Summer Mock AIME 2019 - Solutions

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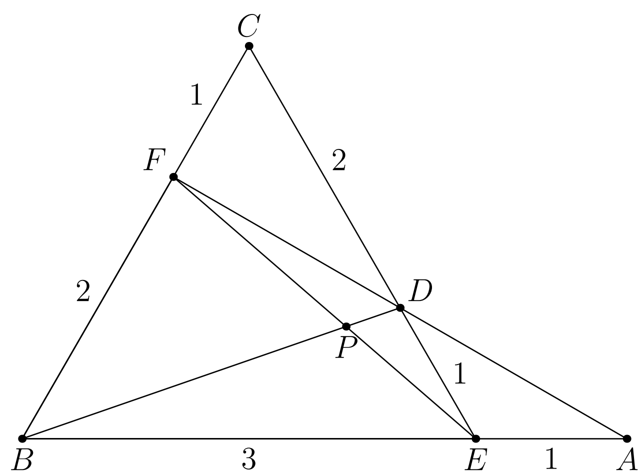
Problem 1

There exists a not necessarily convex quadrilateral $ABCD$ such that

$$\angle B = \angle C = 60^\circ \quad \text{and} \quad \angle A = 30^\circ.$$

Lines AB and CD intersect at E, lines AD and BC intersect at F, and EF meets BD at P. If $CF = AE = 1$, then EP^2 can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find the value of $m+n$.

Proposed by cosmicgenius



We see that since $\angle D = 360^\circ - 30^\circ - 30^\circ - 60^\circ = 240^\circ > 180^\circ$, so $ABCD$ is concave. We have that

$$\angle FCD = 60^\circ \quad \text{and} \quad \angle DFC = \angle 60^\circ + 30^\circ = 90^\circ,$$

so $\triangle FCD$ is a $30-60-90$ triangle. Thus $CD = 2$ and $\angle ADE = \angle CDF = 30^\circ$. So $\triangle DEA$ is isosceles, and $DE = 1$.

Since $\angle B = \angle C = 60^\circ$ and $CE = 3$, $\triangle BCE$ is an equilateral triangle with side length 3. Thus $BF = 2$ and $BE = 3$.

By the Law of Cosines on $\triangle FBE$,

$$EF = \sqrt{2^2 + 3^2 - 2 \cdot 2 \cdot 3 \cos 60^\circ} = \sqrt{7}.$$

Then by Ceva's theorem on $\triangle FBE$,

$$1 = \frac{FC}{CB} \cdot \frac{BA}{EA} \cdot \frac{EP}{FP} = \frac{1}{3} \cdot \frac{4}{1} \cdot \frac{EP}{FP}.$$

Thus

$$\frac{EP}{FP} = \frac{3}{4},$$

and

$$\frac{EP}{EF} = \frac{3}{7}.$$

Therefore

$$EP^2 = \frac{9}{49} \cdot 7 = \frac{9}{7},$$

and the answer is $9 + 7 = \boxed{016}$. ■

Problem 2

Let $f(x) = \log_2(x)$ for all $x > 0$. Find the sum of all x for which

$$x^{95+f\left(x^{f(x)-18}\right)} = 2^{126}$$

holds true.

Proposed by jj_ca888

Our solution mainly use identity $\log_n m^k = k \log_n m$.

In order to do this, we take the f of both sides (since $2^{126} > 0$). Thus

$$f\left(x^{95+f\left(x^{f(x)-18}\right)}\right) = 2^{126} \implies \left(95 + f\left(x^{f(x)-18}\right)\right) f(x) = 126.$$

Using the identity again on $f\left(x^{f(x)-18}\right)$, we obtain $(f(x)-18)f(x)$. Putting this all together, we have

$$(95 + (f(x) - 18)f(x))f(x) = 126.$$

After expanding, we have $f(x)^3 - 18f(x)^2 + 95f(x) - 126 = 0$. However, by using the Rational Root Theorem, we find that $f(x) = 2, 7, 9$ which must be the only solutions since this is a cubic equation. Therefore $x = 2^2, 2^7, 2^9$, and our desired answer is $2^2 + 2^7 + 2^9 = \boxed{644}$. ∎

Problem 3

Let $(a_1, a_2, \ldots, a_8)$ be a permutation of the set $\{1, 2, 3, \ldots, 7, 8\}$. We are given that for all integers $3 \le i \le 6$, the quantity $a_{i-2} + a_i + a_{i+2}$ is divisible by 3. How many such permutations are there?

Proposed by jj_ca888

We split the permutation into two groups, namely

$$(a_1, a_3, a_5, a_7) \quad \text{and} \quad (a_2, a_4, a_6, a_8).$$

Note that in $\{1, 2, 3, \ldots, 7, 8\}$, there are 2 numbers that are 0 mod 3, 3 numbers that are 1 mod 3, and 3 numbers that are 2 mod 3.

Notice that, by the condition, any three adjacent numbers in (a_1, a_3, a_5, a_7) or (a_2, a_4, a_6, a_8) must be all the same, or all different, mod 3. But if some three are all the same, then the last one in the group must also be the same, contradiction. Thus any three adjacent are different.

We see the first and last element of each group are the same mod 3, and each group has at least one of $0, 1, 2$ mod 3. Thus the first elements of the groups must be 1 and 2 mod 3 in some order. The group with first element 1 mod 3 must have the middle two elements $0, 2$ mod 3 in some order, and the other group must have the middle two elements $0, 1$ mod 3 in some order.

We can order the numbers $0, 1, 2$ mod 3 in $2! \cdot 3! \cdot 3!$ ways. We can choose which group has first element 1 mod 3 in 2 ways. We can order the middle two elements of the groups in 2^2 ways. Therefore there are a total of

$$2! \cdot 3! \cdot 3! \cdot 2 \cdot 2^2 = \boxed{576}$$

permutations. ■

Problem 4

Let $k(n)$ be the n^{th} smallest positive integer relatively prime to n. Find the sum of all primes $p \leq 100$ such that the equation

$$k(p+1) + k(p) = k(2p)$$

is satisfied.

