Fixed points, stability and bifurcations in 1D continuous dynamics and the discrete-time logistic map

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Preamble

What can we learn about the nature of the solutions to an ordinary, perhaps non-linear differential equation without actually solving the system? This is the realm of *qualitative* analysis. In general, of course, there are man systems such as the brain or the posture change of are organism which are complex, spatiotemporal processes. But here we ignore spatial variation to isolate the behavior of temporal change. Useful concepts will include *fixed points* (stable and unstable), *limit cycles* (also both stable and unstable), *chaos and strange attractors*, *bifurcations* (the change in the stability or topology of the attractor), *linear & nonlinear* and also *dimensionality.* Here, we limit ourselves to *deterministic* dynamical systems.

Continuous dynamics in 1D

Consider the general dynamics for a 1D state space:

$$\frac{dx}{dt} \equiv \dot{x} = F(x)$$

Of course, we all know how to solve such an equation;

$$\frac{dx}{F(x)} = dt \implies \int_{x_0}^{x_f} \frac{dx}{F(x)} = (t_f - t_0) = \tau,$$

and if we're really lucky then we can write this integral in closed form. But in general it's not going to be easy nor likely even informative, as we generally want $x(t)$ not $t(x)$ (you might try this with a few nonlinear functions). Suppose instead that we make a plot of $F(x)$:

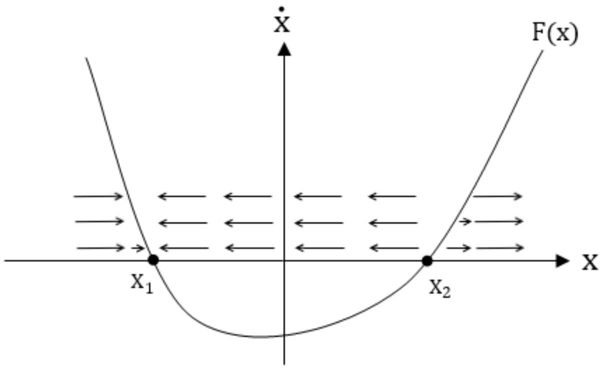

Figure 1: A sketch of a general dynamics in 1D. Here x_1^* (stable) and x_2^* (unstable) are the *fixed points*: points where the dynamics freezes $\dot{x} = F(x^*) = 0$.

When we sketch the velocities, we uncover qualitative features of the dynamics. First we see that there are two fixed points $\dot{x} = F(x^*) = 0$ where the velocity vanishes and thus the system freezes. But we can say more? Imagine displacing the system by a small amount δx from each fixed point. On the left, the velocities point back to the fixed point; the left fixed point is stable. On the right the velocities point away from the fixed point and this point is unstable. Knowing the location and stability of the fixed points we can now describe the asymptotic behavior of *any* 1D system!

Linear stability of 1D dynamics

Although the full dynamics is nonlinear, we can also approach the dynamics analytically by expanding and *linearizing* $F(x)$ about each fixed point. So, let

$$x = x^* + \delta(t)$$

and recall the general form of the taylor expansion around x^*

$$F(x) = \overbrace{F(x^*)}^{0} + \left.\frac{dF}{dx}\right|_{x=x^*} \overbrace{(x - x^*)}^{\delta} + \overbrace{\frac{1}{2}\left.\frac{d^2F}{dx^2}\right|_{x=x^*} (x - x^*)^2 + \mathcal{O}(\delta^3)}^{\text{higher order terms}}$$

The first term is 0 since x^* is a fixed point, $F(x^*) = 0$ by definition of a fixed point in 1D. Rewriting $\dot{x} = F(x)$ by keeping only the linear order in perturbations δ about the fixed point

$$\dot{\delta} = \left.\frac{dF}{dx}\right|_{x=x^*} \delta$$

and let's define $\lambda \equiv \left.\frac{dF}{dx}\right|_{x=x^*}$. Remember, λ is independent of x (and of course t.) The general solution to our linear equation is

$$\delta(t) = \delta(0)e^{\lambda t}$$

Whether the perturbations $\delta(t)$ grow exponentially (an unstable fixed point) or decay exponentially (a stable fixed point) depends on the sign of λ and thus on the slope of $F(x)$ at the fixed point. From a graph such as Figure 1 it is easy to read the location of a fixed point and the sign of the slope of $F(x)$ at that point, which is why the graphical analysis is so powerful. Apart from the marginal case of $\lambda = 0$ (for which higher-order terms are necessary to understand the behavior), the magnitude of λ sets the (characteristic) timescale of the linearized dynamics.

