2020 Mock USAJMO

Andrew Wen, Anthony Wang, William Yue

Day I

April 7th to April 21st, 2020

Note: For any geometry problem whose statement begins with an asterisk $(*)$, the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

JMO 1. Determine, with proof, whether there exists a positive integer n such that $4^n - 1$ divides $5^n - 1$.

JMO 2. For each integer $n \geq 3$, find the number of ways to color each square black or white in an n by n grid of unit squares such that every rectangle defined by the gridlines with an area that is a multiple of 6 contains an even number of black squares.

JMO 3. $(*)$ Let H be the orthocenter of acute triangle ABC. Points X and Y lie on the circumcircle of triangle $\triangle ABC$ such that H lies on chord XY. Let P and Q be the feet of the altitudes from H onto $\overline{AX}$ and $\overline{AY}$, respectively, and let line PQ intersect line XY at T.

(i) Prove that as chord XY varies, point T moves along a circle Ω.

(ii) Let E and F be the feet of the altitudes from B and C to $\overline{AC}$ and $\overline{AB}$, respectively. Prove that the center of Ω lies on line EF.

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JMO 4. Bob has n stacks of rocks in a row, each with heights randomly and uniformly selected from the set $\{1, 2, 3, 4, 5\}$. In each move, he picks a group of consecutive stacks with positive heights and removes 1 rock from each stack. Find, in terms of n, the expected value of the minimum number of moves he must execute to remove all rocks.

JMO 5. ($*$) Let ABC be a triangle with A-excenter I_A, and let X and Y be the feet of the perpendiculars from B and C to the angle bisectors of $\angle ACB$ and $\angle ABC$, respectively. The circumcircles of $\triangle I_ABX$ and $\triangle I_ACY$ meet again at P, and J is the incenter of $\triangle PXY$. Prove that $\angle BJC = 90^\circ$.

JMO 6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - 2y) + f(x + f(y)) = x + f(f(x) - y)$$

for all real numbers x, y.