A parallel condition

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Issue 2

1 Introduction

This article is on the condition

$$\text{iff } \overline{OI} \parallel \overline{BC}$$

in triangles where O is the circumcenter and I is the incenter of $\triangle ABC$.[1] Obviously not all triangles satisfy this, but the ones that do exhibit some nice properties. As far as conditions go in geometry, this is one of the more common ones.

Since this is not about any particular technique, and just learning some properties, this is formatted as a "find the properties" article, similar to Evan Chen's Mixtilinear Incircle handout. I suggest you try[2] finding and proving as many properties as possible with ruler and compass before you look at the next pages. All properties and problems that I show have synthetic solutions, but feel free to bash.

[1] The abbreviation "iff" stands for "if and only if."

[2] I don't blame you if you don't though, honestly I didn't even think about pulling out paper when reading this line in the Mixtilinear Handout (sorry Evan).

2 Diagrams

I've provided 4 diagrams, 2 with $\overline{OI} \parallel \overline{BC}$ and 2 without. Hopefully this helps you find some more properties on top of the ones you already found.

Notation

- D is the foot from I to $\overline{BC}$
- Fe is the Feuerbach point
- H is the orthocenter of $\triangle ABC$
- H_A is the orthocenter of $\triangle BIC$
- M is the midpoint of BC
- Na is the Nagel point
- N_9 is the nine-point center
- P is the point on $\overline{BC}$ such that $\measuredangle AIP = 90^\circ$
- T is the exsimillicenter of the incircle and circumcircle.
- T_A is the A-mixtilinear touchpoint