Proposed by cosmicgenius

We can check that $k(3) = 4$, $k(2) = 3$, and $k(4) = 7$, so $p = 2$ works. So assume that p is odd.

Since p is prime, we know that all positive integers except multiples of p are relatively prime to p. Thus we can compute the values of $k(p)$ and $k(2p)$.

Since every positive integer less than p is relatively prime to p, and there are $p-1$ of these, $p+1$ must be the p^{th} smallest positive integer relatively prime to p. Thus $k(p) = p+1$.

Every odd positive integer less than $4p$ is relatively prime to $2p$ except p and $3p$ (which must be odd), and there are $2p-2$ of these. The next two positive integers that are relatively prime are $4p+1$ and $4p+3$ is $4p+2$ is divisible by 2. Thus $k(2p) = 4p+3$.

Now we have

$$k(p+1) = (4p+3) - (p+1) = 3p+2 = 3(p+1) - 1.$$

But the amount of positive integers less than $3(p+1)$ relatively prime to $p+1$ is $3\phi(p+1)$, and $3(p+1)-1$ is the last positive integer less than $3(p+1)$ relatively prime ot $p+1$, thus

$$p+1 = 3\phi(p+1).$$

If $p_1, p_2, \ldots, p_\ell$ are the primes dividing $p+1$, then

$$p+1 = 3(p+1)\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_\ell}\right)$$

so

$$\frac{(p_1-1)(p_2-1)\ldots(p_\ell-1)}{p_1p_2\ldots p_\ell} = \frac{1}{3}$$

The denominator must have a factor of 3, so one of the primes is 3. But then the numerator will have a factor of $3-1=2$, so the denominator must have a factor of 2. Thus one of the primes must be 2. WLOG assume $p_1 = 2$, $p_2 = 3$. Then

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{p_3-1}{p_3} \cdot \cdots \cdot \frac{p_\ell - 1}{p_\ell} = \frac{1}{3}$$

If $\ell > 2$, then

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{p_3-1}{p_3} \cdot \cdots \cdot \frac{p_\ell - 1}{p_\ell} < \frac{1}{2} \cdot \frac{2}{3} \cdot 1 \cdot \cdots \cdot 1 = \frac{1}{3},$$

contradiction. So $\ell = 2$, and $p+1 = 2^a3^b$ for some positive integers a, b.

It remains to find all primes such that $p = 2^a3^b - 1$ and $p \leq 100$. We must check $p = 5, 11, 23, 47, 95, 17, 35, 71, 53$. We see that $p = 5, 11, 17, 23, 47, 53, 71$ work. Thus the answer is

$$5 + 11 + 17 + 23 + 47 + 53 + 71 + 2 = \boxed{229}. \blacksquare$$

Problem 5

Let a_n be a recursively defined sequence such that $a_1 = 0$, and for all integers $n > 1$, we have

$$a_{n+1} = (2n+1)a_n + 2n.$$

Compute the last three digits of a_{2019}.

Proposed by jj_ca888

Let $(2m-1)!!$ denote the product

$$\prod_{i=1}^{m}(2i-1)$$

for all positive integers m.

Claim: $a_n = (2n-1)!! - 1$

Proof: We will proceed with induction.

Our base case $n = 1$ is clearly true since $a_1 = 1!! - 1 = 0$, which is just one of the problem conditions.

For our inductive step, assume that $a_k = (2k-1)!! - 1$. We see that

$$a_{k+1} = (2k+1)((2k-1)!! - 1) + 2k = (2k+1)!! - 1$$

as desired. $\square$

Hence, the problem reduces to finding the last three digits of 4037!!. We will do this by taking mod 8 and mod 125. It is easy to see that 4037!! is divisible by 125, so we only need to consider mod 8. Notice the following identity:

$$(8k+1)(8k+3)(8k+5)(8k+7) \equiv 1 \cdot 3 \cdot 5 \cdot 7 \equiv 1 \pmod{8}$$

for all positive integers k. Therefore,

$$4037!! \equiv 4039^{-1} \prod_{i=1}^{1008}(8i+1)(8i+3)(8i+5)(8i+7) \equiv 4039^{-1} \equiv (-1)^{-1} \pmod{8}$$

must be true. But -1 is clearly its own inverse mod 8, so we have $4037!! \equiv 0 \pmod{125}$ and $4037!! \equiv 7 \pmod{8}$. Solving the congruence system yields $4037!! \equiv 375 \pmod{1000}$, so our final answer is

$$a_{2019} \equiv 375 - 1 \equiv \boxed{374} \pmod{1000}. \blacksquare$$

Problem 6

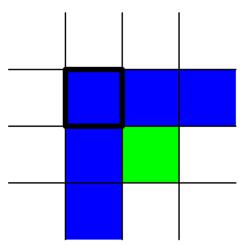
A tortoise starts at the bottom left corner of a 4×4 grid of points. Each point is colored red or blue, and there are exactly 8 of each color. It can only move up or right, and can only move to blue points. If the bottom left corner is always blue, find the number of ways we can color the grid such that the tortoise can reach the top right corner.

Proposed by cosmicgenius

Notice that each normal path of blue squares from the bottom left corner to the top right takes $4 + 4 - 1 = 7$ moves, so we simply wish to place one more blue square (the red squares will then be predetermined).

We can represent each path as a sequence of 3 u's and 3 r's representing whether the tortoise will move up or right.

Define the number of *turning points* of a path as the number of times the tortoise turns, i.e. the amount of times r is followed by u or u is followed by r. We create another valid path by adding the last blue square if and only if it was added at the corner of a turning point.



The outlined square is a turning point, and the green square is its corner.

Let $p(t)$ be the number of paths with t turning points.

Claim: If $t = 2k$, $p(t) = 2\binom{2}{k}\binom{2}{k-1}$. If $t = 2k+1$, $p(t) = 2\binom{2}{k}^2$. *Proof:* We will divide the proof into even and odd cases.

