2020 Mock JMO Solutions

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Problem 1. Determine, with proof, whether there exists a positive integer n such that $4^n - 1$ divides $5^n - 1$.

Proposed by Andrew Wen

Solution (Andrew Wen and William Yue). We will use proof by contradiction. Suppose that for some positive integer n, $4^n - 1 | 5^n - 1$. Since

$$4^n - 1 \equiv 1 - 1 \equiv 0 \pmod 3$$

we see that 3 divides $4^n - 1$ so it must also divide $5^n - 1$. Therefore,

$$5^n - 1 \equiv 2^n - 1 \equiv 0 \pmod 3$$

so n must be even.

Let $n = 2m$ for some positive integer m. We get that $16^m - 1 | 25^m - 1$, and since

$$16^m - 1 \equiv 1 - 1 \equiv 0 \pmod 5$$

it follows that $5 | 16^m - 1$ so $5 | 25^m - 1$, which is clearly a contradiction, as desired. ■

Problem 2. For each integer $n \geq 3$, find the number of ways to color each square black or white in an n by n grid of unit squares such that every rectangle defined by the gridlines with an area that is a multiple of 6 contains an even number of black squares.

Proposed by Anthony Wang, Andrew Wen, and William Yue

Solution (Anthony Wang). We claim that the answer is $\boxed{32}$, for all such n.

Call the left 3 squares of the top row and the left 2 squares of the second highest row *special*. We claim that there exists a bijection between all possible colorings of special squares and all possible colorings of the whole grid satisfying the given conditions.

Assign each square a value, 1 if it is black and 0 otherwise. Let (x, y) be the value of the square at the xth highest row and the y leftmost column. Work over $\mathbb{F}_2$. The given condition is equivalent to saying that the sum of the values of any rectangle of area 6 is 0, since any rectangle with area divisible by 6 can be dissected into rectangles of area 6.

We begin with the following claim:

Claim 1: If a grid satisfies the given condition, then $(x, y) = (x + 3r, y + 3s)$ for all integers x, y, r, s where the two are both defined.

Proof: It suffices to show that $(x, y) = (x + 3, y)$. The proof is simple, assign the squares values as follows, where $a = (x, y)$,

a	b	c
d	e	f
g	h	i
j	k	ℓ

We wish to show that $a = j$. But

$$b + c + e + f + h + i = 0 \quad \text{and} \quad e + f + h + i + k + \ell = 0$$

by the condition, and similarly

$$d + e + f + g + h + i = 0.$$

Adding the first two gives,

$$b + c + k + \ell = 0,$$

thus

$$0 = b + c + d + e + f + g + h + i + k + \ell = 0 - a + 0 - j = -a - j.$$

Hence $a + j = 0$, so $a = j$, as desired. $\square$

Now this means that the board is periodic in 3×3 squares.

Claim 2: The coloring of the special squares uniquely determine the coloring of the top 3×3, and this 3×3 square satisfies the conditions.

Proof: Again, assign the squares values as follows, where the special squares are colored gray.

a	b	c
d	e	f
g	h	i

Now it is easy that by the condition,

$$b+c+e+f+h+i=0 \quad \text{and} \quad d+e+f+g+h+i=0,$$

so summing them gives $g=b+c+d$. Now similarly, we obtain

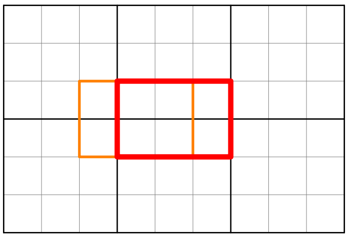
$$a+b+c+d+e+f=0 \quad \text{and} \quad a+b+d+e+g+h=0,$$

which give $f=a+b+c+d+e$, and $h=a+b+d+e+g=a+c+e$. Now similarly, we can compute

$$i=d+e+f+g+h=b+g+e.$$

Now it remains to check that $b+c+e+f+h+i=0$, since our last 3 steps using the other 3 rectangles are reversible. But since $g=b+c+d$, $b+c+e+f+h+i$ and $d+e+f+g+h+i$ sum to 0, so we are done. $\square$

Now it is easy to see by periodicity that every 6×1 rectangle satisfies the conditions, so it suffices to show that every 2×3 rectangle not completely in a 3×3 square equivalent to the top left 3×3 also satisfies the condition.



