

Quantum State Local Distinguishability via Convex Optimization

PhD Defense

Alessandro Cosentino

University of Waterloo

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Outline

State distinguishability problem

Cone programming framework

Applications

- Maximally entangled states

- Unextendable product sets

Conclusions

State distinguishability problem

Quantum state distinguishability problem

An instance of the problem is specified by an ensemble of N states

$$\{(p_1, \rho_1), \dots, (p_N, \rho_N)\},$$

The problem

- ▶ An index $k \in \{1, \dots, N\}$ is picked at random with prob. p_k
- ▶ The state ρ_k is prepared on a register X
- ▶ Alice is asked to identify k by means of measurements on X

Observations

- ▶ *Classical* states are always completely distinguishable
- ▶ *Orthogonal* quantum states are perfectly distinguishable

Quantum state *local* distinguishability problem

In the ensemble

$$\{(p_1, \rho_1), \dots, (p_N, \rho_N)\},$$

the states $\rho_1, \dots, \rho_N \in \mathcal{D}(\mathcal{X} \otimes \mathcal{Y})$ are now bipartite states.

The problem

- ▶ An index $k \in \{1, \dots, N\}$ is picked at random with prob. p_k
- ▶ The state ρ_k is prepared on a register X shared by two parties
- ▶ Alice and Bob are asked to identify k by Local Operations and Classical Communication (LOCC) on X

Motivations

- ▶ Understanding nonlocality/entanglement
- ▶ Applications to *secret sharing*, *data hiding*, *channel capacity*
- ▶ LOCC protocols are interesting from a physical point of view

Quantum state local distinguishability problem

Typical assumptions

- ▶ Sets of *orthogonal pure* states
- ▶ Perfect distinguishability (with probability 1)
- ▶ *Uniform* probability of selection ($p_1 = \dots = p_N = 1/N$)

Distinguishable

- ▶ *any* 2 pure states
[Walgate et al., '00]
- ▶ *any* 3 maximally entangled states in $\mathbb{C}^3 \otimes \mathbb{C}^3$
[Nathanson - '05]
- ▶ ...

Indistinguishable

- ▶ *any* set of 3 or 4 Bell states
[Ghosh et al, '01]
- ▶ 9 *product* states in $\mathbb{C}^3 \otimes \mathbb{C}^3$
[Bennett et al. - PRA, 1999]
- ▶ ...

Quantum measurements and LOCC

N -outcome bipartite measurement

$$\mu : \{1, \dots, N\} \rightarrow \text{Pos}(\mathcal{X} \otimes \mathcal{Y})$$

$$\sum_{k=1}^N \mu(k) = \mathbb{1}_{\mathcal{X} \otimes \mathcal{Y}}$$

LOCC measurements

- ▶ Difficult object to handle mathematically
- ▶ A cumbersome formal definition
- ▶ Many kinds of LOCC

$$\text{LOCC}_1 \subsetneq \text{LOCC}_r \subsetneq \text{LOCC}_{r+1} \subsetneq \text{LOCC}_N \subsetneq \text{LOCC} \subsetneq \overline{\text{LOCC}} \subsetneq \text{SEP} \subsetneq \text{PPT}$$

(from Laura Mančinska's PhD thesis)

Separable and PPT measurements

Separable operators

$$P = \sum_{k=1}^M Q_k \otimes R_k \in \text{Sep}(\mathcal{X} : \mathcal{Y}), \quad \begin{array}{l} M > 0, Q_1, \dots, Q_M \in \text{Pos}(\mathcal{X}), \\ R_1, \dots, R_M \in \text{Pos}(\mathcal{Y}) \end{array}$$

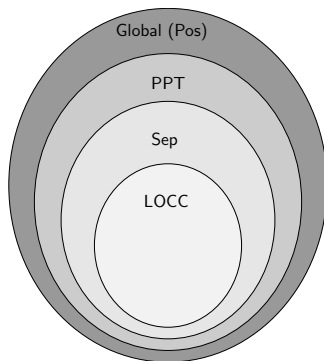
PPT (Positive-partial-transpose) operators

$$\text{PPT}(\mathcal{X} : \mathcal{Y}) = \{P \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y}) : \mathsf{T}_{\mathcal{X}}(P) \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y})\}$$

