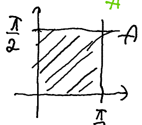
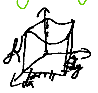
Seminar 13

Monday, January 16, 2023 6:06 PM

ex.

1) a) $\iint_A \cos x \sin y \, dx dy$, $A = \underbrace{[0, \frac{\pi}{2}]}_{x} \times \underbrace{[0, \frac{\pi}{2}]}_{y}$





$I = \int_0^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{2}} \cos x \sin y \, dx \right) dy =$

$\int_0^{\frac{\pi}{2}} \cos x \sin y \, dx = \sin y \int_0^{\frac{\pi}{2}} \cos x \, dx = \sin y \sin x \Big|_0^{x=\frac{\pi}{2}} = \sin y$

x - int
y - const

$I = \int_0^{\frac{\pi}{2}} \sin y \, dy = -\cos y \Big|_0^{\frac{\pi}{2}} = +1$

2) b)

$\iint_R \frac{1}{(x+y)^2} dx dy$ and $A = \underbrace{[1,2]}_{x} \times \underbrace{[0,1]}_{y}$

$I = \int_0^1 \left(\int_1^2 \frac{1}{(x+y)^2} dx \right) dy$

$\int_1^2 \frac{1}{(x+y)^2} dx = \int_1^2 (x+y)^{-2} dx = \frac{(x+y)^{-1}}{-1} \Big|_{x=1}^{x=2} = -\frac{1}{y+2} + \frac{1}{y+1}$

$I = \int_0^1 \left(\frac{1}{y+1} - \frac{1}{y+2} \right) = \ln|y+1| \Big|_0^1 - \ln|y+2| \Big|_0^1 = \ln 2 - \ln 3 + \ln 2 = \ln 4 - \ln 3 = \ln \frac{4}{3}$

second option (dy 1st)

$\int_1^2 \left(\int_0^1 \frac{1}{(x+y)^2} dy \right) dx = I$

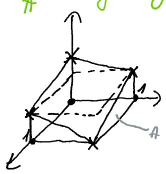
$\int_0^1 \frac{1}{(x+y)^2} dy = \frac{(x+y)^{-1}}{-1} \Big|_{y=0}^{y=1} = -\frac{1}{x+1} + \frac{1}{x}$

$I = \int_1^2 \left(\frac{1}{x} - \frac{1}{x+1} \right) dx = \ln|x| \Big|_1^2 - \ln|x+1| \Big|_1^2 = \ln 2 - \ln 1 - \ln 3 + \ln 2 = \ln 4 - \ln 3 = \ln \frac{4}{3}$ (same value)
Fubini

2) $A = [0,1] \times [0,1]$

Sketch the solid and find its value.

a) $\iint_A (2 - x - y) \, dx dy$



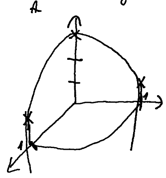
$I = \int_0^1 \left(\int_0^1 2 - x - y \, dx \right) dy$

$\int_0^1 (2 - x - y) dx = 2x - \frac{x^2}{2} - yx \Big|_{x=0}^{x=1} = 2 - \frac{1}{2} - y = \frac{3-y}{2}$

$I = \int_0^1 \frac{3-y}{2} dy = \int_0^1 \left(\frac{3}{2} - y \right) dy = \frac{3}{2} y - \frac{y^2}{2} \Big|_{y=0}^{y=1} = \frac{3}{2} - \frac{1}{2} = 1$

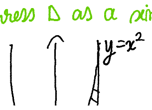
b) $\iint_A (2 - x^2 - y^2) dx dy = \int_0^1 \left(\int_0^1 (2 - x^2 - y^2) dx \right) dy = \int_0^1 \left(2x - \frac{x^3}{3} - y^2 x \Big|_0^1 \right) dy = \int_0^1 \left(2 - \frac{1}{3} - y^2 \right) dy = \frac{5}{3} y - \frac{y^3}{3} \Big|_0^1 = 2 - \frac{1}{3} - \frac{1}{3} = 2 - \frac{2}{3} = \frac{4}{3}$

$f(x,y) = -(x^2 + y^2)$



ex. 3 $D \subseteq \mathbb{R}^2$ bounded by $y = x^2$, $x = 2$, $y = 0$

Express D as a simple set with respect to the x and y-axis



D is a simple set with respect to the y-axis $\Leftrightarrow$

$\exists\, a, b \in \mathbb{R},\ a < b,\ \exists\, \varphi_1, \varphi_2$ continuous functions:

$D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}$

$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 2,\ 0 \le y \le x^2\}$

simple set with respect to the y-axis

$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 4,\ \sqrt{y} \le x \le 2\}$

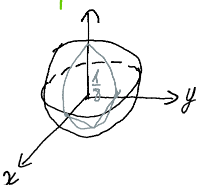
b) Compute $\iint_D xy\,dxdy$ in two ways

$I = \int_0^2 \int_0^{x^2} xy\,dydx = \int_0^2 \left(x\frac{y^2}{2}\Big|_0^{x^2}\right)dx = \int_0^2 \frac{x^5}{2}\,dx = \frac{x^6}{12}\Big|_0^2 = \frac{64}{12} = \frac{16}{3}$

$I = \int_0^4 \int_{\sqrt{y}}^2 xy\,dx\,dy = \int_0^4 \left(\int_{\sqrt{y}}^2 xy\,dx\right)dy = \int_0^4 \left(y\cdot\frac{x^2}{2}\Big|_{\sqrt{y}}^2\right)dy = \int_0^4 \left(2y - \frac{y^2}{2}\right)dy = y^2 - \frac{y^3}{6}\Big|_0^4 = 16 - \frac{32}{3} = \frac{16}{3}$

ex. 4a) Prove that $\iiint_{B(0,1)} 1\,dx\,dy\,dz = \frac{4}{3}\pi$

$B(0,1)$ – open ball around $(0,0)$ radius 1



$x^2+y^2+z^2=1$

$x^2+y^2=1$

$y = \pm\sqrt{1-x^2}$

$-1 \le x \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}$

cube: $-1 \le x \le 1$

$-1 \le y \le 1$

$V = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy\,dx = 8\int_0^1 \left(\frac{1-x^2}{2}\left(\frac{y}{\sqrt{1-x^2}}\cdot\sqrt{1-\frac{y^2}{1-x^2}} + \arcsin\frac{y}{\sqrt{1-x^2}}\right)\right)\Big|_{y=0}^{y=\sqrt{1-x^2}} dx = 4\int_0^1$

take the positive eigth and multiply it

from this shit

$t = \arcsin\frac{y}{a}$

$\int \sqrt{1-x^2-y^2}\,dy = \int \sqrt{a^2-y^2}\,dy = \int \sqrt{\underbrace{a^2-a^2\sin^2 t}_{a^2(1-\sin^2 t)}}\,a\cos t\,dt = \int a^2\cos^2 t\,dt = a^2\int \cos^2 t\,dt = a^2\int \frac{\cos 2t+1}{2}\,dt = \frac{a^2}{2}\left(\frac{\sin 2t}{2} + t\right) + C$

$a = 1-x^2$

$\cos^2 t - \sin^2 t = \cos 2t$

$\cos^2 t = \cos 2t + \sin^2 t \,/ + \cos^2 t$

$\cos^2 t = \frac{\cos 2t + 1}{2}$

$a\sin t = y$

$t = \arcsin\frac{y}{a}$

$= \frac{1-x^2}{2}\left(\sin t\cdot\cos t + \arcsin\frac{y}{a}\right) + C = \frac{1-x^2}{2}\left(\frac{y}{a}\sqrt{1-\sin^2 t} + \arcsin\frac{y}{a}\right) + C$

THIS IS A GOOD EXAMPLE FOR THE EXAMPLE

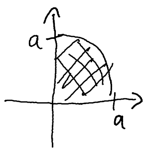
ex.5 By changind the order of integration, evaluate the following:

a) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-y^2}\,dy\,dx$

$0 \le x \le a$

$0 \le y \le \sqrt{a^2-x^2} \,\Big|^2 \Rightarrow 0 \le y^2 \le a^2 - x^2 \,\Big| + x^2$

$x^2 \le \underbrace{x^2+y^2 \le a^2}_{\text{circle}}$



$0 \le y \le a$

$x^2 \le a^2 - y^2$

$0 \le x \le \sqrt{a^2-y^2}$

$I = \int_0^a \int_0^{\sqrt{a^2-y^2}} 1\cdot\sqrt{a^2-y^2}\,dx\,dy = \int_0^a \sqrt{a^2-y^2}\cdot\sqrt{a^2-y^2}\,dy = \int_0^a (a^2-y^2)\,dy = \left(a^2y - \frac{y^3}{3}\right)\Big|_0^a = a^3 - \frac{a^3}{3}$