Areas

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Issue 1

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1 Introduction

When we think about areas, what pops into our minds?

Formulas. Yes, the many area formulas we've developed. Base times height, sine angle formula, incenter area formula, Heron's, Brahmagupta's, Bretschneider's. We've developed an infinite arsenal as to calculating areas in every single form.

Yet, these are all formulas that have been devised to **use lengths** to **calculate areas**. In fact, the vast majority of geometrical studies interpret areas as an **application** – something that come out of a lot of length manipulations, and is only used for "impure", real life purposes. In fact, even when areas are used as a tool – such as in the proof for Menelaus and Ceva – they are not really thought of outside of such instances, and in such proofs are merely dismissed as "tricky steps."

What this article seeks to demonstrate is that areas are much more than the *results* of length manipulations. In fact, length manipulations are often much better described simply as the result of area manipulations, and using such areas to describe lengths can perhaps be even **more** versatile and natural than the descriptions the other way around. This article will thus demonstrate how the deceptively simple formulas for triangle areas can be used to great effect, in describing both awkward lengths and areas that would be otherwise difficult to describe.

> *Remark 1.* $[X]$ will denote the area of polygon X.

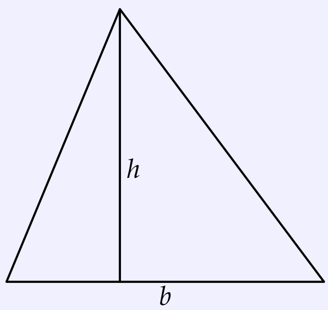
2 Developing Technology

Before we investigate how to use triangle areas, we'll first establish some basic properties of the areas themselves.

For the purposes of this article, we will only be concerning ourselves really with one description of triangle areas:

Theorem 2 (Triangle Area Formula)
Given a triangle with base length b and height length opposite to base h,

$$[\text{triangle}] = \frac{1}{2}bh.$$



Yep, that's all we'll be using today. Personally, I like to think of this formula as a linear **function** in b and h (base and height, respectively), which outputs the area. This is useful because this helps you understand how the area will change, as you fix one value and vary the other.

2.1 Varying the Base

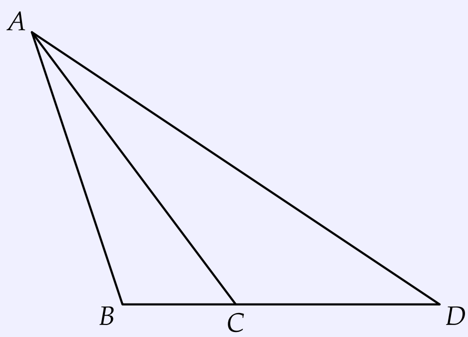
Note that as an immediate consequence of the above theorem, we have:

Theorem 3 (Base Variation)
Given a point A and a line passing through points B, C, and D but not A, we have that

$$\frac{[ABC]}{[ABD]} = \frac{BC}{BD},$$

or (equivalently)

$$\frac{[ABC]}{BC} = \frac{[ABD]}{BD}.$$



This property is essentially what allows us to prop up area ratios given any length ratio: as you can see above, each such length ratio can translate to a triangle area ratio as well, given one other external point.

A few basic corollaries, to get you started:

Corollary 4
If we have a triangle ABC, and M is the midpoint of BC, then ABM and ACM have the same area.

Corollary 5
Consider a triangle ABC. Let points D and E be on sides AB and AC, respectively. Then we have that

$$\frac{[ADE]}{[ABC]} = \left(\frac{AD}{AB}\right)\left(\frac{AE}{AC}\right).$$

Exercise 6. Prove the above corollary with the sine area formula.

Remark 7. If we let DE be parallel to BC, then we have

$$\frac{[ADE]}{[ABC]} = \left(\frac{AD}{AB}\right)^2.$$

This can be proven by applying the above ratio theorem **twice**.

One thing to note here is how you can vary the base from **any side**. This is one of the reasons why I think triangle area ratios are very powerful – instead of length ratios which work in one direction, triangle ratios give you **three** directions at any time.

