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Centrifugal and Tidal Break Up of a Sun like Star In the Presence of a Compact Companion

Aniket Sanghi[1]

[1] *National Public School Koramangala, KA 560047, India*

ABSTRACT

This paper presents theoretical calculations for the critical rotational velocity at which the stellar matter will start breaking - up for the three layers of the Sun (Core, Radiation Zone, and Photosphere). We also study the effect of the rotational velocity on the hypothetical critical distance of the Sun from a compact star (2 Solar Mass Neutron Star). The calculations involve a force balance equation between the Gravitational Force on one side and Gas Pressure, Centrifugal Force, and Radiation Pressure on the other side. The calculations on Gas Pressure uses the Polytropic model of the Sun which is considered an n = 3 star by the Eddington Standard Model of Stellar Structure. The calculation can be applied to other stars knowing the density profiles and standard numerical data. *It is to be noted that our calculations utilise only the standard numerical data without the use of calculus to obtain the approximate values in this paper.* The upper limit for rotational velocities for the different layers are found to lie in the range of 200 $\mathrm{km\,s^{-1}}$ - 350 $\mathrm{km\,s^{-1}}$. We observe that the radiation pressure plays a negligible role in influencing the critical rotational velocities of matter in all three layers of the Sun. We also find that the critical rotational velocity of the inner layers is smaller in magnitude than for the outer layers. Our calculations also show that the distance from the Sun at which solar matter will start flying - off the surface, in the presence of a compact companion, increases as the rotational velocity of the Sun increases and this critical distance is of the order of the Solar Radii.

Keywords: equation of state — stars: neutron — stars: solar type

1. INTRODUCTION

The Sun has been an object of interest for centuries and continues to be the most observed star. Extensive studies have been conducted in order to primarily develop a stellar model that effectively describes the physical properties, energy transport mechanisms, and stability of a stellar system. The central work in this respect has been the development of the Standard Solar Model which is a mathematical treatment of the Sun as a spherical ball of gas in varying states of ionisation such that the hydrogen in the deep interior exists as completely ionised plasma. The Standard Solar Model has described stellar structure through differential equations derived from fundamental physical principles. The model is also constrained by boundary conditions such as luminosity, radius, age and composition of the Sun. An interesting question that can be investigated utilising the Standard Solar Model is what the critical velocity would be for different layers of the Sun before matter starts to break up and escape. These critical velocities can be further utilised to study the stability of the Sun in the presence of a hypothetical Compact star. In a broader context, the methodology of the calculation can be applied to simple astrophysical binary systems to study the critical stability conditions. This paper presents the theoretical calculations on these very aspects. Three layers of the Sun are considered in the calculations presented: Core, Radiation Zone, and Photosphere. To begin with a general idea, a very crude calculation can be performed to identify the range to be adhered to in the calculations presented later in the paper. This crude calculation can be performed directly by balancing Centrifugal Force against Gravitational Force without gas and radiation pressure considerations as -

$$\Delta m\,\omega^2 R_\odot = \frac{GM_\odot \Delta m}{R^2}$$

Substituting $G = 6.67\times10^{-11}\ \mathrm{Nm^2kg^{-2}}$, $M_\odot = 2\times10^{30}$ kg, and $R_\odot = 7\times10^8$ m and then solving for ω, we get $\omega \approx 6.24\times10^{-4}\mathrm{rad\,s^{-1}}$. This translates to a rotational velocity of $v \approx 436.5\ \mathrm{km\,s^{-1}}$ using the equation $v = r\omega$. This is a limiting assumption that has been used to check the correctness of future calculations. The correct estimate is obtained by considering Gas Pressure calculated using the Polytropic equation of state for an $n = 3$ star, radiation pressure calculated as intensity over speed of light, and equations relating to the so-

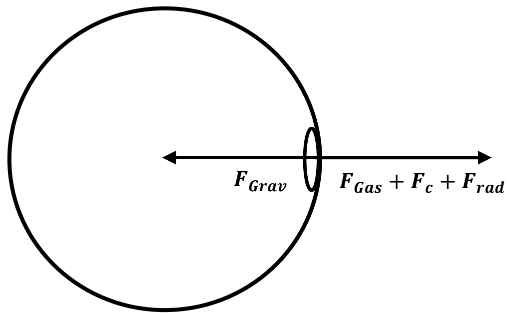

Figure 1. Depiction of the various forces acting on surface layer mass Δm of the Sun.

lution of the Lane – Emden equation (Lane, Jonathan Homer (1870)), which reads as –

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^n=0$$

Where ξ is a dimensionless radius and θ is related to density ρ, and thus pressure for central density ρ_c by -

$$\rho=\rho_c\theta^n$$

The velocities obtained for each of the mentioned layers are then applied hypothetically as the velocity of the Sun to calculate the critical distance at which imbalance in forces caused by the strong tidal force of an imaginary Compact star would start breaking up solar matter at the surface.