Case 1: $t = 2k$.

We wish to find the number sequences of 3 u's and 3 r's such that there are k "groups" of one letter and $k+1$ of another. The r case is the same as the u case, so WLOG there are k groups of u. Then if R represents groups of r's and U represents groups of u's, the sequence must be in the form

$$RUR \dots R.$$

We must distribute 3 u's to k groups and 3 r's to $k+1$ groups, so there are $\binom{2}{k}\binom{2}{k-1}$ ways by Stars and Bars. We must multiply by 2 for the r case, so there are $2\binom{2}{k}\binom{2}{k-1}$ ways.

Case 2: $t = 2k+1$.

This is almost identical to the $t = 2k$ case, except we must 3 u's to k groups and 3 r's to $k+1$ groups, and we multiply by 2 to choose if a u or an r is first.

We have exhausted all cases, so we are done. $\square$

Now for the problem, we can simply find the number of working colorings (paths with the extra blue square) and add them all up. Each path can have between 1 and 5, inclusive, turning points, so for each t from 1 to 5, there are $p(t)$ paths and $16-7-t=9-t$ other squares that are not turning point corners. Thus there are

$$p(t)(9-t)$$

paths with an extra blue square that are not turning point corners.

If the extra blue square is at one of the turning point corners, it will be counted again by another path, thus we must divide each one of those by 2. Thus there are

$$\frac{1}{2}tp(t)$$

paths with an extra blue square that are turning point corners.

We can calculate, based on the claim, that $p(1)=2, p(2)=4, p(3)=8, p(4)=4, p(5)=2$. So in total there are

$$\begin{aligned}\sum_{t=1}^{5} p(t)(9-t)+\frac{1}{2}tp(t) &= \sum_{t=1}^{5}\frac{1}{2}p(t)(18-t)\\ &= \frac{1}{2}(2\cdot 17+4\cdot 16+8\cdot 15+4\cdot 14+2\cdot 13)\\ &= 17+32+60+28+13\\ &= \boxed{150}\end{aligned}$$

ways, so we are done. ■

Problem 7

Consider the functions

$$f_1(x) = 2x$$
$$f_2(x) = 8x + 3$$

Find the number of positive integers $n < 131072$ which can be written in the form $f_{a_1}(f_{a_2}(\dots f_{a_k}(1)\dots))$ for some positive integer k and sequence $(a_1, a_2, ...a_k)$ where each a_i is 1 or 2.

Proposed by sriraamster

The key idea is to convert to binary strings. Note that applying f_1 adds the string 0 to the end of the previously inputted number, and applying f_2 adds the string 011 to the end of the previously inputted number.

Let s_n be the number of n-digit strings satisfying the given conditions. We observe the recurrence relation $s_n = s_{n-1} + s_{n-3}$, since we can either add a 0 or a 011 to the end of a $n-1$ or $n-3$ length string respectively.

Now, we have that $s_1 = 1$, $s_2 = 1$, $s_3 = 1$, so $s_4 = 2$, $s_5 = 3$, $s_6 = 4$, $s_7 = 6$, $s_8 = 9$, $s_9 = 13$, $s_{10} = 19$, $s_{11} = 28$, $s_{12} = 41$, $s_{13} = 60$, $s_{14} = 88$, $s_{15} = 129$, $s_{16} = 189$, and $s_{17} = 277$. But since we must have a positive number of f_1's or f_2's, we cannot have 1, i.e. we cannot have 1 digit numbers.

Now, adding all of them up, we get $\boxed{870}$. ■

Problem 8

Suppose that the polynomial $P(x) = x^{2019} + 20x^{2018} - 19x + 4$ has complex roots $r_1, r_2, r_3 \ldots r_{2019}$. Consider the monic polynomial $Q(x)$ with degree 2019 such that $Q\left(r_i + \frac{1}{r_i}\right) = 0$ for $i = 1, 2, 3, \ldots 2019$. The value of $\frac{Q(0)}{Q(1)}$ can be expressed as a common fraction $\frac{m}{n}$ for relatively prime positive integers m and n. Find the value of $m + n$.

Proposed by jj_ca888

We know that based on the roots of $Q(x)$, we can write

$$Q(x) = \prod_{i=1}^{2019} \left(x - \left(r_i + \frac{1}{r_i}\right)\right)$$

Thus, we need to evaluate the expression

$$V = \frac{\prod_{i=1}^{2019} \left(-\left(r_i + \frac{1}{r_i}\right)\right)}{\prod_{i=1}^{2019} \left(1 - \left(r_i + \frac{1}{r_i}\right)\right)}.$$

Multiplying top and bottom by $-r_i^2$ and further simplifying, we obtain

$$V = \frac{\prod_{i=1}^{2019} r_i^2 + 1}{\prod_{i=1}^{2019} r_i^2 - r_i + 1} = \frac{\prod_{i=1}^{2019} (i - r_i)(-i - r_i)}{\prod_{i=1}^{2019} (-e^{\frac{2\pi i}{3}} - r_i)(-e^{\frac{4\pi i}{2}} - r_i)}.$$

We know that

$$P(x) = \prod_{i=1}^{2019} (x - r_i),$$

so the expression we are looking for is actually

$$V = \frac{P(i)P(-i)}{P(-e^{\frac{2\pi i}{3}})P(-e^{\frac{4\pi i}{3}})}.$$

We can easily compute $P(i) = -20i - 16$ and $P(-i) = 20i - 16$ so the numerator is 656. The denominator, however, is a bit trickier to find. For simplicity's sake, define $\omega = e^{\frac{2\pi i}{3}}$. Since ω is a third root of unity, we can conclude $\omega^3 = 1$ and $\omega^2 + \omega = -1$. We may use these properties to simplify our expression.

$$P(-\omega) = -1 + 20\omega^2 + 19\omega + 4 = \omega^2 - 16$$
$$P(-\omega^2) = -1 + 20\omega + 19\omega^2 + 4 = \omega - 16$$

Therefore, our answer is

$$V = \frac{656}{(\omega - 16)(\omega^2 - 16)} = \frac{656}{1 - 16(\omega + \omega^2) + 256} = \frac{656}{273}$$

hence our final answer of $656 + 273 = \boxed{929}$. ■

Problem 9

Suppose that $p(n)$ denotes the product of the digits of n. Let S be the sum of all positive integers n such that

$$p(n) = n - 210$$

Find the remainder when S is divided by 1000.

