Complexity Lab Module I: Exercises

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1 Introduction to Complexity through Conway's Game of Life

Conway's Game of Life (often simply called "Life") is a zero-player game, meaning its evolution is determined by its initial state, requiring no further input. It takes place on an infinite two-dimensional grid of square cells, each of which can be in one of two states: **alive** or **dead**. Every cell interacts with its eight neighbors (horizontally, vertically, or diagonally adjacent). At each step in time (generation), the following transitions occur simultaneously for all cells:

- Any **live cell** with fewer than two live neighbours **dies**, as if by underpopulation.
- Any **live cell** with two or three live neighbours **lives on** to the next generation.
- Any **live cell** with more than three live neighbours **dies**, as if by overpopulation.
- Any **dead cell** with exactly three live neighbours **becomes a live cell**, as if by reproduction.

The initial configuration of live and dead cells constitutes the "seed" of the system.

1. **Understanding the Basic Rules and Patterns:**
   (a) Familiarize yourself with an online Game of Life simulator (many are freely available, e.g., on Wikipedia).
   (b) Experiment with small, well-known initial patterns. Observe and describe their behavior:
      - **Still Lifes:** e.g., Block, Beehive, Loaf. How do these patterns behave over time? Why are they called "still lifes"?
      - **Oscillators:** e.g., Blinker, Toad, Pulsar. Describe their period and how they change.
      - **Spaceships:** e.g., Glider, Lightweight Spaceship. How do these patterns move? What are their key characteristics?

   (c) Choose one simple pattern (e.g., a Blinker or a Glider) and manually simulate its first 3-5 generations on a small grid. Clearly show the state of the grid at each step.

2. **Emergent Behavior and Unpredictability:**
   (a) **The R-Pentomino:** The R-pentomino is a small 5-cell initial configuration famous for its long and complex evolution before stabilizing. Input an R-pentomino into a simulator and run it for at least 1000 generations.
      - Describe the types of structures that emerge during its evolution (e.g., gliders, blocks, blinkers).
      - How many generations does it take for the pattern to stabilize (if it does)? What does the final stable pattern look like?
      - Discuss how this complex, long-term behavior emerges from such a simple initial state and simple rules. What makes it "unpredictable" without running the simulation?

   (b) **Slight Perturbations:** Start with a well-known pattern (e.g., a Glider). Introduce a single live cell nearby, or change the state of one cell in the pattern slightly.
      - Observe the impact of this small change on the long-term evolution. Does the pattern stabilize differently? Does it die out? Does it lead to chaotic growth?
      - Relate your observations to the concept of **sensitivity to initial conditions**, a hallmark of complex systems. You can optionally consider this exercise with a pattern known as a "Methuselah" (a pattern that takes a very long time to stabilize, like the R-pentomino) and observe how small changes drastically alter its long-term evolution.

2 The Tent Map

Consider the Tent Map, defined as:

$$x_{n+1} = f(x_n) = \begin{cases} \alpha x_n & \text{if } 0 \leq x_n < 1/2 \\ \alpha(1 - x_n) & \text{if } 1/2 \leq x_n \leq 1 \end{cases}$$

where α is a parameter and we restrict $0 \leq x_n \leq 1$.

1. **Finding Fixed Points:** Find the fixed points of the Tent Map for values of α where $0 < \alpha \leq 2$.
2. **Stability Analysis of Fixed Points:** Determine the stability of each fixed point found in part 1.
3. **Investigating Periodic Points (2-Cycles):** A 2-cycle consists of two distinct points x_a and x_b such that $f(x_a) = x_b$ and $f(x_b) = x_a$. This is equivalent to finding fixed points of the second iterate of the map, $x^* = f(f(x^*))$. For $\alpha = 2$, find any 2-cycles of the Tent Map.
4. **Bifurcation Diagram:** Plot the bifurcation diagram for the Tent Map as α increases from 0 to 2. This diagram should show the long-term behavior of x_n for different values of α. You can generate this by iterating the map many times for each α value and plotting the latter points of the trajectory (to discard transients).
5. **Lyapunov Exponent:** Compute and plot the Lyapunov exponent (λ) for the Tent Map as α increases from 0 to 2. Recall that for a 1D map, the Lyapunov exponent is given by $\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(x_i)|$. Interpret your results: What does the sign of the Lyapunov exponent tell you about the dynamics of the Tent Map in different ranges of α? How does it relate to the behavior observed in the bifurcation diagram?

3 Lorenz System

In the course of studying atmospheric convection, Edward Lorenz developed a 3D system which could exhibit unpredictable dynamics

$$\begin{aligned}\dot{x} &= \sigma(y - x),\\ \dot{y} &= \rho x - y - xz,\\ \dot{z} &= xy - bz,\end{aligned}$$

where $\{\sigma, b, \rho\}$ are parameters. These equations can be derived from the more general Navier-Stokes fluid equations, for example see the MIT lecture notes by D Rothman, which also provides insight into the meaning of the parameters (e.g. ρ is related to the Rayleigh number which controls fluid turbulence). While this system of equations looks simple, the solutions can be remarkably complex and we'll want to start with some direct analytical understanding.

1. As with other nonlinear dynamical systems, to understand the evolution we start by looking for simple attracting trajectories such as fixed points or limit cycles. Let's fix two of the three parameters $\{\sigma = 10, b = \frac{8}{3}\}$ and see happens as a function of ρ. First, verify that the origin is a fixed point. Now recall our discussion on linear stability analysis to deduce the asymptotic dynamics for $\rho < 1$? What happens to the stability of the origin fixed point at $\rho > 1$? For $\rho > 1$ (but not too large) are there new fixed points? Can you identify the bifurcation that occurs when ρ increases past unity?
2. Using Python, MATLAB or a related language, construct a procedure to numerically integrate the Lorenz equations. An Euler integration scheme is the simplest but also the least accurate method so while it's fine place to start, I'd also like you to move to a more sophisticated technique like Runge-Kutta. Use the integrator to verify your analytical analysis in the preceding question.
3. Use numerical simulations to analyze the Lorenz dynamics with the original parameter choice of Lorenz $\{\sigma = 10, b = \frac{8}{3}, r = 28\}$. Can you demonstrate that these dynamics are chaotic (for example, by measuring the maximum Lyapunov exponent)? To get a feeling for the accuracy of your numerical integrator you can also compare against the provided high-resolution reference data (with initial condition $x = 1; y = 1; z = 1$).
4. The Lorenz equations are a wonderful laboratory in which to explore many phenomena in nonlinear dynamics, not just chaos. Indeed, for anyone with a deeper interest there is a also an entire book by Colin Sparrow. As an example of this diversity show that the Lorenz dynamics lead to a limit-cycle attractor for parameter values $\{\sigma = 10, b = \frac{8}{3}, \rho = 400\}$.
5. Chaotic systems are often associated with strange attractors. That is, the attracting set of the chaotic dynamical system can have unusual properties, including a non-integer dimension. Read chapter 11.5 in Strogatz which discusses a method, the correlation dimension, for estimating the dimension of an attracting point set from data. Generate a toy example point set by randomly placing points along a 1D interval and show that the correlation dimension is $d = 1$. Now compute the correlation dimension of the Lorenz attractor for the canonical parameter choices and show that $d \approx 2.06$. Note also that there are multiple ways to measure the dimension, each with a slightly different value, but all suggesting a strange attractor. Use your dimension estimation technique to show that the dimension of a limit-cycle attractor is $d = 1$. Attracting points are perhaps too easy with dimension $d = 0$.