2. CALCULATIONS

Force Balance Equation

Based on Figure 1. the force balance equation for any layer of the Sun can be written as a sum of the forces due to gas pressure (F_{Gas}), centrifugal force (F_c), and the radiation pressure (F_{rad}) balancing the gravitational force (F_{Grav}) -

$$F_{Gas}+F_c+F_{rad}=F_{Grav}$$

Which is expanded as,

$$P\Delta A+\Delta mR\omega^2+P_{rad}\Delta A=\frac{GM\Delta m}{R^2} \tag{1}$$

Where, P is the internal gas pressure, Δm is a small element of mass, ΔA is the surface area of Δm, R is distance of Δm from centre of the Sun (for surface Δm, R is radius of the Sun), ω is the Angular velocity of the Sun, G is the Universal Gravitational Constant, and M is the mass of the Sun enclosed within the sphere of radius R. Now, $\Delta m=\rho\Delta V=\rho\Delta A\Delta r$, where ρ is the local density of the layer where Δm is situated and Δr is the thickness of the layer under consideration. Substituting this and rearranging the above equation to obtain ω, we get,

$$\omega=\sqrt{\frac{GM\rho\Delta r-PR^2}{\rho R^3\Delta r}-\frac{P_{rad}}{\rho\Delta rR}} \tag{2}$$

Gas Pressure Calculation

The Polytropic Equation of State (Horedt, G.P. (2004)) can be written as,

$$P=k\rho^{\frac{n+1}{n}} \tag{3}$$

Where,

$$k=\frac{4\pi GR^2\rho_c^{\frac{n-1}{n}}}{(n+1)\,\xi^2} \tag{4}$$

Substituting (4) in (3) and $n=3$ (for Solar-mass main sequence stars), we get,

$$P=\frac{\pi GR^2\rho_c^{\frac{2}{3}}}{\xi^2}\rho^{\frac{4}{3}} \tag{5}$$

Where, ρ_c is the density of the centre of the Sun, R and ρ have the standard meanings as mentioned earlier, and ξ is the rescaled R using a constant α as -

$$\xi=\frac{R}{\alpha}$$

Radiation Pressure Calculation

The numerical value of radiation pressure (Shankar R. (1980))(Carroll, Bradley W. & Dale A. Ostlie (2003))(Jackson, John David (1999))(Kardar, Mehran (2007)) for a given surface is calculated as –

$$P_{rad}=\frac{4\sigma T^4}{3c} \tag{6}$$

Where, σ is the Stefan–Boltzmann Constant equal to 5.67×10^{-8} $\mathrm{Wm^{-2}K^{-4}}$, c is the speed of light equal to 3×10^8 $\mathrm{m\,s^{-1}}$, and T is the temperature of the layer under consideration.

Photosphere

The photosphere is the first outer surface layer of the Sun. The magnitude of its distance from the centre of the Sun is approximately equal to the radius of the Sun. The local density ρ of the Photosphere is about 10^{-6} $\mathrm{kg\,m^{-3}}$, Δr is the thickness of the Photosphere and is about 500 km, the central density ρ_c is about 1.6×10^5 $\mathrm{kg\,m^{-3}}$, and M is the mass enclosed and is equal to the mass of the sun. According to the Lane–Emden solution, for boundary condition (Here Photosphere is the boundary of the Sun), $\theta(\xi)=0$. The

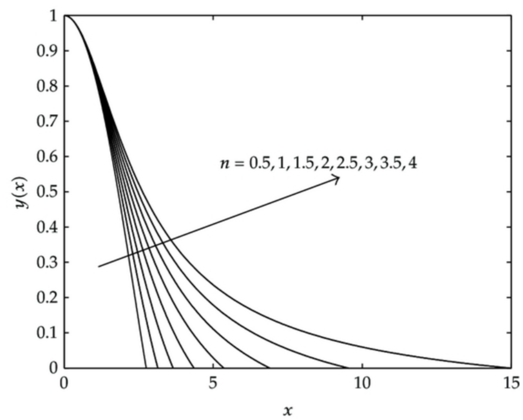

Figure 2. $\theta(\xi)$ vs ξ. Graphical solutions (Motsa, Sandile & Shateyi, Stanford (2012)) to the Lane-Emden Equation for different values of Polytropic index n. Here θ is a quantity related to the variation in density as $\rho = \rho_c \theta^n$.