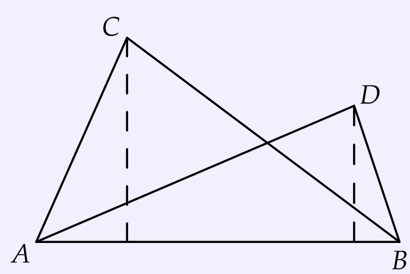
2.2 Varying the Altitude

Again, an immediate consequence of the first theorem:

Theorem 8 (Altitude Variation)
Given two points A, B and two other points C, D, we have that:

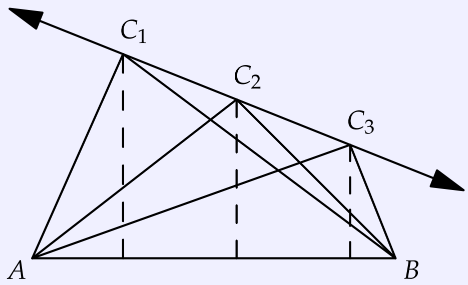
$$\frac{[ABC]}{[ABD]} = \frac{\text{dist}(C, AB)}{\text{dist}(D, AB)},$$

where dist(X, YZ) denotes the length of the altitude dropped from X to YZ.



Perhaps the most important interpretation of this would be the following:

Theorem 9 (Linearity of Area)
Given a line segment AB and a point C moving linearly along the line, the area of $[ABC]$ also moves linearly.



This thus sets up the second tool for us to use with areas: while the Base Variation theorem lets us **create area ratios from lengths**, the Linearity of Area theorem now lets us **move areas given points**. This versatile movement from these properties, is what (as we will see) makes area ratios so powerful.

A few corollaries:

Corollary 10
Let AB be a line segment, and let C and D be two points on the same side of AB. Then if M is the midpoint of CD, then $[ABM]$ is the arithmetic mean of $[ABC]$ and $[ABD]$.

Exercise 11. What happens if C and D are on different sides?

Corollary 12
Let AB be a line segment, and l a line parallel to AB. Then for all points C on l, we have that $[ABC]$ is fixed.

This property is especially important, as it allows us to move the triangle area freely along parallel lines.

Also note that these two properties are not as useful as the first on the AMC contests, but will get much more useful later on. Let's now look at a few more advanced applications of the above.

3 Length Examples

Example 13
Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

Walkthrough.

1. Let's scale ABC's area to equal 1. Using ratios, calculate the area of $B'BC$.
2. Similarly find the area of $BB'C'$ and finish.

Example 14 (One direction of Ceva)
Let ABC be a triangle, and P a point in the triangle, such that AP, BP, CP intersect BC, CA, AB at X, Y, and Z respectively. Prove:

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

Walkthrough. This is the opposite of the previous problem.

1. Let's think of the obvious first, and try and just prop up areas. Can you compute the fraction $\frac{[BPX]}{[CPX]}$ as a length ratio?

2. The equation now has a bunch of areas, but doesn't exactly simplify nicely. Let's try to instead use the area BPC and its cyclic variants as that's more symmetric – can you find a ratio between some of ($[BPC]$, $[CPA]$, $[APB]$) that equals $\frac{BX}{XC}$?

3. Finish.

This final example is black magic:

Example 15 (APMO 2013/1)
Let ABC be an acute triangle with altitudes AD, BE, and CF, and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC, OE dissect the triangle ABC into three pairs of triangles that have equal areas.

Walkthrough.

1. What are the pairs?

2. Instead of manipulating the bases in this problem, we'll be manipulating the **heights**. Let's add the diameter AA' of (ABC). How do the areas of OBF and $A'BF$ compare?

3. Prove that CF and $A'B$ are parallel. What more can we do with the areas now?

4. Finish.

Moral
I think it's important to note here that a lot of the work here seems *manipulative* – that is, the constructions and ratios you're propping up are almost like algebraic manipulations. In general, I think a good way to put this area technique is as a sort of "visual manipulation" – where you can always switch between ratios on a line and between areas, and doing so may or may not help just like any "algebraic manipulation" in the context of an algebra problem. In addition, just like any algebraic manipulation, areas alone won't finish the trick – you need to push with other ideas (most notably synthetic observations) in order to really use everything to its fullest extent.

4 Describing Areas

Now that we've looked at how triangle areas relate to **lengths**, let's now investigate how triangle areas relate to **other areas**.

