PQS

Meeting 2022-02-07 1pm

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Approach

- **Attempting to analytically solve the NLSE for a PQS**
- Assume the effect of nonlinearity is only a change in propagation constant, not intensity
- Look for solutions of the form $u(z)\exp(i\mu z)$ satisfying

$$\mu u + \frac{|\beta_4|}{24}\frac{d^4 u}{d\tau^4} + \gamma u^3 = 0$$

$$\underbrace{u'''' + 4\sigma^4 u}_{L} + \underbrace{\Gamma u^3}_{N} = 0, \qquad \sigma^4 := \frac{6\mu}{|\beta_4|}, \Gamma := \frac{24\gamma}{|\beta_4|}$$

Approach

$$u^{(0)} = A\frac{\cos\sigma x}{\cosh\sigma x} + B\frac{\sin\sigma x}{\sinh\sigma x}, \quad u^{(n)} = \sum_{k=0}^{2n+1} c_k \left(\frac{\cos\sigma x}{\cosh\sigma x}\right)^{2n+1-k} \left(\frac{\sin\sigma x}{\sinh\sigma x}\right)^{k}$$

- In the tails, $u^3 \ll u$, so consider a perturbation of the linear part of the ODE

1. Let $u^{(0)}$ be a solution to $L\left[u^{(0)}\right] = 0$, up to first order of $1/\cosh, 1/\sinh$
2. Then define $u^{(1)}$ such that $L\left[u^{(0)} + u^{(1)}\right] + N\left[u^{(0)}\right] = 0$ up to third orders of $1/\cosh^3, 1/\sinh^3$
3. Repeat for higher orders to (hopefully) converge to a closed-form solution

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Coefficients of u1

$$
\begin{aligned}
c_0 &= \tfrac{A}{4} - \left(2A^3 + 9A^2B + 6AB^2 + 9B^3\right)\varphi = 0.15371 \\
c_1 &= -\tfrac{B}{4} + 3\left(3A^3 + 2A^2B + 3AB^2 - 2B^3\right)\varphi = -0.253315 \\
c_2 &= \tfrac{A}{4} - 3\left(2A^3 + 3A^2B - 2AB^2 + 3B^3\right)\varphi = 0.137759 \\
c_3 &= -\tfrac{B}{4} + \left(9A^3 - 6A^2B + 9AB^2 - 2B^3\right)\varphi = -0.259133
\end{aligned}
$$

where

$$\varphi := \frac{\Gamma}{1280\sigma^4}$$

Higher Coefficients

We can find higher order solutions by solving

$$L\left[\sum_{j=0}^{n} u^{(j)}\right] + N\left[\sum_{j=0}^{n-1} u^{(j)}\right] \equiv 0$$

up to $(2n+1)$th order.

Linear terms appear in most of the coefficients:

Function	$u^{(1)}$	$u^{(2)}$	$u^{(3)}$	$u^{(4)}$	$u^{(5)}$	$u^{(6)}$
Order	3	5	7	9	11	13
Coefficients	A/4	A/8	5A/64	7A/128	21A/512	33A/1024
	-B/4	-B/8	-5B/64	-7B/128	-21B/512	-33B/1024
	A/4	...	-A/64	-A/64	-7A/512	-3A/256
	-B/4	...	B/64	B/64	7B/512	3B/256
		-A/8	-A/64	...	A/256	5A/1024
		B/8	B/64	...	-B/256	-5B/1024
			5A/64	A/64	A/256	...
			-5B/64	-B/64	-B/256	...
				-7A/128	-7A/512	-5A/1024
				7B/128	7B/512	5B/1024
					21A/512	3A/256
					-21B/512	-3B/256
						-33A/1024
						33B/1024

Higher Coefficients

Function	$u^{(1)}$	$u^{(2)}$	$u^{(3)}$	$u^{(4)}$	$u^{(5)}$	$u^{(6)}$
Order	3	5	7	9	11	13
Coefficients	A/4	A/8	5A/64	7A/128	21A/512	33A/1024

If we take the denominators to be increasing as 4^k, then the numerators are $1,2,5,14,42,132$ which are **Catalan numbers**:

$$C(n) = \frac{(2n)!}{n!(n+1)!}$$

Higher Coefficients

Coefficients:

