NICE Journal

NICE Committee

Second Edition

Preface

Hello everyone!

This is the second NICE Journal, and we're glad that you've decided to read us. Below is a short list of notes that may enhance your reading experience.

Chapter Authors and Descriptions

You may be interested in knowing the authors of the journal entries or some information about them. Here they are:

Entry Name	Author	Description
Algebraic Manipulation	Elijah Liu	Basic algebraic manipulations covering simple ideas like variables and SFFT.
Roots of Unity Filter	Raymond Feng & Dylan Yu	How to count only a specific group of coefficients and when to use the filter.
Incenter Miquel	Kazi Aryan Amin	Some rich configurations arising from the foot of altitude in a contact triangle.
Multiplicative Functions	Paul Hamrick	Number-theoretic functions satisfying $f(mn) = f(m)f(n)$, e.g. ϕ and τ.
Titu's Favorite Factoring Trick	Arul Kolla	The factorization of $x^3 + y^3 + z^3 - 3xyz$ and its uses.

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- Kazi Aryan Amin
- Sanjana Das
- Raymond Feng
- Paul Hamrick
- Arul Kolla
- Elijah Liu
- Dylan Yu

Final words before you start reading

Again, we're very excited that you're taking your time to read this body of work. Let us know if you have any suggestions, and have fun learning math!

Sincerely,
Dylan Yu

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1 Algebraic Manipulation

You might want to review factoring and Vieta's formulas.

1.1 Introduction

Let's start with two relatively simple techniques:

Fact 1 (Completing the Square). Start with arbitrary quadratic $ax^2 + bx + c = 0$. Then, one can rearrange that to $a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} = 0$.

Proof. Observe the following manipulations:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= 0 \\ a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right) &= 0 \\ a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right) &= 0 \\ a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} &= 0 \\ a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} &= 0 \end{aligned}$$

as shown. □

It is best not to memorize the result of this, but rather the steps; as completing the square is a very useful technique.

This result can be extended to our beloved quadratic formula:

Theorem 2 (Quadratic Formula)

In an arbitrary quadratic $ax^2 + bx + c = 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Proof. Continue with the "completing the square" from before. Then:

$$\begin{aligned} a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}&=0 \\ \left(x+\frac{b}{2a}\right)^2&=\frac{b^2-4ac}{4a^2} \\ x+\frac{b}{2a}&=\pm\frac{\sqrt{b^2-4ac}}{2a} \\ x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \end{aligned}$$

□

Remark 3. If we let $b'=\frac{b}{2}$, then we can rewrite the quadratic formula as

$$x=\frac{-b'\pm\sqrt{b'^2-ac}}{a}.$$

This representation has fewer coefficients, so it's a reasonable idea to keep this in the back of your head.

Now, on to the actual content.

1.2 Variable Assignment and Flexibility

The great thing about variables is that it is very easy to assign them in problems. Let's take a look at this Euclid problem as an example:

Example 4 (Euclid 2021/2.c)
Qing is twice as old as Rayna. Qing is 4 years younger than Paolo. The average age of Paolo, Qing, and Rayna is 13. Determine their ages.

The idea is to take advantage of the structure of the problem (all the phrases can be represented as equations and the ages as unknowns). This isn't really regarded as a technique, but it works wonders – when you're stuck, try assigning variables randomly and delving further.

A corollary of this is that variable substitutions are very easy to perform, and if you can do one, it is advisable to do so (if it's beneficial, of course).

I'm also a huge fan of using Vieta's formulas backwards when given the chance. This is best illustrated in the following example:

Example 5 (Euclid 2014/3)
Determine all ordered pairs (a, b) that satisfy the following system of equations:

$$a + b = 16$$
$$\frac{1}{a} + \frac{1}{b} = \frac{4}{7}$$

Walkthrough. Very simple, but also very practical.

1. Find ab.
2. Construct the polynomial $x^2 - (a+b)x + ab = 0$ via reverse Vieta's and solve.

Let's also take a look at this comical example:

Example 6 (blackpenredpen)
Solve the equation $\sqrt{5-x} = 5 - x^2$.

Walkthrough. Let's look at the restrictions for the problem.

1. Show that $|x| \leq \sqrt{5}$. This will help check for extraneous roots.
2. Here comes the trick: square both sides and treat 5 as the variable. You'll end up with two quadratics.
3. Solve the quadratics and check the bounds.

1.3 Simon's Favourite Factoring Trick

Also known as "completing the rectangle." Let's start with the canonical example:

Example 7 (AoPS)
Find all pairs of positive integers (m, n) that satisfy $mn + 3m - 8n = 59$.

Walkthrough. If you've never seen this before, you should just read the solution.

1. Factor to get $m(n+3) - 8n = 59$.
2. Find a way to turn the "$-8n$" part to $-8(n+3)$ and get $(m-8)(n+3) = 35$.
3. Note m, n are positive integers: do casework to finish.

The main reason why this is a useful trick is because m and n have to both be integers. If they had to be rational numbers, or even real numbers, it would be much harder or downright impossible to get precise values for m and n.

1.4 Various Useful Identities

Below is a list of useful algebraic identities, in no particular order. Proofs are not included as they will bloat the handout (although it's nice to explore them for yourselves!)

- SFFT: $xy + jx + ky = a \implies (x+k)(y+j) = a + jk$
- $a^2 \pm 2ab + b^2 = (a \pm b)^2$
- $a^2 - b^2 = (a+b)(a-b)$
- $a^3 \pm b^3 = (a \pm b)\left(a^2 \mp ab + b^2\right)$
- TFFT: $a^3 + b^3 + c^3 - 3abc = (a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right)$
- Sophie-Germain Identity: $a^4 + 4b^4 = \left(a^2+2ab+2b^2\right)\left(a^2-2ab+2b^2\right)$
- $x - y$ is always a factor of $x^n - y^n$ for nonnegative integer n
- $x + y$ is always a factor of $x^n + y^n$ for odd positive n

1.5 Power Laddering

Let's take this problem that I definitely did not steal from Instagram:

Example 8 (Instagram)

Without using a calculator, evaluate $\dfrac{x^4 - 6x^3 - 2x^2 + 18x + 23}{x^2 - 8x + 15}$ when $x = \sqrt{19 - 8\sqrt{3}}$.

Walkthrough. The nested radical looks funny.

1. Show that $\sqrt{19 - 8\sqrt{3}} = 4 - \sqrt{3}$.
2. Form a quadratic with $4 - \sqrt{3}$ and its conjugate $4 + \sqrt{3}$. Use it to show the denominator of the expression in question is 2.
3. Multiply the quadratic you formed in the above step by x^2 and subtract it off the numerator. Note that this does not affect your answer because you are subtracting 0.
4. Multiply the quadratic by $2x$ and do the same thing. Then again by multiplying the quadratic with some constant. You'll be left with a constant in the numerator and you can finish from there.

1.6 Problems

1.6.1 Easier problems

Problem 1 (Fermat 2019/24). Consider the quadratic equation $x^2-(r+7)x+r+87=0$ where r is a real number. This equation has two distinct real solutions x which are both negative exactly when $p<r<q$ for some real numbers p and q. Find the value of p^2+q^2.

Problem 2 (Own). Determine the sum of all possible x values such that x is a positive integer and $xy=317+2x+3y$ for some integer y.

Problem 3 (Own). Let p, q, and r be the roots of $x^3-16x^2+84x-112=0$. Find $p^3+q^3+r^3-3pqr$.

Problem 4 (Own). If $x+\dfrac{3}{x}=1$, find x^5-x^2.

Problem 5. Express ϕ^n in terms of ϕ and the Fibonacci numbers, where n is a positive integer and ϕ is the larger solution to $x^2-x-1=0$.

Problem 6 (Jeffrey Qin). Let x, y, and z be real numbers such that $2x^2+4y^2+9z^2+4xy+2x+6z=-2$. The sum of all possible values of $|x+y+z|$ is $\dfrac{a}{b}$, a fraction to the simplest form. Compute $a+b$.

Problem 7 (AIME 1987/14). Compute the following (without a calculator, of course):

$$\frac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}$$

1.6.2 Harder problems

These problems are a step up from the previous ones, so I recommend you gather a couple friends (and maybe some pizza) and tackle them together!

Problem 8 (Purple Comet HS 2019/19). Find the remainder when $\prod_{n=3}^{33} \left|2n^4 - 25n^3 + 33n^2\right|$ is divided by 2019.

Problem 9 (Purple Comet HS 2018/22). Positive integers a and b satsify $a^3 + 32b + 2c = 2018$ and $b^3 + 32a + 2c = 1115$. Find $a^2 + b^2 + c^2$.

Problem 10 (NICE Spring 2021/13, David Altizio). Suppose x and y are nonzero real numbers satisfying the system of equations

$$\begin{aligned} 3x^2 + y^2 &= 13x, \\ x^2 + 3y^2 &= 14y. \end{aligned}$$

Find $x + y$.

2 Roots of Unity Filter

2.1 Lecture notes

2.1.1 Statement

A fun story: roots of unity filter (ROUF) was said to be invented by Nikolai Nikolov when he solved IMO 1995/6 in contest via roots of unity. He was awarded a special prize (which is given to people with beautiful solutions) for this discovery.

Theorem 9 (Roots of Unity Filter)
For integers m and n, if we define $\omega = e^{\frac{2\pi i}{n}}$, then the following holds:

$$\sum_{k=0}^{n-1} \left(\omega^k\right)^m = \begin{cases} 0 & n \nmid m \\ n & n \mid m \end{cases}.$$

Proof. Two proofs.

Solution via symmetric sums It suffices to show the result for $0 \leq m < n$, since we may take the exponent mod n without changing the value of the sum. The result is obvious for $m = 0$, so discard that case. Now, as $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are roots of the polynomial $x^n - 1$, this means that all symmetric sums of the nth roots of unity, except for their product, are 0. However, note that the given sum is of degree less than n, so it can be written in terms the symmetric sums without using the product. But all these symmetric sums are 0, so the entire sum evaluates to 0, as desired.

Solution via induction Strong induct on n. The result is obvious for $n = 1$ and n a prime (why?), then split the nth roots of unity into equally spaced groups of size $\frac{n}{\gcd(n,m)}$ and apply the inductive hypothesis to finish. □

2.1.2 Why is ROUF useful?

The reason: if we have some function[1] $P(x)$, then as the name suggests, by taking $P(1) + P(\omega) + P(\omega^2) + \cdots + P(\omega^{n-1})$, the theorem essentially allows us to *filter* out all coefficients except for those which are a multiple of n. This is why ROUF is usually written in the following form:

[1] Usually a rational function, i.e. a polynomial divided by a polynomial.

Corollary 10 (Alternate Form of ROUF)
Let n be a positive integer, and let $\omega = e^{\frac{2\pi i}{n}}$. For any polynomial $P(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ (with finitely many terms), then

$$a_0 + a_n + a_{2n} + \ldots = \frac{1}{n}\left(P(1) + P(\omega) + P(\omega^2) + \ldots + P(\omega^{n-1})\right).$$

Proof. Left as an exercise; just use the first form of ROUF. □

Remark 11 (Which root of unity?). The nth root of unity we use doesn't have to be $\omega = e^{\frac{2\pi i}{n}}$; it can be ω^k for any k relatively prime to n, but I've never seen a question where ω won't work when ω^k will.

Here is a prototypical example:

Example 12
Evaluate

$$\binom{10}{0} + \binom{10}{2} + \binom{10}{4} + \binom{10}{6} + \binom{10}{8} + \binom{10}{10}.$$

Walkthrough.

1. What should we have as the value of n here? What is our $P(x)$? (Hint: use binomial theorem!)

2. Show that the desired sum is equivalent to $\frac{(1+1)^{10}+(1-1)^{10}}{2}$, and conclude.

The above example uses the 2nd roots of unity, but -1 and 1 aren't enough to justify the use of complex numbers. We'll see a better use of roots of unity in action below.

Example 13
Evaluate

$$\binom{10}{2} + \binom{10}{5} + \binom{10}{8}.$$

Walkthrough.

1. What should we have as the value of n here?

2. Why doesn't the $P(x)$ in our previous problem work? There is an additional reason besides "the common differences are different."

3. How can we edit $P(x)$ so that the coefficients of the terms with degree a multiple of n are precisely $\binom{10}{2}, \binom{10}{5}, \binom{10}{8}$? (Hint: consider dividing by a power of x.)

4. Prove that if $\omega = e^{\frac{2\pi i}{3}}$, then the desired sum is equivalent to

$$\frac{1 \cdot (1+1)^{10} + \omega(1+\omega)^{10} + \omega^2(1+\omega^2)^{10}}{3}.$$

Of course, both of the above examples were small enough that they could just be bashed out, but the walkthroughs just serve to illustrate the process of applying the roots of unity filter; the answer extraction is not really the interesting part.

Exercise 14 (Generating function for binomial ROUF). Show that

$$\sum_{k\geq 0} \binom{N}{m+kn} x^{m+kn} = \frac{1}{n}\sum_{k=1}^{n} (\omega^k)^{-m}(1+\omega^k \cdot x)^N,$$

where ω is an nth root of unity.

Moral
The real power of ROUF comes when evaluating how many ways for something to happen such that some parameter is a multiple of some number n. These problems can be tackled using a generating function and then applying ROUF.

For example, the examples above could have instead been rephrased as:

- **Reworded from example 12**: How many ways are there to choose a subset with even size out of 10 distinct objects?
- **Reworded from example 13**: How many ways are there to choose a subset of k objects out of a set of 10 distinct objects such that $k \equiv 2 \pmod 3$?

Although the computations may become very messy in the following problem, can you see the idea behind the solution for it?

Illustrate an idea How many ways are there to choose a subset of k objects out of a set of 2021 distinct objects such that $k \equiv 2 \pmod 3$ or $k \equiv 1 \pmod 4$?

It's just two-set PIE: add the ways for each modulo, then subtract the overlap.

2.1.3 Themes of ROUF

ROUF is technically an algebraic technique, but its applications to C/G/N are what actually make it useful. In particular, the idea is often to express a non-algebraic problem via the subject's relation to the roots of unity, then use ROUF to do the computation.

Common themes of ROUF: for geometry, it can appear when we're dealing with a regular polygon (because the roots of unity are a regular polygon on the unit circle). For combinatorics and number theory, we'll see it when we're asked to count something constrained by a number-theoretic property, e.g. the large amount of IMO 1995/6 copies.

2.2 Examples

This next problem is a repeat of the ones before but just for insurance:

Example 15
Compute
$$\sum_{k\geq 0}\binom{1000}{3k}.$$

Walkthrough. Let $f(n)$ equal 1 when $3 \mid n$ and 0 otherwise, and let $\omega = e^{\frac{2\pi i}{3}}$ be a 3rd root of unity.

1. Characterize $f(n)$ with ω.
2. Use $1+\omega+\omega^2 = 0$ to simplify your answer and finish. For example, we can arrange the expression to get $1+\omega = -\omega^2$.

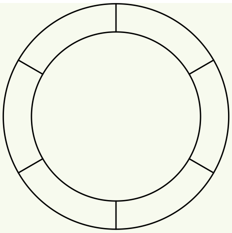
This next example doesn't require roots of unity filter, but when 6 tenors and 8 basses is replaced with a and b of them, respectively, the method becomes more helpful.

Example 16 (AMC 12A 2021/15)
A choir director must select a group of singers from among his 6 tenors and 8 basses. The only requirements are that the difference between the number of tenors and basses must be a multiple of 4, and the group must have at least one singer. Let N be the number of different groups that could be selected. What is the remainder when N is divided by 100?

Walkthrough. Let $f(x,y) = (1+x)^8(1+y)^6$.