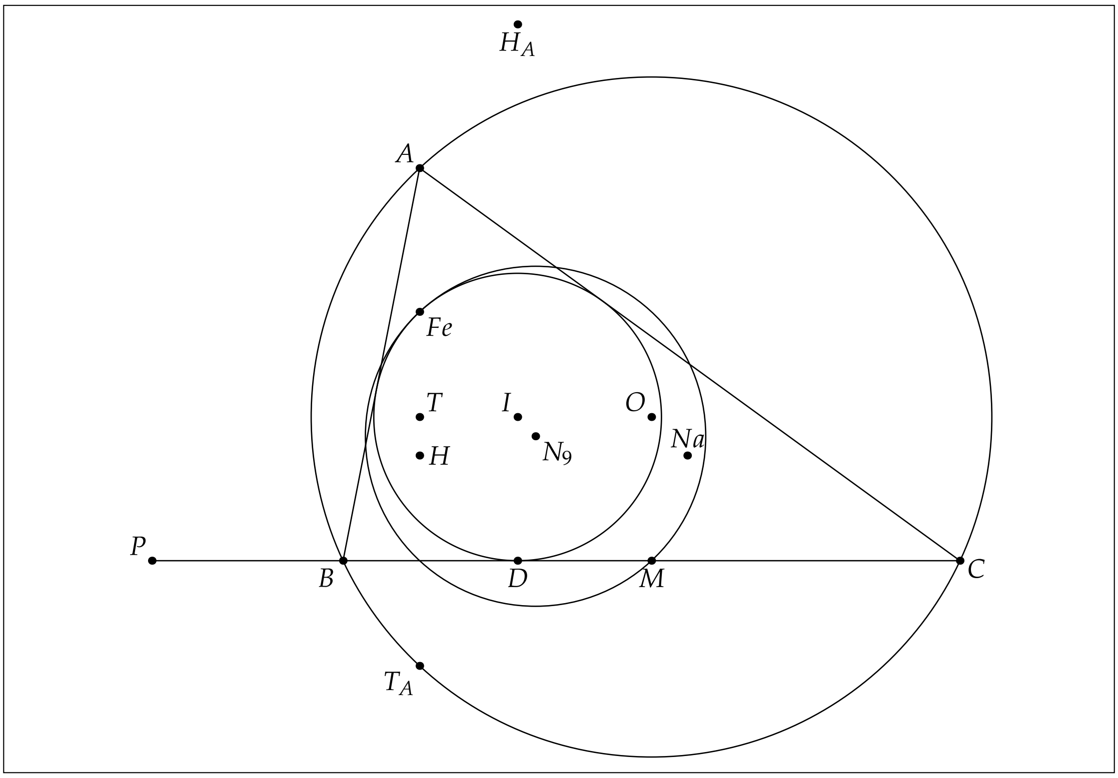
2.1 Things to look out for

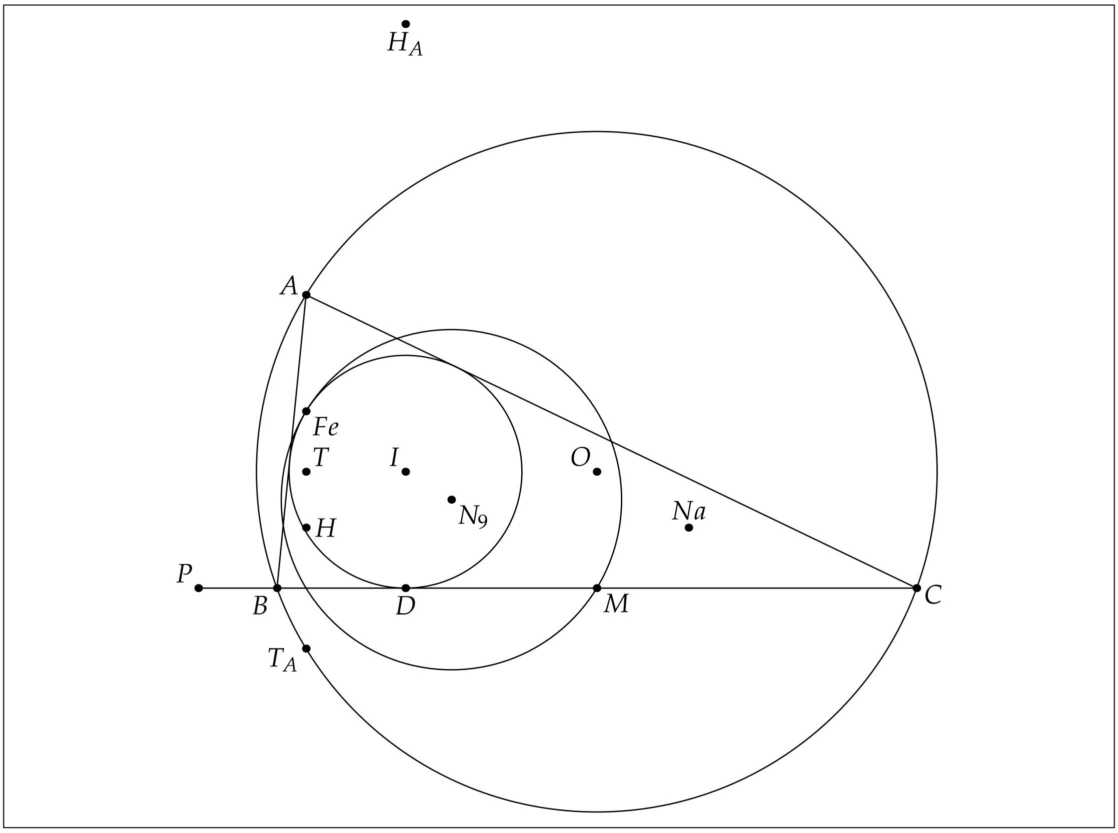
The following are true when $OI \parallel BC$ and false otherwise (so properties that are true when this condition isn't satisfied aren't included).

- There are at least seven nontrivial collinearities among the labelled points. Five of them consist of three collinear points and one consists of four.
- There is at least one set of concurrent lines (which meet outside the triangle).
- There is at least one nontrivial cyclic quadrilateral among the labelled points.
- There are at least three pairs of parallel lines among the lines formed using the labelled points.
- There are at least 3 pairs of perpendicular lines among the lines formed using the labelled points.
- One of the labelled points is the intersection of a circle (whose diameter is one of the line segments formed using two of the labelled points) and a line formed using the labelled points.
- There is one pair of line segments with equal lengths formed using the labelled points.
- There is one pair of isogonal lines with respect to an angle formed using the labelled points.