WLOG it is horizontal, i.e. it covers 2 rows and 3 columns. Notice that shifting the box horizontally does not affect the blocks wrt the top 3×3 square it covers, thus move it so that its left and right edges align with the left and right edges of a 3×3 (In the above case the orange rectangle becomes the red one).

If it is completely inside a 3×3, we are done. Otherwise (like in the case of the above), we wish to show that the top rows and bottom rows of the top left 3×3 square is 0. But this is easy, using the notation from **Claim 2**, we wish to prove that $a+b+c+g+h+i=0$. Now

$$a+b+c+d+e+f=0 \quad \text{and} \quad d+e+f+g+h+i=0,$$

and adding them gives the desired.

Thus the coloring of the five special squares uniquely determine the coloring of the whole grid, and each of these colorings work, so the answer is $2^5=32$, as desired. $\blacksquare$

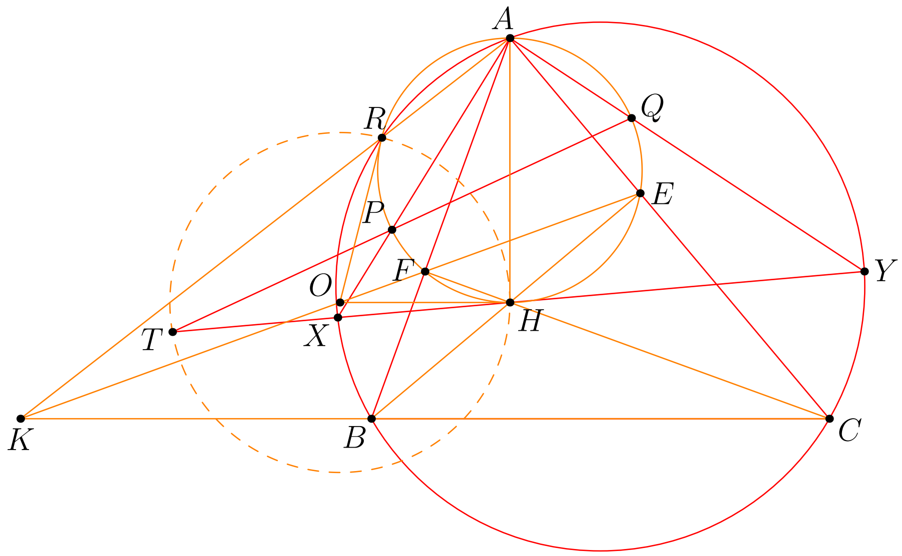
Problem 3. Let H be the orthocenter of acute triangle $\triangle ABC$. X and Y are points on the circumcircle of triangle $\triangle ABC$ such that H lies on chord XY. Then, let P and Q be the feet of the altitudes from H onto AX and AY, respectively, and let line PQ intersect line XY at T.

(i) Prove that as the chord XY containing H varies, point T traces out part of a circle Ω.

(ii) Prove that the center of Ω lies on line EF, where E and F are the feet of the altitudes from B and C to AC and AB, respectively.

Proposed by William Yue

Solution (Andrew Wen and Anthony Wang).