Bifurcations

How can a 1D dynamical system qualitatively change its behavior, since all solutions end up at fixed points (or infinity)? The answer is by changing the fixed points. We consider various examples of co-dimension-1 bifurcations, these are bifurcations produced by a single control parameter, α. The bifurcation parameter α could be an external parameter or could result from the effective fast dynamics of other degrees of freedom in the system. Below we review the 1D bifurcations controlled by a single bifurcation parameter.

Saddle-node bifurcation

$$\dot{x} = \alpha + x^2 \tag{1}$$

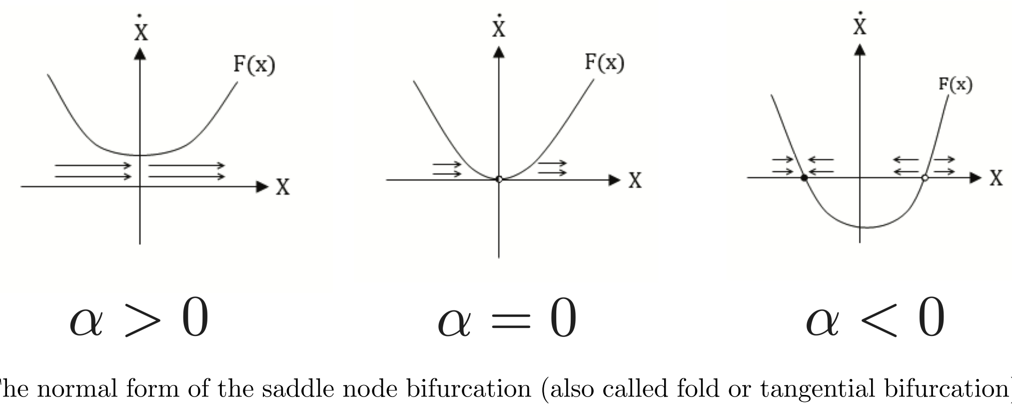

Figure 2: The normal form of the saddle node bifurcation (also called fold or tangential bifurcation) $\dot{x} = \alpha + x^2$ for distinct values of the bifurcation parameter α. (a) $\alpha > 0$: no fixed points, solutions zoom to infinity. (b) $\alpha = 0$: (single) fixed point is half-stable so that the stability (and asymptotic solution) depends on the sign of the perturbation. (c) $\alpha < 0$: two fixed points with differing stability. In all figures the arrows point along direction of motion.

We can collapse the above three figures into a *bifurcation diagram* which summarizes the locations and stability of the fixed points. In a *bifurcation diagram* the position of the fixed point x is plotted as a function of the parameter α (see Figure 3). The exact form of the lines in this plot comes from the solution $F(x^*; \alpha) = 0$ which is $x^* = \pm\sqrt{-\alpha}$ for the saddle-node bifurcation. The two fixed points approach one another as a function of α and annihilate at $\alpha = 0$.

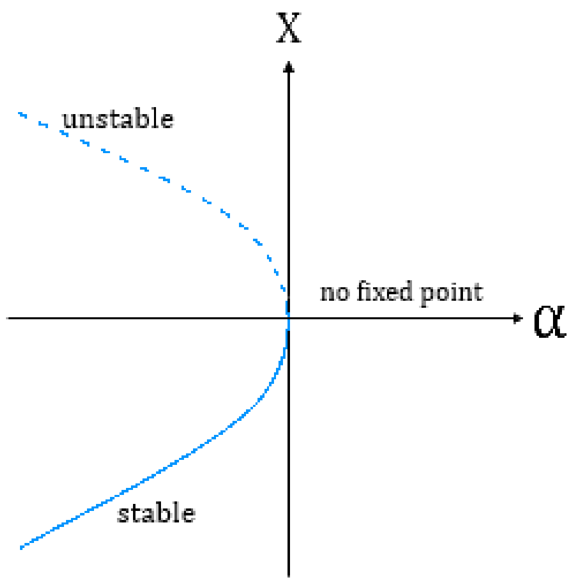

Figure 3: Saddle node bifuration diagram.