This is more simple than it sounds. Basically, what I'm advocating for:

> **Moral**
> Think of polygonal areas as the composition of many triangle areas.

There's more to this that I want to emphasize though – how do we tell which triangle areas to pick? Aside from all the theory that we've developed above, there's one particular indicator that I would like to highlight:

> **Moral**
> Areas are best calculated when given right angles. In particular, if you have a base and a height, you have a triangle area laid right out for you!

In fact, I would go so far as to recommend you **cut up your triangles specifically to utilize this base-height property**. A few examples of what I mean:

Example 16 (Inradius Area Formula)
Let ABC be a triangle, and r its inradius. Prove that if p is the perimeter of the triangle,

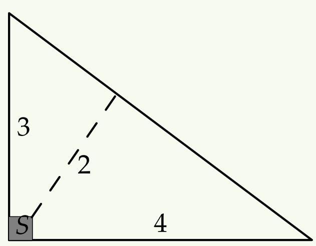
$$[ABC] = \frac{pr}{2}$$

Walkthrough.

1. What does the inradius condition tell us about the I-height of IBC (I being the incenter)?
2. Cut up the triangle area and win.

Finally, perhaps my favourite AMC problem of all time:

Example 17 (AMC 12A 2018/17)
Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



Walkthrough.

1. That 2 is really weird. Do you see any lines perpendicular to the line of length 2, that we could use to calculate an area?
2. Given the area of the triangle above, how can we calculate the rest of the area?

5 Problems

5.1 Easy Problems

Problem 1 (AMC 10B 2005/23). In trapezoid $ABCD$ we have $\overline{AB}$ parallel to $\overline{DC}$, E as the midpoint of $\overline{BC}$, and F as the midpoint of $\overline{DA}$. The area of $ABEF$ is twice the area of $FECD$. What is AB/DC?

Problem 2 (AMC 10B 2004/18). In the right triangle $\triangle ACE$, we have $AC = 12$, $CE = 16$, and $EA = 20$. Points B, D, and F are located on AC, CE, and EA, respectively, so that $AB = 3$, $CD = 4$, and $EF = 5$. What is the ratio of the area of $\triangle DBF$ to that of $\triangle ACE$?

Problem 3 (AMC 10B 2017/21). In $\triangle ABC$, $AB = 6$, $AC = 8$, $BC = 10$, and D is the midpoint of $\overline{BC}$. What is the sum of the radii of the circles inscribed in $\triangle ADB$ and $\triangle ADC$?

Problem 4 (AMC 10A 2018/24). Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

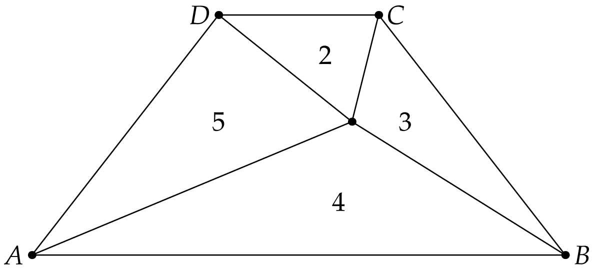
Problem 5 (AMC 12B 2018/13). Square $ABCD$ has side length 30. Point P lies inside the square so that $AP = 12$ and $BP = 26$. The centroids of $\triangle ABP$, $\triangle BCP$, $\triangle CDP$, and $\triangle DAP$ are the vertices of a convex quadrilateral. What is the area of that quadrilateral?

Problem 6 (AIME II 2019/7). Triangle ABC has side lengths $AB = 120$, $BC = 220$, and $AC = 180$. Lines ℓ_A, ℓ_B, and ℓ_C are drawn parallel to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively, such that the intersection of ℓ_A, ℓ_B, and ℓ_C with the interior of $\triangle ABC$ are segments of length 55, 45, and 15, respectively. Find the perimeter of the triangle whose sides lie on ℓ_A, ℓ_B, and ℓ_C.

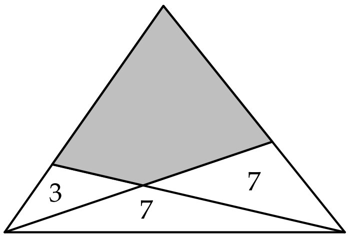
Problem 7 (AMC 10B 2004/20). In $\triangle ABC$ points D and E lie on BC and AC, respectively. If AD and BE intersect at T so that $\frac{AT}{DT} = 3$ and $\frac{BT}{ET} = 4$, what is $\frac{CD}{BD}$?