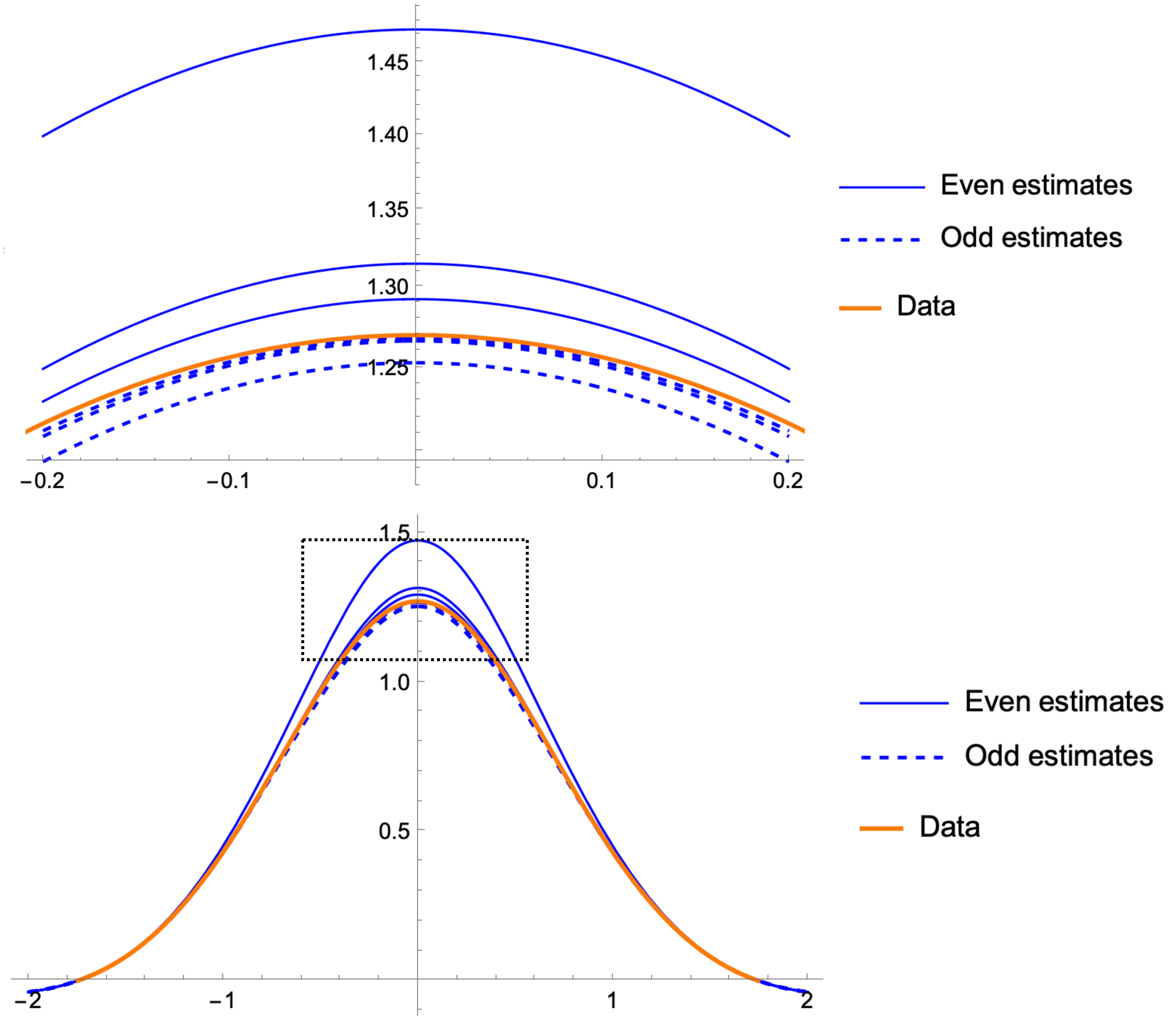
c[0] → 0.157039	d[0] → 0.0903383	e[0] → 0.0609978	f[0] → 0.0448517	g[0] → 0.0348072	h[0] → 0.0280441
c[1] → -0.256215	d[1] → -0.128514	e[1] → -0.0804295	f[1] → -0.0563353	g[1] → -0.0422631	h[1] → -0.0332103
c[2] → 0.140487	d[2] → 0.0271894	e[2] → 0.00526498	f[2] → -0.00152474	g[2] → -0.00390647	h[2] → -0.00469792
c[3] → -0.262153	d[3] → 0.00377046	e[3] → 0.0168306	f[3] → 0.0162891	g[3] → 0.0141419	h[3] → 0.0120898
_	d[4] → -0.0630543	e[4] → -0.0190443	f[4] → -0.00858966	g[4] → -0.00390552	h[4] → -0.0014936
_	d[5] → 0.132236	e[5] → 0.0142077	f[5] → -0.000842331	g[5] → -0.00440549	h[5] → -0.00521731
_	_	e[6] → 0.0366884	f[6] → 0.0133825	g[6] → 0.00733371	h[6] → 0.00430593
_	_	e[7] → -0.0830518	f[7] → -0.015152	g[7] → -0.00337559	h[7] → 0.000378755
_	_	_	f[8] → -0.0244042	g[8] → -0.00985882	h[8] → -0.00593467
_	_	_	f[9] → 0.0583147	g[9] → 0.0136075	h[9] → 0.00456695
_	_	_	_	g[10] → 0.0176156	h[10] → 0.00757355
_	_	_	_	g[11] → -0.0438274	h[11] → -0.0118415
_	_	_	_	_	h[12] → -0.0134343
_	_	_	_	_	h[13] → 0.0344878

Higher Coefficients

Just the linear terms:

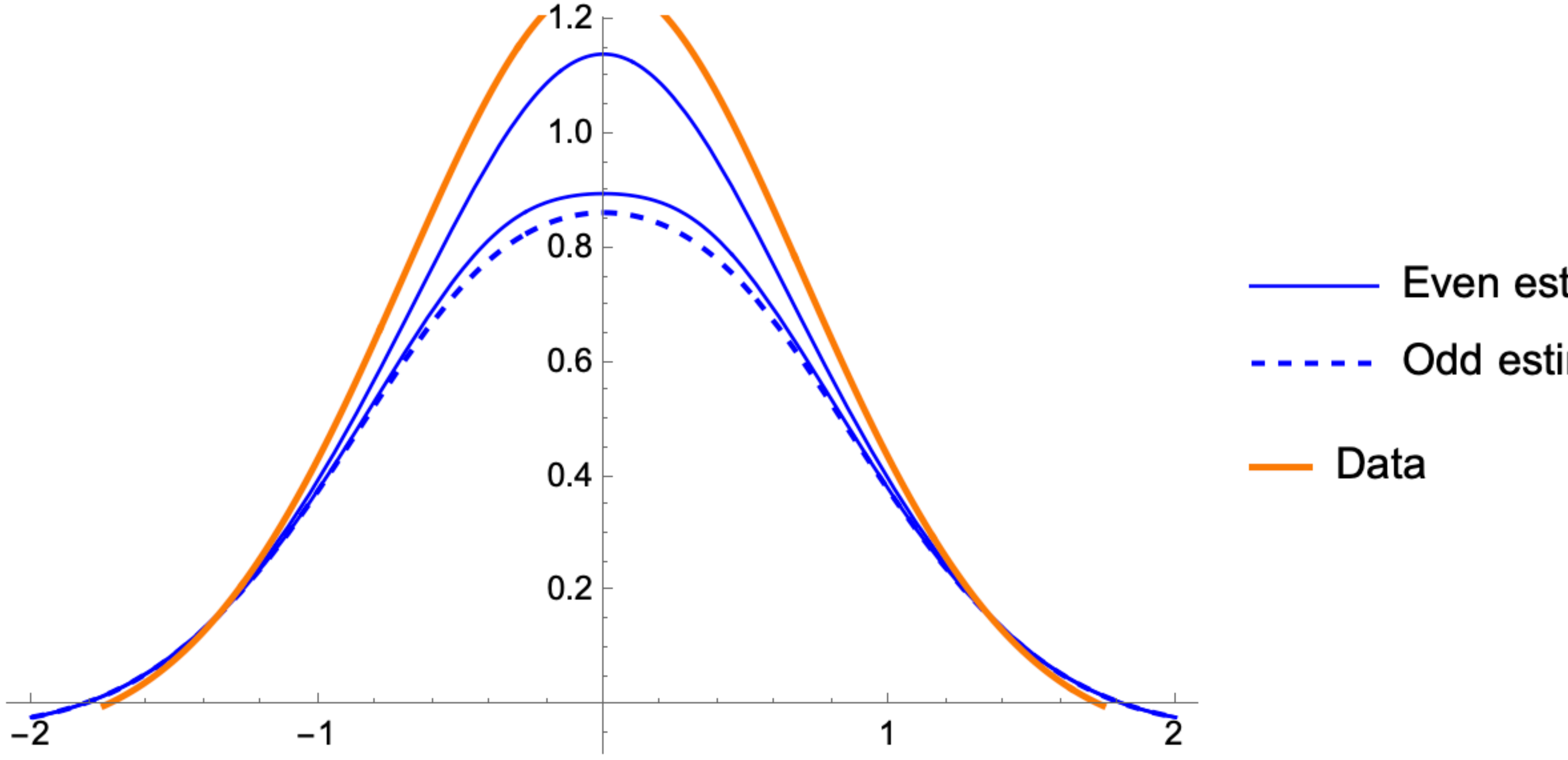
$$
\begin{pmatrix}
0.111352 & 0.055676 & 0.0347975 & 0.0243583 & 0.0182687 & 0.014354 \\
-0.257043 & -0.128521 & -0.0803258 & -0.056228 & -0.042171 & -0.0331344 \\
0.111352 & 0 & -0.0069595 & -0.0069595 & -0.00608956 & -0.00521963 \\
-0.257043 & 0 & 0.0160652 & 0.0160652 & 0.014057 & 0.0120489 \\
\_ & -0.055676 & -0.0069595 & 0 & 0.00173988 & 0.00217484 \\
\_ & 0.128521 & 0.0160652 & 0 & -0.00401629 & -0.00502036 \\
\_ & \_ & 0.0347975 & 0.0069595 & 0.00173988 & 0 \\
\_ & \_ & -0.0803258 & -0.0160652 & -0.00401629 & 0 \\
\_ & \_ & \_ & -0.0243583 & -0.00608956 & -0.00217484 \\
\_ & \_ & \_ & 0.056228 & 0.014057 & 0.00502036 \\
\_ & \_ & \_ & \_ & 0.0182687 & 0.00521963 \\
\_ & \_ & \_ & \_ & -0.042171 & -0.0120489 \\
\_ & \_ & \_ & \_ & \_ & -0.014354 \\
\_ & \_ & \_ & \_ & \_ & 0.0331344
\end{pmatrix}
$$

Higher estimates



Taylor expansion

- Process
  - 1. Substitute u0 + u1 into the DE
  - 2. Taylor expand the result about 0, to the order x^2
  - 3. Equate the coefficients of x^0 and x^2 to 0 to solve for A, B
- Result:
  - {A -> 0.242423, B -> 0.898534}
  - (Our current estimates are
    A -> 0.445408, B -> 1.02817

