Z Transform

$$X(z) = Z\{x(k)\} = \sum_{k=0}^{\infty} x(k) \cdot z^{-k}$$

- Given a sequence $\{f(k)\}$ i.e. $f(0),\ \ f(1),\ \ f(2),\ \ f(3),\ \cdots$

$$Z\{f(k)\} = f(0) + f(1)\cdot z^{-1} + f(2)\cdot z^{-2} + f(3)\cdot z^{-3} + \ldots = \sum_{k=0}^{\infty} x(k)\cdot z^{-k}$$

- $Z\{\delta(k)\},\ \delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \geq 1 \end{cases}$ - impulse sequence

$$Z\{\delta(k)\} = \delta(0)\cdot z^{0} + \delta(1)\cdot z^{-1} + \delta(2)\cdot z^{-2} + \delta(3)\cdot z^{-3} + \ldots = 1\cdot z^{0} + 0\cdot z^{-1} + 0\cdot z^{-2} = 1$$

- $Z\{u(k)\},\ u(k) = 1,\ k \geq 0$

$$Z\{u(k)\} = \sum_{k=0}^{\infty} u(k)\cdot z^{-k} = u(0)\cdot z^{0} + u(1)\cdot z^{-1} + u(2)\cdot z^{-2} + u(3)\cdot z^{-3} + \ldots = 1\cdot z^{0} + 1\cdot z^{-1} + 1\cdot z^{-2} =$$

Geometric series when $k \to \infty, |r| < 1\ is:\ a + a\cdot r + a\cdot r^{2} + a\cdot r^{3} + \ldots = \sum_{k=0}^{\infty} a\cdot r^{-k} = \dfrac{a}{1-r},\ |r| < 1$

$$\frac{1}{2} + \left(\frac{1}{2}\right)^{2} + \left(\frac{1}{2}\right)^{3} + \left(\frac{1}{2}\right)^{4} + \ldots =$$

$$a = 1,\ r = \frac{1}{2}$$

$$S = \frac{a}{1-r} = \frac{1}{1-1/2} = 2$$

$$a = 1,\ r = \frac{1}{z} = z^{-1}$$

$$Z\{u(k)\} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

- $Z\{\sigma(k-m)\} = \sum_{k=0}^{\infty} \sigma(k-m)\cdot z^{-k}$

$$\begin{cases} k - m = l \\ k = l + m \\ k \to \infty \Rightarrow l \to \infty \\ k \to 0 \Rightarrow l \to -m \end{cases}$$

$$= \sum_{l=-m}^{\infty} \sigma(l)\cdot z^{-l-m} = z^{-m} \sum_{l=-m}^{\infty} \sigma(l)\cdot z^{-l} = z^{-m}\cdot Z\{\sigma(k)\}$$

- $Z\{x(k+1)\} = \sum_{k=0}^{\infty} x(k+1)\cdot z^{-k} = z\cdot \sum_{k=0}^{\infty} x(k+1)\cdot z^{-k-1} = z(x(1)\cdot z^{-1} + x(2)\cdot z^{-2} + \ldots) =$

$$= z(\underbrace{x(0) + x(1)\cdot z^{-1} + x(2)\cdot z^{-2} + \ldots}\underbrace{- x(0)\cdot z^{0}}) = z(\underbrace{Z\{x(k)\}}_{X(z)} - x(0)) = z\cdot X(z) - z$$

$\ldots$

$$Z\{x(k+\delta)\} = z^{\delta}(Z\{x(k)\} - x(0)\cdot z^{0} - x(1)\cdot z^{-1} - \ldots - x(\delta-1)\cdot z^{1-\delta}) = z^{\delta}\underbrace{Z\{x(k)\}}_{X(z)} + h(x(0), x(1), \ldots, x(\delta-1))$$

Inverse Z Transform

- $y(k+1) - a \cdot y(k) = 0,\ \ y(0) = y_0 \Rightarrow$ 1st order different equation using z transform

$$Z\{y(k+1)\} = Z\{a \cdot y(k)\} \Leftrightarrow z(Y(z) - y(0)) = a \cdot Y(z) \Leftrightarrow (z-a)Y(z) = y_0 z \Rightarrow Y(z) = \frac{z}{z-a} y_0$$

To find *y(k)* we need to the inverse z transform

$$y(k) = Z^{-1}\left\{\frac{z}{z-a} y_0\right\} = y_0 Z^{-1}\left\{\frac{z}{z-a}\right\} = y_0 Z^{-1}\left\{\frac{1}{1-az^{-1}}\right\} = y_0 \cdot a^k$$

- $x(k+2) + 3x(k+1) + 2x(k) = 0,\ \ x(0) = 0,\ \ x(1) = 1$

$$Z\{y(k+2)\} + 3Z\{y(k+1)\} + 2Z\{y(k)\} = 0 \Leftrightarrow z^2 X(z) - z^2 x(0) - zx(1) + 3zX(z) - 3zx(0) + 2X(z) = 0$$

$$z^2 \cdot X(z) - z^2 \cdot 0 - z + 3 \cdot z \cdot X(z) - 3 \cdot 0 + 2 \cdot X(z) = 0$$

$$(z^2 + 3 \cdot z + 2)X(z) = z \Rightarrow X(z) = \frac{z}{z^2 + 3 \cdot z + 2}$$

To find *x(k)* we need the inverse z transform

$$x(k) = Z^{-1}\left\{\frac{z}{z^2 + 3 \cdot z + 2}\right\} = Z^{-1}\left\{\frac{z}{(z+2)(z+1)}\right\}$$

$$\frac{X(z)}{z} = \frac{1}{(z+2)(z+1)} = \frac{A}{z+2} + \frac{B}{z+1} = \frac{A(z+1) + B(z+2)}{(z+2)(z+1)}$$

Find *A*: put $z = -2 \Rightarrow 1 = -A \Rightarrow A = -1$

Find *B*: put $z = -1 \Rightarrow 1 = B \Rightarrow B = 1$

$$x(k) = -Z^{-1}\left\{\frac{z}{z+2}\right\} + Z^{-1}\left\{\frac{z}{z+1}\right\} = -Z^{-1}\left\{\frac{1}{1+2z^{-1}}\right\} + Z^{-1}\left\{\frac{1}{1+z^{-1}}\right\} = -(-2)^k + (-1)^k$$

References

[1] https://www.khanacademy.org/