Problem 8 (AMC 12A 2009/20). Convex quadrilateral $ABCD$ has $AB = 9$ and $CD = 12$. Diagonals AC and BD intersect at E, $AC = 14$, and $\triangle AED$ and $\triangle BEC$ have equal areas. What is AE?

Problem 9 (AMC 12B 2021/17). Let $ABCD$ be an isosceles trapezoid having parallel bases $\overline{AB}$ and $\overline{CD}$ with $AB > CD$. Line segments from a point inside $ABCD$ to the vertices divide the trapezoid into four triangles whose areas are $2, 3, 4$, and 5 starting with the triangle with base $\overline{CD}$ and moving clockwise as shown in the diagram below. What is the ratio $\frac{AB}{CD}$?



Problem 10 (AMC 10B 2006/23). A triangle is partitioned into three triangles and a quadrilateral by drawing two lines from vertices to their opposite sides. The areas of the three triangles are 3, 7, and 7, as shown. What is the area of the shaded quadrilateral?



Problem 11 (AMC 10A 2008/18). A right triangle has perimeter 32 and area 20. What is the length of its hypotenuse?

Problem 12 (Eric Shen). Let ABC be a triangle with $AB = 9$, $BC = 10$, $CA = 11$. Consider the circle tangent to BC inside of the triangle such that the center of the circle is a distance of 2 away from both AB and AC. Compute the radius of this circle.

Problem 13 (HMMT February Geometry 2018/5). In the quadrilateral $MARE$ inscribed in a unit circle ω, AM is a diameter of ω, and E lies on the angle bisector of $\angle RAM$. Given that triangles RAM and REM have the same area, find the area of quadrilateral $MARE$.

Problem 14 (HMMT February Team 2019/1). Let $ABCD$ be a parallelogram. Points X and Y lie on segments AB and AD respectively, and AC intersects XY at point Z. Prove that

$$\frac{AB}{AX} + \frac{AD}{AY} = \frac{AC}{AZ}.$$

Problem 15 (PUMaC Geometry 2018/A5). Let $\triangle ABC$ be a triangle with side lengths $AB = 9, BC = 10, CA = 11$. Let O be the circumcenter of $\triangle ABC$. Denote $D = AO \cap BC, E = BO \cap CA, F = CO \cap AB$. If $\frac{1}{AD} + \frac{1}{BE} + \frac{1}{FC}$ can be written in simplest form as $\frac{a\sqrt{b}}{c}$, find $a + b + c$.

Problem 16 (AIME II 2018/12). Let $ABCD$ be a convex quadrilateral with $AB = CD = 10$, $BC = 14$, and $AD = 2\sqrt{65}$. Assume that the diagonals of $ABCD$ intersect at point P, and that the sum of the areas of $\triangle APB$ and $\triangle CPD$ equals the sum of the areas of $\triangle BPC$ and $\triangle APD$. Find the area of quadrilateral $ABCD$.

5.2 Challenge Problems

For those of you vying for a challenge this issue, here you go!

Problem 17 (APMO 2004/2). Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH and COH is equal to the sum of the areas of the other two.

Problem 18 (IMO 2007/4). In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

Problem 19 (CMO 2020/2). $ABCD$ is a fixed rhombus. Segment PQ is tangent to the inscribed circle of $ABCD$, where P is on side AB, Q is on side AD. Show that, when segment PQ is moving, the area of $\triangle CPQ$ is a constant.

Problem 20 (CMOQR 2017/8, edited). Find all quadrilaterals $ABCD$ such that for all points P in its interior, $[PAB] + [PCD] = [PBC] + [PDA]$.

Problem 21 (ISL 2007/G5). Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle. Let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C', respectively. Assume that the points A, B, C, A', B', C' are distinct, and lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

Problem 22 (Russia MO 2001/10.7). Points A_1, B_1, C_1 inside an acute-angled triangle ABC are selected on the altitudes from A, B, C respectively so that the sum of the areas of triangles ABC_1, BCA_1, and CAB_1 is equal to the area of triangle ABC. Prove that the circumcircle of triangle $A_1B_1C_1$ passes through the orthocenter H of triangle ABC.