Proposed by: jj_ca888

Clearly, the product of the digits of any positive integer is nonnegative, so it follows that $n \geq 210$.

Claim: $n < 1000$

Proof: For the sake of contradiction, suppose that n is a k digit number ($k \geq 4$) with leftmost digit $m \neq 0$. We know that since the leftmost digit of n is m, the inequality $10^{k-1}m \leq n$ must be true. Furthermore, we also know that $9^{k-1}m \geq p(n)$ since the maximum possible product of the rest of the $k-1$ digits is attained when each of them is 9. Combining these two inequalities gives

$$n - p(n) \geq (10^{k-1} - 9^{k-1})m.$$

Since $k \geq 4$ and $m \geq 1$, it follows that $n - p(n) \geq 10^3 - 9^3 = 271$. However, we are given in the problem that $n - p(n) = 210$, contradiction. $\square$

Now, suppose that we let $n = 100a + 10b + c$, where $a \geq 2$ since $n \geq 210$. The problem statement tells us that

$$abc = 100a + 10b + c - 210,$$

which can be rearranged into

$$(ab - 1)(ac - 10) = 100a^2 - 210a + 10.$$

Since $b, c \leq 9$, we have the following inequality:

$$100a^2 - 210a + 10 = (ab - 1)(ac - 10) \leq (9a - 1)(9a - 10).$$

Solving this quadratic inequality for a in integers gives $0 \leq a \leq 5$, but we already know that $n \geq 210$ so in fact we only have to consider cases where $2 \leq a \leq 5$.

Case 1: $a = 2$

This gives

$$-5 = (2b - 1)(c - 5)$$

so the only integer solutions are when $(b, c) = (1, 0), (3, 4)$. In this case $n = 210, 234$.

Case 2: $a = 3$

This gives

$$280 = (3b - 1)(3c - 10)$$

so the only integer solutions are when $(b, c) = (7, 8)$. In this case $n = 378$.

Case 3: $a = 4$

This gives

$$385 = (4b - 1)(2c - 5)$$

so the only integer solutions are when $(b, c) = (9, 8)$. In this case $n = 498$.

Case 4: $a = 5$

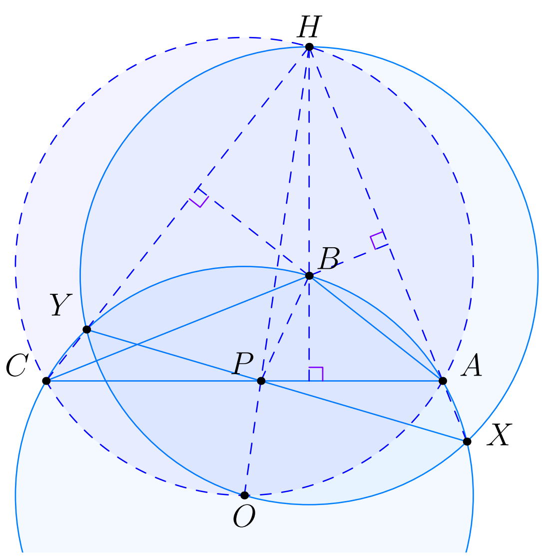
A quick check shows that there are no solutions satisfying all our requirements.

Therefore, our final answer is $210 + 234 + 378 + 498 \equiv \boxed{320} \pmod{1000}$.
■

Problem 10

Triangle $\triangle ABC$ has side lengths $AB = 3$, $BC = 5$, and $CA = 7$. Define H as the orthocenter of $\triangle ABC$. Then, the circle with center B and radius BH intersects the circumcircle of $\triangle ABC$ at two distinct points X and Y. If XY meets AC at P, the length of BP can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find the value of $m + n$.

Proposed by jj_ca888



Let O be the circumcenter of $\triangle ABC$, and H be the orthocenter of $\triangle ABC$. Let R be the circumradius of $\triangle ABC$.

A quick Law of Cosines check shows that $\angle ABC = 120°$. We have that B is the orthocenter of $\triangle AHC$ by the perpendicularities established. Since the reflection of B over AC lies on the circumcircle of $\triangle AHC$ we know that $\angle AHC = 60°$.

A quick and simple angle chase shows that $\angle AOC = 120°$ so therefore, we have $\angle AOC + \angle AHC = 180°$, hence $AHCO$ cyclic.

Claim: $HXOY$ is cyclic.

Proof: The radius of the circumcircle of $AHCO$ is also the radius of the circumcircle of $\triangle AOC$, which is R. It is well known that $BH = 2R \cos \angle AHC$, so

$$BH = 2R \cdot \frac{1}{2} = R.$$

Thus we have $BX = BY = BH = BO = R$, and we are done. $\square$

Now we can use radical axis theorem on the circumcircles of $\triangle ABC$, $AHCO$, and $HXOY$, which tells us that XY, AC, and HO are concurrent at a point, and this point must be P. We see that since O is the midpoint of arc AOC in the circumcircle of $AHCO$, HP must bisect angle $\angle AHC$. Then by multiple applications of the Law of Sines and the Angle Bisector Theorem, we get

$$\frac{AP}{PC} = \frac{AH}{HC} = \frac{\sin HCA}{\sin HAC} = \frac{\sin(\angle BCA + 30°)}{\sin(\angle BAC + 30°)}.$$

Using the sine addition formula and the Law of cosines, we get

$$\cos\angle BAC = \frac{11}{14} \implies \sin\angle BAC = \frac{5\sqrt{3}}{14}$$
$$\cos\angle BCA = \frac{13}{14} \implies \sin\angle BCA = \frac{3\sqrt{3}}{14}.$$

Therefore we conclude that $\frac{AP}{PC} = \frac{11}{13} \implies (AP, PC) = (\frac{77}{24}, \frac{91}{24})$. Now by Stewart's Theorem, $BP = \frac{49}{24}$, so our desired answer is $\boxed{073}$. ■

Problem 11

Fred the frog is on the first lilypad of a row of 16 lilypads, and he can jump to any lilypad. However, if he jumps to the n^{th} lilypad, he will never jump the k^{th} lilypad ever if and only if $k < n$ or n divides k. Find the number of ways he can execute a sequence of jumps such that he ends at the last lilypad.