Separable/PPT measurements

$$\begin{array}{ll} \mu(k) \in \text{Sep}(\mathcal{X} : \mathcal{Y}), \text{ or} & \\ \mu(k) \in \text{PPT}(\mathcal{X} : \mathcal{Y}) & \text{(for each } k \in \{1, \dots, N\}) \end{array}$$

Classes of measurements



- ▶ A bound on PPT/Sep reflects into a bound on LOCC
- ▶ Optimizing over Sep is NP-hard
- ▶ Optimizing over PPT is computationally easy

Cone programming framework

Semidefinite program for global state distinguishability

- One of the first applications of SDP in quantum [Eldar, Megretski, Verghese, '03]

Primal (Global)

Dual (Global)

$$\max: \sum_{k=1}^N p_k \langle \rho_k, \mu(k) \rangle,$$

$$\text{s.t.: } \sum_{k=1}^N \mu(k) = \mathbb{1}_{\mathcal{X}},$$

$$\mu : \{1, \dots, N\} \rightarrow \text{Pos}(\mathcal{X})$$

$$\min: \text{Tr}(H)$$

$$\begin{aligned} \text{s.t.: } H - p_k \rho_k &\in \text{Pos}(\mathcal{X}) \\ &\quad (k = 1, \dots, N), \\ H &\in \text{Herm}(\mathcal{X}). \end{aligned}$$

- *Weak duality*: any H feasible solution of the dual gives an upper bound on the optimal value of the primal

General cone program for bipartite state distinguishability

<u>Primal</u>	<u>Dual</u>
$\max: \sum_{k=1}^N p_k \langle \rho_k, \mu(k) \rangle,$	$\min: \operatorname{Tr}(H)$
$\text{s.t.: } \sum_{k=1}^N \mu(k) = \mathbb{1}_{\mathcal{X} \otimes \mathcal{Y}},$	$\text{s.t.: } \boxed{H - p_k \rho_k \in \mathcal{C}^*}$
$\boxed{\mu : \{1, \dots, N\} \rightarrow \mathcal{C}}$	$(k = 1, \dots, N),$
	$H \in \operatorname{Herm}(\mathcal{X} \otimes \mathcal{Y}).$

- ▶ $\mathcal{C} \subseteq \operatorname{Pos}(\mathcal{X} \otimes \mathcal{Y})$ is a closed convex cone
- ▶ \mathcal{C}^* is the cone dual to \mathcal{C} , that is,

$$\mathcal{C}^* = \{Y \in \operatorname{Herm}(\mathcal{X} \otimes \mathcal{Y}) : \langle X, Y \rangle \geq 0 \text{ for all } X \in \mathcal{C}\}$$

- ▶ $\operatorname{Pos}(\mathcal{X})$ is self-dual: $\operatorname{Pos}(\mathcal{X}) = (\operatorname{Pos}(\mathcal{X}))^*$

Cone program for PPT distinguishability

Primal (PPT)

$$\begin{aligned} \text{max:} \quad & \sum_{k=1}^N p_k \langle \rho_k, \mu(k) \rangle, \\ \text{s.t.:} \quad & \sum_{k=1}^N \mu(k) = \mathbb{1}_{\mathcal{X}}, \\ & \mu : \{1, \dots, N\} \rightarrow \text{PPT}(\mathcal{X} : \mathcal{Y}) \end{aligned}$$

Dual (PPT)

$$\begin{aligned} \text{min:} \quad & \text{Tr}(H) \\ \text{s.t.:} \quad & H - p_k \rho_k \in \text{PPT}^*(\mathcal{X} : \mathcal{Y}) \\ & \quad (k = 1, \dots, N), \\ & H \in \text{Herm}(\mathcal{X}). \end{aligned}$$

