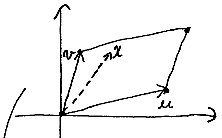
Lecture 14

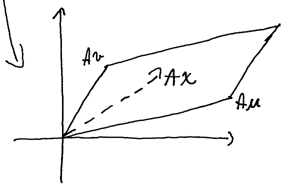
Friday, January 20, 2023 10:11 AM

- Linear Algebra Facts

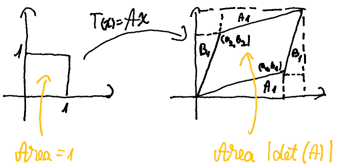
$T: \mathbb{R}^2 \to \mathbb{R}^2$ map, $T(x) = Ax$ } Linear transformation
A is a 2x2 matrix.



$x = \underbrace{\alpha u + \beta v}_{\text{parallelogram}}$, $\alpha, \beta \in [0,1]$, $Ax = \underbrace{\alpha Au + \beta Av}_{\text{parallelogram}}$

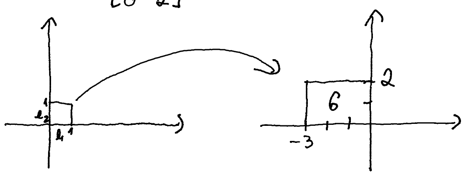


- T transforms (maps) parallelograms into parallelograms
- How does T scale areas?



ex. $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

ex. $A = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$



$\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

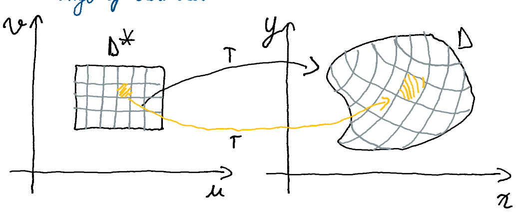
$\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

det A = −6, negative determinate - it does a reflection, changes orientation

|det A| = |−6| = 6

Remark: A is 3x3 matrix, |det (A)| → volume scales

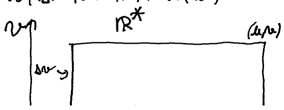
- Change of variables

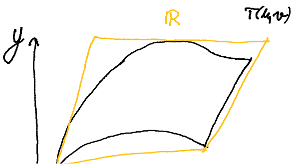


$T: D^* \to D$ bijective and of class C^1 (continous and has a continous derivative)
(it's a non-linear transformation, but)

- $(x, y) = T(u, v)$

1. if T is linear, $T(x) = Ax$
$A(D) = |\det(A)| \cdot A(D^*)$





$u = u_0 + \Delta u$

$v = v_0 + \Delta v$

$$T(u,v) \simeq \underbrace{T(u_0,v_0) + T'(u_0,v_0)\cdot\begin{bmatrix}\Delta u\\ \Delta v\end{bmatrix}}_{\text{linear approximation}}$$

$$T' = \begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix} = J$$

Jacobian Matrix

Rectangle R^* $\xrightarrow{T'}$ Parallelogram $R = T'(R^*)$

Area $= \Delta u \cdot \Delta v$

Area $= \underbrace{|\det(J)|}_{\text{Jacobian determinant}}\Delta u\,\Delta v$

- $J = T' = \begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix}$

$T'\cdot\begin{bmatrix}\Delta u\\ 0\end{bmatrix} = \Delta u\cdot\begin{bmatrix}\frac{\partial x}{\partial u}\\ \frac{\partial y}{\partial u}\end{bmatrix}$ → one edge (side) in R

$T'\cdot\begin{bmatrix}0\\ \Delta v\end{bmatrix} = \Delta v\cdot\begin{bmatrix}\frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial v}\end{bmatrix}$ → the other edge (side) in R

$$A(R) = \iint_R dx\,dy = \iint_{R^*} |\det J|\cdot du\,dv = A(T(R^*))$$

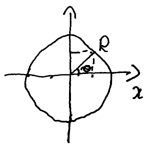
Theorem: Let $D, D^* \subseteq \mathbb{R}^2$, $T: D^* \to D$ bijective and of class C^1

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v))\,|\det(J)|\,du\,dv$$

how the area rescales when you do the transformation

- Polar coordinates

$D = \{(x,y) \in \mathbb{R}^2 \mid \underbrace{x^2+y^2}_{\text{circle}} \leq R^2$