Proposed by: cosmicgenius

Define the *odd part* of an integer n to be the largest odd integer that divides k.

Notice that we must simply construct a set of numbers with largest term 16 such that no term is divisible by another. The only possibility for term a being divisible by b is if the odd term of a is divisible by the odd term of b and $\nu_2(a) \geq \nu_2(b)$ (i.e. there are at least as many powers of 2 in a as in b). So odd parts of each term must be unique because one of $\nu_2(a) \geq \nu_2(b)$ and $\nu_2(b) \geq \nu_2(a)$ must be true, so a divides b or b divides a, contradiction.

So if the odd parts are unique, we only have two possibilities for odd terms dividing each other, since odd terms must be odd integers from 1 to 16; either we have an odd part of 3 and an odd part of 9, or we have an odd part of 3 or 5 and we have an odd part of 15. Since we cannot have any powers of 2 (they would divide 16), all of the odd part choices (which power of two to go with it) are all independent of each other except these two, so we can simply multiply them together in the end.

We proceed with casework on whether $3, 5$ are in the set:

Case 1: 3 is in the set.

Then neither 9 nor 15 can be in the set, and the number with odd part 5 can be $5, 10$ or not in the set, so 3 ways.

Case 2: 5 is in the set but not 3.

Then 9 can either be in or not in the set, but 15 cannot. The number with odd part 3 can either be $6, 12$ or not in the set so $2 \cdot 3 = 6$ ways.

Case 3: Neither 3 nor 5 are in the set.

Then, both $9, 15$ can be in the set. The number with odd part 3 can be $6, 12$ or not in the set, while the number with odd part 5 can be 10 or not in the set. Thus there are $2 \cdot 2 \cdot 3 \cdot 2 = 24$ ways.

In total, for $3, 5, 9, 15$ we have $3 + 6 + 24 = 33$ ways.

Now to finish, we simply compute the number of ways for each odd part other than $3, 5, 9, 15$ since they are independent of each other (and we can never have a power of 2).

Odd part of 7: We can have $7, 14$ or not in the set, so 3 ways.

Odd part of 11: We can have 11 or not in the set, so 2 ways.

Odd part of 13: We can have 13 or not in the set, so 2 ways.

Thus there are $33 \cdot 3 \cdot 2 \cdot 2 = \boxed{396}$ ways in total. ■

Problem 12

Compute the number of ordered pairs of positive integers (m, n) with $m + n \leq 64$ such that there exists at least one complex number z such that $|z| = 1$ and $z^m + z^n + \sqrt{2} = 0$.

Proposed by jj_ca888

Let τ denote 2π.

Claim: (m, n) works if and only if $8\gcd(m, n)|m + n$.

Proof: Let $a = \frac{m}{\gcd(m,n)}$, $b = \frac{n}{\gcd(m,n)}$, and $x = z^{\gcd(m,n)}$. Then,

$$x^a + x^b = z^m + z^n = -\sqrt{2} \in \mathbb{R},$$

so x^a and x^b must be conjugates of each other, and both must have real part $-\frac{\sqrt{2}}{2}$. Thus

$$\{x^a, x^b\} = \left\{e^{\frac{3\tau i}{8}}, e^{\frac{5\tau i}{8}}\right\}.$$

But since $x^a = \overline{x^b}$, $x^a = \frac{1}{x^b}$, thus $x^{a+b} = 1$ and x must be a $a + b^{\text{th}}$ root of unity.

We have that x^a, another $a + b^{\text{th}}$ root of unity, is an 8^{th} root of unity, so we must have $8|a + b$, i.e. $8\gcd(m, n)|m + n$.

To show that (m, n) works when $8|a + b$, we see that

$$1 = \gcd(a, b) = \gcd(a, a + b).$$

Thus a^{-1} exists mod 8. Let k be the value of a^{-1} mod 8 (where $0 < k < 8$). Thus if $x = e^{\frac{3k\tau i}{8}}$, we have that

$$x^a = e^{\frac{3ak\tau i}{8}},$$

but

$$3ak \equiv 3 \pmod 8.$$

Thus

$$x^a = e^{\frac{3\tau i}{8}},$$

and $x^b = e^{\frac{5\tau i}{8}}$, so we are done. $\square$

All that is left to do is to count the number or ordered pairs (m, n) there are such that $8\gcd(m.n)|m + n$. Suppose that we let $m + n = 8k$, where k can be any positive integer between 1 and 8 inclusive. Note that

$$8\gcd(m, n)|m + n \implies \gcd(m, 8k - m)|k \implies \gcd(m, 8k)|k$$

Thus, if p is a prime greater than 2, then we must have

$$\min(v_p(m), v_p(8k)) = \min(v_p(m), v_p(k)) \leq v_p(k)$$

since p cannot divide 8. However, note that this is always true because if $v_p(m) \leq v_p(k)$ we have

$$\min(v_p(m), v_p(k)) = v_p(m) \leq v_p(k)$$

which is always true, and if $v_p(m) > v_p(k)$, we have

$$\min(v_p(m), v_p(k)) = v_p(k) \leq v_p(k)$$

which is also always true. Now, suppose that $p = 2$, and in order for $\gcd(m, 8k)$ to divide k, it must follow that

$$\min(v_2(m), v_2(8k)) = \min(v_2(m), 3 + v_2(k)) \le v_2(k)$$

If $v_2(m) > v_2(k)$, then this is obviously impossible, so it must necessarily follow that $v_2(m) \le v_2(k)$. Hence, it suffices to find the number of ordered pairs of positive integers (m, k) such that $k \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, $m < 8k$, and $v_2(m) \le v_2(k)$.