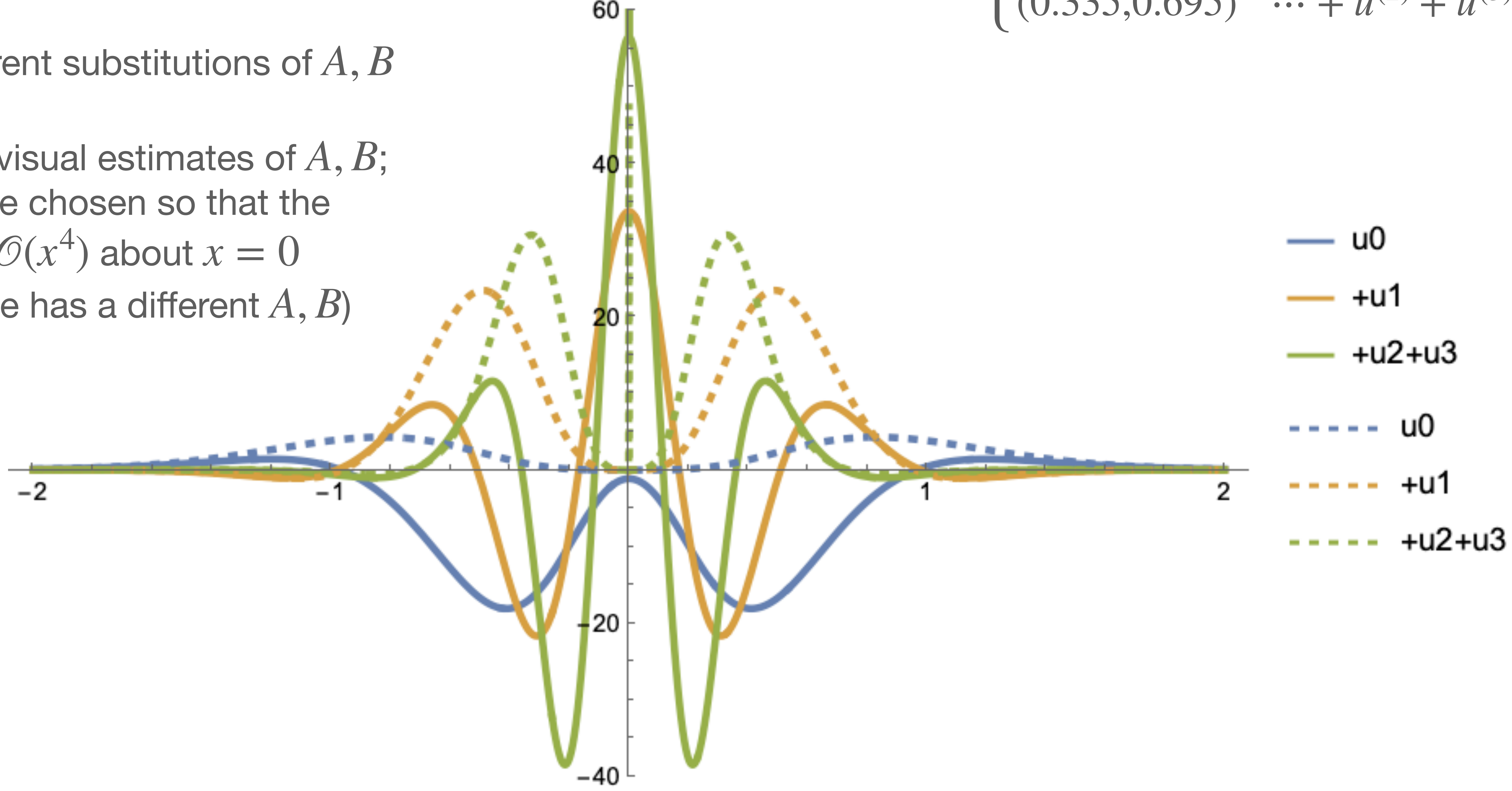


Remainder

$$(A, B) = \begin{cases} (0.445, 1.028) & \text{visual} \\ (0.0333, 1.15) & u^{(0)} \\ (0.242, 0.899) & \cdots + u^{(1)} \\ (0.335, 0.695) & \cdots + u^{(2)} + u^{(3)} \end{cases}$$

