2020 Mock USAJMO

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Day I

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Note: For any geometry problem whose statement begins with an asterisk ($*$), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

JMO 1. Determine, with proof, whether there exists a positive integer n such that $4^n - 1$ divides $5^n - 1$.

JMO 2. For each integer $n \geq 3$, find the number of ways to color each square black or white in an n by n grid of unit squares such that every rectangle defined by the gridlines with an area that is a multiple of 6 contains an even number of black squares.

JMO 3. $(*)$ Let H be the orthocenter of acute triangle ABC. Points X and Y lie on the circumcircle of triangle $\triangle ABC$ such that H lies on chord XY. Let P and Q be the feet of the altitudes from H onto $\overline{AX}$ and $\overline{AY}$, respectively, and let line PQ intersect line XY at T.

(i) Prove that as chord XY varies, point T moves along a circle Ω.

(ii) Let E and F be the feet of the altitudes from B and C to $\overline{AC}$ and $\overline{AB}$, respectively. Prove that the center of Ω lies on line EF.