Seminar 11

Monday, December 19, 2022 6:19 PM

1) poză 19.12.2022

2) Find the second order Taylor polinomial

a) $f(x,y)=\sin(x+2y)$ at $(0,0)=(x_0,y_0)$

$$T_2(x,y)=f(x_0,y_0)+\langle\nabla f(x_0,y_0),(x-x_0,y-y_0)\rangle+(x-x_0,y-y_0)\cdot H_f(x_0,y_0)\cdot(x-x_0,y-y_0)^t$$

$$H_f(x,y)=\begin{pmatrix}\frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x\partial y}(x,y)\\ \frac{\partial^2 f}{\partial y\partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y)\end{pmatrix}$$

$f(x,y)=\sin(x+2y)$

$$\begin{cases}\frac{\partial f}{\partial x}(x,y)=\cos(x+2y)\\ \frac{\partial f}{\partial y}(x,y)=2\cos(x+2y)\end{cases}\Rightarrow \nabla f(0,0)=\left(\frac{\partial f}{\partial x}(0,0),\frac{\partial f}{\partial y}(0,0)\right)=(1,2)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y)=-\sin(x+2y)$$

$$\frac{\partial^2 f}{\partial y\partial x}(x,y)=-2\sin(x+2y)=\frac{\partial^2 f}{\partial x\partial y}$$

$$T_2(x,y)=f(0,0)+\langle(1,2),(x,y)\rangle+0$$

$$H_f(0,0)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

b) $f(x,y)=e^{x+y}$ at $(0,0)$ and $(1,-1)$

$$\frac{\partial f}{\partial x}(x,y)=e^{x+y}=\frac{\partial f}{\partial y}(x,y)\Rightarrow \nabla f(0,0)=(1,1)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y)=e^{x+y}=\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$$

$$H_f(0,0)=\begin{pmatrix}1&1\\1&1\end{pmatrix}=H_f(1,-1)$$

$$T_2(x,y)=f(0,0)+\langle(1,1),(x,y)\rangle+(x,y)\cdot H_f(0,0)\cdot(x,y)^t=$$

$$=1+x+y+(x,y)\cdot\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=1+x+y+(x+y\quad x+y)\cdot\begin{pmatrix}x\\y\end{pmatrix}=$$

$$=1+x+y+x^2+xy+xy+y^2=$$

$$=(x+y)^2+x+y+1$$

$$T_2(x,y)=f(1,-1)+\langle(1,1),(x-1,y+1)\rangle+(x-1,y+1)\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}x-1\\y+1\end{pmatrix}=$$

$$=\cancel{1}+x-\cancel{1}+y+1+(x+y\quad x+y)\begin{pmatrix}x-1\\y+1\end{pmatrix}=1+x+y+(x+y)^2$$

3) $D=\begin{pmatrix}d_1 & & 0\\ & \ddots & \\ 0 & & d_n\end{pmatrix}$

$f: \mathbb{R} \to \mathbb{R}$
$f(x) = \frac{1}{2} \cdot x D x^t$

Prove that $\nabla f(x) = D x^t$ $H(x) = D$; $D_m f(x) = ?$ two ways

$$f(x) = f(x_1, \dots, x_m) = \frac{1}{2}(x_1, \dots, x_m) \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_1 d_1 & x_2 d_2 & \dots & x_m d_m \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \frac{1}{2}(x_1^2 d_1 + x_2^2 d_2 + \dots + x_m^2 d_m)$$

$$\nabla f(x) = \nabla f(x_1, \dots, x_m) = (x_1 d_1, \ x_2 d_2, \dots, x_m d_m) = D \cdot x^t = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$$H f(x_1, \dots, x_m) = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ 0 & 0 & \dots & d_m \end{pmatrix} = D$$

$$D_v f(x) = \lim_{t \to 0} \frac{f(x + t \cdot v) - f(x)}{t} = \lim_{t \to 0} \frac{f(x_1 + t \cdot v_1, \dots, x_m + t \cdot v_m) - f(x_1, \dots, x_m)}{t} =$$

$$= \lim_{t \to 0} \frac{1}{2t}\left((x_1 + t v_1)^2 \cdot d_1 + \dots + (x_m + t \cdot v_m)^2 \cdot d_m) - x_1^2 d_1 - \dots - x_m^2 d_m\right) =$$

$$= \lim_{t \to 0} \frac{1}{2t}(2t x_1 v_1 d_1 + t^2 v_1^2 d_1 + \dots 2t x_m \cdot v_m \cdot d_m + t^2 \cdot v_m^2 d_m) =$$

$$= \lim_{t \to 0} \frac{t}{2}\underbrace{(v_1^2 d_1 + \dots + v_m^2 d_m)}_{\in \mathbb{R}} + x_1 v_1 d_1 + \dots + x_m v_m d_m =$$

$$= 0 + \langle \nabla f(x_1, \dots, x_m), (v_1, \dots, v_m) \rangle = \langle \nabla f(x), v \rangle \to \text{directional derivative}$$

5) Find and classify the critical points

a) $f(x, y) = x^2 - y^2$ $\quad (x_0, y_0)$ critical point if $\nabla f(x_0, y_0) = (0, 0)$

$\nabla f(x, y) = (2x, -2y) = (0, 0) \Rightarrow C = \{(0, 0)\}$ critical poi

$$H f(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow H f(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

