The Vicsek model

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1 Introduction

Minimal statistical mechanical models such as the Ising model continue to provide insight into a wide range of equilibrium systems. The Vicsek model[1] fulfills a role as the prototypical model of active matter. Like the Ising model, it has initiated a generation of research, spawned similar 'Vicsek-type' models and even caused a controversy over the nature of its phase transition, which took over a decade to be conclusively resolved.

2 Mathematical description

2.1 Microscopic behaviour

The Vicsek model is essentially an off-lattice version of the XY model in which spins are replaced with self-propelled particles, pushing the system far from equilibrium. Particles are described by their position $\mathbf{x}_i$ and their velocity $\mathbf{v}_i$, which has a fixed magnitude v_0. A particle's direction of motion, or 'heading', is parameterised by an angle θ such that $\mathbf{v}_i = v_0(\cos\theta_i\,,\,\sin\theta_i)$. The position and heading of the i^{th} particle are updated at each timestep according to the following procedures:

$$\mathbf{x}_i(t+\Delta t) = \mathbf{x}_i(t) + \Delta t\,\mathbf{v}_i(t)\,, \tag{1}$$

$$\theta_i(t+\Delta t) = \arg\left\{\sum_{j\in\mathcal{J}_i^R} e^{\imath\theta_j(t)}\right\} + \eta\,\xi_i(t)\,. \tag{2}$$

Here, $\mathcal{J}_i^R$ denotes the set of particles within a circular region of radius R around the i^{th} particle, and ξ_i is randomly sampled from the uniform distribution $\mathcal{U}(-\frac{1}{2},\frac{1}{2})$.

The dynamics of the system depend on the relative strengths of the two terms on the r.h.s of Eq. (2). The first of these terms acts to align particles with their neighbours, and evidently depends on both the radius of interaction R and the average particle density ρ. Though perhaps less immediately apparent, the tendency for particles to align also depends on v_0, since this influences the number of consecutive timesteps in which two particles are within R of each other.

Conversely, the second term acts to upset this alignment, with the noise amplitude η playing a role analogous to the temperature in equilibrium systems.

2.2 Emergent behaviour

At any instant, a given particle only interacts with 'neighbours' within a radius R, yet correlations in particle motion can occur over scales many orders of magnitude larger. Just as long-range correlations in spin alignments occur in ferromagnetic systems below the Curie temperature, the

[1] T. Vicsek *et al.*, Phys. Rev. Lett., **75**, 1226 (1995)

Vicsek model exhibits this 'emergent' behaviour if $\eta < \eta_c$, where η_c is the critical point for a given set of parameters (v_0, R, ρ). The phase transition which occurs as η crosses η_c is known as the 'flocking transition', due to the resemblance of the ordered state to a flock of birds.

The appropriate order parameter for the Vicsek model is the sum of particle velocity vectors (in ferromagnetic systems, it is the sum of spin vectors – the magnetisation).

$$V = \frac{1}{Nv_0}\left|\sum_i^N \mathbf{v}_i\right| \tag{3}$$

With the normalisation factor $\frac{1}{Nv_0}$, V can take values between 0, for a perfectly disordered system, and 1, for a system where every particle is travelling in the same direction.

For systems in which the fluctuation-dissipation theorem is valid, the variance of the order parameter is directly related to the susceptibility χ of the system, which describes the response of the system to perturbations.

$$\chi \sim \mathrm{Var}(V) = \langle (V - \langle V \rangle)^2 \rangle \tag{4}$$

where $\langle V \rangle$ denotes the expectation value of V. The Vicsek model is far from equilibrium and hence the fluctuation-dissipation theorem does not hold, but some authors refer to the quantity $\mathrm{Var}(V)L^2$ as the susceptibility and it behaves in much the same way, diverging at η_c.

The Binder cumulant U is the kurtosis of the order parameter, which works out as

$$U = 1 - \frac{\langle V^4 \rangle}{3\langle V^2 \rangle^2} \tag{5}$$

By running several simulations of different sizes but the same density, η_c may be identified as the intersection of Binder cumulants.