2.2 Diagrams with the condition satisfied

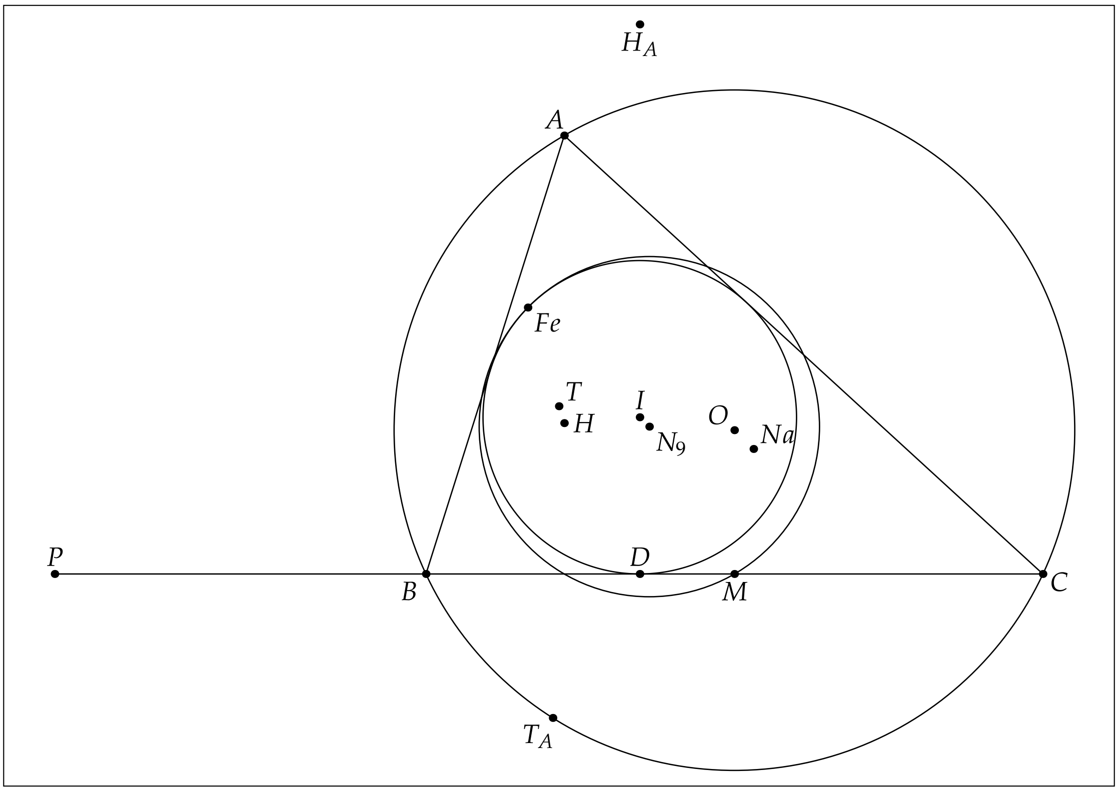
2 diagrams with $\overline{OI} \parallel \overline{BC}$.