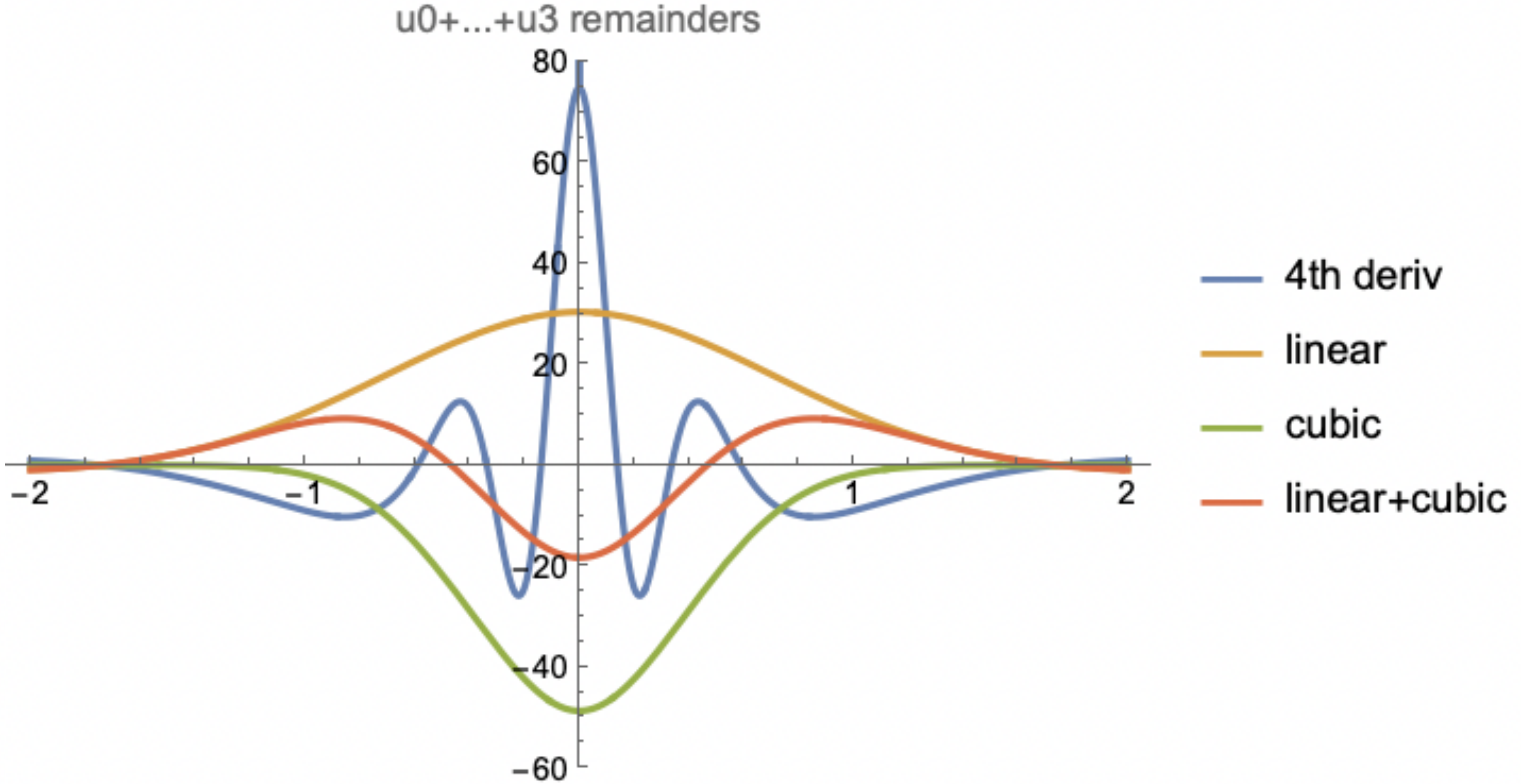
Plotting $\frac{d^4u}{dt^4} + 4\sigma^4 u + \Gamma u^3$ for different u functions and different substitutions of A, B

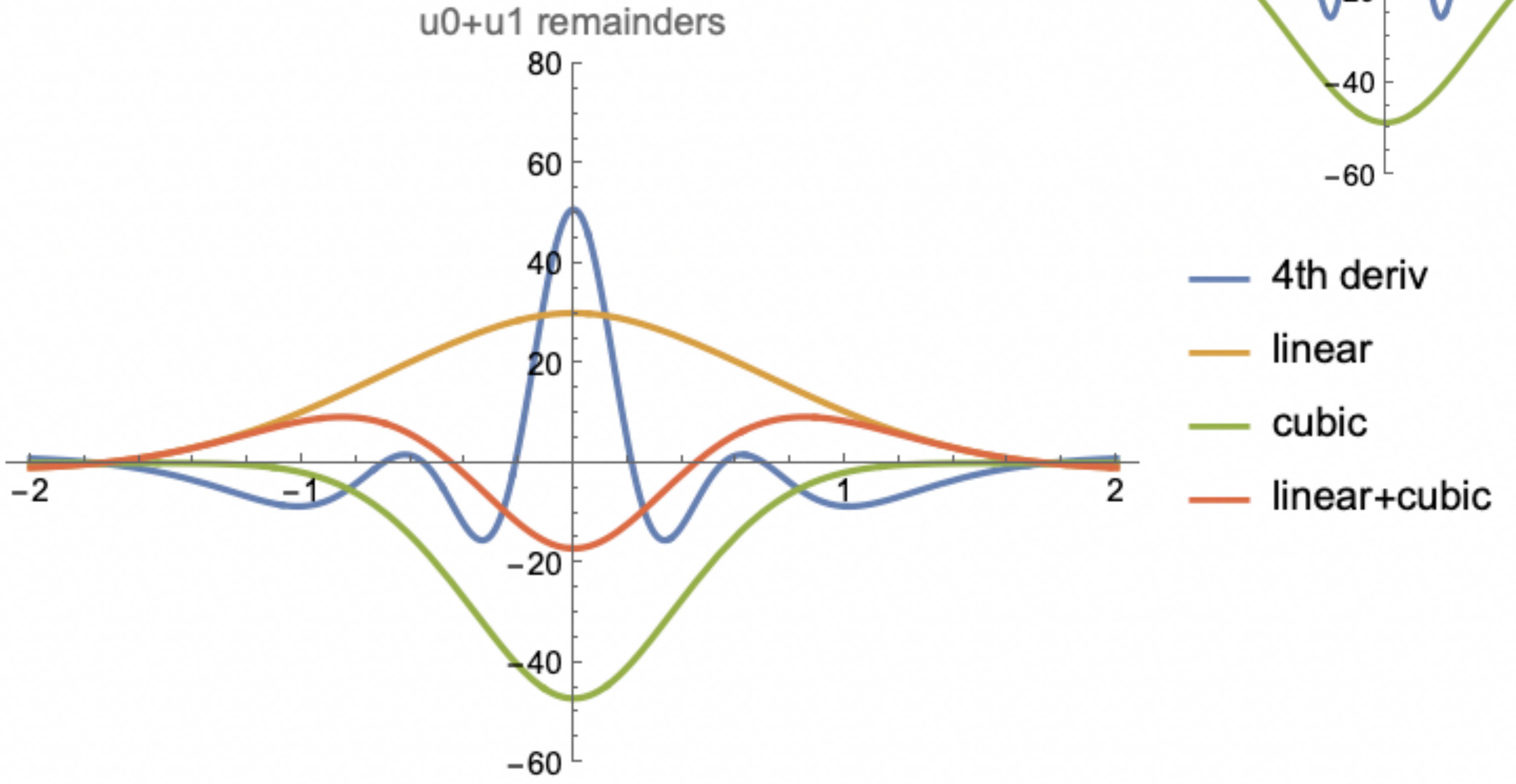
Solid lines are our visual estimates of A, B; dashed lines are chosen so that the remainder is $\mathcal{O}(x^4)$ about $x = 0$
(Each dashed line has a different A, B)