(i) Consider the Miquel point of complete quadrilateral $XPQY$, the second intersection of (AXY) and (APQ), which we denote R. By Miquel point properties, R must lie on (TYQ), hence

$$\angle RTH = \angle RTY = 180^\circ - \angle RQY = \angle RQA = \angle RHA,$$

since clearly, (APQ) has diameter AH. Because A, B, C, H, R in this problem are fixed, angle $\angle RHA$ is fixed, hence $\angle RTH$ is fixed, so as D varies on minor arc BC, T traces out a portion of some circle Ω, as desired. ■

(ii) From Part (i), we see that $\angle RTH$ is always equal to $\angle RHA$, so we see that T must always lie on a circle passing through R, tangent to AH at point H. Thus, Ω passes through R and is tangent to AH at H. Hence if O is the center of Ω, by radius, $OR = OH$, and since OH is tangent to (ARH) at H, we must have OR is tangent to (AEF) at R. It remains to show that $(EF; HR) = -1$.

But notice that by Radical Axis Theorem on (AEF), $(BFEC)$, and (ABC), AR, EF, and BC concur at a point, call it K. Then by Ceva-Menalaus,

$$-1 = (C, B; AH \cap BC, K) \stackrel{A}{=} (E, F; H, R),$$

as desired. pebis ■

Problem 4. Bob has n stacks of rocks in a row, each with heights randomly and uniformly selected from the set $\{1, 2, 3, 4, 5\}$. In each move, he picks a group of consecutive stacks with positive heights and removes 1 rock from each stack. Find, in terms of n, the expected value of the minimum number of moves he must execute to remove all rocks.

Proposed by Anthony Wang

Solution (Anthony Wang). Let the heights of the stacks start at $a_1, a_2, \ldots, a_n$ in that order.

The key is to look at the sequence

$$a_1, a_2 - a_1, a_3 - a_2, \ldots, a_n - a_{n-1}, -a_n.$$

We claim that the minimum number of moves is precisely half the sum of the absolute values of the terms in this sequence.

It can be seen that the terms in this sequence sum to 0, and that each move consists of subtracting 1 from one of these on the left and adding 1 to another on the right, i.e. "moving" 1 rightward.

Now notice that since these terms sum to 0, the sum of the positive terms is equal to the sum of the absolute values of the negative terms, thus we wish to show that the minimum number of moves is the sum of the positive terms.

But notice that in order remove all rocks, our sequence must contain all 0's, hence we must "move" all the 1's from the positive terms to the negative terms so that they cancel out. Thus it is easy to see that we require at most as many moves as half the sum of the absolute values of the terms, since it is exactly the number of 1's in the positive terms.

Now we must show that there exists a sequence of moves that keeps the prefix sums of our sequence nonnegative. But this is easy, simply use the leftmost positive term to "feed" the leftmost negative term, and switch to the next positive term if one becomes 0. Then the prefix sum of the sequence up to before the first negative is clearly nonnegative, and the subsequent sums are all invariant under this move, so we are done.

Now it remains to calculate the expected value. But it is easy to see that two randomly picked numbers from $\{1, 2, 3, 4, 5\}$ have an expected absolute difference of 1.6 through pure inspection, and the expected values of $|a_1|$ and $|-a_n|$ are both 3. Thus the expected value of the minimum number of moves is

$$\frac{1}{2}(1.6(n-1) + 3 + 3) = 0.8n + 2.2,$$

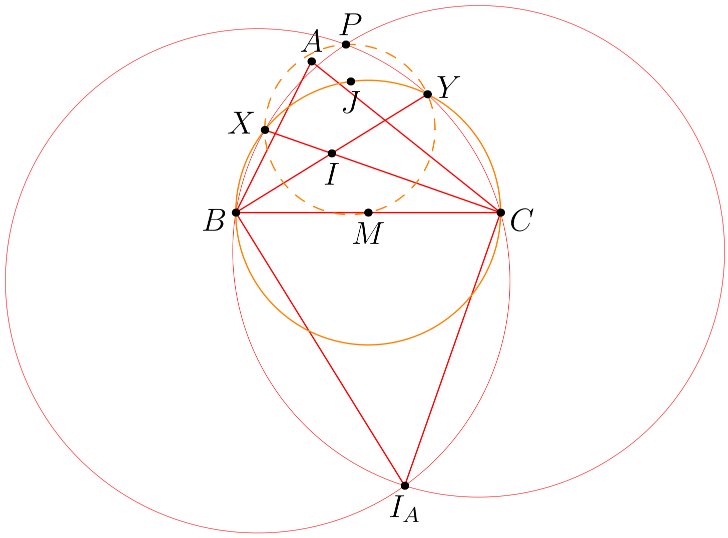
and we are done.