Transcritical bifurcation

$$\dot{x} = \alpha x - x^2 \tag{2}$$

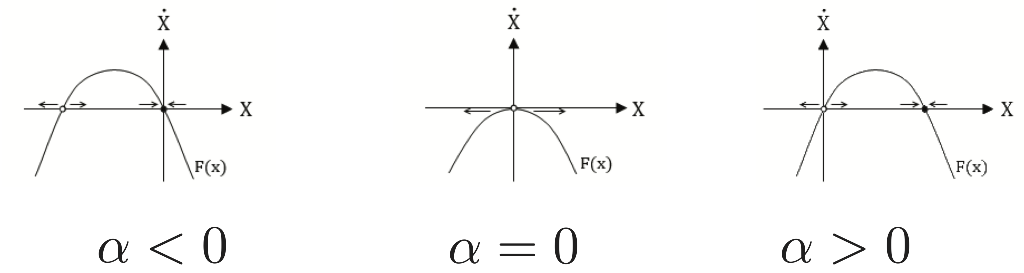

Figure 4: Transcritical bifurcation

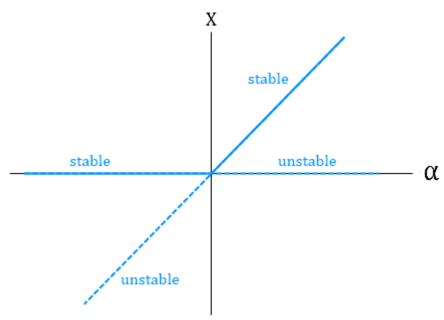

Figure 5: Transcritical bifurcation diagram.

Pitchfork bifurcation (supercritical)

$$\dot{x} = \alpha x - x^3$$

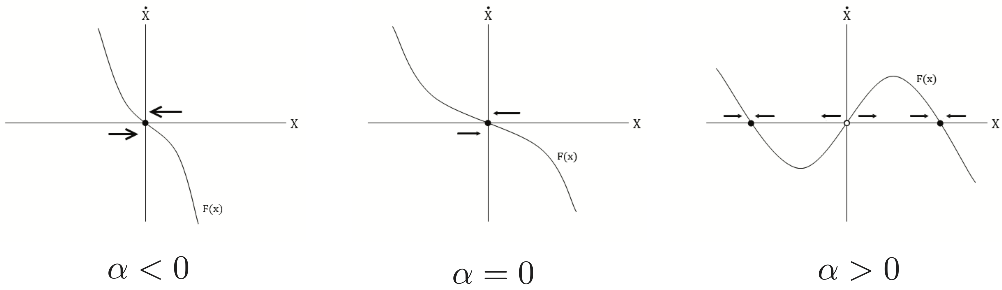

Figure 6: Supercritical bifurcation

We can imagine the supercritical bifurcation as resulting from the overdamped (hence first order) dynamics of a particle in a potential V

$$V(x) = -\frac{1}{2}\alpha x^2 + \frac{1}{4}x^4.$$

At $\alpha = 0$ the transition of V is similar to a second-order phase transition (Figure 7).

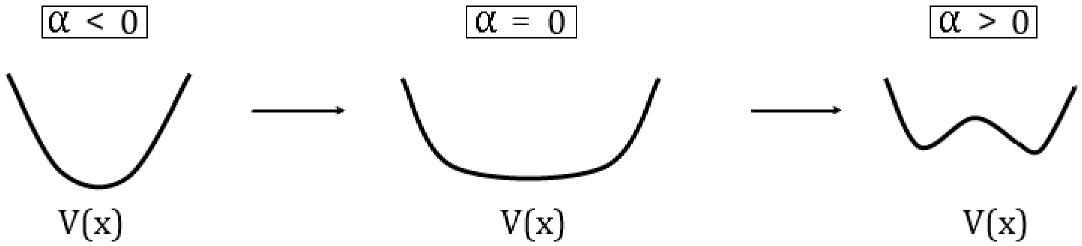

Figure 7: Change in the potential energy landscape $V(x)$ due to altering values of α (supercritical pitchfork bifurcation).

Pitchfork bifurcation (subcritical)

$$\dot{x} = \alpha x + x^3 \tag{3}$$

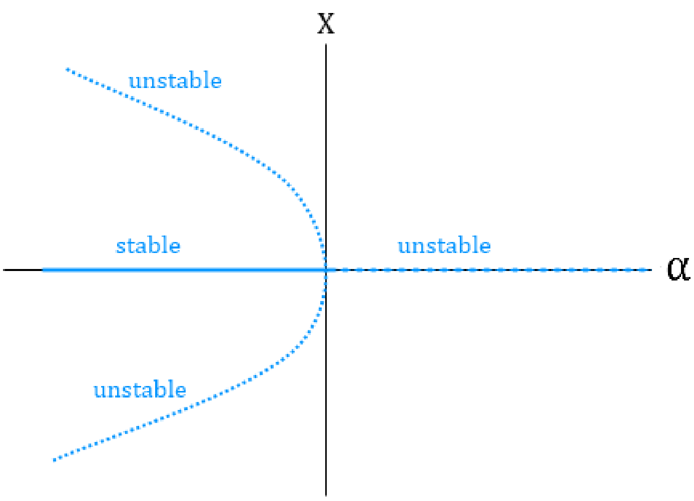

Figure 8: Bifurcation diagram of the subcritical pitchfork bifurcation.

