Laplace transform

$\mathscr{L}\{\mathrm{f}(t)\} \longrightarrow \mathrm{F}(s)$

$\mathscr{L}\{\mathrm{f}(t)\} = \int_0^\infty e^{-st} \cdot f(t)dt$

- $\mathscr{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \cdot dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = -\frac{1}{s}\left(\underbrace{\lim_{t\to\infty} e^{-st}}_{s>0} - \lim_{t\to 0} e^{-st}\right) = -\frac{1}{s}(0-1) = \frac{1}{s}$
- $\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}dt = \int_0^\infty e^{(a-s)t}dt = \frac{1}{a-s}e^{(a-s)t}\Big|_0^\infty = \frac{1}{a-s}\left(\lim_{t\to\infty} e^{(a-s)t} - \lim_{t\to 0} e^{(a-s)t}\right) = \frac{1}{a-s}(0-1) = \frac{1}{s-a}$

  $a - s > 0 \Rightarrow no\ \ limit$

  $a - s < 0 \Rightarrow 0$
- $\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st} \cdot f'(t)dt =$

  Integration by parts $\int u \cdot v' = u \cdot v - \int u' \cdot v$

  $u = e^{-st} \qquad u' = -s \cdot e^{-st}$

  $v' = f'(t) \qquad v = f(t)$

$= e^{-st} \cdot f(t)\Big|_0^\infty - \int_0^\infty \left(-e^{-st} \cdot f(t)\right)dt = e^{-st} \cdot f(t)\Big|_0^\infty + s\underbrace{\int_0^\infty e^{-st} \cdot f(t)dt}_{L\{\mathrm{f(t)}\}} = \lim_{t\to\infty}\left(e^{-st}f(t)\right) - \lim_{t\to 0}\left(e^{-st}f(t)\right) + s \cdot L\{f(t)\}$

$= 0 - f(0) + s \cdot \mathscr{L}\{f(t)\}$

$\mathscr{L}\{f'(t)\} = s \cdot \mathscr{L}\{f(t)\} - f(0) \Rightarrow \mathscr{L}\{f(t)\} = \frac{1}{s}(\mathscr{L}\{f'(t)\}) + f(0)$

$\mathscr{L}\{f''(t)\} = s \cdot \mathscr{L}\{f'(t)\} - f'(0) = s(s \cdot \mathscr{L}\{f(t)\} - f(0)) - f'(0) = s^2\mathscr{L}\{f(t)\} - sf(0)) - f'(0)$

$\dots$

$\mathscr{L}\{f^{(n)}(t)\} = s^n \cdot \mathscr{L}\{f(t)\} - h\{f(0), f'(0), \dots f^{(n-1)}(0)\}$

- $\mathscr{L}\{t\} = \int_0^\infty e^{-st} \cdot t\,dt = -\frac{1}{s}e^{-st} \cdot t\Big|_0^\infty - \int_0^\infty 1 \cdot \left(-\frac{1}{s}e^{-st}\right) =$

  $u = t \qquad v' = e^{-st}$

  $u' = 1 \qquad v = -\frac{1}{s}e^{-st}$

$= \lim_{t\to\infty}\left(-\frac{t}{s}e^{-st}\right) - \lim_{t\to 0}\left(-\frac{t}{s}e^{-st}\right) + \frac{1}{s}\underbrace{\int_0^\infty e^{-st}}_{L\{1\}} = 0 + 0 + \frac{1}{s}L\{1\} \Rightarrow$

$\Rightarrow \mathscr{L}\{t\} = \frac{1}{s^2}$

Inverse Laplace transform

F(s)= $\mathscr{L}$\{f(t)\} $\Leftrightarrow$ f(t)= $\mathscr{L}^{-1}$\{F(s)\}

- $\mathscr{L}^{-1}\left\{\dfrac{2s+1}{s^2+s-6}\right\}$

$$\frac{2s+1}{s^2+s-6}=\frac{2s+1}{(s-3)(s+2)}=\frac{A}{s-3}+\frac{B}{s+2}=\frac{A(s+2)+B(s-3)}{(s-3)(s+2)}$$

$$2s+1=A(s+2)+B(s-3)$$

Find *A*: put $s=3\Rightarrow 2\cdot 3+1=A\cdot 5+B\cdot 0\Rightarrow A=\dfrac{7}{5}$