If $H_f(x_0, y_0)$ positive definite $\Rightarrow (x_0, y_0)$ local min. point
neg. def $\Rightarrow$ max. point
indefinite $\Rightarrow$ saddle point

$$H = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mm} \end{pmatrix} \quad \Delta_k = \text{determinate} = \begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} \end{vmatrix}$$

if $\Delta_k > 0$ $k \in \{1, \dots, m\} \Rightarrow H$ pos. definite $\Leftrightarrow \lambda_i > 0$

if $(-1)^k \Delta_k > 0$ $\Rightarrow H$ neg. definite $\Leftrightarrow \lambda_i < 0$

If $H = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\det H < 0 \Rightarrow$ indefinite, only works for 2x2

$$\left.\begin{aligned} \Delta_1 &= 2 > 0 \\ \Delta_2 &= -4 < 0 \end{aligned}\right\} \Rightarrow \text{indefinite} \Rightarrow (0,0) \text{ saddle point}$$

b) $f(x, y) = x^3 - 3x + y^2$

$\nabla f(x, y) = (3x^2 - 3, \ 2y) = (0, 0) \Rightarrow$

$$\Rightarrow \begin{cases} 3x^2 - 3 = 0 \\ 2y = 0 \end{cases} \Rightarrow \begin{cases} x^2 - 1 = 0 \\ y = 0 \end{cases} \Leftrightarrow \begin{cases} x = \pm 1 \\ y = 0 \end{cases}$$

$C = \{(1, 0), (-1, 0)\}$ critical points

$$H f(x, y) = \nabla^2 f(x, y) = \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix}$$

$$H_f(1,0)=\begin{pmatrix}6 & 0\\ 0 & 2\end{pmatrix}$$

$\Delta_1=6>0$
$\Delta_2=12>0$ positive definite $\Rightarrow (1,0)$ local minimum point

$$H_f(-1,0)=\begin{pmatrix}-6 & 0\\ 0 & 2\end{pmatrix}$$

$$\left.\begin{array}{l}\Delta_1=-6<0\\ \Delta_2=-12<0\end{array}\right\}\Rightarrow \text{indefinite}$$
$\Rightarrow (-1,0)$ saddle point

another version

$Q(h_1,\dots,h_n)=(h_1,\dots,h_n)\,H_f(h_1,\dots,h_n)^t$
If $Q<0\ \forall\ (h_1,\dots,h_n)\in\mathbb{R}^n\setminus\{0_n\}$ neg. def.
If $Q>0 \Rightarrow$ pos. def.
If $\exists\, a,b\in\mathbb{R}^n: Q(a)<0\quad Q(b)>0 \Rightarrow$ indefinite

$$H_f(-1,0)=\begin{pmatrix}-6 & 0\\ 0 & 2\end{pmatrix}$$

$Q(h_1,h_2)=-6h_1^2+2h_2^2$

$$\left.\begin{array}{l}Q(1,0)<0\\ Q(0,1)>0\end{array}\right\}\Rightarrow H_f(-1,0)\ \text{indefinite}$$

c) $f(x,y)=x^3+y^3-6xy$

$\nabla f(x,y)=(3x^2-6y,\ 3y^2-6x)=(0,0)$

$$\begin{cases}x^2-2y=0\\ y^2-2x=0 \Rightarrow x=\dfrac{y^2}{2}\end{cases}$$

$\Rightarrow \dfrac{y^4}{4}-2y=0\ /\cdot 4$

$y(y^3-8)=0$

$y\in\{0,2\}$

$\Rightarrow C=\{(0,0),(2,2)\}$ critical points

$$H_f(x,y)=\begin{pmatrix}6x & -6\\ -6 & 6y\end{pmatrix}$$

$H_f(0,0)=\begin{pmatrix}0 & -6\\ -6 & 0\end{pmatrix}$ det $H_f(0,0)<0 \Rightarrow$ indefinite

$Q(h_1,h_2)=-12h_1h_2=(h_1\ h_2)\begin{pmatrix}0 & -6\\ -6 & 0\end{pmatrix}\begin{pmatrix}h_1\\ h_2\end{pmatrix}$ $\Rightarrow (0,0)$ saddle point

$$\left.\begin{array}{l}Q(1,1)<0\\ Q(-1,1)>0\end{array}\right\}\Rightarrow \text{indefinite}$$

choose random values

$H_f(2,2)=\begin{pmatrix}12 & -6\\ -6 & 12\end{pmatrix}$ $\left.\begin{array}{l}\Delta_1=12>0\\ \Delta_2=144-36>0\end{array}\right\}\Rightarrow$ pos. def.
$\Rightarrow (2,2)$ local minimum point

6) $f(x)=ax+b$, $a=?$, $y=?$

$$\sum_{i=1}^{m}|f(x_i)-y_i|^2=\sum_{i=1}^{m}(ax_i+b-y_i)^2\overset{\text{we}}{\underset{\text{define}}{=:}}g(a,b)$$

$$\nabla g(a,b)=\left(2\sum_{i=1}^{m}x_i(ax_i+b-y_i),\ 2\sum_{i=1}^{m}(ax_i+b-y_i)\right)=(0,0)$$

$$\Rightarrow b=\frac{\sum y_i}{m}-\frac{a\sum x_i}{m}=\bar{y}-a\bar{x}$$

$$a = \frac{\sum (y_i - \bar{y}) \sum (x_i - \bar{x})}{\sum (x_i - x)^2}$$