The Logistic Map

In contrast to continuous-time systems, discrete-time maps $x_{n+1} = f(x_n)$ can show remarkably complex behavior in 1D. First, it's worth reminding yourself about the types of behavior present in a 1D continuous dynamical system. Can there be oscillations (how do you know)? Can there be chaos (how do you know)? How do you describe the general behavior?

Now specifically consider the logistic map, a discrete version of the logistic equation for population growth, and a model dynamics for 1D maps:

$$x_{n+1} = rx_n(1 - x_n), \tag{4}$$

with $0 \leq x \leq 1$ and $0 \leq r \leq 4$. While simple in form, the logistic map contains fixed points, cycles and even chaos, with bifurcations between these variaous attractors as we vary the parameter r.

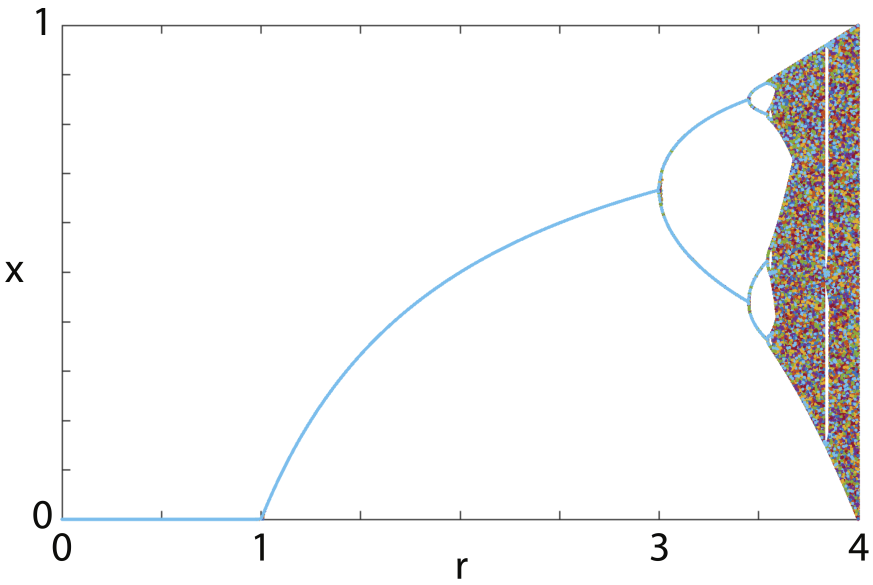

Figure 9: Bifurcation diagram of the logistic map, Eq. 4 For each value of r, we generate 1000 trajectories with random initial conditions $0 \leq x_0 \leq 1$ and plot the last point of an evolution with $n = 500$ steps (thus avoiding any early time transients). Visible on the diagram are fixed points for $r \leq 3$, limit cycles for $r \geq 3$ and chaotic dynamics first appearing around $r \sim 3.6$ (the exact value is $r = 3.569\ldots$, and can be computed by a variety of methods).

In 1D maps, fixed points satisfy $x_{n+1} = x_n$. This results in a quadratic equation for the logistic equation

$$x^* = rx^*(1 - x^*)$$

Though we generally expect two fixed points due to the quadratic equation, why is the number smaller within certain parameter ranges? Now solve the quadratic equation to show

$$\begin{aligned} x^* &= 0 \\ x^* &= 1 - 1/r \end{aligned}$$

Note that $x^* = 1 - 1/r$ only exists for $r \geq 1$.

Fixed-point stability in 1D maps

As with continuous systems, around any fixed point x^* a small perturbation δ will obey a linear equation. To find this equation let $x = x^* + \delta$ and expand $F(x) = F(x^*) + \frac{df}{dx}\Big|_{x=x^*}(x - x^*) + \ldots$. In this perturbation approximation the map dynamics are $\delta_{n+1} + x^* = F(x^*) + \frac{df}{dx}\Big|_{x=x^*}\delta_n$, which is equivalent to

$$\delta_{n+1} = \frac{df}{dx}\bigg|_{x=x^*}\delta_n,$$

since $x^* = F(x^*)$ for any fixed point. If we let $\lambda = \frac{df}{dx}\big|_{x^*}$ then the general solution for the perturbation is

$$\delta_n = \lambda^n \delta_0.$$

Perturbations will thus grow if $|\lambda| > 1$, the sign of unstable dynamics.