1. Show that coefficient a_{mn} of $x^m y^n$ in the expanded form of f is the number of groups of m basses and n tenors. Thus, we want to sum up all values of a_{mn} for which $4 \mid m-n$, except for $a_{00} = 1$.
2. For what value of y (in terms of x) can we change $x^m y^n$ to x^{m-n}? Let $g(x) = f(x, \bullet)$, where $\bullet$ is the new value of y.
3. Apply ROUF on g with the 4th roots of unity.

Example 17 (AIME II 2016/12)
The figure below shows a ring made of six small sections which you are to paint on a wall. You have four paint colors available and you will paint each of the six sections a solid color. Find the number of ways you can choose to paint the sections if no two adjacent sections can be painted with the same color.



Walkthrough. Suppose that the colors are $0, 1, 2, 3$. Clearly the difference between the colors in adjacent sections is $1, 2$, or 3 modulo 4. Define the number at each border between sections to be this difference.

1. Use a generating function to represent each border.
2. What is the generating function for all 6 borders then? In this function, the coefficient of x^n should represent the total number of colorings where the colors' numbers are increased by n as we go around the ring.
3. If we go around the ring and the colors have increased by n, what must n be divisible by to ensure that the color of the section we started with is still the right color? Let the number n is divisible by be m.
4. Apply ROUF on $A(x)$ with the mth roots of unity to finish.

See the solutions for other interesting ways of solving the problem, one using linear algebra, and another using chromatic polynomials.

2.3 Further Reading

1. Things Fourier, Evan Chen
2. Dirichlet's theorem on APs, Evan Chen
3. Cauchy Integral Formula, `Altheman`
4. Generating Functions in CP, `zscoder`
5. More Examples, Kevin Sun
6. Jacobsthal numbers, Wikipedia

2.4 Problems

Silly example:

Problem 11 (AMC 10B 2021/16). Call a positive integer an *uphill* integer if every digit is strictly greater than the previous digit. For example, $1357, 89,$ and 5 are all uphill integers, but $32, 1240,$ and 466 are not. How many uphill integers are divisible by 15?

Problem 12 (BMT Analysis 2015/7). Evaluate $\sum_{k=0}^{37}(-1)^k\binom{75}{2k}$.

Problem 13 (AMC 12A 2017/25). The vertices V of a centrally symmetric hexagon in the complex plane are given by

$$V=\left\{\sqrt{2}i,-\sqrt{2}i,\frac{1}{\sqrt{8}}(1+i),\frac{1}{\sqrt{8}}(-1+i),\frac{1}{\sqrt{8}}(1-i),\frac{1}{\sqrt{8}}(-1-i)\right\}.$$

For each j, $1\le j\le 12$, an element z_j is chosen from V at random, independently of the other choices. Let $P=\prod_{j=1}^{12}z_j$ be the product of the 12 numbers selected. What is the probability that $P=-1$?

Problem 14. Three regular 7-sided dice, two regular 5-sided dice, and one regular 4-sided die are rolled. The probability that the 6 dice sum to a number divisible by 3 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

Problem 15 (IMC 1999/8). We roll a regular die n times. What is the probability that the sum of the numbers shown is a multiple of 5?

Problem 16 (AIME I 2018/12). For every subset T of $U=\{1,2,3,\ldots,18\}$, let $s(T)$ be the sum of the elements of T, with $s(\emptyset)$ defined to be 0. If T is chosen at random among all subsets of U, the probability that $s(T)$ is divisible by 3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

Problem 17 (HMMT February Team 2015/7). Let $f:[0,1]\to\mathbb{C}$ be a *nonconstant* complex-valued function on the real interval $[0,1]$. Prove that there exists $\varepsilon>0$ (possibly depending on f) such that for any polynomial P with complex coefficients, there exists a complex number z with $|z|\le 1$ such that $|f(|z|)-P(z)|\ge\varepsilon$.

Problem 18 (HMMT February Combinatorics 2012/10). Jacob starts with some complex number x_0 other than 0 or 1. He repeatedly flips a fair coin. If the nth flip lands heads, he lets $x_n=1-x_{n-1}$, and if it lands tails he gets $x_n=\frac{1}{x_{n-1}}$. Over all possible choices of x_0; what are all possible values of the probability that $x_{2012}=x_0$?

Problem 19 (Titu). For positive integers n, define

$$f(n) = \sum_{k=0}^{n-1} \cos^{2n}\left(\frac{k\pi}{n}\right).$$

Compute

$$\sum_{k=2}^{\infty} \frac{f(k)}{k \cdot 2^k}.$$

Problem 20 (PUMaC Live 2018/4.3). Let $0 \le a, b, c, d \le 10$. For how many ordered quadruples (a, b, c, d) is $ad - bc$ a multiple of 11?

Problem 21 (HMIC 2021/3). Let A be a set of $n \ge 2$ positive integers, and let $f(x) = \sum_{a \in A} x^a$. Prove that there exists a complex number z with $|z| = 1$ and $|f(z)| = \sqrt{n-2}$.

Problem 22 (Putnam 1974/B6). Let S be a set with 1000 elements. Find a, b, c, the number of subsets R of S such that $|R| \equiv 0, 1, 2 \pmod 3$, respectively. Find a, b, c if $|S| = 1001$ instead.

Problem 23 (MOP 1999). There are n points on a unit circle such that the product of the distances from any point on the unit circle to the given points is at most 2. Prove that the given n points must be vertices of a regular n-gon.

3 Incenter Miquel

Basic knowledge of inversion and projective geometry is assumed.

3.1 Notation

We will introduce some notation that we will use for the rest of the article. Although we would eventually be forced to use some of the point names for different purposes, we hope the purpose is clear from the context.

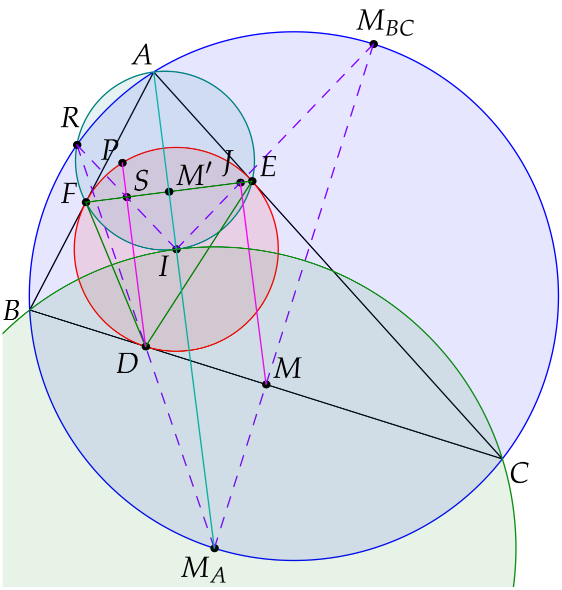
Notation

- We denote the circle passing through the n concyclic points $A_1, \ldots, A_n$ as $(A_1 \cdots A_n)$, the *circumcircle*. If there are only two points (AB), then it denotes the circle with diameter AB.
- For some $\triangle ABC$, let M_A denote the midpoint of arc BC not containing A in (ABC). Furthermore, let M_{BC} denote the midpoint of arc BAC in (ABC). Similarly define M_B, M_C, M_{CA}, M_{AB}.
- Let M denote the midpoint of side BC in $\triangle ABC$.
- Let O denote the *circumcenter* of $\triangle ABC$.
- Let I denote the *incenter* of $\triangle ABC$.
- Let $\triangle DEF$ be the *contact triangle* of $\triangle ABC$ (with the convention $D \in BC, E \in AC$, and $F \in AB$).
- Let A' denote the A-*antipode* in (ABC), and let D' denote the D-antipode in (DEF).
- Let $R = (AI) \cap (ABC)$ (the intersection of the two circles) distinct from A.
- Let $S \in EF$ (S lies on the line EF) satisfy $DS \perp EF$ (DS and EF are perpendicular).
- Let $P = DS \cap (DEF)$ distinct from D, and $Q = AP \cap (DEF)$ distinct from P.
- Let $T = AP \cap (ABC)$ distinct from A.

- Let $G = AR \cap BC$.
- Let $J \in EF$ satisfy $MJ \perp EF$. Furthermore, let M' be the midpoint of EF, and $X_1 = EF \cap BC$. Finally,let $Z = AR \cap EF$.
- Let the A-*mixtillinear incircle* touch AB and AC at K and L.

The diagram featured on the title page conveys all of these points.

3.2 Spiral Similarity at R



Consider the spiral similarity taking EF to BC. Note the similarity is centered at R by using the Miquel point of $EFBC$. Now, this implies it sends $(AFIE)$ to (ABC), so as I is the midpoint of arc EF, I goes to M_A. Similarly, A goes to M_{BC}.

Now, we can easily angle chase to get

$$\angle PFE = \angle PDE = \angle SDE = 90^\circ - \angle SED = 90^\circ - \angle FED = \frac{1}{2}\angle B = \angle IBC$$

and similarly with $\angle PEF$ gives that $\triangle PFE \sim \triangle IBC$. In particular, if the spiral similarity f centered at R maps $F \mapsto B$ and $E \mapsto C$, then

$$f(P) = I \qquad f(S) = D \qquad f(I) = M_A$$

However, we know that $SD \parallel IM_A$ (both are perpendicular to EF). Since R is the spiral center taking SI to DM_A, hence it is also the center of a spiral similarity g which takes SD to IM_A.So g is a homothety for the two mentioned lines. Inverting about the incircle, we note that we switch R and S, so R, S, I are collinear. Thus, we have the following fact:

Lemma 18
In the said configuration, R, D, M_A are collinear.

Proof. From our above discussion on g being a homothety and R, S, I being collinear, their images are also collinear, so R, D, M_A are collinear. □

In addition, we have the following claim:

Lemma 19
I, M_{BC}, J are all collinear.

Proof. We will use phantom points. Define $IM_{BC} \cap EF = J_\perp$. We note that

$$\frac{MM_{BC}}{MM_A} = \frac{AM'}{M'I} = \frac{IJ_\perp}{J_\perp M_{BC}}$$

where the first relation follows by the first spiral similarity mentioned and the second one follows by using Thales' theorem on $EF \parallel AM_{BC}$. Thus, we get that $MJ_\perp \parallel AI$, but as $AI \perp EF$, we have $MJ_\perp \perp EF$. Thus, $J = J_\perp$. □

The reader might want to take a look at the following exercises:

Exercise 20 (USAJMO 2014/6). Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and let E, F be the tangency points of γ with $\overline{CA}$ and $\overline{AB}$, respectively. Let U, V be the intersections of line EF with line MN and line MP, respectively, and let X be the midpoint of arc BAC of Γ.

(a) Prove that I lies on ray CV.

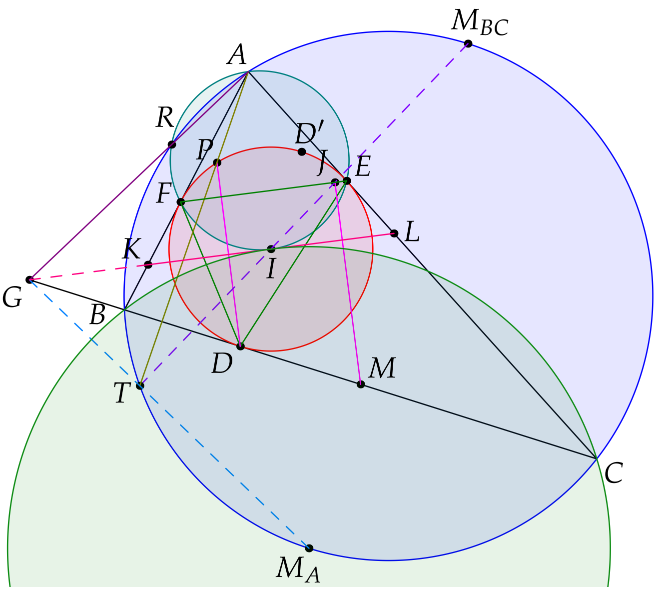
(b) Prove that line XI bisects $\overline{UV}$.

The above problem is incinerated by the Iran lemma and our mentioned claim.

Exercise 21 (EMMO Seniors 2016). In $\triangle ABC$, let point D be the tangency point of incircle ω with side BC. Analogously define E and F for sides AC and AB respectively. Let P be the feet of the perpendicular from D to EF. Let N be the midpoint of arc $\overset{\frown}{ABC}$ of circumcircle Γ of $\triangle ABC$. Prove that the lines AP and ND concur on Γ.

Exercise 22 (CMIMC Tiebreaker 2018/G3). Let ABC be a triangle with incircle ω and incenter I. The circle ω is tangent to BC, CA, and AB at D, E, and F respectively. Point P is the foot of the angle bisector from A to BC, and point Q is the foot of the altitude from D to EF. Suppose $AI = 7$, $IP = 5$, and $DQ = 4$. Compute the radius of ω.

3.3 The Mixtillinear Touchpoint



T wasn't named by accident: let's look at the following claim:

Lemma 23
T is the A-mixtillinear touchpoint.

Proof. Note that P and D' are reflections of each other in AI, as $PD' \parallel EF$. This means that AP and AD' are isogonal, so as AD' is the A-nagel cevian. This implies the mixtillinear touchpoint lies on AP, so T is indeed the said touchpoint. □

Remark 24. This means that T, I, J, M_{BC} are collinear.

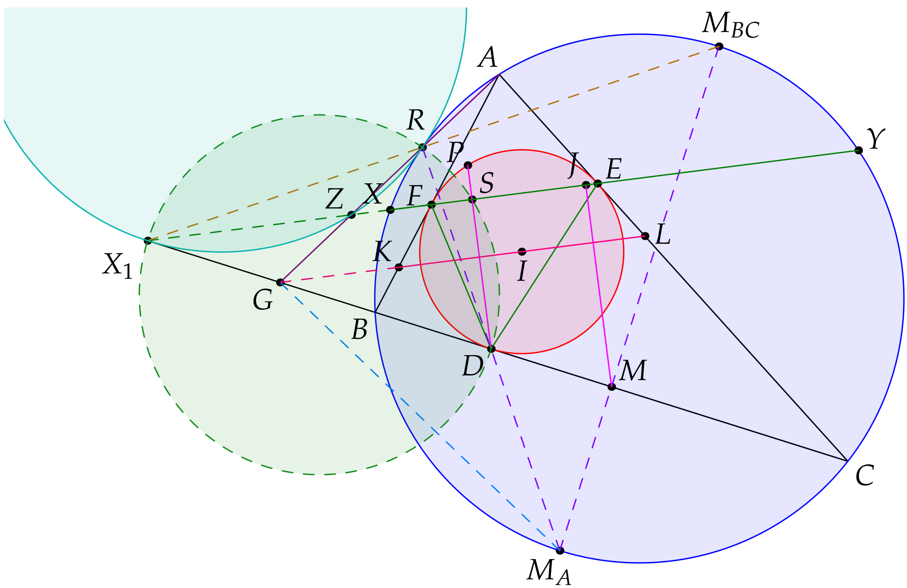
Lemma 25
Recall that $G := AR \cap BC$. Then

$$G \equiv RA \cap BC \cap KL \cap TM_A$$

Proof. By using radical axes on $(AI), (BIC), (ABC)$, we see that AR, KL, BC all concur. As $AR \cap BC = G$, we have that $G \in KL$. Similarly, using radical axis on $(IM_A), (BIC)$, and (ABC), we see that BC, KL, TM_A concur. As $G = BC \cap KL$, we have that $G \in TM_A$, which completes the proof. □

> ***Remark 26.*** Note that $(GRIDT)$ is cyclic.

3.4 More Interesting Properties



Now, let's talk about X_1. We have the following concurrencies and concyclities:

Lemma 27
X_1, R, M_{BC} are collinear and X_1, R, S, D are concyclic.