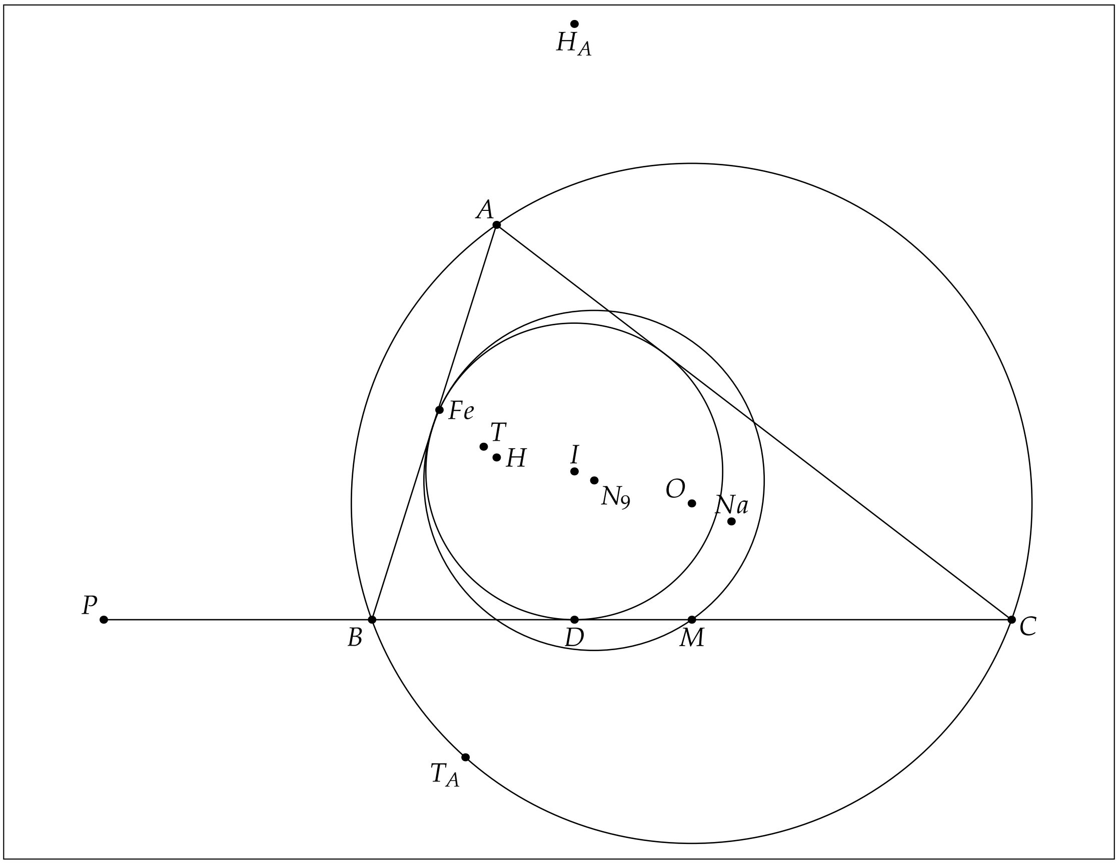




2.3 Diagrams with the condition unsatisfied

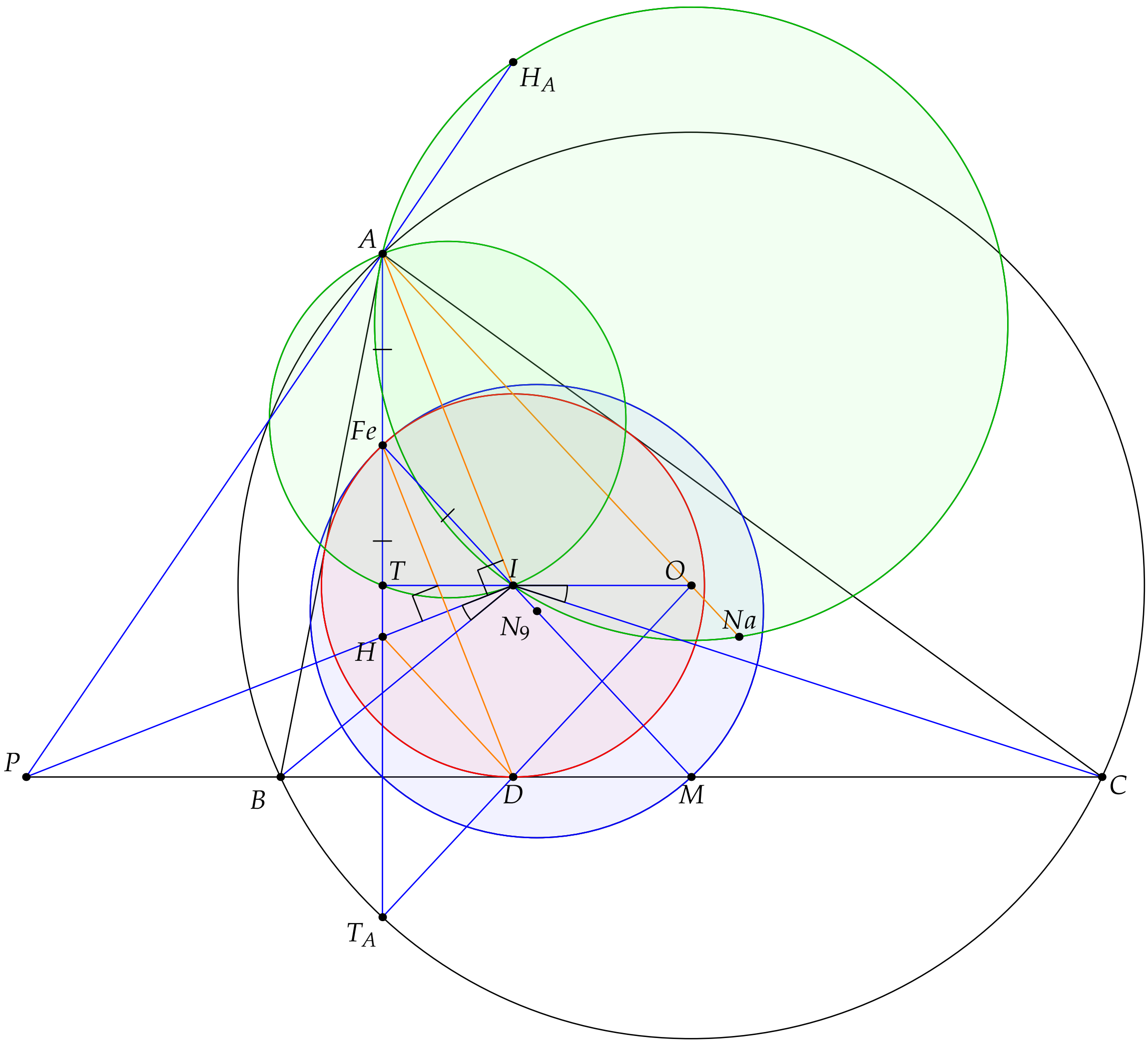
2 diagrams without $\overline{OI} \parallel \overline{BC}$.





3 Properties

I've found 17 properties that hold iff $OI \parallel BC$. This list is by no means exhaustive, so your list may be a little different. I suggest going in order, as some properties require the previous properties to be true. A short hint to help you out with actually proving these though: since I found and proved the properties on the next page, there is an implicit bias towards configurations and projective geometry ☺. In the next section I give hints to all of these.



1. $\overline{HD} \parallel \overline{ANa}$.
2. $T = (AI) \cap \overline{AH}$.
3. Points A, H, T_A are collinear.
4. Points A, O, Na are collinear.
5. $FeA = FeI$.
6. Points A, I, H_A, Na are concyclic.
7. Points T_A, D, O are collinear.
8. Points Fe, I, N_9, M are collinear.
9. Point Fe bisects segment AH.
10. $\overline{HD} \parallel \overline{AO}$.
11. $\overline{FeD} \parallel \overline{AI}$.
12. Points A, H_A, P are collinear.
13. Points H, I, P are collinear (alternatively $\overline{AI} \perp \overline{HI}$).
14. Lines $\overline{IH}, \overline{IO}$ are isogonal with respect to $\angle BIC$.
15. Lines $\overline{AH_A}, \overline{HI}, \overline{BC}$ concur (notice how this differs from 12,13).
16. $\overline{FeD} \perp \overline{HI}$.
17. $\overline{AFe} \perp \overline{IO}$

4 Hints

Here "normally" refers to a property that holds for any triangle ABC.