One Semidefinite program for PPT distinguishability

Primal (PPT)

$$\begin{aligned} \max: \quad & \sum_{k=1}^N p_k \langle \rho_k, \mu(k) \rangle, \\ \text{s.t.}: \quad & \sum_{k=1}^N \mu(k) = \mathbb{1}_{\mathcal{X}}, \\ & \mu : \{1, \dots, N\} \rightarrow \text{PPT}(\mathcal{X} : \mathcal{Y}) \end{aligned}$$

Dual (PPT)

$$\begin{aligned} \min: \quad & \text{Tr}(H) \\ \text{s.t.}: \quad & H - p_k \rho_k \in \boxed{\text{PPT}^*(\mathcal{X} : \mathcal{Y})} \\ & (k = 1, \dots, N), \\ & H \in \text{Herm}(\mathcal{X}). \end{aligned}$$

Decomposable operators

- By definition,

$$\text{PPT}^*(\mathcal{X} : \mathcal{Y}) = \{S + T_{\mathcal{X}}(R) : S, R \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y})\}$$

- Choi-isomorphic to *decomposable maps*

$$\Phi(Y) = \Psi(Y) + (T \circ \Xi)(Y), \quad \Psi, \Xi \text{ completely positive}$$

Cone program for separable distinguishability

Primal (Separable)

$$\begin{aligned} \max: \quad & \sum_{k=1}^N p_k \langle \rho_k, \mu(k) \rangle, \\ \text{s.t.}: \quad & \sum_{k=1}^N \mu(k) = \mathbb{1}_{\mathcal{X} \otimes \mathcal{Y}}, \\ & \mu : \{1, \dots, N\} \rightarrow \text{Sep}(\mathcal{X} : \mathcal{Y}). \end{aligned}$$

Dual (Separable)

$$\begin{aligned} \min: \quad & \text{Tr}(H) \\ \text{s.t.}: \quad & H - p_k \rho_k \in \text{Sep}^*(\mathcal{X} : \mathcal{Y}) \\ & \quad (k = 1, \dots, N), \\ & H \in \text{Herm}(\mathcal{X}). \end{aligned}$$

Cone program for separable distinguishability

Primal (Separable)

$$\begin{aligned} \max: \quad & \sum_{k=1}^N p_k \langle \rho_k, \mu(k) \rangle, \\ \text{s.t.}: \quad & \sum_{k=1}^N \mu(k) = \mathbb{1}_{\mathcal{X} \otimes \mathcal{Y}}, \\ & \mu : \{1, \dots, N\} \rightarrow \text{Sep}(\mathcal{X} : \mathcal{Y}). \end{aligned}$$

Dual (Separable)

$$\begin{aligned} \min: \quad & \text{Tr}(H) \\ \text{s.t.}: \quad & H - p_k \rho_k \in \boxed{\text{Sep}^*(\mathcal{X} : \mathcal{Y})} \\ & (k = 1, \dots, N), \\ & H \in \text{Herm}(\mathcal{X}). \end{aligned}$$

Block-positive operators

- ▶ Choi representation of positive linear mappings:

$$\Phi(Y) \in \text{Pos}(\mathcal{X}), \quad \text{for every } Y \in \text{Pos}(\mathcal{Y})$$

- ▶ $\text{Sep}^*(\mathcal{X} : \mathcal{Y}) \setminus \text{Pos}(\mathcal{X} \otimes \mathcal{Y})$ are the *entanglement witnesses*

Benefits of the cone programming formulation

Analytically

- ▶ To prove a set is *not perfectly* distinguishable, we find H s.t.
 1. $\text{Tr}(H) < 1$
 2. $H - p_k \rho_k \in \text{Sep}^*(\mathcal{X} : \mathcal{Y})$, for all $k \in \{1, \dots, N\}$
- ▶ Many families of operators in $\text{Sep}^*(\mathcal{X} : \mathcal{Y})$ are known
- ▶ Actual numerical bounds, as opposed to qualitative statements