Proof. Notice that (SX_1D) is the S-Apollonian circle of $\triangle SBC$, so taking a spiral similarity of R gives

$$\frac{RB}{RC} = \frac{FB}{EC} = \frac{BS}{SC}$$

which implies that R is also on the S-Appolonius incircle. Thus, X_1, R, S, D are concyclic. In addition, DX_1 is the diameter, so we have $DR \perp RX_1$, As $DR = DM_A \perp RM_{BC}$, we see that X_1, R, M_{BC} are collinear. □

Now, let's turn to Z. We have the following claim:

Lemma 28
$\triangle RDX_1$, $\triangle RIG$, and $\triangle RSZ$ are similar.

Proof. Notice that $\measuredangle RIG = \measuredangle RDG = \measuredangle RDX_1$ and from our above discussion, $\measuredangle X_1RD = \measuredangle GRI = \frac{\pi}{2}$. Thus the first two triangles are similar. Furthermore, $SZ = EF \parallel IG$, which means that the last two triangles are similar. $\square$

We have the following claim:

Lemma 29
GI is tangent to (X_1GR).

Proof. Note that $\measuredangle IGR = \measuredangle GX_1R$. $\square$

We proceed with our next claim:

Lemma 30
Let f is the spiral similarity at R discussed before. Then

$$f(Z) = X_1.$$

Proof. To see this, just note that $Z \in EF$. Thus, as $f(EF) = BC$, we see that $f(Z) \in BC$. However, we also have that $\measuredangle X_1RZ = \measuredangle IRD$, so as $\measuredangle IRD$ is the angle of the spiral similarity, $f(Z) = X_1$. $\square$

Let's see another tangent line:

Lemma 31
BC is tangent to (RZX_1).

Proof. Note that $\measuredangle(EF, BC) = \measuredangle(X_1R, RZ)$, which immediately implies the tangency at $X_1 = EF \cap BC$. $\square$

To wrap this section up, we complete it with this final claim:

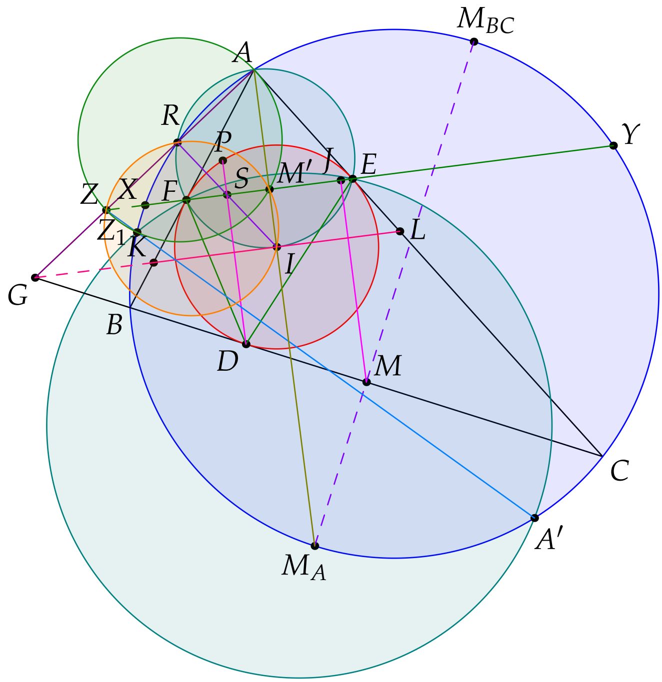
Lemma 32
If K_1 be the midpoint of DX_1, then $ZK_1 \perp OI$.

Proof. Note that $ZR \cdot ZA = ZE \cdot ZF$ imples Z lies on the radical axis of (ABC) and (DEF). Also by midpoint of Harmonic Bundle Lemma, we have

$$K_1D^2 = K_1B \cdot K_1C \implies K_1 \text{ lies on the radical axis of } (ABC) \text{ and } (DEF)$$

which implies the perpendicularity. $\square$

3.5 Involving the A-Antipode



This configuration is heavily based on ELMO SL 2019/G3. Let $ZA' \cap (ABC) = Z_1$.

The main claim is the following:

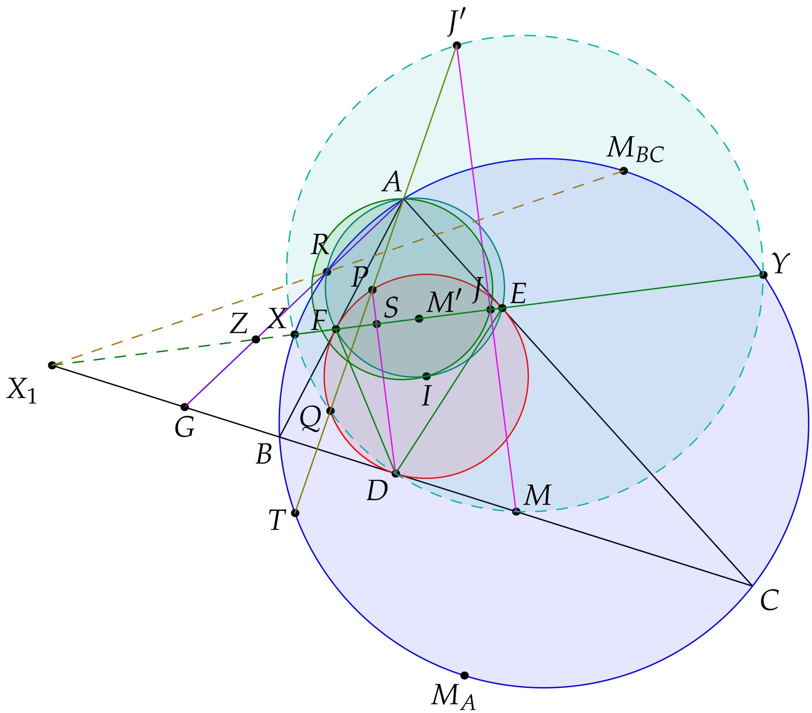
Lemma 33
S lies on the radical axis of (AZZ_1) and $(A'EF)$.

Proof. Note that $Z_1 \in (A'EF)$, $ZRM'I$ is cyclic, $ZZ_1M'A$ is cyclic, and S is the orthocenter of $\triangle AZI$. Thus, $ZS \cdot SM' = IS \cdot SR = SE \cdot SF$, because the reflection of S over M' lies on (AI) and the inversion about the incircle sends S to R and A to M', so PoP on (AI) yields the desired. $\square$

Exercise 34 (ELMO SL 2019/G3). Let $\triangle ABC$ be an acute triangle with incenter I and circumcenter O. The incircle touches sides BC, CA, and AB at D, E, and F respectively, and A' is the reflection of A over O. The circumcircles of ABC and $A'EF$ meet at G, and the circumcircles of AMG and $A'EF$ meet at a point $H \neq G$, where M is the midpoint of EF. Prove that if GH and EF meet at T, then $DT \perp EF$.

Exercise 35 (ELMO 2010/6). Let ABC be a triangle with circumcircle ω, incenter I, and A-excenter I_A. Let the incircle and the A-excircle hit BC at D and E, respectively, and let M be the midpoint of arc BC without A. Consider the circle tangent to BC at D and arc BAC at T. If TI intersects ω again at S, prove that SI_A and ME meet on ω.

3.6 A very interesting concyclicity



We present the following beautiful lemma:

Lemma 36
If $EF \cap (ABC) = \{X, Y\}$, then we have $(XYQDM)$ is concyclic.

Proof. We first prove that $M \in (DXY)$. Note that $RM_{BC}DM$ is cyclic by the shooting lemma. Note that

$$X_1D \cdot X_1M = X_1R \cdot X_1M_{BC} = X_1X \cdot X_1Y$$

Showing that $Q \in (DXY)$ is a bit more trickier:

Claim —

$$Q \in (DXY)$$

We will introduce two very special points which are very closely related to the standard orthocenter configuration, namely Queue and Ex Points. Surprisingly, the will aid a lot of our proofs that follow in this article. We wil now define them.

Remark 37 (The A-Queue point of $\triangle ABC$). If we let B', C' to be the feet of altitudes from B, C onto CA, AB, with $BB' \cap CC' \equiv H$, then the A-Queue point of $\triangle ABC$, Q_A is defined as $Q_A \equiv (AB'HC'') \cap (ABC)$.

Remark 38 (The A-Ex point of $\triangle ABC$). If we let B', C' to be the feet of altitudes from B, C onto CA, AB, with $BB' \cap CC' \equiv H$, then the A-Ex point of $\triangle ABC$, Q_A is defined as $X_A \equiv B'C' \cap BC$.

Remark 39. Q_A and X_A are inverses of each other in the circle $(A, \sqrt{AB' \cdot AC})$

If the reader is interested, they should have a look at The Ex-points and the Queue-points Part One and Part Two for more interesting properties.

Approach 1; Using Orthic Axis. Note that $-1 = (SQ : EF) \implies Q$ is the D-Queue point of triangle DEF. Let $DQ \cap EF = X_D$.

Then it is well known that X_D is the D-Ex point in $\triangle DEF$. So X_D lies on the orthic axis of $\triangle DEF$, which is the radical axis of (DEF) and the circumcircle of its tangential triangle, that is (ABC). Thus we have

$$X_D X \cdot X_D Y = X_D E \cdot X_D F = X_D Q \cdot X_D D$$

This implies $Q \in (DXY)$. $\square$

Remark 40. Radical axes on $(ARSM')$, (AI),and $(DQSM')$ give that $Z = X_D$ (!) So Z is the D-Ex point of $\triangle DEF$

Approach 2; Coaxility Lemma. We use phantom points. Let $(DXY) \cap (DEF) = Q' \neq D$. First note that since OI is the Euler Line of $\triangle DEF$, (say by an incircle inversion).This implies that (ABC), (DEF),and the nine point circle of $\triangle DEF$ are coaxal. Thus, by the forgotten lemma of coaxality, we have

$$\frac{\operatorname{Pow}_{(DEF)} M'}{\operatorname{Pow}_{(ABC)} M'} = \frac{M'F \cdot M'E}{M'X \cdot M'Y} = \frac{SF \cdot SE}{SX \cdot SY} = \frac{\operatorname{Pow}_{(DEF)} S}{\operatorname{Pow}_{(ABC)} S}$$

Now, we also have

$$\frac{M'F \cdot M'E}{M'X \cdot M'Y} = \frac{\operatorname{Pow}_{(DEF)} M'}{\operatorname{Pow}_{(DQ'XY)} M'}$$

and similarly for S, implying that $(DSM'), (DQ'XY)$, and (DEF) are coaxal. Thus, $Q' \in (DSM') \equiv (DM')$. We present two ways to finish from the second approach:

First Finish. Note that Q' becomes the D-Queue point of $\triangle DEF$. So $Q = Q'$ (!) $\square$

Second Finish. We want AD' and AQ' are isogonal. In other words, if $K = AM_{BC} \cap M'Q'$, then we need $-1 = (KM'; Q'D')$. This is again equivalent to proving that $(M'K)$ is orthogonal to the incircle.

To do this note that $A \in (M'K)$, so since we have $IM' \cdot IA = r^2$ (where r is the radius of the incircle), we are done! $\square$

$\square$

$\square$

Having done this, lets take a look at some contest problems destroyed by the lemma. We will suitably change the notation in the problems presented to suit the notation used so far in the article.

Example 41 (Taiwan TST 2019/2/2)
Prove that if MJ hits (DXY) at J', then $J' \in AT$.

Solution. Angle chase:

$$\measuredangle MJ'Q = \measuredangle MDQ = \measuredangle BDQ = \measuredangle DPQ.$$

$\square$

Example 42 (MR O451)
Recall that $J = IM_{BC} \cap EF$. Prove that the radical axis of (AIJ) and (DXY) passes through T.

Solution. Note that we have $\measuredangle QIT = \measuredangle BDQ = \measuredangle QJ'J$. This implies $QJ'JI$ is cyclic. So T lies on the radical axis of (AIJ) and (DXY). $\square$

Example 43
Prove that the line IM_{BC} is the perpendicular bisector of QD.

Solution. This follows from the fact that TA and TD are isogonal in $\measuredangle BTC$ and $IQ = ID$. $\square$

These exercises will be left for the reader.

Exercise 44. Prove that AS,BC, IM_{BC} concur

Exercise 45. Prove that AS, DM_{BC} concur on (ABC).

Exercise 46. Prove that the midpoint of M_BM_C is the center of $(XQDMY)$.

From now on, we will call $AS \cap BC = U$ and $AS \cap (ABC) \cap V \neq A$.

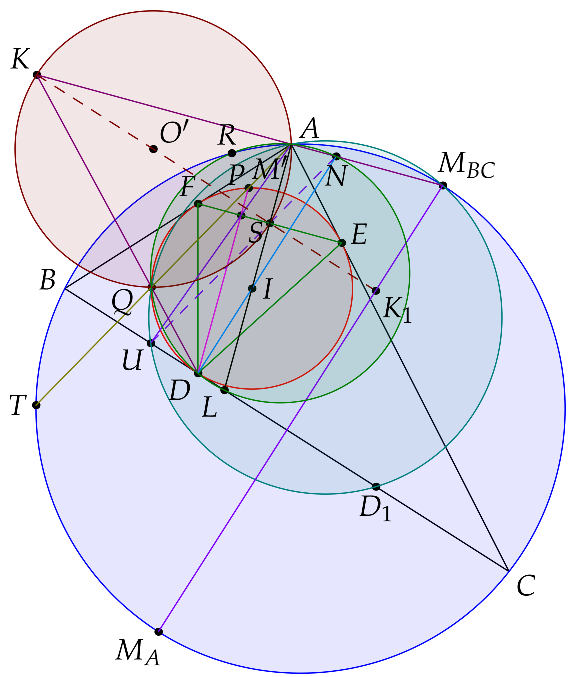
Exercise 47. Prove that $RBVC$ is a harmonic quadrilateral.

3.7 Some Nice Contest Examples

3.7.1 A problem related to IMO 2019/6

Example 48
Let ABC be a triangle with $AC > AB$. Let $N = DI \cap AM_{BC}$. D_1 is the BC-extouch point. Suppose that $U' \equiv (AM_{BC}D_1) \cap BC$. Show that U', M', and N are collinear.

Solution. Consider the following figure:



In the notation, you can probably guess the first claim:

Claim —

$$U \equiv U'$$

Proof. Note that

$$\measuredangle UM_{BC}A = \measuredangle TM_{BC}A = \measuredangle TCA = \pi - \measuredangle B - \measuredangle CAT$$

and

$$\measuredangle UD_1A = \measuredangle BD_1A = \pi - \measuredangle B - \measuredangle BAD_1$$

Now, we note AD_1, AT are isogonal in $\measuredangle BAC$, so $\measuredangle UD_1A = \measuredangle UM_{BC}A$, so indeed $U' \equiv U$. □

All poles and polars are taken with respect to (DEF). Now note that, by La Hire, $U, M'N$ are collinear if and only if their polars are concurrent.

It is easy to note that the polar of M' is AM_{BC}. By a previous example, we have that the polar of U is DQ. Also the polar of N is just the line through M′ perpendicular to DI. (Remark that by La hire M' lies on the polar of N).

Let $K = DQ \cap AM_{BC}$. Note that $KAM'Q$ is cyclic with diameter (KM').We need to prove that $KM' \perp DI$. Note that $M'Q$ and IT are parallel, so we have

$$\angle AKM' = \angle AQM' = \angle ATI = \angle AM_AM_{BC}$$

So if KM' cuts M_AM_{BC} at some point K_1, then KAK_1M_A is cyclic, so KM' is perpendicular M_AM_{BC} which gives $KM' \perp DI$. □

Now we present some easy facts for the reader to ponder on. Let O' denote the midpoint of KM'

Exercise 49. Prove that O', Q, U are collinear

Exercise 50. Prove that $(M'K)$ is orthogonal to (DEF)

The point N is famous as the IMO 2019/6 concurrency point. We present an auxiliary lemma which is helpful in solving the IMO problem itself.