1. Parallelograms!
2. Draw in M_A, the circumcenter of $\triangle BIC$. Note that T, D, M_A are collinear. Where do $\overline{DM_A}$ and $\overline{AH}$ normally intersect?
3. Points A, T, T_A are normally collinear.
4. T, Na are normally isogonal conjugates.
5. Let D' be the reflection of D over I. $\overline{FeD'}$ normally bisects AI.
6. Use the fact A, T, H, T_A are collinear to angle chase.
7. Transform the collinearity of the points A, O, Na.
8. Same as hint 7. In addition use the parallelogram from hint 1.
9. Points Fe, N_9, M are collinear and Fe is on the 9-Point Circle. In addition, you can prove that normally points H, T, Fe are collinear.
10. Homothety at A with factor $\frac{1}{2}$.
11. Let D' be the reflection of D over $\overline{AI}$. Note that normally M, D', Fe are collinear.
12. Use Fontene's First Theorem to prove that $\overline{AH_A}, \overline{BC}$, and the tangent to the incircle at Fe concur.
13. Reflect H over I.
14. Let N_{9A} be the nine-point center of $\triangle BIC$. Prove that $\overline{IO}, \overline{IN_{9A}}$ are isogonal. Project a bundle containing N_{9A} to link the collinearity of the points H, I, N_{9A} to a previous fact.
15. My favorite proof is using the Feuerbach hyperbola. If you don't want to, then let I' be the reflection of I over $\overline{BC}$. Prove that $\overline{AH_A}, \overline{I'M_A}, \overline{IN_{9A}}$ normally concur.
16. Same as hint 12.
17. Draw the perpendicular from A to $\overline{IO}$ and draw $2 \cdot fe - a$ (where fe, a are the complex number representations of Fe, A respectively).

5 Problems

A short set of contest (and contest-like) problems.

Problem 1 (Taiwan MO 2021/4). Let I be the incenter of triangle ABC and let D the foot of altitude from I to BC. Suppose the reflection point D' of D with respect to I satisfies $AD' = ID'$. Let Γ be the circle centered at D' that passes through A and I, and let $X, Y \neq A$ be the intersection of Γ and AB, AC, respectively. Suppose Z is a point on Γ so that AZ is perpendicular to BC. Prove that AD, $D'Z$, XY are concurrent.

Problem 2 (HMMT February Team 2011/A.12). Let ABC be a triangle, and let E and F be the feet of the altitudes from B and C, respectively. If A is not a right angle, prove that the circumcenter of triangle AEF lies on the incircle of triangle ABC if and only if the incenter of triangle ABC lies on the circumcircle of triangle AEF.

Problem 3 (Trig Lemma). Show that $\cos(B) + \cos(C) = 1$ if and only if $\overline{OI} \parallel \overline{BC}$.

Problem 4 (HMMT Feburary Team 2021/9). Let scalene triangle ABC have circumcenter O and incenter I. Its incircle ω is tangent to sides BC, CA, and AB at D, E, and F, respectively. Let P be the foot of the altitude from D to EF, and let line DP intersect ω again at $Q \neq D$. The line OI intersects the altitude from A to BC at T. Given that $OI \parallel BC$, show that $PQ = PT$.

Problem 5 (LMAO 2020/5). Let ABC be a triangle with incenter I, orthocenter H, and circumcenter O. Lines $\overline{HI}$, $\overline{AO}$ meet at F, and lines $\overline{AI}$, $\overline{BC}$ meet at G. If $\overline{OI}$ is parallel to $\overline{BC}$, show that the circumcenter of $\triangle AFG$ lies on $\overline{OI}$.

Problem 6 (Krishna Pothapragada). Let ABC be a scalene triangle, I be its incenter, and M be the midpoint of BC. If the perpendicular to $\overline{AI}$ from I meets $\overline{AB}$, $\overline{AC}$ at B', C', prove that $\overline{MB'}$ is tangent to $(AB'C')$ if and only if $\overline{OI} \parallel \overline{BC}$.

6 Selected Solutions

6.1 Solution 3 (Trig Lemma)

By Carnot's[3] we have

$$\frac{r}{R} = \cos(A) + \cos(B) + \cos(C) - 1.$$

But $r = ID = OM = R\cos(A)$, so

$$\cos(A) = \cos(A) + \cos(B) + \cos(C) - 1$$

as desired.

> *Remark 1.* For contest purposes, the more useful form of this lemma is $\frac{r}{R} = \cos(A)$.

