solving the DE around $x = 0$? We have our function set up so that it is a fit in the tails, and we're forcing it to fit at the peak where we know it does not give a good match.

Note this is what the remainder to the DE looks like when substituting in our current estimates for A, B and using $u^{(0)} + u^{(1)}$:



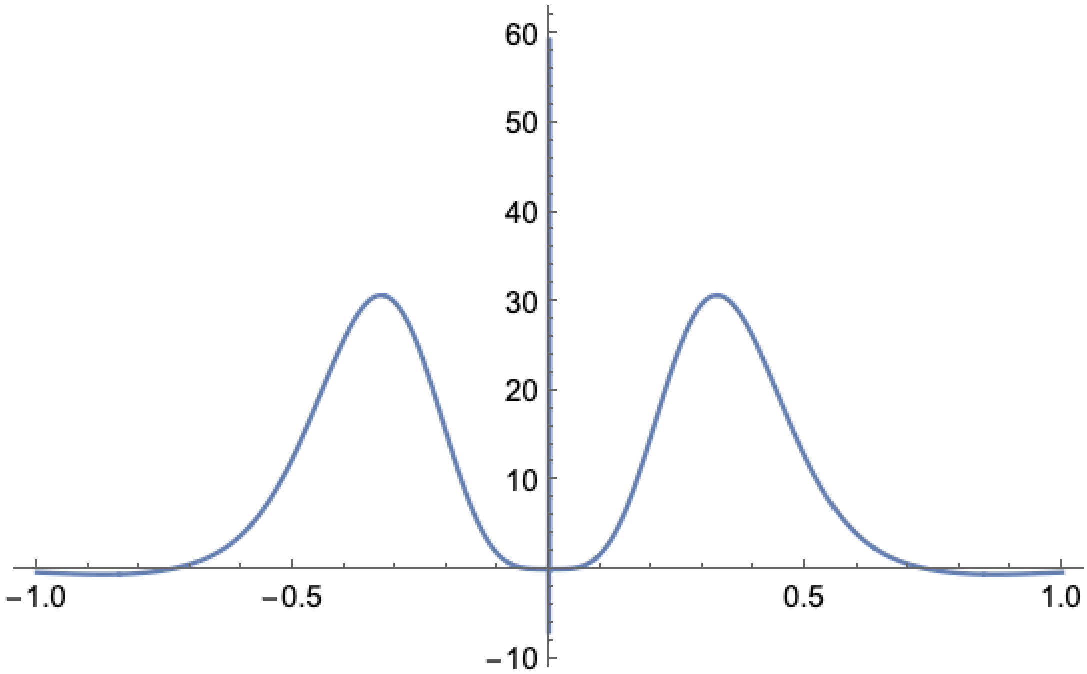
If we set A, B such that the Taylor series is 0 up to x^2 near $x = 0$, then we get



- So we are getting a very localised "fit" with this approach"
- And the "best estimates" don't give a particularly good approximation either near 0
    - So this may not be the right approach

> Your (second) figure is very pretty and it really illustrates what I had hoped it would achieve. However, as the (first) figure shows it is not the right answer! The other day I mentioned a variational approach. In effect it comes down to minimising the area of the square of these figures. It would seem to me that that area in the first figure is not noticeably larger than that of the second one, so I would guess that a variational approach would not work well either.

For the $u^{(0)} + \cdots + u^{(3)}$ estimate, we get



- Which is flat and 0 at $x = 0$, as expected

B.3 Outcomes

- Perhaps we have gone as far as we can with this approach, we might need a new function or idea for getting these coefficients

> BTW, the amplitude of the solution should be larger than $\sqrt{\mu/\gamma}$.

> It is not clear to me how to proceed from here. Once you commit to the u^(0) ansatz you do not have a lot of choice, since the general form of u^(1) is in essence determined by (u^(0))^3. In other words, if one would want to do this with a different ansatz

> it seems to me that it would have to be done immediately at the u^(0) level. One possibility would be to replace cos/cosh by cos/(1+a cosh). However, this does not work for the sin/sinh case as you would get a divergence somewhere. Applying it only to the cos/cosh term breaks the symmetry between the sine and the cosine-like terms I fear.

B.4 To Do

- [ ] What else can we do to try and get A and B?
    - We can maybe get the amplitude using fixed point analysis
    - But the phase is confusing
    - [ ] Can we do a variational approach?
        - Like the Taylor expansion approach but it acts globally to minimise the RHS of the DE
        - But doesn't give an exact approach, and it emphasises the centre rather than the tails
- [ ] Is there a better functional form we can use for $u^{(0)}$?
    - The tail looks like $e^{\pm\sigma x}\cos(\sigma x+\phi)$ (found by dropping the nonlinear term)
    - Even function
    - Doesn't diverge at origin

$$u'''' + 4\sigma^4 u = 0$$

$$u = \sum a_j e^{\pm\sigma t} \begin{matrix}\cos\sigma t\\ \sin\sigma t\end{matrix}$$

leading edge: $e^{\sigma t}(A\cos\sigma t + B\sin\sigma t)$

$$= e^{\sigma t} A\cos(\sigma t+\varphi)$$

u_0: $Ae^{\sigma t}\cos\sigma t \rightarrow$

u_1: $\underline{\Gamma A^3} e^{3\sigma t}(a\cos 3\sigma t + b\sin 3\sigma t + c\cos\sigma t + d\sin\sigma t)$.

...

- [x] ~~Check Long's numerical solution by plugging it into the DE and see if it fits~~

- ~~Be careful with taking fourth derivatives, each derivative adds noise~~

- [x] ~~Repeat remainder DE plots with~~ $\cancel{+u2+u3}$ ~~etc~~