DSA – Seminar 2 – Complexity (Algorithm Analysis)

1. TRUE or FALSE?
    a. $n^2 \in O(n^3)$ - True
    b. $n^3 \in O(n^2)$ – False
    c. $2^{n+1} \in \Theta(2^n)$ – True
    d. $2^{2n} \in \Theta(2^n)$ - False
    e. $n^2 \in \Theta(n^3)$ – False
    f. $2^n \in O(n!)$ - True
    g. $\log_{10}n \in \Theta(\log_2 n)$ - True
    h. $O(n) + \Theta(n^2) = \Theta(n^2)$ - True
    i. $\Theta(n) + O(n^2) = O(n^2)$ – True
    j. $O(n) + O(n^2) = O(n^2)$ – True
    k. $O(f) + O(g) = O(\max\{f,g\})$ – True
    l. $O(n) + \Theta(n) = O(n)$ – By definition true, but $\Theta(n)$ should be used in such cases
    m. $(n + m)^2 \in O(n^2 + m^2)$ – True - because $(n+m)^2 < 3*(n^2+m^2)$
    n. $3^n \in O(2^n)$ – False
    o. $\log_2 3^n \in O(\log_2 2^n)$ – True

2. Complexity of search and sorting algorithms

Algorithm	Time Complexity				Extra Space Complexity
	Best C.	Worst C.	Average C.	Total	
Linear Search	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$	$O(n)$	$\Theta(1)$
Binary Search	$\Theta(1)$	$\Theta(\log_2 n)$	$\Theta(\log_2 n)$	$O(\log_2 n)$	$\Theta(1)$
Selection Sort	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(1)$ – in place
Insertion Sort	$\Theta(n)$	$\Theta(n^2)$	$\Theta(n^2)$	$O(n^2)$	$\Theta(1)$ – in place
Bubble Sort	$\Theta(n)$	$\Theta(n^2)$	$\Theta(n^2)$	$O(n^2)$	$\Theta(1)$ – in place
Quick Sort	$\Theta(n \log_2 n)$	$\Theta(n^2)$	$\Theta(n \log_2 n)$	$O(n^2)$	$\Theta(1)$ – in place
Merge Sort	$\Theta(n \log_2 n)$	$\Theta(n \log_2 n)$	$\Theta(n \log_2 n)$	$\Theta(n \log_2 n)$	$\Theta(n)$- out of place

3. Analyze the time complexity of the following two subalgorithms:

```
subalgorithm s1(n) is:
    for i ← 1, n execute
        j ← n
        while j ≠ 0 execute
            j ← [j/2]
        end-while
    end-for
```

```
end-subalgorithm
```

- The *for* loop is repeated n times.
- The *while* loop is repeated $\log_2 n$ times. (how many times can we divide n to get to 0)
- $T(n) \in \Theta(n * \log_2 n)$

```
subalgorithm s2(n) is:
      for i ← 1, n execute
            j ← i
            while j ≠ 0 execute
                  j ← [j/2]
            end-while
      end-for
end-subalgorithm
```

- The *for* loop is repeated n times.
- The *while* loop is repeated $\log_2 i$ times.
- $T(n) = \log_2 1 + \log_2 2 + \log_2 3 + \ldots + \log_2 n = \log_2 n! \Rightarrow n \log_2 n$ (Stirling's approximation)
- $T(n) \in \Theta(n * \log_2 n)$

4. Analyze the time complexity of the following two subalgorithms:

```
subalgorithm s3(x, n, a) is:
      found ← false
      for i ← 1, n execute
            if xi = a then
                  found ← true
            end-if
      end-for
end-subalgorithm
```

$$\left.\begin{array}{l} BC: \theta(n) \\ WC: \theta(n) \end{array}\right\} \Rightarrow \Theta(n)$$

```
subalgorithm s4(x, n, a) is:
      found ← false
      while found = false and i ≤ n execute
            if xi = a then
                  found ← true
            end-if
            i ← i + 1
      end-while
end-subalgoritm
```

BC: $\Theta(1)$
WC: $\Theta(n)$

AC: there are n+1 possible cases (element is found on one of the n positions and the case when element is not found. We suppose that all of these cases have equal probability – even if this might not always be the case in real life).