Numerically

- ▶ We can approximate the Sep cone program by a hierarchy of semidefinite programs
- ▶ A MATLAB implementation is now in N. Johnston's QETLAB

Applications

Entanglement cost of distinguishing Bell states

- ▶ Max. prob. to LOCC-distinguish the 4 Bell states is $1/2$
- ▶ Max. prob. to LOCC-distinguish any set of 3 Bell states is $2/3$

Entanglement cost

Alice and Bob are given as a resource a 2-qubit entangled state

$$|\tau_\varepsilon\rangle = \sqrt{\frac{1+\varepsilon}{2}} |0\rangle|0\rangle + \sqrt{\frac{1-\varepsilon}{2}} |1\rangle|1\rangle, \quad \varepsilon \in [0, 1]$$

- ▶ Up to local unitaries, it characterizes all 2-qubit states
- ▶ 4 Bell states: *perfect* LOCC-disting. iff $\varepsilon = 0$ [HSSH, '03]
- ▶ 3 Bell states: *perfect* PPT-disting. iff $\varepsilon \leq 1/3$ [YDY, '14]
- ▶ 3 Bell states: what about separable and LOCC?

Entanglement cost of distinguishing Bell states

Theorem (new)

The maximum probability of distinguishing 3 Bell states is

$$\frac{1}{3} \left(2 + \sqrt{1 - \varepsilon^2} \right).$$

Corollary

1. *Max. ent. state ($\varepsilon = 0$) is necessary for perfect discrimination*
2. *Teleportation is optimal*

Corresponding map

A new positive map from $L(\mathbb{C}^2 \oplus \mathbb{C}^2)$ to $L(\mathbb{C}^2 \oplus \mathbb{C}^2)$

Local distinguishability of maximally entangled states

- ▶ N max. ent. states in $\mathbb{C}^d \otimes \mathbb{C}^d$ (what's the role of d ?)
- ▶ A significant case because of the dual role of entanglement

	PPT	LOCC	References
$N = 2$, any d	—	<i>all</i> dist.	[Walgate et al., 2000]
$N = 3 = d$	—	<i>all</i> dist.	[Nathanson, 2005]
$N = 4 = d$	<i>some</i> indist.	—	[Yu, Duan, and Ying, 2012]
$4 < N < d$	<i>some</i> indist.	—	<i>this thesis</i>
$N > d$	<i>all</i> indist.	—	[Yu, Duan, and Ying, 2012, Duan et al. 2009, Ghosh et al. 2004]

Yu–Duan–Ying states

Theorem (YDY, '12)

The following 4 orthogonal max. ent. states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ are not perfectly distinguishable by PPT (and therefore LOCC):

$$\phi_1 = \psi_0 \otimes \psi_0$$

$$\phi_2 = \psi_1 \otimes \psi_1$$

$$\phi_3 = \psi_2 \otimes \psi_1$$

$$\phi_4 = \psi_3 \otimes \psi_1$$

Theorem (new)

The optimal probability of distinguishing YDY by LOCC is 3/4.

Corresponding map

Breuer-Hall positive linear maps from $L(\mathbb{C}^4)$ to $L(\mathbb{C}^4)$

Small sets of locally indistinguishable max. ent. states

	PPT	LOCC	References
$N = 2$, any d	—	<i>all</i> dist.	[Walgate et al., 2000]
$N = 3 = d$	—	<i>all</i> dist.	[Nathanson, 2005]
$N = 4 = d$	<i>some</i> indist.	—	[Yu, Duan, and Ying, 2012]
$4 < N < d$	<i>some</i> indist.	—	<i>this thesis</i>
$N > d$	<i>all</i> indist.	—	[GKRS '04, DFXY '09, YDY '12]

Theorem (new)

There exists an indistinguishable set of $N < d$ maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$