Lemma 51
If $L = AI \cap BC$, then $(ARQDLN)$ is a cyclic hexagon

Proof. Note that $NADL$ is cyclic due to obvious reasons. By Shooting Lemma,

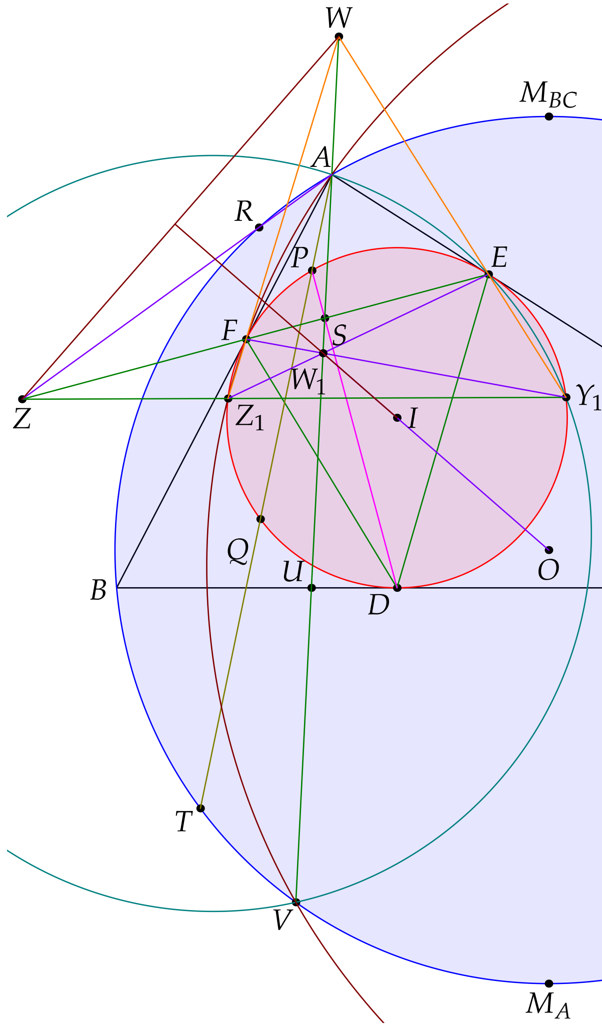
$$M_AL \cdot M_AA = M_AD \cdot M_AR \implies ARDLN \text{ is cyclic}$$

To finish, note that $\measuredangle AQD = \measuredangle PQD = \measuredangle RDB = \measuredangle ALD$. □

3.7.2 A beautiful Isogonal Mittenpunkt problem

Example 52 (Fake USAMO 2020/3)
Let $(AVE) \cap (DEF) = \{E, Y_1\}$ and $(AVF) \cap (DEF) = \{F, Z_1\}$. Prove that FY_1, EZ_1, OI are concurrent.

Solution. We present a motivated and rather "easy-to-come-up-with" solution for this very beautiful and difficult problem. However, given the artillery we've developed so far, this is going to be a walk in a park!



The presence of so many circles motivates looking at radical axes. Indeed,note that by

Radical axes on (AVE), (AVF) and (DEF), we have that FZ_1, EY_1 and AV concur at a point, say W.

Since FZ_1Y_1E seems to be our chief quadrilateral, we will introduce $W_1 = FY_1 \cap EZ_1$ and $Z' = Y_1Z_1 \cap EF$. The reader might have already guessed what's going on. Since EFZ_1Y_1 is a cyclic quadrlateral, we have $IW_1 \perp Z'W$. So we need to prove that $OI \perp Z'W$. However note that because we have $WY_1 \cdot WE = WV \cdot WA$, this implies that W lies on the radical axis of (ABC) and (DEF).

Hence,we realize that Z' must also lie on this radical axis. But since $Z' \in EF$, so Z' must actually be Z. So we now seek to prove that $Z' = Z$. It turns out that proving $Z' = Z$, or equivalently Z', R, A are collinear, is just a matter of cross ratio chasing. All poles and polars are taken with respect to the incircle.

Remark that we have, by definition,

$$-1 = (A, I; E, F) \overset{R}{=} (Z, S; E, F)$$

By La Hire, we also know that A lies on the polar of Z'. So the polar of Z' is just AV. So S lies on the polar of Z'. Hence,we may write

$$-1 = (Z', S; E, F)$$

completing the problem. □

We now leave the following exercise to the reader, which justifies the title of this problem.

Exercise 53. Prove that W_1 is the Isogonal Mittenpunkt Point (X_{57}) of $\triangle ABC$.

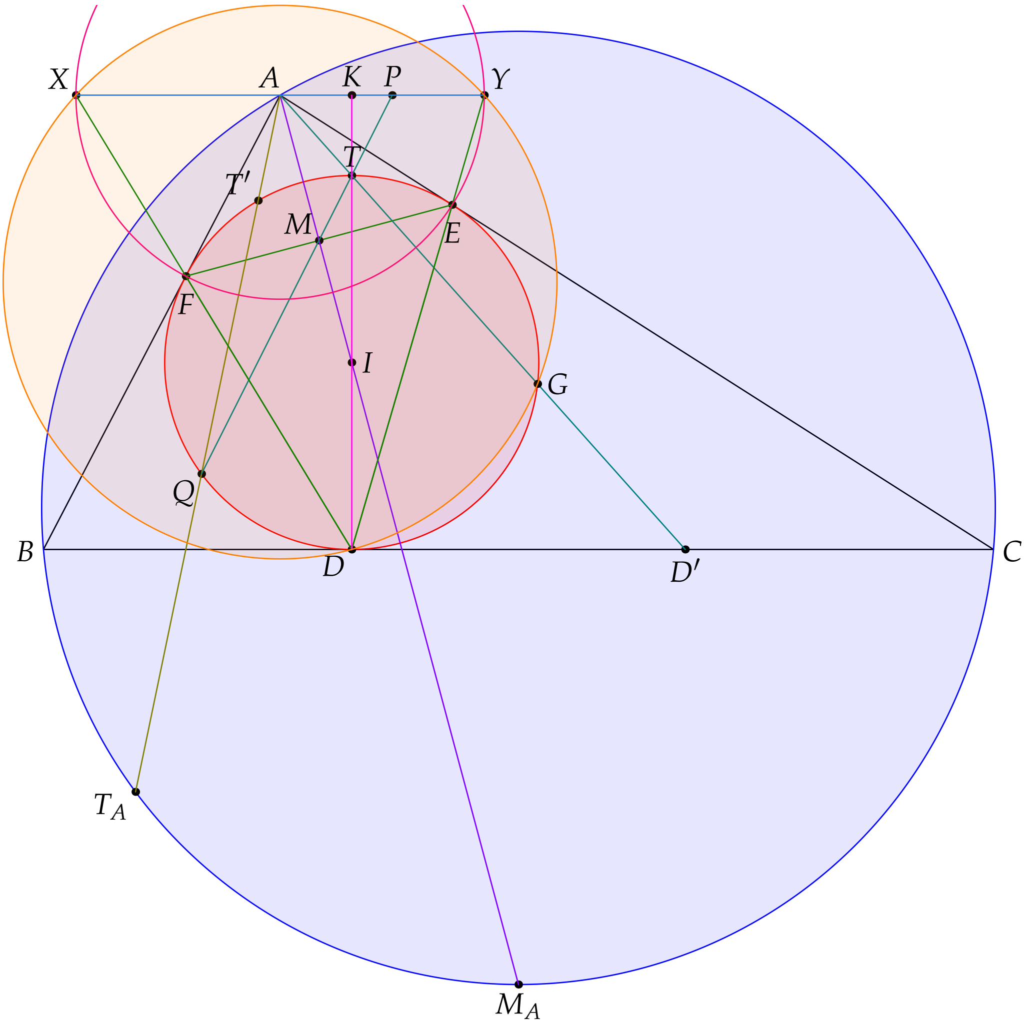
3.8 Changing an incenter problem to an orthocenter one

It turns out that since Q is the D-queue point in $\triangle DEF$, so it might be useful to rephrase the problem in terms of the the contact triangle. In some rather nice cases (like the problem about to be presented), it turns out that we can completely convert the problem into an orthocenter configuration, which might be easier to handle. So now we present a really beautiful problem.

We will stick to the original wording of the problem this time:

Example 54 (CAMO 2020/3)
Let ABC be a triangle with incircle ω, and let ω touch $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Point M is the midpoint of $\overline{EF}$, and T is the point on ω such that $\overline{DT}$ is a diameter. Line MT meets the line through A parallel to $\overline{BC}$ at P and ω again at Q. Lines DF and DE intersect line AP at X and Y respectively. Prove that the circumcircles of $\triangle APQ$ and $\triangle DXY$ are tangent.

Solution. Let T' be the reflection of T in AI. Let $AT \cap (DEF) = G$. It is obvious that G, Q are reflections of each other about AI.



We claim that G is the desired concurrency point We will prove a series of claims:

Claim — $EFXY$ is cyclic with center A.

Proof. Note that we have

$$\measuredangle FED = \measuredangle BDF = \measuredangle FXY$$

This implies that $EFXY$ is cyclic. Next, note that we also have

$$\measuredangle AXD = \measuredangle FDB = \pi - \frac{B}{2}$$

We also have

$$\measuredangle AFX = \measuredangle XFA = \measuredangle BFD = \pi - \frac{B}{2}$$

Similarly, we can prove that $AX = AE = AF = AY$. The claim is hence proved. $\square$

Claim — T is the orthocenter of $\triangle DXY$

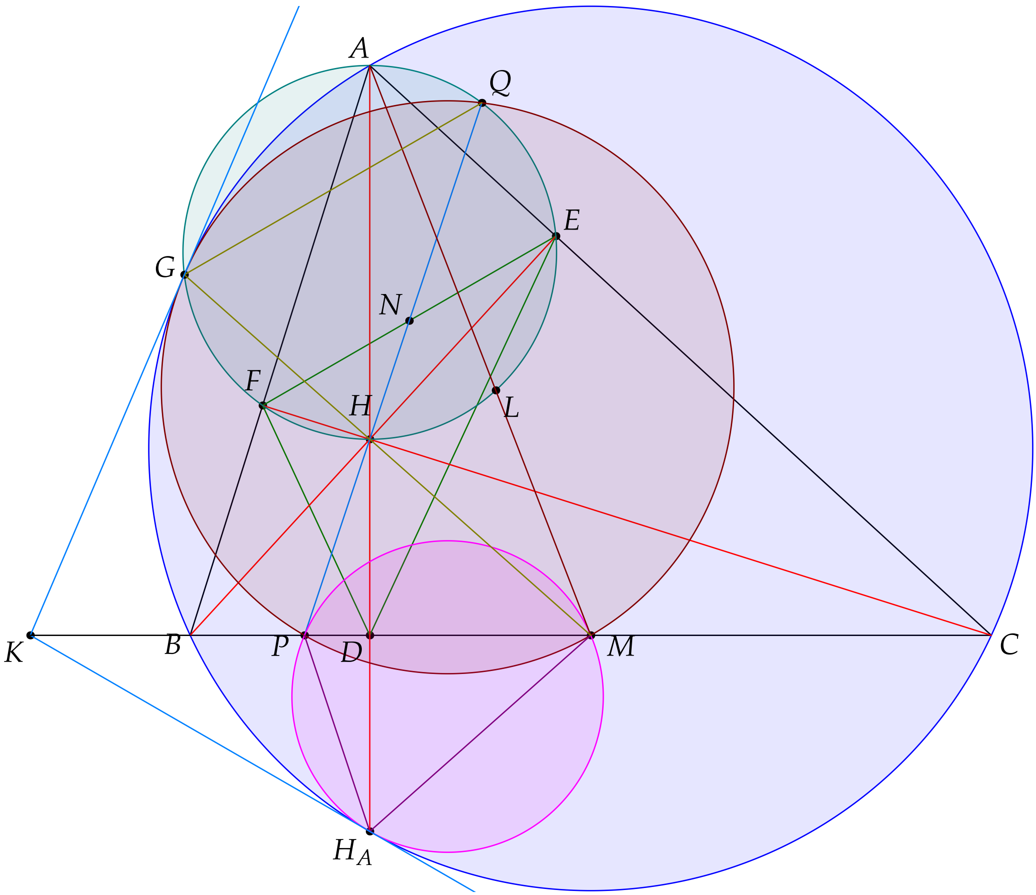
Proof. Note that we have $\measuredangle XED = \measuredangle TED = \frac{\pi}{2}$ This implies that X,T,E are collinear.Similarly Y,T,F are collinear. $\square$

Claim — $G \in (DXY)$

Proof. Let D^* denote the D-antipode in (DXY). We have $\measuredangle DGT = \measuredangle DGA = \frac{\pi}{2}$. But since A, T, D^* are collinear, we have that $\measuredangle DGD^* = \frac{\pi}{2}$. Thus, $G \in (DXY)$. Hence, the claim is proven. $\square$

Now note that G is the D-Queue point in $\triangle DXY$. So we completely rephrase the problem in terms of $\triangle DXY$ as follows.

Restated in terms of $\triangle DXY$ Let $\triangle ABC$ be a triangle with orthic triangle as $\triangle DEF$. Let G and Q be the D-Queue points of $\triangle ABC$ and $\triangle AEF$. Let I denote the midpoint of AH. Let $QH \cap BC = P$. Prove that (MPQ) and (ABC) are tangent to each other at G, where M is the midpoint of BC.



Let M and N denote the midpoints of BC and EF respectively. Note that H, N and Q are collinear.

First note that since $AQDP$ and $AGDM$ are cyclic quadrilaterals, hence, we can use a simple Power of Point trick.

We have

$$PH \cdot HQ = AH \cdot HD = HG \cdot HM$$

so $MPQG$ is cyclic. Remark that we had proved earlier that G, Q are reflections of each other in MI. In particular, we have $EF \parallel GQ$. Let the tangent to G at (ABC) hit BC at K. Let H_A be the reflection of H in BC.

Since we have $-1 = (GH_A; BC)$, hence KH_A is tangent to (ABC) at H_A. Further, since (BHC) and (ABC) are reflections of each other about BC, hence KH is tangent to(ABC). Hence KH is anti-parallel to BC and hence parallel to EF.

Now to finish off, we will proceed by radical axes. We want that KH_A is tangent to (MPH_A). Thus, we we basically need $\measuredangle KH_AP = \measuredangle BMH_A$.

We will just do an angle chase. Note that G is the Miquel point of $BFEC$ and the spiral similarity centered at G that sends $FE \mapsto BC$ also sends $N \mapsto M$.