As an example, now use these ideas to show for the logistic equation that the fixed point $x^* = 0$ is stable for $r < 1$ and unstable for $r \geq 1$. For what value of r does the fixed point $x^* = 1 - 1/r$ first become unstable?

2D dynamics: fixed points, stability and limit cycles

Greg J Stephens

2D continuous dynamics

In a two-dimensional continuous system, the dynamics are given by

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \tag{1}$$

where in general both $f(x,y)$ and $g(x,y$ are nonlinear functions. 2D dynamics are especially useful for gaining intuition, since the dynamics is rich, yet the *phase plane* or (equivalently) *state space* can be fully visualized. Fixed points $\{x^*, y^*\}$ in 2D are determined by the solution to the *system* of equations,

$$\begin{aligned} \dot{x} = 0 &\rightarrow f(x,y) = 0 \\ \dot{y} = 0 &\rightarrow g(x,y) = 0 \end{aligned} \tag{2}$$

and we refer to each curve as a *nullcline*. The fixed points are determined as the intersection of the two nullclines. An example phase space, including null nullclines, fixed points, and linear stability is appended to these notes.

Linear analysis and classification

How do we classify the dynamics near a fixed point? As in 1D we write our dynamics as a small perturbation about the fixed point, but this time for both x and y,

$$\begin{aligned} x &= x^* + \delta_x(t) \\ y &= y^* + \delta_y(t) \end{aligned}$$

and recall the general form of the Taylor expansion around x^*

$$\begin{aligned} f(x,y) &= \overbrace{f(x^*,y^*)}^{0} + \left.\frac{\partial f(x,y)}{\partial x}\right|_{x,y=x^*,y^*} \delta_x + \left.\frac{\partial f(x,y)}{\partial y}\right|_{x,y=x^*,y^*} \delta_y + \dots \\ g(x,y) &= \overbrace{g(x^*,y^*)}^{0} + \left.\frac{\partial g(x,y)}{\partial x}\right|_{x,y=x^*,y^*} \delta_x + \left.\frac{\partial g(x,y)}{\partial y}\right|_{x,y=x^*,y^*} \delta_y + \dots \end{aligned}$$

So our equation for the dynamics of the perturbations is now a system of equations

$$\begin{aligned} \frac{d\delta_x}{dt} &= \left.\frac{\partial f(x,y)}{\partial x}\right|_{x,y=x^*,y^*} \delta_x + \left.\frac{\partial f(x,y)}{\partial y}\right|_{x,y=x^*,y^*} \delta_y \\ \frac{d\delta_y}{dt} &= \left.\frac{\partial g(x,y)}{\partial x}\right|_{x,y=x^*,y^*} \delta_x + \left.\frac{\partial g(x,y)}{\partial y}\right|_{x,y=x^*,y^*} \delta_y \end{aligned}$$

which we compactly write

$$\frac{d\vec{\delta}}{dt} = J(x^*,y^*)\vec{\delta}$$

where J is the *Jacobian* matrix evaluated at the fixed point. In our linear analysis, since we evaluate the Jacobian at a fixed point, the dynamics is governed by solutions to this linear matrix equation.

Linear dynamics in 2D, generally

Since the stability of a fixed point in 2D is governed by the Jacobian matrix evaluated at the fixed point, let' think about 2D linear dynamics through a general coupling matrix W,

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \frac{d\vec{x}}{dt} = W\vec{x}, \qquad W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \quad \text{(coupling matrix)}$$

Since W_{12} and W_{21} are generally nonzero, these dynamics are coupled between x and y. You have likely already solved such a coupled system of equations in a previous math or physics class but we will also review the solution here.
How do we solve such dynamics? Let U be a unitary matrix of constant coefficients so that $U\,U^T = I$. Then

$$\frac{d\vec{x}}{dt} = W\,U\,U^T\vec{x}$$

or

$$U^T\,\frac{d\vec{x}}{dt} = U^T\,W\,U\,U^T\vec{x}$$

and since U is simply a matrix on constants, we can bring it inside the time derivative

$$\frac{d}{dt}\left(U^T\vec{x}\right) = U^T\,W\,U\,\left(U^T\vec{x}\right)$$

so this motivates the definition of a new variable

$$\vec{z} = U^T\vec{x}$$

What is the advantage of introducing this matrix U? Well, we are free to choose U so that it *diagonalizes* the original coupling matrix W. The dynamics of z are then characterized by the eigenvalues of W. The eigenvalues are obtained by solving the characteristic equation $\mathrm{Det}(W - \lambda I) = 0$:

$$\mathrm{Det}\begin{pmatrix} W_{11}-\lambda & W_{12} \\ W_{21} & W_{22}-\lambda \end{pmatrix} = 0.$$

Solving for λ yields

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \quad \lambda_1 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2},$$

where we have expressed the eigenvalues in terms of τ and Δ, respectively the trace and determinant of W.