6.2 Solution 6 (Krishna Pothapragada)

Two solutions:

Solution via trig bash Let r be the inradius and R be the circumradius of $\triangle ABC$. Then

$$\begin{aligned}
\tan(A) &= \frac{\sin(A)}{\cos(A)} \\
&= \frac{2\sin(\frac{A}{2})\cos(\frac{A}{2})}{\cos(A)} \\
&= \frac{2 \cdot \frac{r}{IA} \cdot \frac{r}{IB'}}{\cos(A)} \\
&= \frac{2r^2}{IA \cdot IB' \cdot \cos(A)} \\
&= \frac{2r^2}{IA \cdot IB \cdot \cos(A)} \cdot \frac{IM \cdot IA}{2Rr} \\
&= \frac{IM}{IB'} \cdot \frac{r}{R\cos(A)} \\
&= \tan(\angle IB'M) \cdot \frac{r}{R\cos(A)},
\end{aligned}$$

so the condition is equivalent to $R\cos(A) = r$, as desired.

Solution via projective Let E, F be the feet from I to $\overline{AC}, \overline{AB}$. Let H, H', H_A be the orthocenters of $\triangle ABC, \triangle AEF, \triangle BIC$, respectively.

> **Claim** — Points H, H', H_A are collinear.

Proof. We use complex numbers (for notation purposes) and we denote the complex numbers of the points with the corresponding lowercase. (e.g. point A is denoted with a). Line $\overline{HH'}$ is perpendicular to the line passing through $\frac{b+e}{2}$ and $\frac{c+f}{2}$ (a property of the Newton-Gauss line). Let N be the midpoint of EF. We have that $\overline{HH'}$ is perpendicular to the line passing through N and the midpoint of E and E', where $e' = b + e - c$. Similarly, $\overline{HH'}$ is also perpendicular to the line passing through N and the midpoint of F and F', where $f' = c + f - b$.

[3] Carnot has many theorems, this one is the one I am referring to.

Claim — $\overline{H_A N}$ is the perpendicular bisector of $E'F'$.

Proof. Clearly N is the midpoint of $E'F'$ (by complex numbers). Let X be the point such that $x = b + c - d$. Note that the perpendicular bisector of $E'X$ is the line through B perpendicular to $\overline{CI}$. By symmetry, H_A is the circumcenter of the points $\triangle E'F'X$, as desired. □

By projecting through H_A then N, we have

$$(H', I; N, P_\infty) \overset{H_A}{=} -1 \overset{N}{=} \left(\frac{b-c}{2} + e, P_\infty; E', E\right).$$

Thus, $\overline{H_A H'}$ is perpendicular to the line between N and $\frac{b-c}{2} + e$, as desired. □

Let's return to the problem. We will use the following property of the Schiffler point, which we state without proof:[4]

Lemma
The Euler lines of $\triangle ABC, \triangle BIC, \triangle AIC, \triangle AIB$ concur at a point Sc on the rectangular hyperbola through A, B, C, I.

Introduce Na. We have that $\overline{H_A Na} \parallel \overline{AI}$ by a reflection across M. Let M_A be the circumcenter of $\triangle BIC$. Under an inversion with respect to the circle with center A and radius AI, M_A goes to H' iff the problem condition. Thus, $(A, P_\infty); (H', M_A); (I, I)$ is an involution iff the problem condition. Projecting this from H_A onto the Feuerbach hyperbola means that $(A, Na); (H, Sc); (I, I)$ is an involution iff the problem condition. This means A, O, Na are collinear iff the problem condition, or $\overline{OI} \parallel \overline{BC}$ iff the problem condition, as desired.

Remark 2. This solution, while accurate, is probably impossible to come up with in-contest. It is similar to how I constructed the problem in the first place, and I leave it here for comedic value. Please trig bash instead, as presented above.

Remark 3. The second claim above solves ISL 2009/G3 after showing G, H_A, N are collinear (using G as given in the ISL problem).

[4] If you're interested, the proof is by an application of Pascal's on the Feuerbach Hyperbola.