So we have

$$\measuredangle KH_AP = \measuredangle KHP = \measuredangle FNH = \measuredangle GQH = \measuredangle GQN = \measuredangle NGQ = \measuredangle GNF = \measuredangle GMB$$

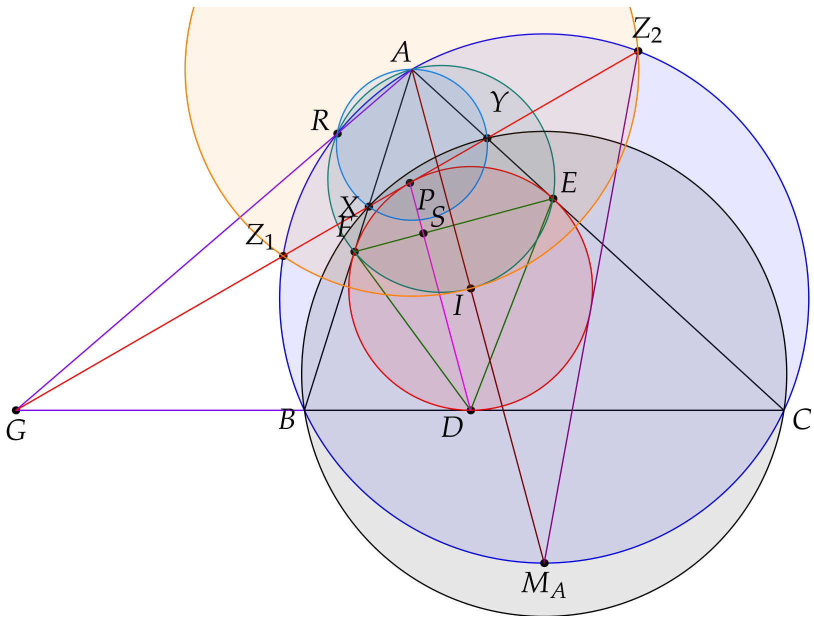
$$\measuredangle GMB = \measuredangle HMK = \measuredangle KMH_A = \measuredangle BMH_A$$

so by radical axes on (ABC), (PMH_A), (MPQ), we win! □

3.9 An induced bicentric quadrilateral and its properties

We start off this section by introducing some new points.

Define $Y = AC \cap BS$ and $X = AB \cap CS$. We investigate various properties of the points X and Y.



Lemma 55
$XYBC$ has an incircle.

Proof. Note that we have

$$-1 = (\overline{SY} \cap (I), \overline{SB} \cap (I); F, D)$$

Projecting this through S onto $(I) \implies PY$ is tangent to (I). Similarly, we can show for X and conclude, XY is tangent to (I) at P. □

Lemma 56
$XYBC$ has a circumcircle.

Proof.

$$\angle BXY = 180^\circ - \angle FIP = 180^\circ - 2\angle FDS = 2\angle DFE = \angle DIE = 180^\circ - \angle YCB$$

□

So yes, the title of the section does make sense. $XYCB$ is our very own induced bicentric quadrilateral.

Lemma 57
$RAYX$ is a cyclic quadrilateral.

Proof using phantom points. Redefine: $(AXY) \cap (ABC) = R'$ and $XY \cap BC = G'$. By Radical Axes, $R' \in AG'$. Notice R' is the Miquel Point of $BXYC \implies R', S, I$ are collinear and $\overline{R'SI} \perp AG'$. This is enough to imply that $R' = R$ and $G' = G$. □

Now we have a look at a few examples:

Example 58 (HMMT November Team 2016/10)
Let R_1 denote the point on (AXY) such that $AR_1 \perp BC$. Prove that S, I, R_1 are collinear.

Solution. Remark that since XY antiparallel to $BC \implies A$-antipode in (AXY) is R_1. The rest is a trivial angle chase. Details are left to the reader. □

Example 59 (SORY 2019/6, Aditya Khurmi)
Prove that if the tangents from M_A to (DEF) meet (ABC) at Z_1, Z_2, then Z_1, X, Y, Z_2 are collinear. Z_1 lies closer to B than C.

Solution. We will use a small fact that inversion with center A and radius AI swaps the A-mixtilinear incircle and the incircle of $\triangle ABC$. We will also use another small claim, which is indeed trivial. Yet we will state it.

Claim (Rephrased Fact 5) — In any triangle $\triangle XYZ$ with K as the midpoint of arc YZ not containing X in (XYZ), the incenter I of the triangle is the intersection of XK and (KYZ) which lies closer to X.

This is just a trivial application of Fact 5.

We now show that, if (A, AI) meets the circumcircle of $\triangle ABC$ at Z_1', Z_2', then $Z_1'Z_2'$ is tangent to the incircle at S. This combined with the "Claim" implies the conclusion, as then $Z_1'Z_2'$ is just YX and MZ_1', MZ_2' are tangents to the incircle. This is true as A is midpoint of arc $Z_1'Z_2'$ As I is the incenter of $\triangle MZ_1'Z_2'$, therefore $\triangle MZ_1'Z_2'$ and $\triangle ABC$ would share the same incircle.

Perform an inversion with center A and radius AI. It swaps the A-mixtilinear incircle with the incircle. So, P and T are swapped. Now the conclusion trivially follows as (ABC) (image of $\overline{Z_1'Z_2'}$) is tangent to the A-mixtilinear incircle (image of the incircle) at T (image of S). The problem is thus solved. □

The reader might want to have a look at the following exercises:

Exercise 60 (OMO Spring 2020/15). Let ABC be a triangle with $AB = 20$ and $AC = 22$. Suppose its incircle touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at D, E, and F respectively, and P is the foot of the perpendicular from D to $\overline{EF}$. If $\angle BPC = 90^\circ$, then compute BC^2.

Exercise 61. Prove that (AXY) is tangent to the A-mixtilinear incircle at a point on AD.

Exercise 62 (Due to math-pi-rate). Let tangents from M_A to (I) hit (I) at U_1 and V_1.Prove that $\triangle PU_1V_1$ and $\triangle DEF$ share the same nine point circle.

3.10 Problems

We finally present a bunch of practice problems for the reader. Try invoking the machinery we have developed throughout the article. The problems are quite nice and we encourage the reader to try them. Enjoy!

Remark 63. A few of the following problems might require knowledge the "Iran incenter lemma." If the reader is unfamiliar with the lemma, then the article here is a fantastic one to learn it.

Problem 24 (CJMO 2019/3). Let I be the incenter of $\triangle ABC$, and M be the midpoint of $\overline{BC}$. Let Ω be the nine-point circle of $\triangle BIC$. Suppose that $\overline{BC}$ intersects Ω at a point $D \neq M$. If Y is the intersection of $\overline{BC}$ and the A-intouch chord, and X is the projection of Y onto $\overline{AM}$, prove that X lies on Ω, and the intersection of the tangents to Ω at D and X lies on the A-intouch chord of $\triangle ABC$.

Note. The nine-point circle of $\triangle ABC$ is the circumcircle of its medial triangle, and if the incircle touches $\overline{AC}$ and $\overline{AB}$ at E and F, respectively, then $\overline{EF}$ is the A-intouch chord.

Problem 25 (GGG 2.3, Andrew Wu). Acute, scalene $\triangle ABC$ is given with circumcircle Γ and intouch triangle DEF; D, E, F lie on $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. L is the midpoint of arc $\overset{\frown}{BAC}$. G is the intersection of $\overleftrightarrow{EF}$ and $\overleftrightarrow{BC}$; T lies on $\overline{EF}$ with $\overline{DT} \perp \overline{EF}$. P lies on $\overleftrightarrow{LG}$ with $\overleftrightarrow{PT} \parallel \overleftrightarrow{BC}$. K lies on Γ such that $\overline{KT}$ bisects $\angle EKF$; K and A are not on the same side of $\overleftrightarrow{EF}$. If A' is the antipode of A on Γ, then show that $\overleftrightarrow{A'K}$, the line passing through T

and the midpoint of $\overline{DG}$, and the perpendicular from P to $\overleftrightarrow{EF}$ concur.

Problem 26 (GGG 2.6, Andrew Wu). Let ABC be an acute, scalene triangle with incenter I and circumcircle Γ; A' is the antipode of A on Γ, and L is the midpoint of arc $\widehat{BAC}$. The incircle of $\triangle ABC$ meets $\overline{BC}$ at D. Suppose that $\overleftrightarrow{A'I}$ and $\overleftrightarrow{BC}$ meet at P, and that $\overleftrightarrow{AP}$ and $\overleftrightarrow{DI}$ meet at Q; $X \neq L$ is the intersection of $\overleftrightarrow{LQ}$ and Γ. Show that the circumcircles of $\triangle XDQ$ and $\triangle IDP$ are orthogonal.

Problem 27 (GGG 3.3, Andrew Wu). Let ABC be a scalene triangle with incenter I and incircle ω. Let K be a point lying on the circumcircle of triangle ABC such that $\angle AKI = 90^\circ$, and suppose that ω meets $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F, respectively. Suppose that $\overleftrightarrow{KD}$ meets ω at points D and X. Show that the line joining the circumcenters of triangles BFX and CEX is parallel to $\overline{BC}$.

Problem 28 (GGG 3.6, Andrew Wu). Let ABC be a scalene triangle with incenter I and circumcircle Ω; L is the midpoint of arc BAC and A' is diametrically opposite A on Ω. D is the foot of the perpendicular from I to $\overline{BC}$. $\overleftrightarrow{LI}$ meets $\overleftrightarrow{BC}$ and Ω at X and Y, respectively, and $\overleftrightarrow{LD}$ meets Ω again at Z. $\overleftrightarrow{XZ}$ meets $\overleftrightarrow{A'I}$ at T. $\overleftrightarrow{DI}$ meets the circle with diameter $\overline{AI}$ again at P. Show that the second intersection between $\overleftrightarrow{PT}$ and the circle with diameter $\overline{AI}$ lies on $\overline{AY}$.

Problem 29 (Romania JBMO TST 2019/3). Let ABC a triangle, I the incenter, D the contact point of the incircle with the side BC and E the foot of the bisector of the angle A. If M is the midpoint of the arc BC which contains the point A of the circumcircle of the triangle ABC and $\{F\} = DI \cap AM$, prove that MI passes through the midpoint of EF.

Problem 30 (Iran TST 2008/10). In the triangle ABC, $\angle B$ is greater than $\angle C$. T is the midpoint of the arc BAC from the circumcircle of ABC and I is the incenter of ABC. E is a point such that $\angle AEI = 90^\circ$ and $AE \parallel BC$. TE intersects the circumcircle of ABC for the second time in P. If $\angle B = \angle IPB$, find the angle $\angle A$.

Problem 31 (Sharygin CR 2020/21). The diagonals of bicentric quadrilateral $ABCD$ meet at point L. Given are three segments equal to AL, BL, CL. Restore the quadrilateral using a compass and a ruler.

Problem 32 (trumpeter's PoTD Forum 1/10/16). $\triangle ABC$ has its incircle ω tangent to BC, CA, AB at D, E, F respectively. Let ℓ be the line through A parallel to BC, and let DE, DF intersect ℓ at M, N respectively. Let P be the intersection of NE with MF, and let Γ be the circumcircle of $\triangle NPM$; finally, let AD intersect Γ at a point X such that X, P are not on the same side of ℓ. Prove that P, X are antipodal points with respect to Γ.

Problem 33 (tworigami). Let ABC be a triangle with incircle centered at I and tangent to BC, AC, AB at D, E, F respectively. Let K denote the foot of the altitude from D to EF, and let Ω denote the circumcircle of $\triangle BIC$. Points P and Q are chosen on AC and AB respectively so that $\angle AIP = \angle AIQ = 90°$. Let the circumcircle of $\triangle KDP$ intersect Ω at points B_1 and B_2 and define points C_1 and C_2 similarly. Furthermore, let the circle with diameter DK intersect Ω at points A_1 and A_2. Prove that B_1B_2, C_1C_2, A_1A_2 concur on DK.

Problem 34 (buratinogigle). Let ABC be a triangle, incircle touches BC, CA, AB at D, E, F respectively. Prove that orthocenter of triangle DEF is radical center of circles (A, AD), (B, BE), (C, CF).

Problem 35 (POGCHAMP 2020/3). Scalene triangle ABC has incenter I, incircle ω, and circumcircle Ω. Let ω touch $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F respectively. Let P_A be the point on Ω such that $AP_A \perp P_AI$, and let Q_A be the point on ω such that $AI \parallel DQ_A$. Lines AD and P_AQ_A meet at S_A. Points P_B, P_C, Q_B, Q_C, S_B, and S_C are defined analogously. Prove that

$$\frac{DS_A}{AS_A} + \frac{ES_B}{BS_B} + \frac{FS_C}{CS_C} < \frac{9}{2}.$$

Problem 36 (POGCHAMP 2020/4). Let $\triangle ABC$ have incircle ω, and suppose ω touches sides BC, CA, and AB at D, E, and F, respectively. P is the point on ω such that $DP \perp EF$, and let O_B and O_C be the circumcenters of $\triangle BFP$ and $\triangle CEP$, respectively. Prove that $O_BO_C = \frac{1}{2}BC$.

Problem 37. Given a acute, scalene $\triangle ABC$ with incenter I and contact triangle $\triangle DEF$ $R \in (DEF)$ such that $DR \perp EF$. Let $DR \cap EF = P$. Let (Y) and (Z) denote the circumcircles of (BPF) and (CPE).Let U, V denote the points of intersection of EF with (Y) and (Z) respectively, Finally, let EF intersect BI, CI at K, L respectively. Prove that

$$\frac{FL}{FU} = \frac{EV}{EK}.$$

Furthermore, if $AD \cap (I) = X$ and XF, XE intersects (Y) and (Z) at Y', Z' respectively, then prove that $Y'Z' \parallel EF$.

Remark 64. An interested reader should check out *On a rich configuration related to tangent circles in a triangle* by Navneel Singhal.

4 Multiplicative Functions

Basic knowledge of some number theory and sum and product notations is assumed.

4.1 Dictionary

To motivate this handout, here are a few common multiplicative functions, some more trivial than others.

- The ***identity function*** id on the naturals maps each positive integer to itself.
- The ***unit function*** e on the naturals is defined as

$$e(n) = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}.$$

- The ***constant function*** $\mathbf{1}$ mapping each natural to 1.
- The ***divisor function*** $\tau(n)$ on the naturals counts the number of divisors of a given positive integer.

  If the integer n can be written as $\prod_i p_i^{e_i}$ for distinct primes p_i and exponents e_i, then $\tau(n) = \prod_i (e_i + 1)$ by constructive counting on the individual primes.

- The ***sum of divisors function*** $\sigma(n)$ on the naturals taking each positive integer to the sum of its positive integral divisors including itself.

  Similar to the divisor function, if the integer n can be written as $\prod_i p_i^{e_i}$ for distinct primes p_i and exponents e_i, then $\sigma(n) = \prod_i (p_i^{e_i} + p_i^{e_i - 1} + \cdots + 1)$ by considering all possible exponents of p_i in each divisor.

- The ***Euler phi function*** $\phi(n)$ on the naturals taking each integer n to the number of integers less than n and relatively prime to it.

  If the integer n can be written as $\prod_i p_i^{e_i}$ for distinct primes p_i and exponents $e_i \geq 1$, then one can show that

$$\phi(n) = n \cdot \prod_i \left(1 - \frac{1}{p_i}\right).$$

  In particular, when $n = 1$ the product is empty and thus $\phi(1) = 1$.

  This result follows from establishing the result for each prime power p^k and then applying the Chinese Remainder Theorem as many times as is necessary.

- The ***Möbius function*** $\mu(n)$ on the naturals is defined as follows: if n is 1 then $\mu(n) = 1$, if $n = \prod_{i=1}^{k} p_i$ for distinct primes p_i then $\mu(n) = (-1)^k$, and if n is divisible by a particular prime twice, $\mu(n) = 0$.

  This function is clearly multiplicative by its definition.

- A positive integer n is ***square-free*** if it is divisible by no prime twice, that is, if $\mu(n) \neq 0$.

What all of these functions have in common is that they are *multiplicative*: if f is one of these functions and m, n are relatively prime positive integers, then $f(mn) = f(m)f(n)$. In fact, the first function is *totally multiplicative* because the relatively prime part can be ignored: for all positive integers m, n, we have $\mathrm{id}(mn) = \mathrm{id}(m)\mathrm{id}(n)$. Thus, if we want to understand a multiplicative function, we only have to look at its values on prime powers p^k.