$\blacksquare$

Problem 5. Let $\triangle ABC$ be a triangle with A-excenter I_A. Let X and Y be the feet of the perpendiculars from B and C to the angle bisectors of $\angle ACB$ and $\angle ABC$, respectively. If the circumcircles of I_ABX and I_ACY meet again at P, show that $\angle BJC = 90°$, where J is the incenter of triangle $\triangle PXY$.

Proposed by Andrew Wen

Solution (Andrew Wen and Anthony Wang).



Note that what we are asked to prove is equivalent to J lies on the circle with diameter BC. Let M be the midpoint of BC.

Claim: $PXMY$ is cyclic.

Proof: We simply angle chase. Notice that

$$\begin{aligned}\angle XPY &= \angle XPI_A + \angle YPI_A \\ &= (180° - \angle XBI_A) + (180° - \angle YCI_A) \\ &= 180° - (90° + \angle XBY) + 180° - (90° + \angle XCY) \\ &= 180° - 2 \cdot \frac{1}{2}\angle XMY = 180° - \angle XMY\end{aligned}$$

Therefore, $\angle XMY + \angle XPY = 180°$, proving the cyclicity, as desired. $\square$

Note that now, since $MX = MY$, M is midpoint of arc XY not containing P in $(PXMY)$. Finally, by the Incenter-Excenter Lemma, $MX = MY = MJ$, and we are done. $\blacksquare$

Problem 6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x-2y) + f(x+f(y)) = x + f(f(x)-y)$$

for all $x, y \in \mathbb{R}$.

Proposed by Andrew Wen

Solution (Andrew Wen). We claim that $f(x) = x$ is our only solution. A quick check shows that this does indeed work.

Let $P(x, y)$ denote the assertion. Note that $P(x, x - f(x))$ yields $f(x + f(x - f(x))) = x$, hence f is surjective. $P(x, 0)$ gives $2f(x) = x + f(f(x))$. Now we can prove that f is also injective. Consider a, b such that $f(a) = f(b)$. Then, $b + f(f(b)) = 2f(b) = 2f(a) = a + f(f(a))$ and since $f(a) = f(b)$, taking the f of both sides yields $f(f(a)) = f(f(b))$ so $a = b$, as desired. Therefore, f is a bijection.

The rest of the problem is just abusing the property that $x + f(f(x)) = 2f(x)$. First off, $P(0, 0)$ gives $f(0) = 0$. Next, we consider $P(f(x), f(x) - x)$, which yields

$$f(2x - f(x)) + f(f(x) + f(f(x) - x)) = x + f(f(f(x)) - f(x) + x).$$

Note that $x + f(f(x)) = 2f(x) \implies f(f(x)) - f(x) = f(x) - x$, and using this substitution simplifies the RHS to $x + f(f(x))$, so now

$$f(2x - f(x)) + f(f(x) + f(f(x) - x)) = x + f(f(x)).$$

Plug in $f(y) = x$, and we get:

$$f(2f(y) - f(f(y))) + f(f(f(y)) + f(f(f(y)) - f(y))) = f(y) + f(f(f(y))).$$

Again, we apply $2f(y) - f(f(y)) = y$ and $f(f(y)) - f(y) = f(y) - y$, which simplifies this to:

$$f(f(f(y)) + f(f(y) - y)) = f(f(f(y)))$$

and using injectivity tells us that $f(f(y)) + f(f(y) - y) = f(f(y)) \implies f(f(y) - y) = f(0) = 0$ and applying injectivity once more gives $f(y) - y = 0 \implies f(y) = y$ for all reals y, as desired. $\blacksquare$