Case 1: k = 1

$v_2(1) = 0$ so m must be odd and in the interval $[1, 7]$, hence a total of 4 pairs.

Case 2: k = 2

$v_2(2) = 1$ so m must not be divisible by 4. Furthermore, it is required to be in the interval $[1, 15]$, hence a total of $15 - 3 = 12$ pairs.

Case 3: k = 3

$v_2(3) = 0$ so m must be odd and in the interval $[1, 23]$, hence a total of 12 pairs.

Case 4: k = 4

$v_2(4) = 2$ so m must not be divisible by 8. Furthermore, it is required to be in the interval $[1, 31]$, hence a total of $31 - 3 = 28$ pairs.

Case 5: k = 5

$v_2(5) = 0$ so m must be odd and in the interval $[1, 39]$, hence a total of 20 pairs.

Case 6: k = 6

$v_2(6) = 1$ so m must not be divisible by 4. Furthermore, it is required to be in the interval $[1, 47]$, hence a total of $47 - 11 = 36$ pairs.

Case 7: k = 7

$v_2(7) = 0$ so m must be odd and in the interval $[1, 55]$, hence a total of 28 pairs.

Case 8: k = 8

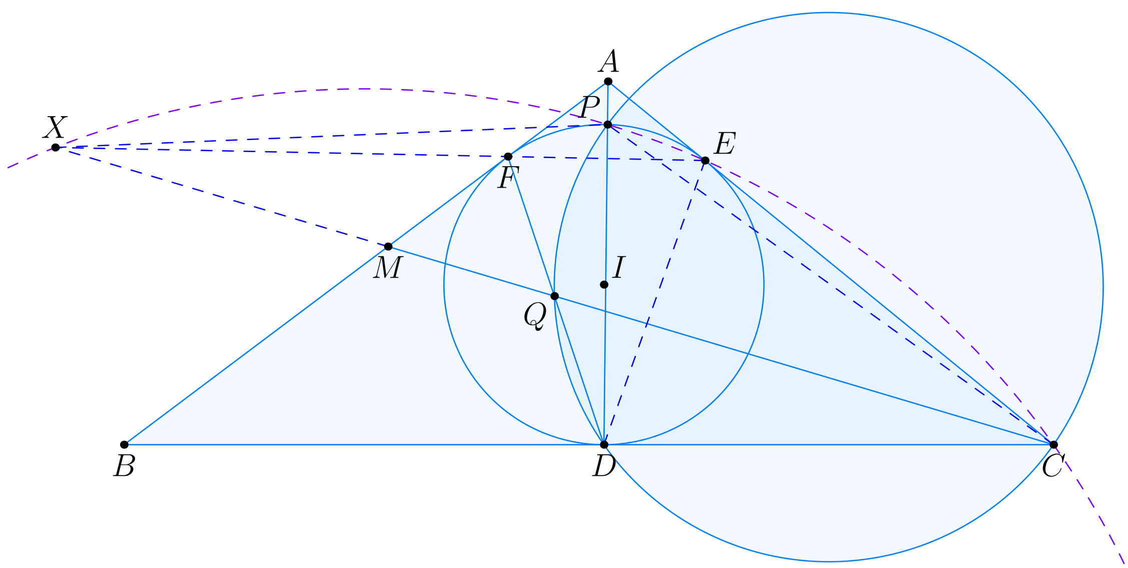
$v_2(8) = 3$ so m must not be divisible by 16. Furthermore, it is required to be in the interval $[1, 63]$, hence a total of $63 - 3 = 60$ pairs.

Therefore, our final sum is $4 + 12 + 12 + 28 + 20 + 36 + 28 + 60 = \boxed{200}$ ordered pairs, as desired. ■

Problem 13

Suppose that triangle $\triangle ABC$ has side lengths $AB = 20$ and $AC = 19$. Furthermore, let the incircle ω of $\triangle ABC$ touch segments BC, CA, AB at points D, E, and F respectively. Let AD hit ω at a point $P \neq D$. Suppose that DF intersects the circumcircle of $\triangle CDP$ at a point $Q \neq D$. Let line CQ intersect AB at point M. If we are given that $\frac{AM}{BM} = \frac{5}{6}$, then the perimeter of triangle $\triangle ABC$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find the value of $m + n$.

Proposed by jj_ca888



Let[1] $EF \cap BC = T$ and $EF \cap CQ = X$. Let[2] I be the incenter of $\triangle ABC$.

Claim 1: $CEPX$ is cyclic.
Proof: Note that $\angle XEP = \angle FEP = \angle FDP = \angle QDP = \angle QCP = \angle XCP$ which implies our claim. $\square$

Claim 2: Q is the midpoint of CX
Proof: Using Law of Sines on PX and PC of $\triangle XPC$ and Ratio Lemma on $\triangle XPC$ and PQ, we obtain

$$\frac{QX}{QC} = \frac{PX}{PC} \cdot \frac{\sin XPQ}{\sin CPQ} = \frac{\sin PCX}{\sin PXC} \cdot \frac{\sin XPQ}{\sin CPQ}.$$

However, recall from our angle chasing in Claim 1, $\angle PCX = \angle ADF$ and $\angle PXC = 180^\circ - \angle PEC = \angle PEA = \angle EDA$, where the last part is obtained by using the fact that $AE^2 = AP \cdot AD \implies \triangle PEA \sim \triangle EDA$. More angle chasing will result in

$$\angle CPQ = 180^\circ - \angle CDQ = \angle BDF,$$

and

$$\angle XPQ = \angle XPC - \angle CPQ = \angle XEC - \angle BDF$$
$$= (90^\circ + \frac{1}{2}\angle A) - (90^\circ - \frac{1}{2}\angle B) = 90^\circ - \frac{1}{2}\angle C = \angle CDE.$$

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[1] T is not on the diagram because it is too far away
[2] Unlike this diagram suggests, I is not on AD!