Remainder

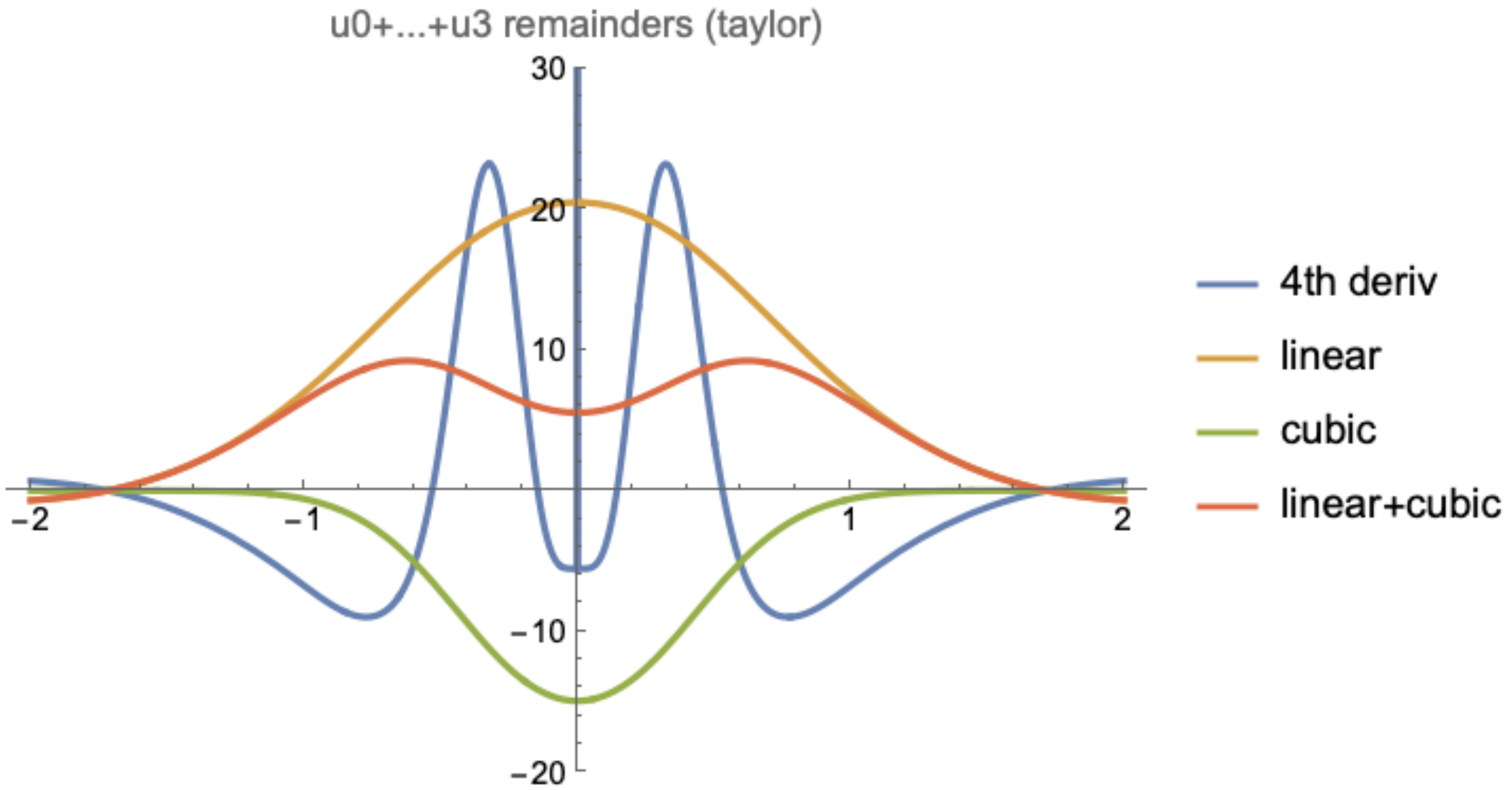
Plotting $\dfrac{d^4u}{dt^4}$, $4\sigma^4 u$, Γu^3 for the visual estimates of A, B