Dynamics in the (τ, Δ)-plane

In 2D it is convenient to represent the linear dynamics using τ and Δ. Usefully, both the trace τ and determinant Δ are invariant to linear transformations! That means that you can compute these from the original Jacobian.

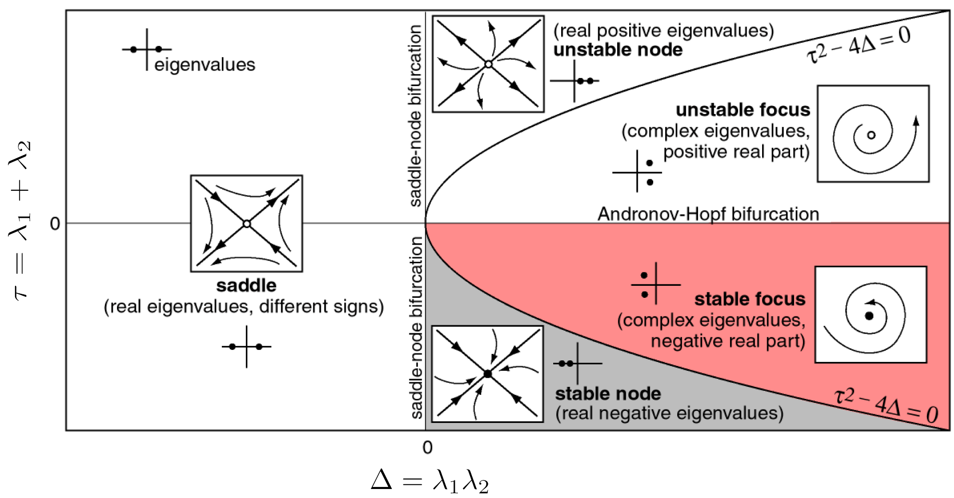

Figure 1: 2D linear dynamics is conveniently described using the trace τ and determinant Δ, the (τ, Δ)-plane. Note that $\lambda = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\Delta})$. Nodes and saddles essentially involve the same linear dynamics (exponential growth or decay, no imaginary eigenvalues) that we saw before in 1D. Spirals, however, are new and point to the creation and destruction of limit cycles.

If $\quad \Delta < 0 \quad \Longrightarrow \lambda_1, \lambda_2 \in \mathbb{R}$ but opposite sign $\Longrightarrow$ saddles.

$$\text{If} \quad \tau^2 > 4\Delta \quad \Longrightarrow \lambda_1, \lambda_2 \in \mathbb{R} \quad \text{but same sign} \quad \Longrightarrow \begin{cases} \text{stable nodes} & \text{if } \lambda_1, \lambda_2 < 0. \\ \text{unstable nodes} & \text{if } \lambda_1, \lambda_2 > 0. \end{cases}$$

$$\text{If} \quad \tau^2 < 4\Delta \quad \Longrightarrow \lambda_1, \lambda_2 \in \mathbb{C} \quad ; \quad \lambda = a \pm b\imath \quad \Longrightarrow \begin{cases} \text{stable spirals} & \text{if } a < 0. \\ \text{unstable spirals} & \text{if } a > 0. \end{cases}$$

If we trace various paths in the (τ, Δ)-plane we recover 2D versions of bifurcations that you have already seen. Fro example from stable node to saddle simply means that we destablized one of the eigenvalues. But look also in the regions where there are spirals. If we cross from stable spirals to unstable spirals this is a bifurcation that is new to 2D: the Poincare–Andronov–Hopf bifurcation that generally signals the creation or destruction of a limit cycle.

Limit cycles

We have previously seen that fixed points are the only possible attractors in 1D continuous dynamics. In 2D a qualitatively new attractor emerges: a limit cycle. An example of such a trajectory is given in Fig. 2.

