Summer Mock AIME 2019*

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1 Rules

1. DO NOT SCROLL DOWN AND/OR LOOK AT THE PROBLEMS UNTIL YOU GIVE YOURSELF THE SIGNAL TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor a penalty for wrong answers.
3. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smart-watches, or computing devices are allowed. No problems on the test will require the use of a calculator.
4. You cannot qualify for the USA(J)MO through this exam.
5. Record all of your answers, and general feedback for the test on a private message to cosmicgenius, jj_ca888, and sriraamster.

*Sadly we will not make a booklet version.

2 Problems

1. There exists a not necessarily convex quadrilateral $ABCD$ such that

$$\angle B = \angle C = 60^\circ \quad \text{and} \quad \angle A = 30^\circ.$$

Lines AB and CD intersect at E, lines AD and BC intersect at F, and EF meets BD at P. If $CF = AE = 1$, then EP^2 can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find the value of $m + n$.

2. Let $f(x) = \log_2(x)$ for all $x > 0$. Find the sum of all x for which

$$x^{95+f\left(x^{f(x)-18}\right)} = 2^{126}$$

holds true.

3. Let $(a_1, a_2, \ldots, a_8)$ be a permutation of the set $\{1, 2, 3, \ldots, 7, 8\}$. We are given that for all integers $3 \leq i \leq 6$, the quantity $a_{i-2} + a_i + a_{i+2}$ is divisible by 3. How many such permutations are there?

4. Let $k(n)$ be the n^{th} smallest positive integer relatively prime to n. Find the sum of all primes $p \leq 100$ such that the equation

$$k(p + 1) + k(p) = k(2p)$$

is satisfied.

5. Let a_n be a recursively defined sequence such that $a_1 = 0$, and for all integers $n > 1$, we have

$$a_{n+1} = (2n + 1)a_n + 2n.$$

Compute the last three digits of a_{2019}.

6. A tortoise starts at the bottom left corner of a 4×4 grid of points. Each point is colored red or blue, and there are exactly 8 of each color. It can only move up or right, and can only move to blue points. If the bottom left corner is always blue, find the number of ways we can color the grid such that the tortoise can reach the top right corner.

7. Consider the functions

$$f_1(x) = 2x$$
$$f_2(x) = 8x + 3$$

Find the number of positive integers $n < 131072$ which can be written in the form $f_{a_1}(f_{a_2}(\dots f_{a_k}(1)\dots))$ for some positive integer k and some sequence $(a_1, a_2, a_3 ... a_{k-1}, a_k)$ where each a_i is 1 or 2.

8. Suppose that the polynomial $P(x) = x^{2019} + 20x^{2018} - 19x + 4$ has complex roots $r_1, r_2, r_3 ... r_{2019}$. Consider the monic polynomial $Q(x)$ with degree 2019 such that $Q\left(r_i + \frac{1}{r_i}\right) = 0$ for $i = 1, 2, 3, ...2019$. The value of $\frac{Q(0)}{Q(1)}$ can be expressed as a common fraction $\frac{m}{n}$ for relatively prime positive integers m and n. Find the value of $m + n$.

9. Suppose that $p(n)$ denotes the product of the digits of n. Let S be the sum of all positive integers n such that

$$p(n) = n - 210$$

Find the remainder when S is divided by 1000.

10. Triangle $\triangle ABC$ has side lengths $AB = 3$, $BC = 5$, and $CA = 7$. Define H as the orthocenter of $\triangle ABC$. Then, the circle with center B and radius BH intersects the circumcircle of $\triangle ABC$ at two distinct points X and Y. If XY meets AC at P, the length of BP can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find the value of $m + n$.

11. Fred the frog is on the first lilypad of a row of 16 lilypads, and he can jump to any lilypad. However, if he jumps onto the n^{th} lilypad, he will never jump onto the k^{th} lilypad if either $k < n$ or n divides k. Find the number of ways he can execute a sequence of jumps such that he ends at the last lilypad.

12. Compute the number of ordered pairs of positive integers (m, n) with $m + n \leq 64$ such that there exists at least one complex number z such that $|z| = 1$ and $z^m + z^n + \sqrt{2} = 0$.

13. Suppose that triangle $\triangle ABC$ has side lengths $AB = 20$ and $AC = 19$. Furthermore, let the incircle ω of $\triangle ABC$ touch segments BC, CA, AB at points D, E, and F respectively. Let AD hit ω at a point $P \neq D$. Suppose that DF intersects the circumcircle of $\triangle CDP$ at a point $Q \neq D$. Let line CQ intersect AB at point M. If we are given that $\frac{AM}{BM} = \frac{5}{6}$, then the perimeter of triangle $\triangle ABC$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find the value of $m + n$.

14. Let a, b, c be positive reals such that $abc + a + b = c$ and

$$\frac{19}{\sqrt{a^2+1}} + \frac{20}{\sqrt{b^2+1}} = 31.$$

The maximum possible value of c^2 can be written in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Find the value of $m + n$.

15. Triangle $\triangle ABC$ has side lengths $AB = 13$, $BC = 21$, and $AC = 20$. A point D is selected on the line BC. The circle with diameter AD intersects AB at X, BC at Y, and AC at Z. Denote I as the incenter of triangle $\triangle XYZ$. The minimum possible value of AI can be written as $\frac{m\sqrt{n}}{p}$ where m and p are relatively prime with n not divisible by the square of any prime. Find the value of $m + n + p$.