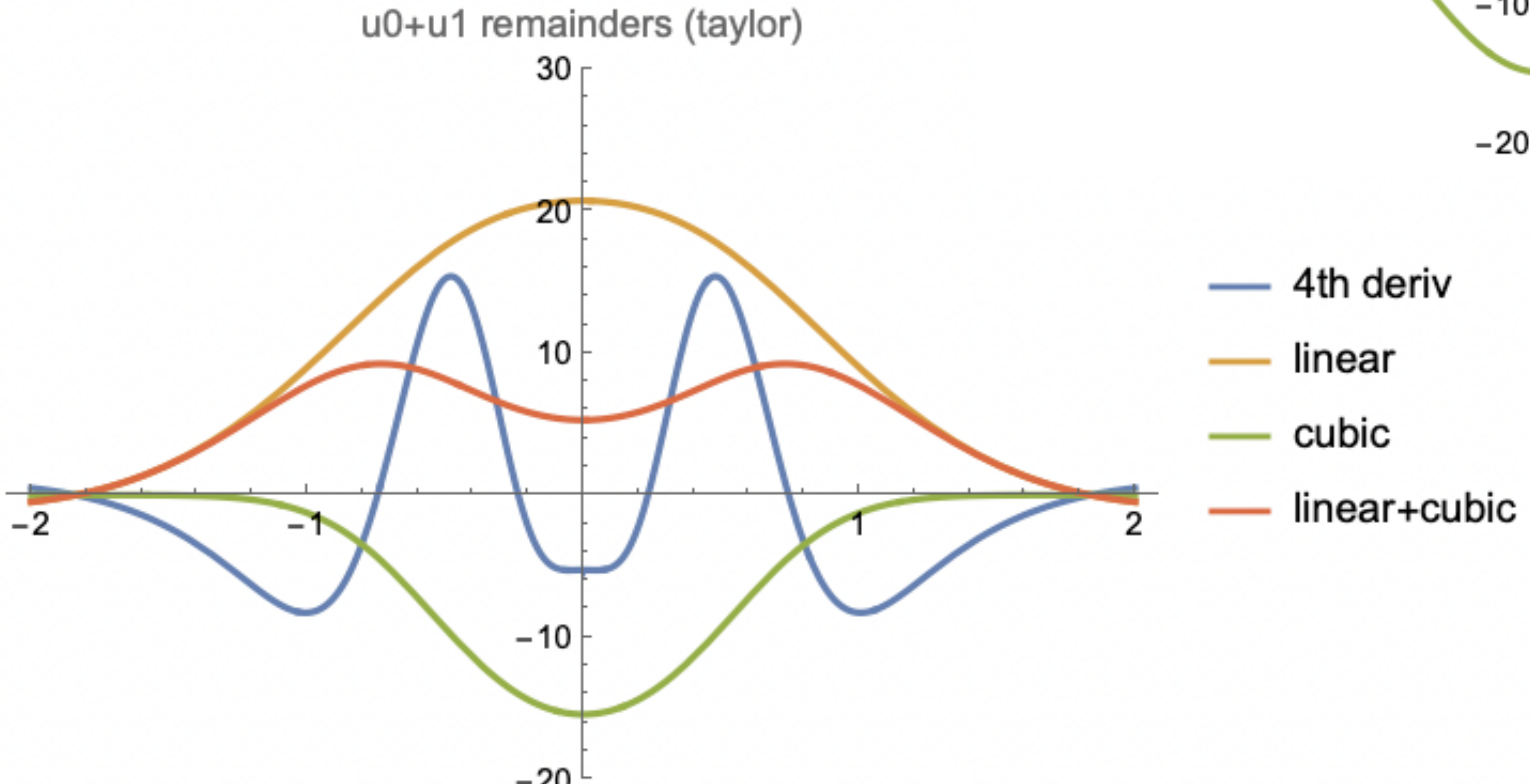




Remainder

Plotting $\frac{d^4u}{dt^4}, \quad 4\sigma^4 u, \quad \Gamma u^3$ for the Taylor estimates of A, B