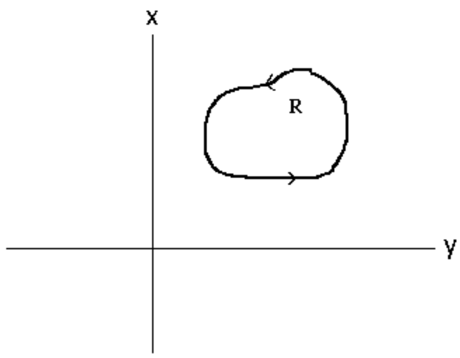

Figure 2: In 2D, closed trajectories in the state space are possible.

A limit cycle is an isolated, closed trajectory. The trajectory is isolated is there are no nearby closed trajectories (stable or unstable).

Poincaré-Bendixson theorem: If a trajectory is confined in $\mathbb{R}^2$ it is either a closed orbit or it will spiral into a closed orbit.

The Poincaré-Bendixson theorem implies that there is no chaos in 2D.
Besides fixed points our qualitative dynamical picture will now also include limit cycles. But first, a bit more on 2D systems.

Hopf Bifurcations

If (x^*, y^*) is a stable fixed point in 2D then the eigenvalues of the linear dynamics about this fixed point must have negative real values, as we have seen above. To destabilize a fixed point at least one eigenvalue must cross zero. Saddle-node, transcritical and pitchfork bifurcations all result from real eigenvalues that change sign. But what if the eigenvalues have have an imaginary component? If the resulting attractor is a limit cycle, then the bifurcation is a Hopf bifurcation. When the limit cycle is stable, then this is a supercritical Hopf bifurcation. Hopf bifurcations are the generic process by which limit cycles are created and destroyed.

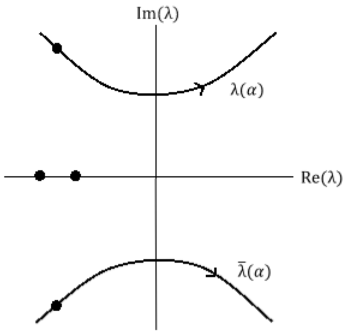

Figure 3: In a Hopf bifurcation, a pair of complex conjugate eigenvalues cross the real axis. The bifurcation parameter is α.

Supercritical Hopf bifurcations

$$\begin{array}{c} \dot{r} = \alpha r - r^3 \\ \dot{\theta} = w + br^2 \end{array} \implies \begin{array}{c} \dot{x} = \alpha x - wy + \mathcal{O}(x^2, y^2) \\ \dot{y} = \alpha y + wx + \mathcal{O}(x^2, y^2) \end{array}$$

$$W = \begin{pmatrix} \alpha & -w \\ w & \alpha \end{pmatrix} \implies \begin{array}{c} (\alpha - \lambda)(\alpha - \lambda) + w^2 = 0 \\ \lambda = \alpha \pm \imath w \end{array}$$

$$r(r^2 - \alpha) = 0,\ \alpha < 0 \implies r = 0$$

From the linearization, perturbations from the origin spiral to $r = o$ when $\alpha < 0$. When $\alpha > 0$ there is another "fixed point" in r, namely $r = \alpha$, which is a limit cycle.

Limit cycles in the logistic map

We have already found the fixed points of the logistic map, what about cycles? In discrete-time maps $x_{n+1} = f(x_n)$, cycles are easily enumerated by their period p, which must be a discrete number of steps. In general, we search for p-cycles by looking for fixed-points of the p-iterated map,

$$f^p[x] = x,$$

since by definition of a p-cycle we return to the same point after iterating the map p times. Let's solve this cycle equation for a 2-cycle.

$$\begin{aligned} x_1^* = &\quad (r + 1 + \sqrt{(r-3)(r+1)}))/(2r) \\ x_2^* = &\quad (r + 1 - \sqrt{(r-3)(r+1)}))/(2r). \end{aligned} \tag{3}$$

You can easily check that this 2-cycle is the stable attractor of the logistic map for values of r just a bit above 3, say $r = 3.2$ (do it!). Notice that this 2-cycle only exists for $r \geq 3$ since otherwise the orbit has an imaginary component. To find higher-period cycles we simply have to solve increasingly complicated algebraic root equations. So how exactly did we find the limit cycle points, Eq.3? Well, the 2-cycle equation $F(F(x)) = x$ for the logistic map is generally 4th order, but we know that two of these solutions should be our previous discovered fixed points. That means that the two cycle equation can be written in the general form

$$x(x - d)(ax^2 + bx + c) = 0,$$

where $d = 1 - 1/r$. Written explicitly, the two-cycle equation is

$$r(rx(1 - x)(1 - rx(1 - x)) = x$$

and comparing the powers yields the polynomial coefficients

$$\begin{aligned} a &= -r^3 \\ b &= r^2(r + 1) \\ c &= -r(r + 1), \end{aligned}$$

from which solutions Eq. 3 directly follow.