3 Further reading

- T. Vicsek *et al.*, Phys. Rev. Lett., **75**, 1226 (1995). This is the original paper in which the Vicsek model was proposed.
- J. Toner and Y. Tu, Phys. Rev. Lett., **75**, 4326 (1995). A theoretical paper released simultaneously with Visek's paper, describing a continuum dynamical model of active matter.
- G. Baglietto and E. V. Albano, Phys. Rev. E, **78**, 021125 (2008). I found this study of the Vicsek model close to its critical point one of the clearest to follow.

4 Illustrative figures

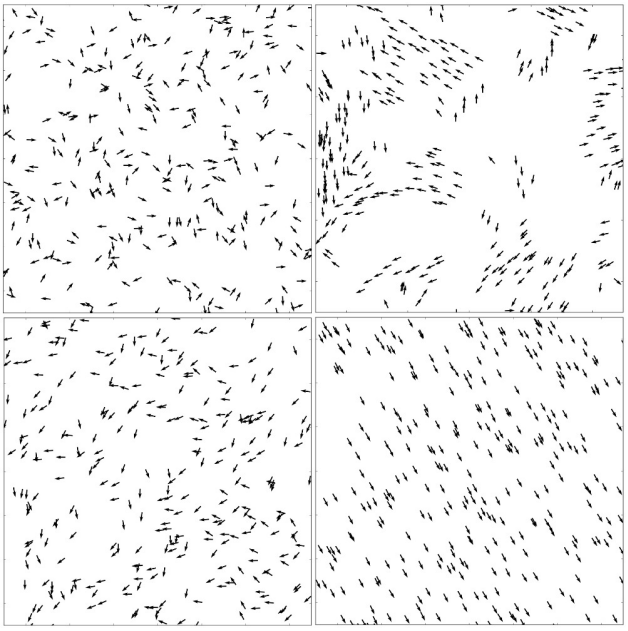

Figure 1: Snapshots from four simulations with $N = 300, v_0 = 0.03$ at different densities and noise amplitudes. **Top left:** high disorder with $L = 5$, $\eta = 5$. **Top right:** low density and noise $L = 25$, $\eta = 0.1$, leading to the formation of smaller coherent groups. **Bottom left:** Higher density and medium noise $L = 7$, $\eta = 2$, there is some correlation but particles move fairly independently. **Bottom right:** highly ordered phase with high density and low noise $L = 5$, $\eta = 0.1$. N.B. These values are the same as those presented by Vicsek *et al.* in their original 1995 paper.

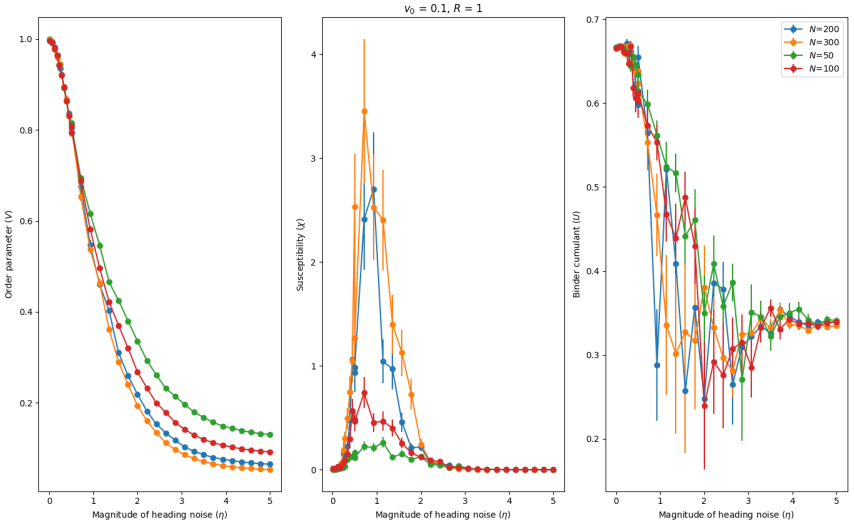

Figure 2: Measurements of the order parameter, susceptibility and Binder cumulant across the flocking transition, for simulations with the same ρ, v_0 and R, but different N. Each point represents the average of 50 simulations, which were each run for 10000 timesteps after an initial 'burn-in' time. Due to large fluctuations, the error bars increase substantially near the critical point, particularly for the Binder cumulant. If one wanted to obtain smooth curves so as to estimate the intersection of Binder cumulants or the critical exponents, then *many* more simulations should be averaged over, with each lasting far longer.