Bojan-Topsy Lucian-Cosmin. Temă EDP

Ex 1. (Ⓤ) $\begin{cases} u_{tt}(x,t) + u_{tx}(x,t) - 2u_{xx}(x,t) = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x) & \\ u_t(x,0) = g(x) & , x \in \mathbb{R}. \end{cases}$

unde $f \in C^2(\mathbb{R})$ și $g \in C^1(\mathbb{R})$.

"Asociem" o ec. car. potrivită în variabila în t. în ec 1:

$\lambda^2 + \lambda - 2 = 0$ $\qquad \lambda_{1,2} = \frac{-1 \pm 3}{2} < \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = -2. \end{matrix}$

$\Delta = 1 - 4(-2) = 9 = 3^2.$

Deci $(\lambda - 1)(\lambda + 2) = 0.$

$(\partial_t - 1\partial_x)(\partial_t + 2\partial_x)u =$

$= (\partial_t - \partial_x)(u_t + 2u_x)$

$= u_{tt} + 2u_{xt} - u_{tx} - 2u_{xx}$

$= u_{tt} + u_{tx} - 2u_{xx}.$

Astfel avem:

$u_{tt} + u_{tx} - 2u_{xx} = 0.$

$(\partial_t - \partial_x)(\partial_t + 2\partial_x)u = 0$

sau.

$(\partial_t + 2\partial_x)\underbrace{(\partial_t - \partial_x)u}_{v} = 0.$ - alegem pe aceasta.

Notăm $v(x,t) = u_t(x,t) - u_x(x,t)$

v-ul verifică o ec. de transport omogenă. $\begin{cases} v_t(x,t) + 2v_x(x,t) = 0, x \in \mathbb{R}, t > 0 \\ v(x,0) = u_t(x,0) - u_x(x,0) = \\ \quad = g(x) - f'(x) \quad (*) \end{cases}$

$0 = (v_x, v_t)(2,1) = \nabla v \cdot \bar{a}$, $\bar{a} = (2,1)$

$= \frac{\partial v}{\partial \bar{a}} \Rightarrow v$ e constantă pe direcția $\bar{a}$

Deci $v(x,t) = v(t \cdot \cancel{(2,1)} + (\cancel{x} x - 2t, 0)) = v(x-2t, 0) \overset{(*)}{=} g(x-2t) - f'(x-2t)$

• u verifică ecuația de transport neomogenă.

$\begin{cases} u_t(x,t) - u_x(x,t) = g(x-2t) - f'(x-2t) \\ u(x,0) = f(x) \end{cases}$

Fixez $x, t$ și consider funcția $w(s) := u(x-s, t+s)$, $s \in \mathbb{R}$.

$$\boxed{\begin{aligned} w'(s) &= -u_x(x-s, t+s) + u_t(x-s, t+s) \\ &= u_t(x-s, t+s) - u_x(x-s, t+s) \\ &= g(x-s-2t-2s) - f'(x-s-2t-2s) \\ &= g(x-2t-3s) - f'(x-2t-3s). \end{aligned}}$$

Pe de altă parte:

$w(0) = u(x, t)$

$w(-t) = u(x+t, 0) = f(x+t)$

$$\int_{-t}^{0} w'(s)\,ds = \int_{-t}^{0} g(x-2t-3s) - f'(x-2t-3s)\,ds.$$

$\parallel$ Leibniz.

$w(0) - w(-t)$

$$u(x,t) - f(x+t) = \int_{-t}^{0} g(x-2t-3s)\,ds - \underbrace{\int_{-t}^{0} f'(x-2t-3s)\,ds}_{\parallel \text{ Leibniz. } f(x-2t) - f(x+t)}.$$

$$\Rightarrow u(x,t) - f(x+t) = \int_{-t}^{0} g(x-2t-3s)\,ds - f(x-2t) + f(x+t)$$

$$\Rightarrow u(x,t) = 2f(x+t) - f(x-2t) + \int_{-t}^{0} g(x-2t-3s)\,ds$$

Ex2. $\begin{cases} u_{tt}(x,t)+u_{tx}(x,t)-2u_{xx}(x,t)=t \\ u(x,0)=e^{-x} \\ u_t(x,0)=\sin x \end{cases}, x\in\mathbb{R}, t>0$

(Căutăm $l=l(t)$ a.î. $v(x,t):=u(x,t)+l(t)$, să verifice. ~~$v_{tt}(x,t)+\ldots=0$~~)

$$v_{tt}+v_{tx}-2v_{xx}=0.$$

Avem $u_{tt}+l''(t)+u_{tx}+\underbrace{\frac{\partial l(t)}{\partial t\partial x}}_{=0}-2u_{xx}=0.$

$$u_{tt}+l''(t)+u_{tx}-2u_{xx}=0$$

$$l''(t)+t=0$$

$$l'(t)=-\int t\,dt=-\frac{t^2}{2}+\underset{\text{iau } C=0}{C}.=-\frac{t^2}{2}.$$

$$l'(t)=-\frac{t^2}{2}.$$

$$l(t)=\int-\frac{t^2}{2}\,dt=-\frac{1}{2}\int t^2\,dt=-\frac{1}{2}\cdot\frac{1}{3}t^3+\underset{\text{iau } 0}{\tilde{C}}$$

$$l(t)=-\frac{t^3}{6}$$

$$v(x,t)=u(x,t)-\frac{t^3}{6}.$$

verificăm:

$$v_{tt}+v_{tx}-2v_{xx}=0$$

$v(x,0)=u(x,0)+\underset{0}{l(0)}=e^{-x}$ ✓

$v_t(x,t)=u_t(x,t)+l'(t)$

$v_t(x,0)=u_t(x,0)+\underset{0}{l'(0)}=\sin x$ ✓

Avem deci sistemul $\begin{cases} v_{tt}+v_{tx}-2v_{xx}=0 \\ v(x,0)=e^{-x} \\ v_t(x,0)=\sin x. \end{cases}$

Care e rezolvată la seminar.