To better understand the bifurcation at $r = 3$, let's reexamine the stability of the fixed point $x^* = 1 - 1/r$. The stability is reflected in $\lambda = \frac{df}{dx}\Big|_{x=x^*} = 2 - r$. So for $1 \leq r \leq 2$ small perturbations from the fixed point decay monotonically. But from $r > 2$, while still stable, the perturbations oscillate (negative λ) as they decay, analogous to the stable spiral we have seen in linear, continuous time dynamics. For $r \geq 3$ the oscillating perturbations no longer decay, the spiral is unstable, and we have the birth of a limit cycle.

Stability of the 2-cycle

A 2-cycle is stable if perturbations around either one of the cycle points $\{x_1^*, x_2^*\}$ do not grow *upon return to the same point*,

$$\left| \frac{d}{dx} f(f(x)) \Big|_{x_1^*,\, x_2^*} \right| < 1.$$

Recall the chain rule for the composition of two functions

$$(h(g(x))' = h'(g(x))g'(x),$$

so that

$$(f(f(x))' = f'(x_1)f'(x_2).$$

For the logistic map

$$f' = r(1 - 2x)$$

so that with the points along the 2-cycle, Eq. 3

$$f'(x_1^*)f'(x_2^*) = -(\sqrt{(r+1)(r-3)} - 1)(\sqrt{(r+1)(r-3)} + 1).$$

The stability condition

$$|f'(x_1^*)f'(x_2^*)| < 1$$

leads to a quadratic equation for r

$$r^2 - 2r - 5 = 0,$$

with the the only positive r solution

$$r^* = 1 + \sqrt{6} \approx 3.45$$

Beyond r^* the 2-cycle is unstable and a new stable attractor emerges, a 4-cycle (which you could show using the same techniques, the algebra is just more involved). The appearance of progressively higher period cycles is an indication of the period-doubling route to chaos of the logistic map.

Quantifying Chaos

Greg J Stephens

The Lyapunov exponent for 1D maps

We start our 1D discrete-time dynamics $x_{n+1} = F(x_n)$ with an initial condition x_0. Now consider a nearby point $x_0 + \delta_0$. When we iterate the map then the separation will change. If

$$|\delta_n| \sim |\delta_0| e^{\lambda n},$$

then λ is known as the Lyapunov exponent for the map. When this exponent is positive, the dynamics is chaotic. In general there are as many Lyapunov exponents as state space dimensions and in these cases we refer to the Lyapunov spectrum. Unfortunately, for dimensions greater than 1, estimating the Lyapunov spectrum is challenging, requiring additional computational treatment. Even in 1D, estimating the Lyapunov exponent from the exponential divergence of nearby trajectories is subtle. For example, the dynamics is bounded by whatever attractors are present and so the exponential divergence cannot hold for arbitrarily long trajectories. However, in the 1D case we can derive an important estimation equation. By definition,

$$\delta_n = f^n(x_0 + \delta_0) - f^n(x_0),$$

so that

$$\lambda = \frac{1}{n} \log \left| \frac{\delta_n}{\delta_0} \right|$$

or

$$\lambda = \frac{1}{n} \log \left| \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right|$$

This gives

$$\lambda = \frac{1}{n} \log |(f^n)'(x_0)|,$$

when $\delta_0 \to 0$. Now recall the chain rule for the composition of two functions

$$(h(g(x))' = h'(g(x))g'(x)$$

and take an example of just two iterations

$$(f(f(x_0)' = f'(x_1)f'(x_0).$$

It's then easy to see how to generalize to n iterations

$$(f^n)'(x_0) = \prod_{i=0}^{n-1} f'(x_i)$$

and thusc

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |(f'(x_i)| \tag{1}$$

For 1D maps the single Lyapunvov exponent is simply computed as the average of the natural logarithm of the absolute value of the Jacobian over all points in the attractor. What is the sign of the Lyapunov exponent for a stable and unstable fixed point, respectively? Why? What about a stable 2-cycle? As a point of terminology note that attractors (even chaotic ones) are stable *as point sets* so that perturbations will flow back onto the attractor. When we use Eq. 1 to calculate the Lyapunov exponent for an attractor, we only consider the sum over points in the attracting set.