plot (Figure 2.) of $\theta(\xi)$ vs ξ for an $n = 3$ star, gives us the required solution $\xi = 6.89$. Substituting these values along with the standard constants in equation (5), we get $P \approx 63.8$ Pa. Now, the temperature T of the photosphere is approximately 5777 K. Substituting this in equation (6) along with the necessary constants, we get $P_{rad} = 0.28$ Pa. Substituting all the required values in equation (2), we obtain the following values:

$$\frac{GM\rho\Delta r - PR^2}{\rho R^3 \Delta r} \approx 2.07 \times 10^{-7} \mathrm{rad}^2\,\mathrm{s}^{-2}$$

$$\frac{P_{rad}}{\rho \Delta r R} \approx 8.0 \times 10^{-10} \mathrm{rad}^2\,\mathrm{s}^{-2}$$

Here, the radiation pressure term is about two orders of magnitude smaller than the gas pressure term and hence can be neglected. Taking the square root of the gas pressure term, we get $\omega = 4.55 \times 10^{-4}$rad s^{-1}. This translates to a rotational velocity of $v \approx 318.5$ km s^{-1} using the equation $v = r\omega$, where $r = R_\odot$.

Radiation Zone

The radiation zone is the second of the inner layers of the Sun extending from $0.25R_\odot$ to $0.6R_\odot$ and hence the R to be used in the equations is $0.6R_\odot$. The local density ρ is about 1.01×10^4 kg m^{-3}, the central density ρ_c is about 1.6×10^5 kg m^{-3}, and the thickness Δr is about 3×10^8 m. Mass of the sphere enclosed by the Radiation Zone is calculated as $M = \rho V$, where $V = \frac{4}{3}\pi d^3$, (where $d = 0.35R_\odot$.). This mass is added to the mass of the core which is about 10^{30}kg to yield a total mass $M = 1.622 \times 10^{30}$ kg. Now, using the equation $\xi = \frac{R}{\alpha}$ and the fact that α is constant, the required ξ value can be calculated from the known boundary condition value as $\xi = 0.6 \times 6.89 \approx 4.13$. Substituting the required values in equation (5) gives the value of $P = 1.39 \times 10^{15}$ Pa. The temperature T of the radiation zone is about 4.5 million K. Substituting this in equation (6) yields $P_{rad} = 1.03 \times 10^{11}$ Pa. Finally, substituting all the required values in equation (2), we obtain the following values:

$$\frac{GM\rho\Delta r - PR^2}{\rho R^3 \Delta r} \approx 3.68 \times 10^{-7} \mathrm{rad}^2\,\mathrm{s}^{-2}$$

$$\frac{P_{rad}}{\rho \Delta r R} \approx 8.09 \times 10^{-11} \mathrm{rad}^2\,\mathrm{s}^{-2}$$

Here, the radiation pressure term is smaller than the gas pressure term by three orders of magnitude and hence can be neglected. Taking the square root of the remaining value we get $\omega = 6.07 \times 10^{-4}$ rad s^{-1}. This translates to a rotational velocity of $v \approx 255$ km s^{-1} using the equation $v = r\omega$, where $r = 0.6R_\odot$.

Core

The core forms the innermost layer of the Sun. It extends to a distance of $0.25R_\odot$ from the centre of the Sun and has an average density $\rho = 8.5 \times 10^4$ kg m^{-3}. The thickness of the core $\Delta r = 1.5 \times 10^8$ m, the central density ρ_c is about 1.6×10^5 kg m^{-3}, and its mass $M = 10^{30}$ kg. Now, using the equation $\xi = \frac{R}{\alpha}$ and the fact that α is constant, the required ξ value can be calculated from the boundary condition value as $\xi = 0.25 \times 6.89 \approx 1.723$. Substituting the required values in equation (5) gives the value of $P = 2.38 \times 10^{16}$ Pa. The temperature of the core of the Sun is about 15 million K. Substituting this in equation (6) yields $P_{rad} = 1.28 \times 10^{13}$ Pa. Finally, substituting all the required values in equation (2), we obtain the following values:

$$\frac{GM\rho\Delta r - PR^2}{\rho R^3 \Delta r} \approx 1.78 \times 10^{-6} \mathrm{rad}^2\,\mathrm{s}^{-2}$$

$$\frac{P_{rad}}{\rho \Delta r R} \approx 5.74 \times 10^{-9} \mathrm{rad}^2\,\mathrm{s}^{-2}$$

Here, the radiation pressure term is smaller than the gas pressure term by two orders of magnitude and hence can be neglected. Taking the square root of the remaining value we get $\omega = 1.33 \times 10^{-3}$ rad s^{-1}. This translates to a rotational velocity of $v \approx 233.5$ km s^{-1} using the equation $v = r\omega$, where $r = 0.25R_\odot$.