Small sets of locally indistinguishable max. ent. states

Generalized Yu–Duan–Ying

$$\begin{array}{ll}\phi_1 = \psi_0 \otimes \psi_0 \otimes \psi_0 & \phi_5 = \psi_1 \otimes \psi_0 \otimes \psi_0 \\ \phi_2 = \psi_0 \otimes \psi_1 \otimes \psi_1 & \phi_6 = \psi_1 \otimes \psi_1 \otimes \psi_1 \\ \phi_3 = \psi_0 \otimes \psi_2 \otimes \psi_1 & \phi_7 = \psi_1 \otimes \psi_2 \otimes \psi_1 \\ \phi_4 = \psi_0 \otimes \psi_3 \otimes \psi_1 & \phi_8 = \psi_1 \otimes \psi_3 \otimes \psi_1\end{array}$$

- ▶ 7 orthogonal maximally entangled states in $\mathbb{C}^8 \otimes \mathbb{C}^8$
- ▶ Maximum probability for any separable measurement is $6/7$
- ▶ Perfectly distinguishable by PPT

Distinguishability of unextendable product sets

Definition (Unextendable product sets – UPS)

An orthonormal collection of product vectors

$$\mathcal{A} = \{u_k \otimes v_k : k = 1, \dots, N\} \subset \mathcal{X} \otimes \mathcal{Y}$$

s.t. there is no (nonzero) product vector $u \otimes v \perp \mathcal{A}$.

Previously known

- ▶ no UPS can be perfectly distinguished by LOCC meas.
- ▶ every UPS can be perfectly distinguished by PPT meas.
- ▶ every UPS in $\mathbb{C}^3 \otimes \mathbb{C}^3$ can be perfectly distinguished by separable meas.

UPS indistinguishable by separable measurements

Theorem (new)

Feng's UPS in $\mathbb{C}^4 \otimes \mathbb{C}^4$ is not perfectly distinguishable by Sep.

$$\begin{aligned} |\phi_1\rangle &= |0\rangle|0\rangle, \\ |\phi_2\rangle &= |1\rangle(|0\rangle - |2\rangle + |3\rangle)/\sqrt{3}, \\ |\phi_3\rangle &= |2\rangle(|0\rangle + |1\rangle - |3\rangle)/\sqrt{3}, \\ |\phi_4\rangle &= |3\rangle|3\rangle, \\ |\phi_5\rangle &= (|1\rangle + |2\rangle + |3\rangle)(|0\rangle - |1\rangle + |2\rangle)/3, \\ |\phi_6\rangle &= (|0\rangle - |2\rangle + |3\rangle)|2\rangle/\sqrt{3}, \\ |\phi_7\rangle &= (|0\rangle + |1\rangle - |3\rangle)|1\rangle/\sqrt{3}, \\ |\phi_8\rangle &= (|0\rangle - |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle)/3. \end{aligned}$$

Proof

- ▶ Not the usual cone program this time...
- ▶ ...but a criterion for perfect separable discrimination of UPS checkable with a *linear program*!

Conclusions

Contributions of the thesis

- ▶ A framework based on cone programming to test local distinguishability of quantum states
- ▶ Precise (often optimal) numerical bounds on the success probability of distinguishing some famous sets of states
- ▶ A novel connection between quantum state distinguishability and the study of positive linear maps

“State discrimination is the most fundamental task in physics.”

— Jon Walgate (Calgary, 2005)

Open questions

- ▶ The other way: new sets of indistinguishable states from known positive maps?
- ▶ What about a cone program for LOCC...

Dual

min: $\text{Tr}(H)$

$$\text{s.t.: } \begin{pmatrix} H - p_1 \rho_1 & & \\ & \ddots & \\ & & H - p_N \rho_N \end{pmatrix} \in \boxed{\text{Meas}_{\text{LOCC}}^*(N, \mathcal{X} : \mathcal{Y})},$$

Can we say anything about $\text{Meas}_{\text{LOCC}}^*(N, \mathcal{X} : \mathcal{Y})$?

- ▶ Other settings (multipartite, two copies, etc.)

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PPT-indistinguishable states via semidefinite programming.
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IEEE Transactions on Information Theory, 2015.