Remainder

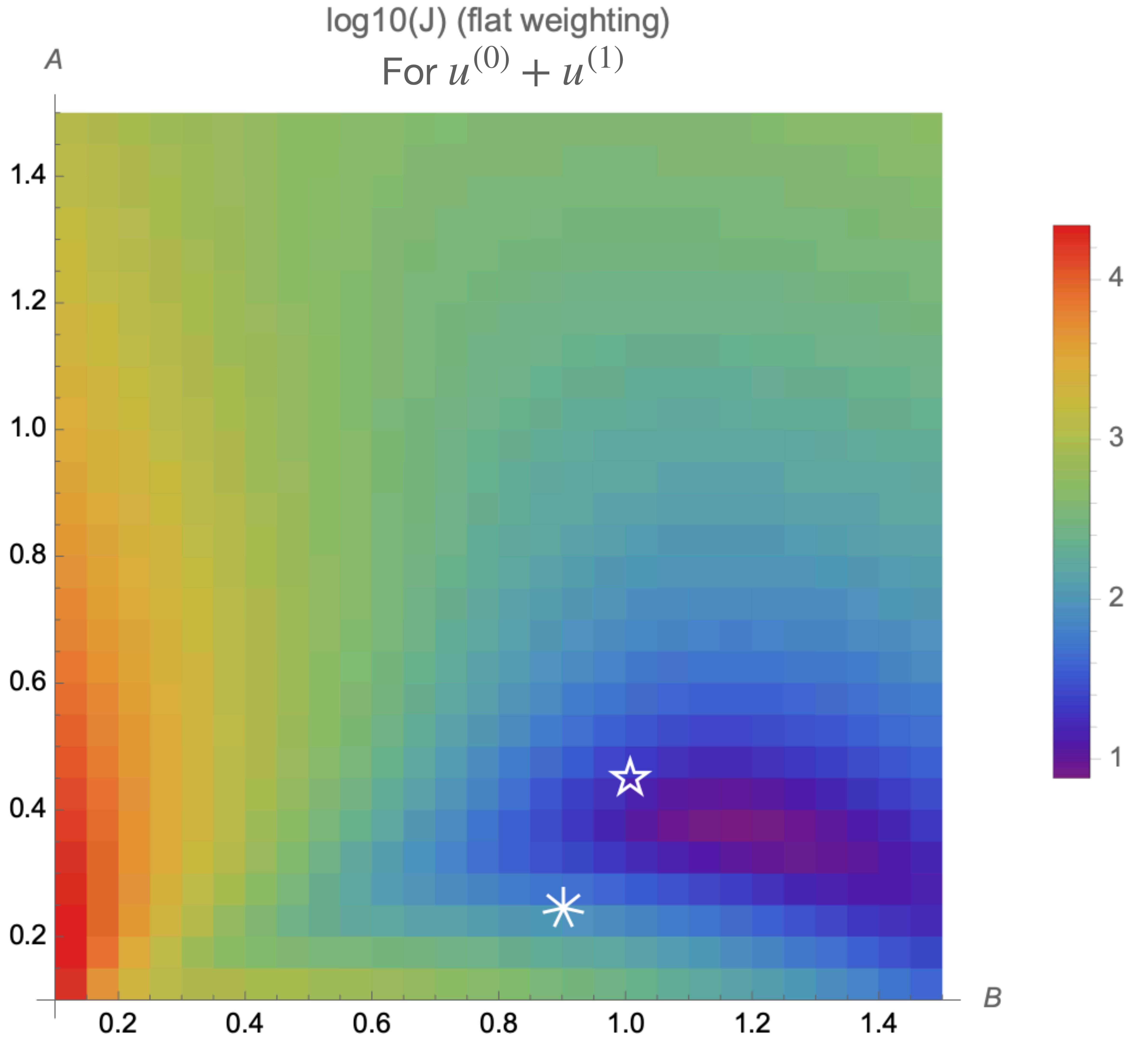
Calculate the squared area under the curve

$$I = \int_{\mathbb{R}} w(x') \left[D\left(x';\ u^{(0)} + u^{(1)}, (A, B)\right)\right]^2 \mathrm{d}x'$$

Where $w(\cdot)$ is chosen so the $w(x) = 0$ for $|x| < \epsilon$ (to prevent divergence in the numeric integration)

$$(A, B) = \begin{cases} (0.445{,}1.028) & \text{visual} \\ (0.0333{,}1.15) & u^{(0)} \\ (0.242{,}0.899) & \cdots + u^{(1)} \\ (0.335{,}0.695) & \cdots + u^{(2)} + u^{(3)} \end{cases}$$

The ⋆ is the visual estimate, the * is the Taylor estimate for $u^{(0)} + u^{(1)}$



Remainder

Calculate the squared area under the curve

$$I = \int_{\mathbb{R}} w(x') \left[D\left(x';\ u^{(0)} + \cdots + u^{(3)}, (A,B)\right)\right]^2 \mathrm{d}x'$$

Where $w(\cdot)$ is chosen so the $w(x) = 0$ for $|x| < \epsilon$ (to prevent divergence in the numeric integration)

$$(A,B) = \begin{cases} (0.445, 1.028) & \text{visual} \\ (0.0333, 1.15) & u^{(0)} \\ (0.242, 0.899) & \cdots + u^{(1)} \\ (0.335, 0.695) & \cdots + u^{(2)} + u^{(3)} \end{cases}$$

The $\star$ is the visual estimate, the $*$ is the Taylor estimate for $u^{(0)} + \cdots + u^{(3)}$