The results are summarized in Figure 3.

Limiting distance from a Compact Star

Hypothetically assuming the existence of a 2 Solar Mass

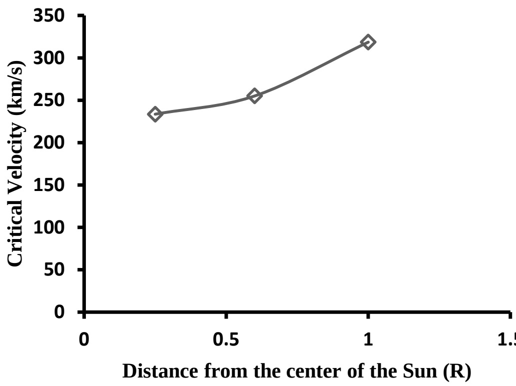

Figure 3. Depicts the critical velocities at which matter would break - up in the different layers of the Sun.

Neutron Star in the vicinity of the Sun, the limiting distance at which matter in the star would start to break up can be calculated considering tidal force effects in addition to forces considered in the previous calculations. This limiting distance in which the Sun, held together only by its own gravity, will disintegrate due to the Neutron Star's tidal forces exceeding the Sun's gravitational self-attraction is defined as the Roche limit of the Sun. (Eric W. Weisstein (2007)) The calculation has been performed for three cases of rotational velocity of the Sun-like star – 2 km s^{-1}, 233.5 km s^{-1}, and 255 km s^{-1}, as calculated in the cases above. (The case involving a rotational velocity of 318.5 km s^{-1} would mean surface break – up due to rotation alone and would give an infinite distance if substituted in the formula derived below).

Force Balance Equation for Compact Star Calculations
Based on Figure 4. the force balance equation can be written as -

$$F_{Gas} + F_c + F_{tidal} = F_{Grav}$$

Which is expanded as,

$$P\Delta A + \Delta m R_\odot \omega^2 + \frac{2G\,(2M_\odot)\,R_\odot \Delta m}{d^3} = \frac{GM\Delta m}{R_\odot^2} \quad (7)$$

Where, P is the internal gas pressure, Δm is small element of mass, ΔA is surface area of Δm, $R_\odot$ is radius of the Sun, ω is the Angular velocity of the Sun, G is the Universal Gravitational Constant, $M_\odot$ is the interior mass of the Sun, and d is the required Roche limit. Now, $\Delta m = \rho \Delta V = \rho \Delta A \Delta r$, where ρ is the local density of the layer where Δm is situated. Substituting this and rearranging the above equation to obtain d, we get,

$$d = \left(\frac{4GM_\odot R_\odot^3 \rho \Delta r}{GM_\odot \rho \Delta r - \rho \Delta r \omega^2 R_\odot^3 - PR_\odot^2} \right)^{\frac{1}{3}} \quad (8)$$

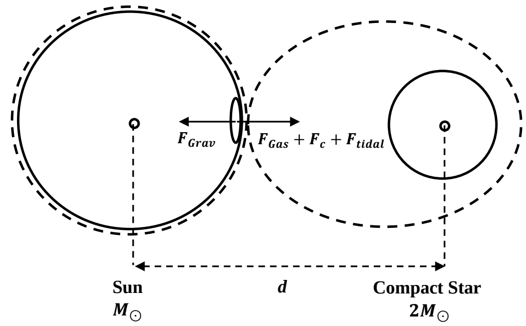

Figure 4. Depiction of a two-star system comprising of the Sun and a Compact Star and the forces acting on the Sun's surface mass Δm. Roche Lobes are teardrop shaped regions bounded by a critical gravitation equipotential within which an orbiting material is bound to the star. In this case the Roche Lobes are arranged such that surface matter of the Sun is just outside the Roche Lobe of the Compact Star.

We know that $v = r\omega$. Thus,

$$\omega^2 = \frac{v^2}{r^2} \quad (9)$$

Case I ($v = 2$ km s^{-1})
Substituting $v = 2$ km s^{-1} and $r = R_\odot = 7 \times 10^8$ m in equation (9), we get $\omega = 2.86 \times 10^{-6}$ rad s^{-1}. Substituting this value of ω and the values of the constants as per surface conditions $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_\odot = 2 \times 10^{30}$ kg, $R_\odot = 7 \times 10^8$ m, $\rho = 10^{-6}$ kg m^{-3}, $\Delta r = 500$ km, $P = 63.8$ Pa (As calculated for the photosphere previously) in equation (8), we get, $d = 1.96 R_\odot$.