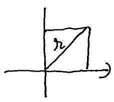
$x^2+y^2 = r^2$

$x = r\cos\theta$

$y = r\sin\theta$

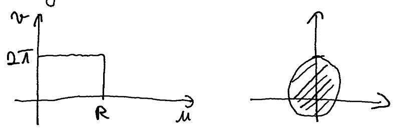
$r \in [0,R]$

$\theta \in [0,2\pi]$



$$J = \begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix} = \begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}$$

$$|\det(J)| = |r\cos^2\theta + r\sin^2\theta| = |r| = r$$



$$\iint_D f(x,y)\,dx\,dy = \int_0^R\int_0^{2\pi} f(r\cos\theta, r\sin\theta)\,r\,d\theta\,dr$$

example $\iint_D e^{x^2+y^2}\,dx\,dy$

$\left(\int e^{x^2}dx = xe^{x^2} - \int x\cdot 2x\cdot e^{x^2}\right)$ → impossible

change of variables

$x = r\cos\theta$

$y = r\sin\theta$ $\quad e^{x^2+y^2} = e^{r^2}$

$x^2+y^2 = r^2$

$$\iint_\Delta e^{x^2+y^2}\,dx\,dy = \int_0^R\int_0^{2\pi} e^{r^2} r\,d\theta\,dr = \int_0^R \underbrace{r e^{r^2}}_{\text{Constant in regards to }\theta}\Big(\underbrace{\int_0^{2\pi} d\theta}_{\text{constant in regards to } r}\Big)dr = \int_0^{2\pi} d\theta \cdot \int_0^R e^{r^2} r\,dr = \pi\cdot\int_0^R 2re^{r^2}dr = \pi e^{r^2}\Big|_0^R = \pi(e^{R^2}-1)$$

example $\iint_\Delta e^{-(x^2+y^2)}\,dx\,dy$

$$\iint_\Delta e^{-(x^2+y^2)}\,dx\,dy = \int_0^R\int_0^{2\pi} e^{-r^2} r\,d\theta\,dr$$

$$= \underbrace{\int_0^{2\pi} d\theta}_{=2\pi}\cdot\int_0^R r\cdot e^{-r^2}dr = \pi\cdot\int_0^R 2r\cdot e^{-r^2}dr = -\pi\cdot e^{-r^2}\Big|_0^R = -\pi(e^{-R^2}-1) = \pi(1-e^{-R^2})$$

Take $R\to\infty$

radius of disc

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy = \pi\left(\lim_{R\to\infty}\pi(1-e^{-R^2})\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}\cdot e^{-y^2}\,dx\,dy = \int_{-\infty}^{\infty} e^{-y^2}\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)dy =$$

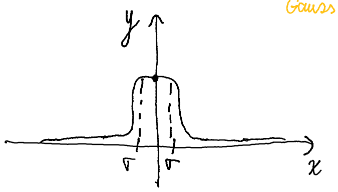
$$= \int_{-\infty}^{\infty} e^{-x^2}dx\cdot\int_{-\infty}^{\infty} e^{-y^2}dy = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 = \pi$$

identical

we proved above

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}}$$ → important in probability and statistics,

Gauss curve



The area underneath of the bell curve is $\sqrt{\pi}$

$\sigma \to 65\%$

$2\sigma \to 84\%$

$3\sigma \to 99...\%$

- Triple integrals

Let $B = [a_1,a_2]\times[b_1,b_2]\times[c_1,c_2]$

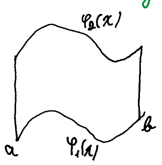
$$\iiint_B f(x,y,z)\,dx\,dy\,dz = \int_{c_1}^{c_2}\int_{b_1}^{b_2}\int_{a_1}^{a_2} f(x,y,z)\,dx\,dy\,dz$$

FUBINI $\longrightarrow = \int_{c_1}^{c_2}\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(x,y,z)\,dy\,dx\,dz$

or any other permutation

- Let $\Delta = \{(x,y,z)\in\mathbb{R}^3 \mid a\le x\le b,\ \varphi_1(x)\le y\le \varphi_2(x),\ \psi_1(x,y)\le z\le \psi_2(x,y)\}$

surfaces



$$\iiint_\Delta f(x,y,z)\,dx\,dy\,dz = \int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)}\left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz\right)dy\right)dx$$

- Variable change in 3Δ

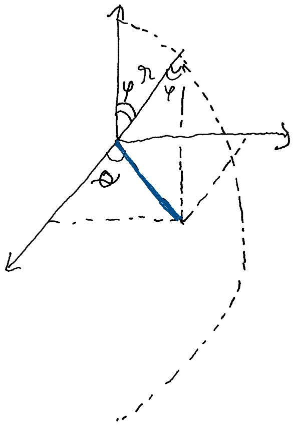
$(x,y,z) = T(u,v,w)$, $T:\Delta^*\to\Delta$ bijective and of class C^1

$$\iiint_\Delta f(x,y,z)\,dx\,dy\,dz = \iiint_{\Delta^*} f(x(u,v,w), y(u,v,w), z(u,v,w))\cdot|\det(J)|\,du\,dv\,dw$$

the volume of the box scales by the determinant of a 3x3 matrix

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

- Polar Coordinates in 3D: sphere



$z \text{(height)} = r\cos\varphi$

$\varphi \in [0, \pi]$

$x = (r\sin\varphi)\cos\theta$

$y = (r\sin\varphi)\sin\theta$

$z = r\cos\varphi$

$r \in [0, R]$

$\theta \in [0, 2\pi]$ left -> right

$\varphi \in [0, \pi]$ top -> bottom