4.2 An instructive example

Example 65 (AMC 12A 2021/25)
Let $f(n) = \dfrac{\tau(n)}{\sqrt[3]{n}}$. Compute the value of N that maximizes $f(N)$.

Walkthrough.

1. Prove f is multiplicative.
2. Look at how f behaves on prime powers p^k with p small and determine the maxima.
3. Prove that for large primes p, a prime power p^k has maximum $f(p^k)$ for $k = 0$.
4. Extract the answer.

The main idea in the above solution was that f being multiplicative gave us the ability to maximize f for each individual prime and thus limit our scope in a useful way.

Exercise 66 (PUMaC Number Theory 2010/A2). Find the largest positive integer n for which $\sigma(n) = 28$.

4.3 Dirichlet Convolution

In the following examples, multiplicativity will be used more deeply, so we need a method to tell if functions are multiplicative. One useful fact is the following:

Theorem 67 (Multiplicative Convolution)
If two functions f and g are multiplicative, then their ***Dirichlet convolution***

$$(f * g)(n) = \sum_{d \mid n} f(d) g\left(\frac{n}{d}\right)$$

is also multiplicative.

Proof. To show this function is multiplicative, note that for relatively prime m and n we have

$$\begin{aligned}(f * g)(n) = \sum_{d \mid mn} f(d) g\left(\frac{mn}{d}\right) &= \sum_{d_1 \mid m} \sum_{d_2 \mid n} f(d_1 d_2) g\left(\frac{m}{d_1} \cdot \frac{n}{d_2}\right) \\ &= \left(\sum_{d_1 \mid m} f(d_1) g\left(\frac{m}{d_1}\right)\right)\left(\sum_{d_2 \mid n} f(d_2) g\left(\frac{n}{d_2}\right)\right) \\ &= (f * g)(m) \cdot (f * g)(n).\end{aligned}$$

□

Here are some examples of Dirichlet convolutions so that the definition does not appear to come out of nowhere:

- $(f * e)(n) = f(n)$ for all multiplicative f
- $\tau(n) = (\mathbf{1} * \mathbf{1})(n)$
- $\sigma(n) = (\mathrm{id} * \mathbf{1})(n)$

This is enough to solve the following problem.

Example 68 (PUMaC Number Theory 2019/A2)
Let f be a function over the naturals so that

- $f(1) = 1$,
- If $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ for distinct primes p_i and nonnegative integers e_i, then $f(n) = (-1)^{e_1 + \cdots + e_k}$.

Find

$$\sum_{i=1}^{2019} \sum_{d \mid i} f(d).$$

Walkthrough.

1. Prove f is multiplicative.

2. What function is $f * 1$ and what is it doing in this problem?

3. Extract the answer.

Here is a slightly harder example based on the same premise (although it avoids citing Dirichlet convolutions).

Example 69 (PUMaC Number Theory 2016/A5)
Let $k = 2^6 \cdot 3^5 \cdot 5^2 \cdot 7^3 \cdot 53$. Let S be the sum of $\frac{\gcd(m,n)}{\operatorname{lcm}(m,n)}$ over all ordered pairs of positive integers (m,n) where $mn = k$. If S can be written in simplest form as $\frac{r}{s}$, compute $r+s$.

Walkthrough.

1. Write $\frac{\gcd(m,n)}{\operatorname{lcm}(m,n)}$ in a way more conducive to multiplicativity shenanigans, that is, breaking the problem into dealing with individual primes.

2. Determine why $\frac{\gcd(m,t/m)^2}{t}$ is a useful expression.

3. Answer the problem for each prime power factor.

4. Finish.

Here is yet another PUMaC example.

Example 70 (PUMaC Number Theory 2015/A6)
Determine the smallest positive integer n for which

$$35 \mid \sum_{t \mid n} \tau(t)^3.$$

Walkthrough.

1. What is the multiplicative function here?

2. Worry about 5 and 7 dividing $\sum_{t|n} \tau(t)^3$ separately, looking at prime powers for n.

3. Extract the minimum.

4.4 Operations with convolutions

Now, we discuss some more theory concerning the Dirichlet convolution. Since $f * e = f$ for all multiplicative f, it must be true that e is the identity. Thus, if $f * g = e$ for functions f and g then f and g are *inverses*.

- Clearly the convolution is commutative.

- Interestingly, it is also associative:

$$(f*(g*h))(n) = ((f*g)*h)(n) = \sum_{d_1 d_2 d_3 = n} f(d_1)g(d_2)h(d_3).$$

- It is also distributive over addition because $f*(g+h) = f*g + f*h$.

Another identity dealing with these functions is

$$\phi * \mathbf{1} = \mathrm{id},$$

for which it suffices to check the result on prime powers:

$$\sum_{d \mid p^k} \phi(d) = p^k - p^{k-1} + p^{k-1} - p^{k-2} + \cdots + p - 1 + 1 = p^k.$$

Remark 71. The idea of characterizing a multiplicative f in terms of its primes shows up heavily in problems involving NT functions.

Theorem 72 (Inverse of μ)
The Möbius function μ is the inverse of $\mathbf{1}$.

Proof. It suffices to check $\mu * \mathbf{1} = e$ on prime powers p^k. Indeed, $k = 0$ yields a sum of 1, $k = 1$ yields a sum of 0, and summands corresponding to divisors p^ℓ with $\ell \geq 2$ are zero. □

As a corollary, $f * \mathbf{1} = g$ implies

$$g * \mu = f * \mathbf{1} * \mu = f * (\mathbf{1} * \mu) = f * e = f,$$

so μ essentially recovers f from g. Moreover, this result does not actually care if f and g are multiplicative, as exemplified below.

Example 73 (ISL 1989/11)
Define a sequence $(a_n)_{n\geq 1}$ so that $\sum_{d\mid n} a_d = 2^n$. Show $n \mid a_n$.

Walkthrough.

1. Determine a_n with Möbius inversion.
2. For a particular prime $p \mid n$ with p dividing n a total of e times, it is enough to show $p^e \mid n$.
3. Take the whole expression modulo p^e and recall Euler's totient theorem, this should finish.

We conclude this section with a final difficult example.

Example 74 (HMMT February Algebra-NT 2019/8)
There is a unique function $f : \mathbb{N} \to \mathbb{R}$ such that $f(1) > 0$ and such that

$$\sum_{d|n} f(d) f\left(\frac{n}{d}\right) = 1$$

for all $n \geq 1$. What is $f\left(2018^{2019}\right)$?

Walkthrough.

1. What would be a good assumption on f?
2. Look at prime powers.
3. The condition naturally rewrites in terms of generating functions.
4. What is the generating function of $\frac{1}{\sqrt{1-x}}$?
5. Extract the answer.

4.5 Problems

Problem 38 (AMC 12B 2021/7). Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

Remark 75. It is **complete overkill** to use multiplicative properties on the problem above. But why not?

Problem 39. Show that $\tau(n) \leq \sqrt{3n}$.

Problem 40 (Italy MO 1998/6). Let $f : \mathbb{N} \to \mathbb{N}$ be a completely multiplicative, increasing function. Prove that if $f(2) = 2$, then $f(n) = n$ for all $n \in \mathbb{N}$. Does this remain true if the word "completely" is omitted?

Problem 41 (NEMO Team 2018/9). Find the sum of all N such that

$$n = 7\tau(n).$$

Problem 42 (Elementary Number Theory, David Burton). Let $F(n) = \sum_{d|n}(f(d))$. Prove that $f(d)$ is multiplicative if and only if $F(n)$ is multiplicative.

Problem 43. For any positive integer n, prove that

$$\sum_{d|n}(\sigma(d)) = \sum_{d|n}\left(\frac{n}{d}\tau(d)\right).$$

Problem 44 (Baltic Way 2019/16). For a positive integer N, let $f(N)$ be the number of ordered pairs of positive integers (a, b) such that the number

$$\frac{ab}{a+b}$$

is a divisor of N. Prove that $f(N)$ is always a perfect square.

Problem 45 (Number Theory, George E. Andrews). Prove that if $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_r^{a_r}$, then

$$\sum_{d|n} d\phi(d) = \left(\frac{p_1^{2a_1+1}+1}{p_1+1}\right) \cdot \ldots \cdot \left(\frac{p_r^{2a_r+1}+1}{p_r+1}\right).$$

Problem 46 (PAMO SL 2018/A5). Let $g : \mathbb{N} \to \mathbb{N}$ be a function satisfying:

- $g(xy) = g(x)g(y)$ for all $x, y \in \mathbb{N}$,
- $g(g(x)) = x$ for all $x \in \mathbb{N}$, and
- $g(x) \neq x$ for $2 \leq x \leq 2018$.

Find the minimum possible value of $g(2)$.

Problem 47. Prove that for any positive integer n, the number of divisors of n of the form $4k + 1$ is larger than the number of divisors of the form $4k + 3$, where k is a nonnegative integer.

Problem 48 (HMMT February Algebra-NT 2021/5). Let N be the product of the first 10 primes, and let

$$S = \sum_{xy|n} \phi(x) \cdot y.$$

Compute $\frac{S}{n}$.

Problem 49 (ISL 2004/N2). The function f from the set $\mathbb{N}$ of positive integers into itself is defined by the equality

$$f(n) = \sum_{k=1}^{n} \gcd(k, n), \qquad n \in \mathbb{N}.$$

(a) Prove that $f(mn) = f(m)f(n)$ for every two relatively prime $m, n \in \mathbb{N}$.

(b) Prove that for each $a \in \mathbb{N}$ the equation $f(x) = ax$ has a solution.

(c) Find all $a \in \mathbb{N}$ such that the equation $f(x) = ax$ has a unique solution.

Problem 50 (AMM 12003, Nikolai Osipov). Given an odd positive integer n, compute

$$\sum_{k=1}^{n} \frac{\gcd(k, n)}{\cos^2\left(\frac{\pi k}{n}\right)}.$$

5 Titu's Favorite Factoring Trick

5.1 "Theory"

This entire article revolves around this single factorization, hereby dubbed "Titu's Favorite Factoring Trick" (you'll see why in the problem set).

Theorem 76 (Titu's Favorite Factoring Trick)
For all real numbers a, b, c,

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)\left(a^2 + b^2 + c^2 - ab - bc - ca\right). \tag{5.1}$$

Proof. By the sum of cubes formula,

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a + b)^3 - 3a^2b - 3ab^2 + c^3 - 3abc \\ &= (a + b + c)^3 - 3(a + b)^2c - 3(a + b)c^2 - 3ab(a + b + c) \\ &= (a + b + c)^3 - 3(a + b)c(a + b + c) - 3ab(a + b + c) \\ &= (a + b + c)\left(a^2 + b^2 + c^2 - ab - bc - ca\right). \end{aligned}$$

□

This seemingly innocent factorization actually results in tons of nontrivial corollaries. For example:

Corollary 77
For all real numbers a, b, c,

$$a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c)\left((a - b)^2 + (b - c)^2 + (c - a)^2\right).$$

Proof. Note that

$$\begin{aligned} (a - b)^2 + (b - c)^2 + (c - a)^2 &= a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ca + a^2 \\ &= 2(a^2 + b^2 + c^2 - ab - bc - ca), \end{aligned}$$

which is twice the last factor of (5.1). □

Corollary 78
For all real numbers a, b, c,

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega),$$

where ω is a complex number satisfying $\omega^2 + \omega + 1 = 0$.

Proof. Just check that

$$(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) = \frac{1}{2}\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)$$

through direct expansion. □

Corollary 79
For all real numbers a, b, c,

$$a + b + c = 0 \implies a^3 + b^3 + c^3 = 3abc.$$

Proof. Since $a^3 + b^3 + c^3 - 3abc = (a + b + c)\left(a^2 + b^2 + c^2 - ab - bc - ca\right)$, if $a + b + c = 0$, then $a^3 + b^3 + c^3 - 3abc = 0$, or $a^3 + b^3 + c^3 = 3abc$. □

This last corollary is extremely useful, and it shows up in a ton of problems. Let's see some examples in action.

Example 80
Reals a, b, c satisfy $abc = 1$ and $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = 0$. Find $a + b + c$.

Walkthrough. This is one of those problems that seems very hard to do if you don't know Titu's Favorite Factoring Trick, but becomes straightforward after seeing it.

1. What do we know if $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = 0$? (Explicit hint: use the corollary above!)
2. Extract the answer.

Example 81 (AHSME 1999/30)
How many ordered pairs of integers (m, n) satisfy $mn \geq 0$ and

$$m^3 + n^3 + 99mn = 33^3?$$

Walkthrough.

1. Rewrite the equation to be in the form $a^3 + b^3 + c^3 - 3abc = 0$.

2. One of the three factorizations above will help us weed out integer solutions quickly. Which one?

3. Extract the answer.

Example 82 (MR S499, Titu Andreescu)
Let a and b be distinct real numbers. Prove that

$$27ab(\sqrt[3]{a}+\sqrt[3]{b})^3 = 1 \text{ if and only if } 27ab(a+b+1) = 1.$$

Walkthrough. This, along with many other problems in the Mathematical Reflections Journal proposed by Titu Andreescu, inspired the titular "Titu's Favorite Factoring Trick".

1. The problem tries to hide the cubes by instead using cube roots. Make the substitution $a = x^3, b = y^3$. What alternative statement appears?

2. The first equation will be a perfect cube, so take its cube root. Which of the equations is (probably) easier to write in the form $a^3 + b^3 + c^3 - 3abc$?

3. Factor the equation.

4. Where does the hypothesis that a and b are distinct real numbers come in to play? Conclude.

5.2 A Potpourri of Problems

You might ask yourself, why so many problems? Well, one major reason for creating this article was just to be a collection of these problems (especially since there isn't much theory), but hopefully you also get to see just how widely this trick is used!

Problems are arranged in *very* roughly increasing order of difficulty.

Problem 51 (GGMT Speed 2020/2). Let p, q, r be the real roots of the polynomial $x^3 - 7x - 13$. Evaluate $p^3 + q^3 + r^3$.

> *Remark 83.* The problem is probably not intended to be solved with Titu's Favorite Factoring Trick, but I think it's funnier that way.

Problem 52 (Vietnam MO 1985/1). Find all pairs of integers (x, y) such that $x^3 - y^3 = 2xy + 8$.

Problem 53 (expii). An integer is called *circular* if it can be expressed as $a^3+b^3+c^3-3abc$ for some integers a, b, and c. Show that the product of any two circular numbers is circular.

Problem 54 (Putnam 2015/B1). An equilateral triangle has all of its vertices on the graph of $x^3+3xy+y^3=1$. Find its area.

Problem 55. Prove that the triangle ABC is isosceles if and only if

$$\sqrt[3]{a-b}+\sqrt[3]{b-c}+\sqrt[3]{c-a}=0.$$

(As usual, $a=BC$, $b=CA$, and $c=AB$.)

Problem 56. Find all ordered pairs of real numbers (x, y) satisfying

$$\ln x+\ln y=\ln\left(x^3+\frac{y^3}{3}+\frac{1}{9}\right).$$

Problem 57 (BMT Algebra 2020/7). Let a, b, c be real numbers such that $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $abc=5$. Find the value of

$$\left(a-\frac{1}{b}\right)^3+\left(b-\frac{1}{c}\right)^3+\left(c-\frac{1}{a}\right)^3.$$

Problem 58 (BMO 2007/2/1). Find the minimum value of $x^2+y^2+z^2$ where x, y, z are real numbers such that $x^3+y^3+z^3-3xyz=1$.

Problem 59. Let x and y be nonzero real numbers satisfying $x^3+y^3+3x^2y^2=x^3y^3$. Find all possible values of $\frac{1}{x}+\frac{1}{y}$.

