# Answersheet: Calculus

**1.** To evaluate the integral $\int_0^1 x^2 e^x \, dx$ using integration by parts, we follow these steps:

1. Identify parts for integration by parts formula: $\int u \, dv = uv - \int v \, du$.
2. Let $u = x^2$ and $dv = e^x \, dx$.
3. Compute $du$ and $v$:
   - $du = 2x \, dx$
   - $v = \int e^x \, dx = e^x$
4. Apply the integration by parts formula:
   $\int_0^1 x^2 e^x \, dx = \left. x^2 e^x \right|_0^1 - \int_0^1 2x e^x \, dx$
5. Evaluate the boundary term:
   $\left. x^2 e^x \right|_0^1 = 1^2 e^1 - 0^2 e^0 = e - 0 = e$
6. Evaluate the remaining integral using integration by parts again:
   - Let $u = 2x$ and $dv = e^x \, dx$.
   - Compute $du$ and $v$:
     - $du = 2 \, dx$
     - $v = e^x$
   - Apply the integration by parts formula:
     $\int_0^1 2x e^x \, dx = \left. 2x e^x \right|_0^1 - \int_0^1 2 e^x \, dx$
   - Evaluate the boundary term:
     $\left. 2x e^x \right|_0^1 = 2 \cdot 1 \cdot e^1 - 2 \cdot 0 \cdot e^0 = 2e - 0 = 2e$
   - Evaluate the remaining integral:
     $\int_0^1 2 e^x \, dx = 2 \int_0^1 e^x \, dx = 2 \left. e^x \right|_0^1 = 2(e-1)$
7. Combine the results:
   $\int_0^1 x^2 e^x \, dx = e - (2e - 2) = e - 2e + 2 = -e + 2$
8. The final answer is:
   $\int_0^1 x^2 e^x \, dx = 2 - e.$

**2.** To find the limit $\lim_{x \to 0} \frac{\sin(3x)}{x}$, we can use the standard limit result $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$. Follow these steps:

1. Rewrite the limit in a form that uses the standard result:
   $\lim_{x \to 0} \frac{\sin(3x)}{x} = \lim_{x \to 0} \frac{\sin(3x)}{3x} \cdot 3$
2. Recognize that $\lim_{x \to 0} \frac{\sin(3x)}{3x} = 1$ because it matches the standard limit form with $3x$ in place of $x$.
3. Therefore, the limit becomes:
   $\lim_{x \to 0} \frac{\sin(3x)}{x} = 3 \cdot 1 = 3$
4. The final answer is:
   $\lim_{x \to 0} \frac{\sin(3x)}{x} = 3.$

3. To determine the derivative of the function $f(x) = x^x$, we use logarithmic differentiation. Follow these steps:
    1. Take the natural logarithm of both sides:
    $\ln(f(x)) = \ln(x^x)$
    2. Simplify the right-hand side using logarithm properties:
    $\ln(f(x)) = x\ln(x)$
    3. Differentiate both sides with respect to $x$:
    $\frac{d}{dx}[\ln(f(x))] = \frac{d}{dx}[x\ln(x)]$
    4. Apply the chain rule to the left-hand side and the product rule to the right-hand side:
    $\frac{1}{f(x)}f'(x) = \ln(x) + 1$
    5. Solve for $f'(x)$:
    $f'(x) = f(x)(\ln(x) + 1)$
    6. Substitute back $f(x) = x^x$:
    $f'(x) = x^x(\ln(x) + 1)$
    7. The final answer is:
    $f'(x) = x^x(\ln(x) + 1)$.

4. To evaluate the limit $\lim_{x\to\infty} \frac{2x^3-x+1}{x^3+3x^2}$, we follow these steps:
    1. Divide the numerator and the denominator by the highest power of $x$ in the denominator, which is $x^3$:
    $\lim_{x\to\infty} \frac{2x^3-x+1}{x^3+3x^2} = \lim_{x\to\infty} \frac{\frac{2x^3}{x^3}-\frac{x}{x^3}+\frac{1}{x^3}}{\frac{x^3}{x^3}+\frac{3x^2}{x^3}}$
    2. Simplify each term:
    $\lim_{x\to\infty} \frac{2-\frac{1}{x^2}+\frac{1}{x^3}}{1+\frac{3}{x}}$
    3. As $x$ approaches infinity, the terms $\frac{1}{x^2}$, $\frac{1}{x^3}$, and $\frac{3}{x}$ approach 0:
    $\lim_{x\to\infty} \frac{2-0+0}{1+0} = \frac{2}{1} = 2$
    4. The final answer is:
    $\lim_{x\to\infty} \frac{2x^3-x+1}{x^3+3x^2} = 2$.

5. To find the second derivative of the function $f(x) = \ln(x^2+1)$, we follow these steps:
    1. Find the first derivative using the chain rule:
    $f'(x) = \frac{d}{dx}[\ln(x^2+1)] = \frac{1}{x^2+1} \cdot \frac{d}{dx}[x^2+1] = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$
    2. Find the second derivative using the quotient rule:
    $f''(x) = \frac{d}{dx}\left(\frac{2x}{x^2+1}\right)$
    3. Apply the quotient rule: $\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$ where $u = 2x$ and $v = x^2+1$:
        - $u' = 2$
        - $v' = 2x$
        - $f''(x) = \frac{(2)(x^2+1)-(2x)(2x)}{(x^2+1)^2}$
    4. Simplify the expression:
    $f''(x) = \frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2} = \frac{2(1-x^2)}{(x^2+1)^2}$
    5. The final answer is:
    $f''(x) = \frac{2(1-x^2)}{(x^2+1)^2}$.