$$T(n) = \sum_{I \in D} P(I) * E(I) = \frac{1}{n+1} + \frac{2}{n+1} + \dots + \frac{n}{n+1} + \frac{n}{n+1} = \frac{n*(n+1)}{2*(n+1)} + \frac{n}{n+1} \in \Theta(n)$$

Total Complexity: O(n)

5. Analyze the time complexity of the following algorithm (x is an array, with elements x_i <= n):

```
Subalgorithm s5(x, n) is:
      k← 0
      for i ← 1, n execute
            for j ← 1, xi execute
                  k ← k + xj
            end-for
      end-for
end-subalgorithm
```

   a. if every $x_i > 0$
   b. if all x_i have value 0
   c. if x_i can be 0

Does the complexity change if we allow values of 0 in the array?

a. Solution:

$$T(x,n) = \sum_{i=1}^{n}\sum_{j=1}^{xi} 1 = \sum_{i=1}^{n} x_i = s \ (sum\ of\ all\ elements)$$

T(n) ϵ Θ (s)

c. if x_i can be 0

Think about an array *x* defined in the following way:

Let $x_i = \begin{cases} 1, if\ i\ is\ a\ perfect\ square \\ 0, otherwise \end{cases}$

In this case: s = √n

T(x, n) ϵ Θ (max {n, s}) = Θ(n + s)

6. Consider the following problems and find an algorithm (having the required time complexity) to solve them :
   a. Given an arbitrary array with numbers $x_1...x_n$, determine whether there are 2 equal elements in the array. Show that this can be done with Θ (n $\log_2$ n) time complexity.
      i. Solution: MergeSort + a linear search for two consecutive equal values
   b. Given an arbitrary array with numbers $x_1...x_n$, determine whether there are two numbers whose sum is *k* (for some given k). Show that this can be done with Θ (n $\log_2$ n) time complexity. What happens if *k* is even and *k/2* is in the array (once or multiple times)?
      i. Solution: MergeSort + for each element x_i a binary search for the value k-x_i
   c. Given an ordered array $x_1...x_n$, in which the elements are distinct integers, determine whether there is a position such that A[i] = i. Show that this can be done with O($\log_2$ n) complexity.
      i. Solution: A variant of binary search

7. Analyze the time complexity of the following algorithm:

```
subalgorithm s6(n) is:
      for i ← 1,n execute
            @elementary operation
      end-for
      i ← 1
      k ← true
      while i <= n - 1 and k execute
            j ← i
            k₁ ← true
            while j <= n and k₁ execute
                  @ elementary operation (k₁ can be modified)
                  j ← j + 1
            end-while
            i ← i + 1
            @elementary operation (k can be modified)
      end-while
end-subalgorithm
```

Best Case: k, k_1 can become false after one iteration => Θ (n) (because of the *for* loop at the beginning)

Worst Case: k, k_1 never becomes false

$$T(n) = n + \sum_{i=1}^{n-1} \sum_{j=i}^{n} 1 = n + \sum_{i=1}^{n-1} n - i + 1 = n + \sum_{i=1}^{n-1} n - \sum_{i=1}^{n-1} i + \sum_{i=1}^{n-1} 1 =$$

$$n + n * (n-1) - \frac{n * (n-1)}{2} + n - 1 \,\epsilon\, \Theta\,(n^2)$$

Average case:

Do it in two steps. First consider the inner for loop (and fix the i from the outer for loop):

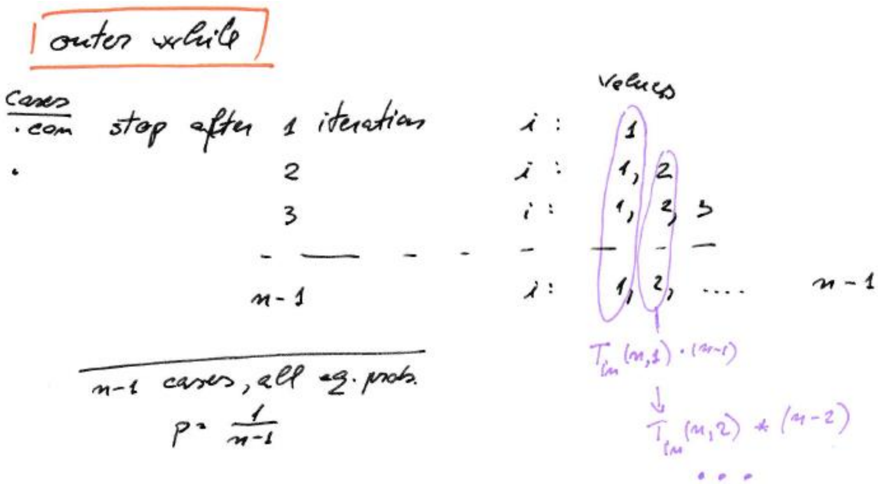
- for a fixed *i*, k_1 can become false after 1,2, ..., n-i+1 iterations.