Therefore, substituting back into our Ratio Lemma, we get

$$\frac{QX}{QC} = \frac{\sin ADF}{\sin EDA} \cdot \frac{\sin CDE}{\sin BDF}.$$

Now, we make use of the fact that DA is the D-symmedian of $\triangle DEF$. This along with angle chasing and Law of Sines tells us that

$$\frac{\sin ADF}{\sin EDA} = \frac{\sin PEF}{\sin PFE} = \frac{PF}{PE} = \frac{DF}{DE} = \frac{\sin DEF}{\sin DFE}.$$

We can angle chase even more to get

$$\angle DEF = \frac{1}{2}\angle FID = \frac{1}{2}(180^\circ - 2\angle IDF) = 90^\circ - \angle IDF = \angle BDF$$

and similarly $\angle DFE = \angle CDE$. Therefore,

$$\frac{QX}{QC} = \frac{\sin ADF}{\sin EDA} \cdot \frac{\sin CDE}{\sin BDF} = \frac{\sin BDF}{\sin CDE} \cdot \frac{\sin CDE}{\sin BDF} = 1.$$

as desired. $\square$

Using Menelaus with collinear points D, Q, and F, we see that

$$\frac{FM}{FB} \cdot \frac{BD}{DC} \cdot \frac{CQ}{QM} = 1 \implies \frac{FM}{DC} = \frac{MQ}{CQ}$$

Since AD, BE, CF concur at the Gergonne point of $\triangle ABC$, it follows that

$$(B, C; D, T) \stackrel{F}{=} (M, C; Q, X) = -1$$

Hence, writing out the cross ratio without directed lengths gives

$$\frac{QC}{QM} = \frac{XC}{XM}$$

Since we have already established that Q is the midpoint of segment CX in Part 1, we know that $XC = 2QC$, therefore, $XM = 2QM$. This also means that $CQ = XQ = XM + QM = 3QM$, so therefore,

$$\frac{FM}{DC} = \frac{MQ}{CQ} = \frac{1}{3}$$

Now we proceed to length bash with the information obtained from the previous part. We let $AE = AF = a$, $BD = BF = b$, and $CD = CD = c$. From the problem conditions and the result from Part 2, we can write

$$a + b = 20$$

$$a + c = 19$$

$$\frac{3a + c}{3b - c} = \frac{5}{6} \implies 11c = 15b - 18a$$

We add $18a + 18b$ to RHS and 360 to LHS and we get

$$11c + 360 = 33b$$

Subtracting the 2nd equation from the first gives $b = c + 1$. Substitution yields

$$11c + 360 = 33c + 33 \implies c = \frac{327}{22}, b = \frac{349}{22}$$

We add the two to get $BC = b + c = \frac{338}{11}$, hence the perimeter of $\triangle ABC$ is $39 + \frac{338}{11} = \frac{767}{11}$, and therefore our final answer is $767 + 11 = \boxed{778}$. $\blacksquare$

Problem 14

Let a, b, c be positive reals such that $abc + a + b = c$ and

$$\frac{19}{\sqrt{a^2+1}} + \frac{20}{\sqrt{b^2+1}} = 31.$$

The maximum possible value of c^2 can be written in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Find the value of $m + n$.

Proposed by jj_ca888

Notice that $abc = c - a - b$ rearranges to

$$c = \frac{a+b}{1-ab}$$

so it makes sense to let $a = \tan A$, $b = \tan B$, and $c = \tan(A+B)$. Since we are given that $a, b, c > 0$, we can assume $A, B > 0$ and $A + B < \frac{\pi}{2}$. Using the identity $\tan^2\theta + 1 = \sec^2\theta$ and the fact that all of a, b, c are positive, we get

$$\frac{19}{\sqrt{a^2+1}} + \frac{20}{\sqrt{a^2+1}} = 31 \iff 19\cos A + 20\cos B = 31 \quad (i)$$

In order to maximize $\tan(A+B) > 0$, we should minimize $\cos(A+B)$.

Let $z_1 = e^{iA}, z_2 = e^{iB}$ and $c_1 = 19z_1 + \frac{20}{z_2}, c_2 = \frac{19}{z_1} + 20z_2$. From (i) we derive

$$19z_1 + \frac{19}{z_1} + 20z_2 + \frac{20}{z_2} = 62 \iff c_1 + c_2 = 62$$

Given this, we must to minimize $\cos(A+B) = \frac{1}{2}\left(z_1z_2 + \frac{1}{z_1z_2}\right)$, which must be real. In addition, if $K = c_1c_2$

$$K = c_1c_2 = \left(19z_1 + \frac{20}{z_2}\right)\left(\frac{19}{z_1} + 20z_2\right) = 761 + 760\cos(A+B)$$

is real, so K is real.

Thus by Vieta's Formula, c_1, c_2 are the roots of the equation

$$x^2 - 62x + K = 0.$$

We claim that $K \geq 961$ with equality at $c_1 = c_2 = 31$. Assume for the sake of contradiction there exist z_1, z_2 such that $K < 961$. Then

$$62^2 - 4K > 62^2 - 4 \cdot 961 = 0,$$

so c_1, c_2 are distinct real numbers. But

$$\text{Re}(c_1) = \text{Re}(19z_1 + 20\overline{z_2}) = 19\text{Re}(z_1) + 20\text{Re}(z_2) = \text{Re}(19\overline{z_1} + 20z_2) = \text{Re}(c_2),$$

Thus $c_1 = c_2$, contradiction. So

$$\cos(A+B) \geq \frac{961-761}{760} = \frac{5}{19} \iff c^2 = \tan^2(A+B) \geq \frac{19^2 - 5^2}{5^2} = \frac{336}{25}$$

We have equality at

$$(z_1, z_2) = \left(\frac{461 + 80i\sqrt{21}}{589}, \frac{25 + 4i\sqrt{21}}{31}\right)$$

i.e.