Find *B*: put $s=-2\Rightarrow -4+1=A\cdot 0+B\cdot(-5)\Rightarrow B=\dfrac{3}{5}$

$$\mathscr{L}^{-1}\left\{\frac{2s+1}{s^2+s-6}\right\}=\frac{7}{5}\,\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\}+\frac{3}{5}\,\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}=\frac{7}{5}e^{3t}+\frac{3}{5}e^{-2t}$$

- $\mathscr{L}^{-1}\left\{\dfrac{s+2}{(s-3)(s^2+9)}\right\}$

$$\frac{s+2}{(s-3)(s^2+9)}=\frac{A}{s-3}+\frac{Bs+C}{s^2+9}=\frac{A(s^2+9)+(Bs+C)(s-3)}{(s-3)(s^2+9)}$$

Find *A*: put $s=3\Rightarrow 3+2=A(9+9)+(B\cdot 3+C)\cdot 0\Rightarrow A=\dfrac{5}{18}$

Find *C*: put $s=0\Rightarrow 0+2=\dfrac{5}{18}\cdot(0+9)+C\cdot(-3)\Rightarrow 2=\dfrac{5}{2}-3\cdot C\Rightarrow 3\cdot C=\dfrac{5}{2}-2\Rightarrow C=\dfrac{1}{6}$

Find *B*: put $s=1\Rightarrow 3=\dfrac{5}{18}\cdot 10+\left(B+\dfrac{1}{6}\right)\cdot(-2)\Rightarrow 3=\dfrac{25}{9}-2\cdot B-\dfrac{1}{3}\Rightarrow B=\dfrac{1}{2}\cdot\dfrac{-27+25-3}{9}\Rightarrow B=-\dfrac{5}{18}$

$$\mathscr{L}^{-1}\left\{\frac{s+2}{(s-3)(s^2+9)}\right\}=\frac{5}{18}\,\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\}-\frac{5}{18}\,\mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}+\frac{1}{6}\,\mathscr{L}^{-1}\left\{\frac{1\cdot 3}{s^2+3^2}\right\}\cdot\frac{1}{3}=$$

$$\mathscr{L}^{1}\left\{\frac{s}{s^2+a^2}\right\}=\cos(at)$$

$$\mathscr{L}^{1}\left\{\frac{a}{s^2+a^2}\right\}=\sin(at)$$

$$=\frac{5}{18}e^{3t}+\frac{-5}{18}\cos(3t)+\frac{1}{18}\sin(3t)$$

- $y''+5y'+6y=0;\;\; y(0)=2;\;\; y'(0)=3$

$$\mathscr{L}\{y''\}+5\,\mathscr{L}\{y'\}+6\,\mathscr{L}\{y\}=0$$

$$s\,\mathscr{L}\{y'\}-y'(0)+5\,\mathscr{L}\{y'\}+6\,\mathscr{L}\{y\}=0$$

$$\mathscr{L}\{y'\}=s\,\mathscr{L}\{f(t)\}-y(0)$$

$$s(s\,\mathscr{L}\{y\}-y(0))-y'(0)+5(s\,\mathscr{L}\{y\}-y(0))+6\,\mathscr{L}\{y\}=0$$

$$s^2\,\mathscr{L}\{y\}-2s-3+5s\,\mathscr{L}\{y\}-10+6\,\mathscr{L}\{y\}=0$$

$$\mathscr{L}\{y\}(s^2+5s+6)-2s-3-10=0$$

$$\mathscr{L}\{y\}=\frac{2s+13}{s^2+5s+6}$$

$$y=\mathscr{L}^{-1}\left\{\frac{2s+13}{s^2+5s+6}\right\}$$

$$\frac{2s+13}{s^2+5s+6}=\frac{2s+13}{(s+2)(s+3)}=\frac{A}{s+2}+\frac{B}{s+3}=\frac{A(s+3)+B(s+2)}{(s+2)(s+3)}$$

$$\begin{cases}A+B=2\Rightarrow A=2-B\\ 3A+2B=13\end{cases}$$

$$\Rightarrow 6-3B+2B=13\Rightarrow B=-7$$

$$\Rightarrow A=9$$

$$\mathscr{L}^{-1}\left\{\frac{2s+13}{s^2+5s+6}\right\}=9\frac{1}{s+2}-7\frac{1}{s+3}=9\cdot e^{-2t}-7\cdot e^{-3t}$$

References

[1] https://www.khanacademy.org/