Case II ($v = 233.5$ km s^{-1})
Substituting $v = 233.5$ km s^{-1} and $r = R_\odot = 7 \times 10^8$m in equation (9), we get $\omega = 3.34 \times 10^{-4}$ rad s^{-1}. Substituting this value of ω and the values of the constants as per surface conditions $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_\odot = 2 \times 10^{30}$ kg, $R_\odot = 7 \times 10^8$ m, $\rho = 10^{-6}$ kg m^{-3}, $\Delta r = 500$ km, $P = 63.8$ Pa (As calculated for the photosphere previously) in equation (8), we get, $d = 2.54 R_\odot$.

Case III ($v = 255$ km s^{-1})
Substituting $v = 255$ km s^{-1} and $r = R_\odot = 7 \times 10^8$ m in equation (9), we get $\omega = 3.64 \times 10^{-6}$ rad s^{-1}. Substituting this value of ω and the values of the constants as per surface conditions $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_\odot = 2 \times 10^{30}$ kg, $R_\odot = 7 \times 10^8$ m, $\rho = 10^{-6}$ kg m^{-3}, $\Delta r = 500$ km, $P = 63.8$ Pa (As calculated for the photosphere previously) in equation (8), we get, $d = 2.76 R_\odot$.

A similar calculation can be performed for various different rotational velocities (50 km s^{-1}, 75 km s^{-1}

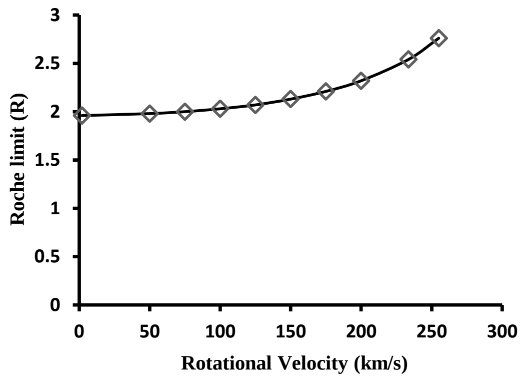

Figure 5. Depicts the Roche limit in the presence of a 2 Solar Mass Neutron Star for various rotational velocities including the cases considered.

$100\,\mathrm{km\,s^{-1}}$, $125\ \mathrm{km\,s^{-1}}$, $150\ \mathrm{km\,s^{-1}}$, $175\ \mathrm{km\,s^{-1}}$, and $200\,\mathrm{km\,s^{-1}}$) to obtain Figure 5.

3. CONCLUSION

Through our calculations we have arrived at two trends. The critical velocity of a layer of the Sun at which the layer matter starts to break up increases from $233.5\ \mathrm{km\,s^{-1}}$ for the core to $255\ \mathrm{km\,s^{-1}}$ for the radiation zone and to $318.5\ \mathrm{km\,s^{-1}}$ for the photosphere. We observe a larger increase in critical velocity in moving from the radiation zone to the photosphere than in moving from the core to the radiation zone. This can be accounted for by the change in gas pressure. The respective contributions of gas pressure in the core and radiation zone differ by a smaller margin that that between the radiation zone and photosphere. As a result, a greater value of centrifugal force is necessary to cause a force imbalance for matter in the photosphere. We have also observed that the radiation pressure term is negligible in all three cases considered, even at temperatures as high as 15 million K. This negligibility can be attributed to the high density and layer thickness in the case of the core and radiation zone and to low surface temperature (5777 K) in the case of the photosphere. Next, the Roche limit of the Sun in the presence of a 2 Solar Mass companion star is also found to increase from $1.96R_\odot$ at the Sun's present velocity to $2.54R_\odot$ at $v = 233.5\ \mathrm{km\,s^{-1}}$ and further to $2.76R_\odot$ at $v = 255\ \mathrm{km\,s^{-1}}$, as with increasing velocity, centrifugal force increases and less contribution is required from the tidal force to cause break − up of matter at the surface. From Figure 5. it is observed that the value of d is found to change at a very slow rate (nearly constant) until $v = 125\,\mathrm{km\,s^{-1}}$. This is because of the very small change in the value of the $\rho\Delta r\omega^2R_\odot^3$ term in the derived equation on approaching $v = 125\ \mathrm{km\,s^{-1}}$ after which the differences in value of the above term become more significant.

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