Probability: $\frac{1}{n-i+1}$

$$\frac{1}{n-i+1} + \frac{2}{n-i+1} + \ldots + \frac{n-i+1}{n-i+1} = \frac{(n-i+1)*(n-i+2)}{2\,(n-i+1)} = \frac{(n-i+2)}{2}$$

For the external *while* loop, *k* can become false after 1, 2, …, n-1 iterations.

We have P(I) * E(I) – in formula, where E(i) is the average number of repetitions for the inner loop (computed above) times how many times we repeat the outer for loop (the instructions which are not part of the inner for loop).



…

$$T(n) = \frac{1}{n-1}\sum_{k=1}^{n-1} T_{in}(n,k) * (n-k) = \dots \quad \in \Theta(n^2)$$

Total complexity: $O(n^2)$

8. Analyze the time complexity of the following algorithm:

```
subalgorithm p(x,s,d) is:
   if s < d then
         m ← [(s+d)/2]
         for i ← s, d-1, execute
               @elementary operation
         end-for
         for i ← 1,2 execute
               p(x, s, m)
         end-for
   end-if
end-subalgorithm
```

```
Initial call for the subalgorithm: p(x, 1, n)
```

- In case of recursive algorithms, the first step of the complexity computation is to write the recurrence relation.

$$T(n) = \begin{cases} 2 * T\left(\frac{n}{2}\right) + n, if\ n > 1 \\ 1, \quad otherwise \end{cases}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$
$$n = 2^k$$
$$T\left(2^k\right) = 2 * T\left(2^{k-1}\right) + 2^k$$
$$2 * T\left(2^{k-1}\right) = 2^2 * T\left(2^{k-2}\right) + 2^k$$
$$2^2 * T\left(2^{k-2}\right) = 2^3 * T\left(2^{k-3}\right) + 2^k$$
$$\ldots$$
$$2^{k-1} * T\,(2) = 2^k * T(1) + 2^k$$

$$T\left(2^k\right) = 2^k * T(1) + k * 2^k = n + n * log_2 n \rightarrow T(n) \in \Theta(n\, log_2 n)$$

9. Analyze the time complexity of the following algorithm:

```
Subalgorithm s7(n) is:
      s ← 0
      for i ← 1, n^2 execute
            j ← i
            while j ≠ 0 execute
                  s ← s + j
                  j ← j - 1
            end-while
      end-for
end-subalgorithm
```

$$T(n) = \sum_{i=1}^{n^2} \sum_{j=1}^{i} 1 = \sum_{i=1}^{n^2} i = \frac{n^2 * (n^2 + 1)}{2} \in \Theta(n^4)$$

10. Analyze the time complexity of the following algorithm:

```
Subalgorithm s8(n) is:
      s ← 0
      for i ← 1, n^2 execute
            j ← i
            while j ≠ 0 execute
                  s ← s + j - 10 * [j/10]
                  j ← [j/10]
            end-while
      end-for
end-subalgorithm
```

- The *while* loop is repeated $log_{10} i$ times (but we report complexities in base 2)
- So we will have: $log_2 1 + log_2 2 + log_2 3 + \ldots + log_2 n^2 = log_2 (n^2)!$
- Striling's approximation tells us that: $log_2 x! = x * log_2 x$
- $log_2(n^2)! = n^2 * log_2 n^2 = 2 * n^2 * log_2 n$ – constants are ignored

- $T(n) \in \Theta(n^2 \log_2 n)$