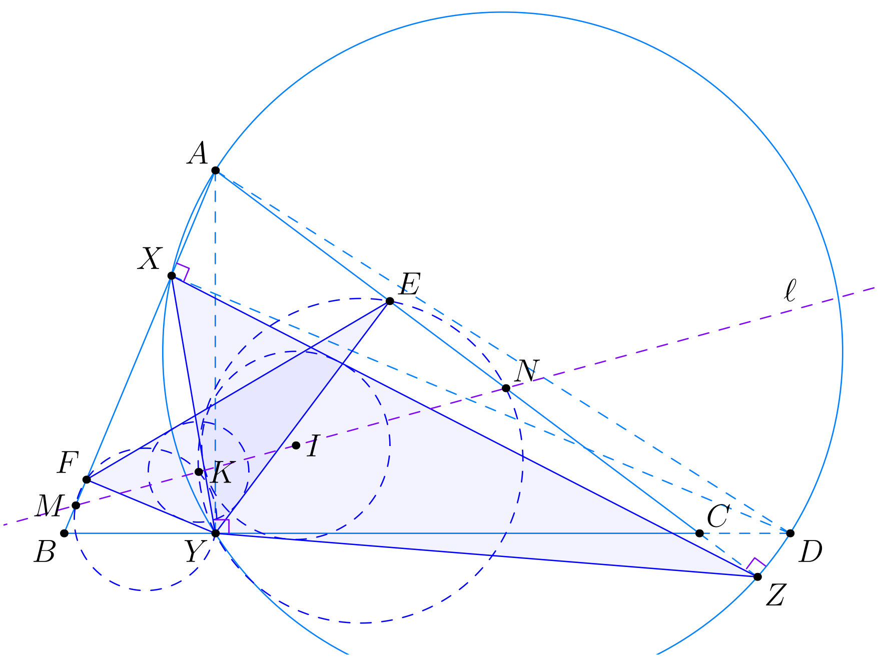
$$(a, b, c) = \left(\frac{80\sqrt{21}}{461}, \frac{4\sqrt{21}}{25}, \frac{4\sqrt{21}}{5}\right).$$

Therefore c^2 is minimized at $\frac{336}{25}$, so $m + n = \boxed{361}$. ■

Problem 15

Triangle $\triangle ABC$ has side lengths $AB = 13$, $BC = 21$, and $AC = 20$. A point D is selected on the line BC. The circle with diameter AD intersects AB at X, BC at Y, and AC at Z. Denote I as the incenter of triangle $\triangle XYZ$. The minimum possible value of AI can be written as $\frac{m\sqrt{n}}{p}$ where m and p are relatively prime with n not divisible by the square of any prime. Find the value of $m + n + p$.

Proposed by jj_ca888



Let E and F as the feet of the perpendiculars from Y to AC and AB, respectively. Denote K as the incenter of $\triangle YEF$. Let the line ℓ be the line through K perpendicular to YK. Suppose that ℓ intersects AB and AC at points M and N respectively.

Since $AXYDZ$ is cyclic with diameter AD, it follows that $\angle AXD = \angle AYD = \angle AZD = 90°$. Therefore, we see that Y is fixed at the foot of the perpendicular from A to BC. Thus E, F, ℓ are fixed. Furthermore, note that no matter where D is, the angles of triangle $\triangle XYZ$ are constant with $\angle XYZ = 180° - \angle A$, $\angle YZX = \angle YAB$, and $\angle ZXY = \angle CAY$ by cyclic quadrilateral $AXYZ$.

Claim 1: The locus of I is ℓ.

Proof: It suffices to show $\angle YKI = 90°$.

$\triangle FYE$ is just $\triangle XYZ$ in the case $D = Y$, thus $\triangle FYE \sim \triangle XYZ$. Therefore, it must also be true that $\triangle FYK \sim \triangle XYI$. Since Y is the center of spiral similarity bringing FK to XI, it is also the spiral center bringing FX to KI. Therefore, it follows that $\triangle YFX \sim \triangle YKI$, and $\angle YFX = \angle YKI = 90°$ as desired. $\square$

Claim 2: YM bisects $\angle BYF$ and YN bisects $\angle CYE$.

Proof: Since $\angle MKY = \angle MFY = 90°$, we know that $MYKF$ is cyclic.

Therefore,

$$\angle MYF = \angle MKF = \angle FKY - 90^\circ = 180^\circ - \frac{1}{2}\angle FYE - \frac{1}{2}\angle EFY - 90^\circ$$

However, by cyclic quadrilateral $AFYE$, we conclude that

$$\frac{1}{2}\angle FYE = 90^\circ - \frac{1}{2}\angle BAC$$
$$\frac{1}{2}\angle EFY = \frac{1}{2}\angle YAC$$

Substituting, we get

$$\angle MYF = \frac{1}{2}(\angle BAC - \angle YAC) = \frac{1}{2}\angle YAB = \frac{1}{2}\angle BYF$$

Similarly, $\angle NYE = \frac{1}{2}\angle CYZ$, as desired. $\square$

Now we just need to find the distance from A to ℓ, which is the minimum length of AI. Some easy length chasing with the angle bisector theorem shows that $BM = 1$ and $CN = 8$, hence $AM = AN = 12$. Therefore, the distance from A to ℓ is $12\cos\left(\frac{1}{2}\angle BAC\right) = \frac{108}{\sqrt{130}} = \frac{54\sqrt{130}}{65}$, so we get a final answer of $54 + 130 + 65 = \boxed{249}$. $\blacksquare$