Problem 23 (Newton-Gauss Line). Let $ABCD$ be a quadrilateral, such that AB and CD intersect at E, and BC and DA intersect at F. Prove that the midpoints of AC, BD, and EF are collinear.

Problem 24 (Folklore). Let $ABCD$ be a quadrilateral circumscribed around a circle with center I. Prove that I lies on the line joining the midpoints of AC and BD.

Problem 25 (IZhO 2021/2). In a convex cyclic hexagon $ABCDEF$, $BC = EF$ and $CD = AF$. Diagonals AC and BF intersect at point Q, and diagonals EC and DF intersect at point P. Points R and S are marked on the segments DF and BF respectively so that $FR = PD$ and $BQ = FS$. The segments RQ and PS intersect at point T. Prove that the line TC bisects the diagonal DB.

Problem 26 (ISL 2019/G5). Let $ABCDE$ be a convex pentagon with $CD = DE$ and $\angle EDC \neq 2 \cdot \angle ADB$. Suppose that a point P is located in the interior of the pentagon such that $AP = AE$ and $BP = BC$. Prove that P lies on the diagonal CE if and only if $[BCD] + [ADE] = [ABD] + [ABP]$.

6 Selected Solutions

6.1 Solution 13

Scale so that $[ABC] = 1$. Now $[B'BC] = \frac{B'B}{AB}[ABC] = 3$, and $[B'BC'] = \frac{BC'}{BC}[B'BC] = 12$, so $[A'B'C] = 3 \times 12 + 1 = \boxed{37}$.

6.2 Solution 14 (One direction of Ceva)

Length chase:

$$\begin{aligned}
&\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} \\
&= \frac{[APB]}{[CPA]} \cdot \frac{[BPC]}{[APB]} \cdot \frac{[CPA]}{[BPC]} \\
&= 1.
\end{aligned}$$

6.3 Solution 15 (APMO 2013/1)

Let A' be the reflection of A across O. Note that $A'B$ is parallel to CF, and $A'C$ is parallel to BE. Thus, we have:

$$\begin{aligned}
[BOF] &= \frac{1}{2}[BFA'] \\
&= \frac{1}{2}[BCA'] \\
&= \frac{1}{2}[ECA'] \\
&= [ECO],
\end{aligned}$$

where the first and last equalities are due to the base-length triangle area formula.

6.4 Solution 16 (Inradius Area Formula)

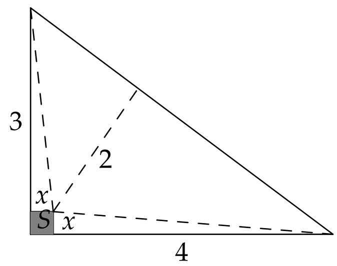
Let the incenter of ABC be I. Then:

$$\begin{aligned}
[ABC] &= \sum_{\text{cyc}} [BIC] \\
&= \sum_{\text{cyc}} \frac{1}{2} r \cdot BC \\
&= \frac{pr}{2}
\end{aligned}$$

6.5 Solution 17 (AMC 12A 2018/17)

The following solution is taken from the first solution of the respective AoPS Wiki page found here.

Solution via splitting the figure Let the square have side length x. Connect the upper-right vertex of square S with the two vertices of the triangle's hypotenuse. This divides the triangle in several regions whose areas must add up to the area of the whole triangle, which is 6.



Square S has area x^2, and the two thin triangle regions have area $\dfrac{x(3-x)}{2}$ and $\dfrac{x(4-x)}{2}$. The final triangular region with the hypotenuse as its base and height 2 has area 5. Thus, we have

$$x^2 + \frac{x(3-x)}{2} + \frac{x(4-x)}{2} + 5 = 6$$

Solving gives $x = \dfrac{2}{7}$. The area of S is $\dfrac{4}{49}$ and the desired ratio is $\dfrac{6 - \frac{4}{49}}{6} = \boxed{\dfrac{145}{147}}$.

Different finish via metasolving Alternatively, once you get $x = \frac{2}{7}$, you can avoid computation by noticing that there is a denominator of 7, so the answer must have a factor of 7 in the denominator, which only $\boxed{\dfrac{145}{147}}$ does.