2020 Mock USAJMO

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Day II

April 7th to April 21st, 2020

Note: For any geometry problem whose statement begins with an asterisk ($*$), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

JMO 4. Bob has n stacks of rocks in a row, each with heights randomly and uniformly selected from the set $\{1, 2, 3, 4, 5\}$. In each move, he picks a group of consecutive stacks with positive heights and removes 1 rock from each stack. Find, in terms of n, the expected value of the minimum number of moves he must execute to remove all rocks.

JMO 5. $(*)$ Let ABC be a triangle with A-excenter I_A, and let X and Y be the feet of the perpendiculars from B and C to the angle bisectors of $\angle ACB$ and $\angle ABC$, respectively. The circumcircles of $\triangle I_A BX$ and $\triangle I_A CY$ meet again at P, and J is the incenter of $\triangle PXY$. Prove that $\angle BJC = 90^\circ$.

JMO 6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - 2y) + f(x + f(y)) = x + f(f(x) - y)$$

for all real numbers x, y.