Problem 60 (RMO/INMO Preparation Thread). Find all real x satisfying

$$2\lfloor x\rfloor^3+\lfloor x\rfloor\{x\}+2\{x\}^3=\frac{1}{108}.$$

Problem 61. Let x, y, z be integers such that

$$(x-y)^2+(y-z)^2+(z-x)^2 = xyz.$$

Prove that $x^3+y^3+z^3$ is divisible by $x+y+z+6$.

Problem 62 (AMSP Admissions Test 2021/C.9). Find all positive integers m, n satisfying

$$\left(m^3+196(28-3m)\right)\left(n^3+256(32-3n)\right) = 2021.$$

Problem 63. Suppose that x, y, z are distinct real numbers such that

$$(x-y)\sqrt[3]{1-z^3}+(y-z)\sqrt[3]{1-x^3}+(z-x)\sqrt[3]{1-y^3} = 0.$$

Prove that

$$(1-x^3)(1-y^3)(1-z^3) = (1-xyz)^3.$$

Problem 64 (Purple Comet HS 2020/16). Let a, b, c be nonzero real numbers satisfying

$$a\sqrt[3]{\frac{a}{b}}+b\sqrt[3]{\frac{b}{c}}+c\sqrt[3]{\frac{c}{a}} = 0.$$

Find the maximum possible value of

$$\left(\frac{a^3}{b^2c}+\frac{b^3}{c^2a}+\frac{c^3}{a^2b}\right)^2.$$

Problem 65 (Purple Comet HS 2021/21). Let a, b, and c be real numbers satisfying the equations

$$a^3+abc = 26$$
$$b^3+abc = 78$$
$$c^3-abc = 104.$$

Find $a^3+b^3+c^3$.

Problem 66 (`arqady`). Let a, b and c be real numbers such that $\sqrt[3]{ab}+\sqrt[3]{ac}+\sqrt[3]{bc} \geq 0$. Prove that $ab+ac+bc \geq 0$.

Problem 67 (Purple Comet HS 2013/27). Suppose a, b, c are real numbers satisfying $a+b+c=5$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{5}$. Find the greatest possible value of $a^3+b^3+c^3$.

Problem 68 (Purple Comet HS 2018/19). Suppose that a and b are positive real numbers such that $3\log_{101}\left(\frac{1{,}030{,}301-a-b}{3ab}\right)=3-2\log_{101}(ab)$. Find $101-\sqrt[3]{a}-\sqrt[3]{b}$.

Problem 69. Find all triples (a, b, c) of integers satisfying

$$\begin{cases} a^3-b^3-c^3=3abc, \\ a^2=2(a+b+c). \end{cases}$$

Problem 70 (Purple Comet HS 2019/23). Find the number of ordered pairs of integers (x, y) such that

$$\frac{x^2}{y}-\frac{y^2}{x}=3\left(2+\frac{1}{xy}\right).$$

Problem 71 (JMC 12 2021/22). For positive reals $a_1, a_2, \ldots, a_n$, the function $\mathcal{D}(a_1, a_2, \ldots, a_n)$ denotes the difference between the arithmetic mean and the geometric mean of those n numbers. Suppose positive real numbers x, y, and z satisfy

$$\begin{cases} xyz=1, \\ \mathcal{D}(\sqrt{x}, \sqrt{y}, \sqrt{z})=\frac{74}{13}, \\ \mathcal{D}(\sqrt[3]{x}, \sqrt[3]{y})+\mathcal{D}(\sqrt[3]{y}, \sqrt[3]{z})+\mathcal{D}(\sqrt[3]{z}, \sqrt[3]{x})=\frac{37}{9}. \end{cases}$$

What is the value of $\mathcal{D}(\sqrt[6]{x}, \sqrt[6]{y}, \sqrt[6]{z})$?

Problem 72 (David Altizio). Suppose ABC is a triangle with angles measures A, B, and C such that

$$\tan A+\tan B+\tan C=4 \quad \text{and} \quad \cot A+\cot B+\cot C=5.$$

Find $\tan^3 A+\tan^3 B+\tan^3 C$.

Problem 73 (MR J479, Titu Andreescu). Let a, b, c be nonzero real numbers, not all equal, such that

$$\left(\frac{a^2}{bc}-1\right)^3+\left(\frac{b^2}{ca}-1\right)^3+\left(\frac{c^2}{ab}-1\right)^3=3\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}-\frac{bc}{a^2}-\frac{ca}{b^2}-\frac{ab}{c^2}\right).$$

Prove that $a+b+c=0$.

> *Remark 84 (Samuel Zhou).* This (the above problem) is the best problem of all time. Its comedic value is astronomical.

Problem 74 (Putnam 2019 A1). Determine all possible values of the expression

$$A^3+B^3+C^3-3ABC$$

where A, B, and C are nonnegative integers.

Problem 75 (MR S546, Titu Andreescu). Find all triples of real numbers (x,y,z) satisfying the system below.

$$\begin{cases} x^3-2xyz+y^3=\frac{1}{2} \\ y^3-2xyz+z^3=1 \\ z^3-2xyz+x^3=-\frac{3}{2} \end{cases}$$

Problem 76. Let a,b,c be the three real roots of x^3+x^2-2x-1. Find $\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}$.

Problem 77 (SMMT Champion 2021/8). Let (x,y,z) be a solution to the following simultaneous system of equations:

$$\frac{x+2z+3}{y-z}+\frac{y+2x+3}{z-x}+\frac{z+2y+3}{x-y}=x+y+z$$
$$\frac{1}{y-z}+\frac{1}{z-x}+\frac{1}{x-y}=\frac{420}{(y-z)(z-x)(x-y)}.$$

Given that $x+y+z$ is not an integer, find $(x-y)^3+(y-z)^3+(z-x)^3$.

Problem 78. Is $512^3+675^3+720^3$ prime?

Problem 79. Let k be an integer and let $n=\sqrt[3]{k+\sqrt{k^2+1}}+\sqrt[3]{k-\sqrt{k^2+1}}+1$. Prove that $n^3-3n^2\in\mathbb{Z}$.

Problem 80 (IWYMIC 2018/1/11). Find all pairs of real numbers (x, y) satisfying the system below.

$$\begin{cases} \sqrt[3]{x} - \sqrt[3]{y} = 12 \\ xy = \left(\dfrac{x+y+8}{6}\right)^3 \end{cases}$$

Problem 81 (HOMC Team 2018/J8). Let a, b, c be real numbers with $a + b + c = 2018$. Suppose x, y, and z are distinct positive real numbers which satisfy $a = x^2 - yz - 2018$, $b = y^2 - zx - 2018$, and $c = z^2 - xy - 2018$. What is

$$\frac{\sqrt{a^3 + b^3 + c^3 - 3abc}}{x^3 + y^3 + z^3 - 3xyz}?$$

Problem 82 (CMIMC NT 2018/8). Find the unique set of positive primes $\{p, q, r\}$ such that

$$\frac{p^3 + q^3 + r^3}{p + q + r} = 249.$$

Problem 83 (CMIMC Algebra-NT 2020/9). Let $p = 10009$ be a prime number. Determine the number of ordered pairs of integers (x, y) such that $1 \le x, y \le p$ and $x^3 - 3xy + y^3 + 1$ is divisible by p.

Problem 84 (MR J469). Prove that

$$(3a + 1)(3b + 1) = 3a^2b^2 + 1$$

if and only if

$$\left(\sqrt[3]{a} + \sqrt[3]{b}\right)^2 = a^2b^2.$$

Problem 85 (USAMTS 3/4/16). Find a polynomial $f(x, y, z)$ in three variables with integer coefficients such that for all integers a, b, c, the sign of $f(a, b, c)$ is the same as the sign of $a + b\sqrt[3]{2} + c\sqrt[3]{4}$.

Problem 86 (Mock AIME I 2015/11, David Altizio). Let a, b, c be complex numbers satisfying the system

$$\begin{cases} a+b+c=6, \\ a^3+b^3+c^3=87, \\ (a+1)(b+1)(c+1)=33. \end{cases}$$

Find $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

Problem 87 (Mock AIME 2017/10, Ankan Bhattacharya). Find the number of triples of integers (a,b,c) satisfying $-100 \le a,b,c \le 100$ and

$$\left(\frac{a+b+c}{3}\right)^3 = abc.$$

Problem 88 (US Algebra Math Open Winter 2015/14). How many functions $f : \mathbb{Z} \to \mathbb{Z}$ are there such that $f(a^3) = f(a)^3$ for all $a \in \mathbb{Z}$ and $f(a^3)+f(b^3)+f(c^3) = 3f(a)f(b)f(c)$ for all $a,b,c \in \mathbb{Z}$ satisfying $a+b+c=0$?

Problem 89 (OMO Spring 2014/18). Find the number of pairs (m,n) of integers with $-2014 \le m,n \le 2014$ such that $x^3+y^3 = m+3nxy$ has infinitely many integer solutions (x,y).

Problem 90 (MR J517). Let $(a_n)_{n\ge 1}$ be a sequence of positive real numbers such that $a_1 = 1, a_2 = 2$ and

$$\frac{a_{n+1}^3+a_{n-1}^3}{9a_n} + a_{n+1}a_{n-1} = 3a_n^2.$$

Find a closed form for a_n.

Problem 91 (Rioplatense MO 2007/3/4). Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the following property: if $x+y+z=0$, then $f(x)+f(y)+f(z)=xyz$.

Problem 92 (MR S393). How many positive integers n exist such that n^2+11 is prime and $n+4$ is a perfect cube?

Problem 93. Find the least real number r such that for each triangle with side lengths a,

b, and c, we have

$$\frac{\max(a,b,c)}{\sqrt[3]{a^3+b^3+c^3+3abc}} < r.$$

Problem 94 (China TST 2019/3/1). Given complex numbers x, y, z satisfying $|x|^2 + |y|^2 + |z|^2 = 1$, prove that

$$\left|x^3+y^3+z^3-3xyz\right| \le 1.$$

Problem 95 (OMO Spring 2018/24). Find the number of ordered triples (a, b, c) of integers satisfying $0 \le a, b, c \le 1000$ for which

$$a^3+b^3+c^3 \equiv 3abc+1 \pmod{1001}.$$

Problem 96. Reals $\alpha, \beta, \gamma \in (0, \pi)$ satisfy

$$\cos\alpha+\cos\beta+\cos\gamma = \cos 2\alpha+\cos 2\beta+\cos 2\gamma = \cos 3\alpha+\cos 3\beta+\cos 3\gamma = 0.$$

Find all possible values of $\sin\alpha+\sin\beta+\sin\gamma$.

Problem 97 (MR J489, Titu Andreescu). Prove that in any triangle ABC,

$$8r(R-2r)\sqrt{r(16R-5r)} \le a^3+b^3+c^3-3abc \le 8R(R-2r)\sqrt{(2R+r)^2+2r^2}.$$

(As usual, $a = BC$, $b = CA$, $c = AB$, r is the inradius, and R is the circumradius.)

Problem 98 (math.SE). Is $\sqrt[4]{2} + \sqrt[3]{3}$ rational?

Problem 99 (Vietnam TST 2008/6). Consider the set $M = \{1, 2, \ldots, 2008\}$. Paint every number in the set M with one of the three colors blue, yellow, red such that each color is used at least once. Define two sets:

$$S_1 = \{(x,y,z) \in M^3 \mid x,y,z \text{ have the same color and } 2008 | (x+y+z)\}.$$
$$S_2 = \{(x,y,z) \in M^3 \mid x,y,z \text{ have three pairwise different colors and } 2008 | (x+y+z)\}.$$

Prove that $2|S_1| > |S_2|$.

Problem 100 (ELMO 2017/6). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers a, b, and c:

- If $a+b+c \geq 0$ then $f(a^3)+f(b^3)+f(c^3) \geq 3f(abc)$.
- If $a+b+c \leq 0$ then $f(a^3)+f(b^3)+f(c^3) \leq 3f(abc)$.

Problem 101 (Putnam 1939/B7). Prove that $a_0^3+a_1^3+a_2^3-3a_0a_1a_2 = 1$, where

$$a_i = \sum_{n=0}^{\infty} \frac{x^{3n+i}}{(3n+i)!}.$$

Editor's Notes

Congrats, you made it to the end!

A short note

You may have noticed that the Incenter Miquel chapter includes its solutions to the examples in the chapter rather than at the end (unlike the other chapters). This is because many of the solutions are well-motivated and thus are worth reading in conjunction with the rest of the material. The other chapters present their motivation in the walkthroughs and therefore the solutions should only be read after the content is introduced.

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That's it for now! See you guys in the next issue.
Dylan Yu

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B Selected Solutions

B.1 Solution 4 (Euclid 2021/2.c)

Let P, Q, and R be the ages of Paolo, Qing, and Rayna respectively. Then, we have the following system of 3 equations:

$$R = \frac{Q}{2}$$
$$P = Q + 4$$
$$\frac{P+Q+R}{3} = 13$$

The answer is $\boxed{(18, 14, 7)}$.

Substitute the first two equations into the third to get that $\frac{Q+4+Q+\frac{Q}{2}}{3} = 13$. We can then simplify that to get $Q = 14$. Then, substitute this into the first two equations to get that $P = 18$ and $R = 7$.

B.2 Solution 5 (Euclid 2014/3)

Clearing denominators, we get that $7(a+b) = 4ab$, or $ab = 28$. Then, applying Vieta's formulas, we note that a and b are the roots of the quadratic $x^2 - 16x + 28 = 0$, which we can easily factor to give the solutions for a and b.

B.3 Solution 6 (blackpenredpen)

Let's look at the restrictions for the problem. Clearly, $5 - x \geq 0$ (you can't take the square root of a negative number and get a real answer). Moreover, $5 - x^2 \geq 0$, because the range of the square root function is strictly nonnegative. Combining these restrictions, we get that $-\sqrt{5} \leq x \leq \sqrt{5}$.

Instead of treating x as the variable, let's square both sides and treat 5 as the variable:

$$5 - x = 5^2 - (2x^2)5 + x^4$$
$$5^2 - (2x^2+1)5 + x^4 + x = 0$$

Use the quadratic formula on 5:

$$5 = \frac{2x^2 + 1 \pm \sqrt{(-2x^2 - 1)^2 - 4(x^4 + x)(1)}}{2}$$
$$5 = \frac{2x^2 + 1 \pm \sqrt{4x^2 - 4x + 1}}{2}$$
$$5 = \frac{2x^2 + 1 \pm (2x - 1)}{2}$$

This gives us two cases, $5 = x^2 + x$ and $5 = x^2 - x + 1$. The final answer is left as an exercise to the reader (you should get only 2 roots).

B.4 Solution 7 (AoPS)

Let's try to group factor naively:

$$mn + 3m - 8n = 59$$
$$m(n + 3) - 8n = 59$$

If only we could get a $n + 3$ factor somewhere. Let's artificially force it to happen:

$$m(n + 3) - 8n = 59$$
$$m(n + 3) - 8n - 24 = 59 - 24$$
$$m(n + 3) - 8(n + 3) = 35$$
$$(m - 8)(n + 3) = 35$$

Thus, 35 is a product of these two factors, and we can do casework to check. The final answer is left as an exercise to the reader.

B.5 Solution 8 (Instagram)

The nested radical looks kind of funny, and it can normally be simplified. Let's do some magic:

$$\begin{aligned}\sqrt{19 - 8\sqrt{3}} &= \sqrt{16 - 8\sqrt{3} + 3} \\ &= \sqrt{4^2 - 2 \times 4 \times \sqrt{3} + \left(\sqrt{3}\right)^2} \\ &= \sqrt{\left(4 - \sqrt{3}\right)^2} \\ &= 4 - \sqrt{3}\end{aligned}$$

So we have simplified the artificially complex condition to $x = 4 - \sqrt{3}$. Then:

$$\begin{aligned} x &= 4 - \sqrt{3} \\ x - 4 &= -\sqrt{3} \\ (x-4)^2 &= \left(-\sqrt{3}\right)^2 \\ x^2 - 8x + 13 &= 0 \end{aligned}$$

Conveniently, the denominator of the expression is $x^2 - 8x + 15$, which is just equal to 2. Now for the numerator, we will "power ladder." Since we know that $x^2 - 8x + 13 = 0$, we can basically multiply both sides by *anything we want* and it will still equal 0. In other words,

$$f(x)(x^2 - 8x + 13) = 0$$

for practically any function f. So let's multiply both sides by $f(x) = x^2$ and subtract that off the numerator of the question to reduce the degree of the numerator. Repeating this process gives us that the numerator is simply equal to $x^2 - 8x + 23$, which is just 10. We have simplified the expression to $\frac{10}{2}$, which is just equal to $\boxed{5}$.

B.6 Solution 12

By binomial theorem,

$$\begin{aligned} (1+1)^{10} &= \sum_{k=0}^{10} \binom{10}{k}, \\ (1-1)^{10} &= \sum_{k=0}^{10} (-1)^k \binom{10}{k}, \end{aligned}$$

and averaging the two quantities gives us the desired result of $\boxed{512}$.

> *Remark 85.* For reference, we applied ROUF on $P(x) = (1+x)^{10}$, i.e. computed $\frac{P(1)+P(-1)}{2}$.

B.7 Solution 13

Let $\omega = e^{\frac{2\pi i}{3}}$. Then by applying ROUF on $P(x) = x^{-2}(1+x)^{10}$, we note that we have shifted the coefficients by two (i.e. the kth coefficient of $(1+x)^{10}$ is now the $k-2$th coefficient of $x^{-2}(1+x)^{10}$), so $\binom{10}{3k+2}$ for nonnegative integers k are now included in our count. Thus,

$$\begin{aligned} \binom{10}{2} + \binom{10}{5} + \binom{10}{8} &= P(1) + P(\omega) + P(\omega^2) \\ &= 1 \cdot (1+1)^{10} + \omega^{-2}(1+\omega)^{10} + \omega^{-4}(1+\omega^2)^{10}. \end{aligned}$$

Using $\omega^3 = 1$, we get that

$$1 \cdot (1+1)^{10} + \omega(1+\omega)^{10} + \omega^2(1+\omega^2)^{10} = 1026,$$

implying the answer is $\frac{1}{3} \cdot 1026 = \boxed{342}$.

B.8 Solution 15

Solution by Evan Chen.

We can rewrite the sum as

$$\sum_{n \geq 0} \binom{1000}{n} f(n)$$

where

$$f(n) = \begin{cases} 1 & n \equiv 0 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

The trick is that we can take

$$f(n) = \frac{1}{3}\left(1^n + \omega^n + \omega^{2n}\right)$$

where $\omega = e^{\frac{2\pi i}{3}}$ is a cube root of unity, satisfying the relation $\omega^2 + \omega + 1 = 0$. Thus, we have

$$\sum_{n \geq 0} \binom{1000}{n} f(n) = \frac{1}{3} \sum_{n \geq 0} \binom{1000}{n} (1 + \omega^n + \omega^{2n}).$$

We can swap the order of summation now and instead consider

$$\sum_{n \geq 0} \binom{1000}{n} f(n) = \frac{1}{3} \sum_{n \geq 0} \binom{1000}{n} + \frac{1}{3} \sum_{n \geq 0} \binom{1000}{n} \omega^n + \frac{1}{3} \sum_{n \geq 0} \binom{1000}{n} \omega^{2n}.$$

By the binomial theorem, the expression in question is

$$\begin{aligned} \sum_{n \geq 0} \binom{1000}{n} f(n) &= \frac{1}{3}\left[(1+1)^{1000} + (1+\omega)^{1000} + (1+\omega^2)^{1000}\right] \\ &= \frac{1}{3}\left[2^{1000} + (-\omega^2)^{1000} + (-\omega)^{1000}\right] \\ &= \frac{1}{3}\left[2^{1000} + \omega + \omega^2\right] \\ &= \frac{1}{3}\left[2^{1000} - 1\right]. \end{aligned}$$

B.9 Solution 16 (AMC 12A 2021/15)

Solution by AoPS user `lawliet163`.

Let $f(x,y) = (1+x)^8(1+y)^6$. By expanding the binomials and distributing, $f(x,y)$ is the generating function for different groups of basses and tenors. That is,

$$f(x,y) = \sum_{m=0}^{8}\sum_{n=0}^{6} a_{mn}x^m y^n$$

where a_{mn} is the number of groups of m basses and n tenors. What we want to do is sum up all values of a_{mn} for which $4 \mid m-n$ except for $a_{00} = 1$. To do this, define a new function

$$g(x) = f(x, x^{-1}) = \sum_{m=0}^{8}\sum_{n=0}^{6} a_{mn}x^{m-n} = (1+x)^8(1+x^{-1})^6.$$

Now we just need to sum all coefficients of $g(x)$ for which $4 \mid m-n$. Consider a monomial $h(x) = x^k$. If $4 \mid k$,

$$h(i) + h(-1) + h(-i) + h(1) = 1+1+1+1 = 4$$

otherwise,

$$h(i) + h(-1) + h(-i) + h(1) = 0.$$

$g(x)$ is a sum of these monomials so this gives us a method to determine the sum we're looking for:

$$\frac{g(i) + g(-1) + g(-i) + g(1)}{4} = 2^{12} = 4096$$

(since $g(-1) = 0$ and it can be checked that $g(i) = -g(-i)$). Hence, the answer is $4096 - 1$ with the -1 for a_{00} which gives $\boxed{95}$.

B.10 Solution 17 (AIME II 2016/12)

Three interesting solutions are presented below. You can alternatively do the problem with easier methods, e.g. PIE, casework, or recursion.

Solution via generating functions and ROUF We use generating functions. Suppose that the colors are $0, 1, 2, 3$. Then as we proceed around a valid coloring of the ring in the clockwise direction, we know that between two adjacent sections with colors s_i and s_{i+1}, there exists a number $d_i \in \{1, 2, 3\}$ such that

$$s_{i+1} \equiv s_i + d_i \pmod 4.$$

Thus, we can represent each border between sections by the generating function $x + x^2 + x^3$, where x, x^2, x^3 correspond to increasing the color number by $1, 2, 3 \pmod 4$, respectively. Thus the generating function that represents going through all six borders is

$$A(x) = (x + x^2 + x^3)^6,$$

where the coefficient of x^n represents the total number of colorings where the colors' numbers are increased by a total of n as we proceed around the ring. But if we go through all six borders, we must return to the original section, which is already colored. Therefore, we wish to find the sum of the coefficients of x^n in $A(x)$ with $n \equiv 0 \pmod 4$.

Thus, the sum of the coefficients of $A(x)$ with powers congruent to $0 \pmod 4$ is

$$\frac{A(1) + A(i) + A(-1) + A(-i)}{4} = \frac{3^6 + (-1)^6 + (-1)^6 + (-1)^6}{4} = \frac{732}{4}.$$

We multiply this by 4 to account for the initial choice of color, so the answer is $\boxed{732}$.

Solution via linear algebra (Allen Wang) Consider the graph K_4 and its corresponding adjacency matrix A. The answer to the problem is therefore the trace of A^6 by definition of matrix multiplication. Since $B = A + I$ is a 4×4 matrix full of 1s, B has a rank of 1, and therefore has 3 eigenvalues of 0. Since the trace of B is 4, the last eigenvalue is 4. Thus, A has eigenvalues $-1, -1, -1, 3$.

The trace of A^6 is the sum of the 6th powers of its eigenvalues, so we find that the trace is $3^6 + 3 = \boxed{732}$, as desired.

Solution via chromatic polynomials The chromatic polynomial for a cycle C_n is $(x-1)^n + (-1)^n(x-1)$, where x is the number of colors and n is the number of sections in the cycle. Clearly $x = 4$ and $n = 6$, so the answer is

$$(4-1)^6 + (-1)^6(4-1) = \boxed{732}.$$

Remark 86. Chromatic polynomials are actually what this problem is based on, and thus not the intended solution. Contest problems that can be destroyed with chromatic polynomials usually succumb to casework as well, but I recommend you learn them as they can be useful for timed competitions.

B.11 Solution 65 (AMC 12A 2021/25)

Note f is multiplicative, so it suffices to determine for each prime p which value of k maximizes

$$f(p^k) = \frac{k+1}{\sqrt[3]{p^k}}.$$

Note that for $p > 8$, $\sqrt[3]{p} > 2$ so $k = 0$ is optimal by the bound $\sqrt[3]{p}^k > 2^k = (1+1)^k \geq 1+k$ for $k \geq 1$. We now compute k for $p \in \{2,3,5,7\}$. Note that

$$f(p^k) > f(p^{k+1}) \iff \frac{k+1}{\sqrt[3]{p}^k} > \frac{k+2}{\sqrt[3]{p}^{k+1}} \iff \sqrt[3]{p} > \frac{k+2}{k+1},$$

so we need to find the first k with

$$\sqrt[3]{p} > \frac{k+2}{k+1}$$

because $\frac{k+2}{k+1}$ is a decreasing function. For $p \in \{5,7\}$, it turns out that $2^3 > p > (3/2)^3 = 27/8$. For $p = 3$, it is the case that $(3/2)^3 = 27/8 > p > (4/3)^3 = 64/27$. For $p = 2$, it is the case that $(4/3)^3 = 64/27 > p > (5/4)^3 = 125/64$. Hence

$$N = 2^3 3^2 5^1 7^1 = \boxed{2520}.$$

For fun, this maximal value $f(N)$ is

$$\frac{\tau(2520)}{\sqrt[3]{2520}} = \frac{4 \cdot 3 \cdot 2 \cdot 2}{2\sqrt[3]{315}} = \frac{24}{\sqrt[3]{315}} \approx 3.527.$$

B.12 Solution 68 (PUMaC Number Theory 2019/A2)

Note $\sum_{d|i} f(d)$ is multiplicative because f is multiplicative and this function is $(f * \mathbf{1})(n)$. Hence it suffices to compute the function for prime powers p^k. We have

$$\sum_{d|p^k} f(d) = (-1)^0 + (-1)^1 + \cdots + (-1)^k,$$

which is 1 if k is even and zero otherwise. Hence the problem simply asks us for the number of squares at most 2019, which is $\lfloor \sqrt{2019} \rfloor = \boxed{44}$.

B.13 Solution 69 (PUMaC Number Theory 2016/A5)

Note that each prime divisor of k functions "independently" in the expression. Hence it is sufficient to compute

$$\sum_{d|t} \frac{\gcd(d,t/d)}{\operatorname{lcm}(d,t/d)} = \sum_{d|t} \frac{\gcd(d,t/d)^2}{t}$$

for each of $t \in \{2^6, 3^5, 5^2, 7^3, 53\}$ and multiply the expressions together.

- $t = 2^6$: The expression is
$$\frac{1+2^2+2^4+2^6+2^4+2^2+1}{2^6} = \frac{106}{2^6} = \frac{53}{2^5}.$$
- $t = 3^5$: The expression is
$$\frac{1+3^2+3^4+3^4+3^2+1}{3^5} = \frac{2\cdot 91}{3^5}.$$
- $t = 5^2$: The expression is
$$\frac{1+5^2+1}{5^2} = \frac{3^3}{5^2}.$$
- $t = 7^3$: The expression is
$$\frac{1+7^2+7^2+1}{7^3} = \frac{2^2\cdot 5^2}{7^3}.$$
- $t = 53$: The expression is
$$\frac{1+1}{53} = \frac{2}{53}.$$

Hence
$$S = \frac{53}{2^5}\cdot\frac{2\cdot 91}{3^5}\cdot\frac{3^3}{5^2}\cdot\frac{2^2\cdot 5^2}{7^3}\cdot\frac{2}{53} = \frac{1}{2}\cdot\frac{13}{3^2}\cdot\frac{1}{7^2} = \frac{13}{882}.$$
The answer is then $13 + 882 = \boxed{895}$.

B.14 Solution 70 (PUMaC Number Theory 2015/A6)

Clearly $\tau(n)^3$ is multiplicative, so $(\tau(n)^3 * \mathbf{1})(n)$ is multiplicative as well. Hence we need to compute this for prime powers p^k. At prime powers,
$$\sum_{t\mid p^k}\tau(t)^3 = \sum_{0\le i\le k}(i+1)^3 = \left(\frac{(i+1)(i+2)}{2}\right)^2.$$
Thus 5 divides this if i is 3 or 4 mod 5 and 7 divides this if i is 5 or 6 mod 7. Then it must be optimal to take $n = 2^5\cdot 3^3 = \boxed{864}$.

B.15 Solution 73 (ISL 1989/11)

By Möbius inversion, $a_n = \sum_{d\mid n}\mu(n/d)2^d$. Let $n = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ for positive integral e_i. Of course, the result is immediate for $n = 1$. For each $p_i^{e_i}$, note the expression disappears by considering instances of $p_i^{e_i}$ and $p_i^{e_i-1}$ in the numerator of 2^d and using Euler's totient theorem.

B.16 Solution 74 (HMMT February Algebra-NT 2019/8)

Note the condition is $f * f = \mathbf{1}$. We require f to be multiplicative: this is clearly okay because then checking $f * f = \mathbf{1}$ follows from checking that for prime powers. We now compute f on prime powers p^k. Let $f(p^k) = g(k)$. We have

$$\sum_{a+b=k} g(a)g(b) = 1 \qquad \forall k, a, b \geq 0.$$

Then let

$$G(x) = \sum_{i=0}^{\infty} g(i)x^i$$

and observe

$$G(x)^2 = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

Hence

$$G(x) = \frac{1}{\sqrt{1-x}}.$$

Then $g(2019)$ is the value of $\frac{1}{2019!}G^{(2019)}(0)$ where 2019 denotes 2019 differentiations. That is,

$$g(2019) = \frac{1}{2019!} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{4037}{2} = \frac{4037!!}{4038!!}.$$

This implies that

$$f(2018^{2019}) = \frac{4037!!^2}{4038!!^2}.$$

B.17 Solution 80

Since $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = 0$, we have $\sqrt[3]{a^3} + \sqrt[3]{b^3} + \sqrt[3]{c^3} = 3abc$ by corollary 1.4. But $3abc = 3$, so $\sqrt[3]{a^3} + \sqrt[3]{b^3} + \sqrt[3]{c^3} = a + b + c = \boxed{3}$.

B.18 Solution 81 (AHSME 1999/30)

Rewrite the equation as

$$m^3 + n^3 - (-33)^3 - 3 \cdot m \cdot n \cdot -33 = 0.$$

Factoring, we have

$$\frac{1}{2}(m + n - 33)\left((m-n)^2 + (m+33)^2 + (n+33)^2\right) = 0.$$

Thus, either $m + n - 33 = 0$ or $(m-n)^2 + (m+33)^2 + (n+33)^2 = 0$. The answer is evidently $\boxed{35}$.

B.19 Solution 82 (MR S499, Titu Andreescu)

Let $a = x^3, b = y^3$. Then the first equation becomes

$$27x^3y^3(x+y)^3 = 1,$$

$$3xy(x+y) = 1,$$

and the second equation becomes

$$27x^3y^3(x^3+y^3+1) = 1,$$

$$(3x^2y)^3 + (3xy^2)^3 + (-1)^3 - 3(3x^2y)(3xy^2)(-1) = 0,$$

and so we clearly must have that

$$3x^2y + 3xy^2 - 1 = 0$$

as desired. Note that there is some unrigorousness to this proof, because we have not shown these steps are bidirectional; this is left as an exercise to the reader.