Seminar 9

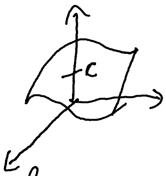
Monday, December 5, 2022 6:05 PM

1. Sketch the level sets

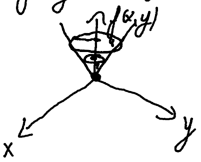
$L_c = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}$

$f(x,y) = \sqrt{x^2+y^2}$

$c \in \{0,1,4\}$



$f(x,y) = \sqrt{x^2+y^2} = 0 \Rightarrow x = y = 0$

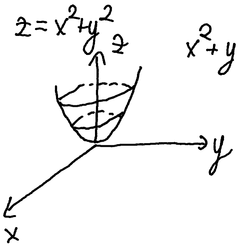


$y = 0 \Rightarrow \sqrt{x^2} = |x|$

$\sqrt{x^2+y^2} = 1 \Rightarrow x^2+y^2 = 1$ – center = (0,0)
radius = 1

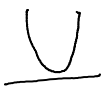
$\sqrt{x^2+y^2} = 4 \Rightarrow x^2+y^2 = 4^2$ – center = (0,0)
radius = 4

$z = x^2+y^2$



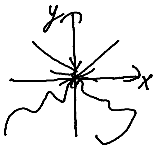
$x^2+y^2 = 4$

$y = 0 \quad x^2$



ex. 2

a) $\lim\limits_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2} = ?$



$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x+y}{x^2+y^2} = \lim\limits_{x\to 0} \frac{x}{x^2} = \lim\limits_{x\to 0} \frac{1}{x}$

↳ doesn't exist, can't be ± infinity ⇒

⇒ the original one doesn't exist

(if one limit doesn't exist, all of it doesn't exist)

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2} \nexists$

b) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

$\left.\begin{array}{l} \lim\limits_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2-y^2}{x^2+y^2} = \lim\limits_{x\to 0} \frac{x^2}{x^2} = 1 \\ \lim\limits_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^2-y^2}{x^2+y^2} = \lim\limits_{y\to 0} \frac{-y^2}{y^2} = -1 \end{array}\right\} \Rightarrow \nexists$ they are different

c) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^3+y^3}{x^2+y^2} = \lim\limits_{x\to 0} \frac{x^3}{x^2} = 0$

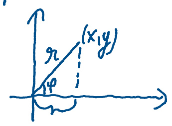
$\lim\limits_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^3+y^3}{x^2+y^2} = \lim\limits_{y\to 0} \frac{y^3}{y^2} = 0$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ x=y}} \frac{x^3+y^3}{x^2+y^2} = \lim\limits_{x\to 0} \frac{2x^3}{2x^2} = 0$

$\sim y$

now, we have to prove the limit is 0

polar coordinates



$$\begin{cases} x = r\cos\varphi \\ y = r\cdot\sin\varphi \\ \cos\varphi = \frac{x}{r} \\ r>0,\ \varphi\in[0,2\pi) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} \frac{(r\cos\varphi)^3+(r\sin\varphi)^3}{(r\cos\varphi)^2+(r\sin\varphi)^2} =$$

$$= \lim_{r\to 0} \frac{r^{\cancel{3}}(\cos^3\varphi+\sin^3\varphi)}{\cancel{r^2}\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{=1}} = \lim_{r\to 0} r(\cos^3\varphi+\sin^3\varphi) = 0$$

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| \le \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{y^3}{x^2+y^2}\right| = \frac{|x|\cdot x^2}{x^2+y^2} + \frac{|y|\cdot y^2}{x^2+y^2} \le \frac{|x|\cdot\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} + \frac{|y|\cdot\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} = |x|+|y| \to 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$$

d) $$\lim_{(x,y)\to(0,0)} \frac{\sin x - \sin y}{x-y} = \lim_{(x,y)\to(0,0)} \frac{\overbrace{\left(\frac{\sin x}{x}\right)}^{\to 1}\cdot x - \overbrace{\left(\frac{\sin y}{y}\right)}^{\to 1}\cdot y}{x-y} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{x-y}{x-y} = 1$$

$$\frac{\sin x - \sin y}{x-y} = \frac{\sin\frac{x-y}{2}\cos\frac{x+y}{2}}{\frac{x-y}{2}} \to 1$$

$$2$$

3) Study the continuity and partial differentiability of $f:\mathbb{R}^2\to\mathbb{R}$

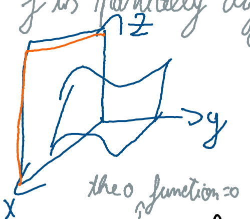
$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

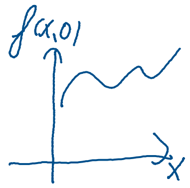
f is continous at $(0,0)$ iff $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2} \neq 0 \Rightarrow \text{the limit is not continous}$$

f is partially differentiable at $(0,0)$ with respect to x iff $\lim_{x\to x_0} \frac{f(x,0)-f(x_0,0)}{x-0}$





$$\lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{f(x,0)}{x} = \lim_{x\to 0} \frac{\frac{x\cdot 0}{x^2+0}}{x} = 0 \Rightarrow f \text{ is part. diff. with respect to } x \text{ at } (0,0)$$

$$f(x,0)=0$$

f partially differentiable with respect to y at $(0,0)$:

$$\lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0} \frac{\frac{0\cdot y}{0\cdot y^2}}{y} = 0 \in \mathbb{R}$$

$\Rightarrow f$ is partially diff. with respect to y at $(0,0)$

4) Compute the first order partial derivatives:

d) $f(x,y,z) = x^2 y \cdot z + y e^z$

$\frac{\partial f}{\partial x}(x,y,z) \underset{y,z\text{-const}}{\overset{x\text{-var}}{=\!=}} 2x \cdot yz + 0$ partial der. w.r.t. x

$\frac{\partial f}{\partial y}(x,y,z) \underset{x,z\text{-const}}{\overset{y\text{-var}}{=\!=}} x^2 z + e^z$

$\frac{\partial f}{\partial z}(x,y,z) \underset{x,y\text{-const.}}{\overset{z\text{-var}}{=\!=}} x^2 y + y \cdot e^z$

a) $f(x,y) = \sqrt{x^2+y^2}$

$\frac{\partial f}{\partial x}(x,y) \underset{y\text{-const.}}{\overset{x\text{-var}}{=\!=}} \frac{\partial}{\partial x}(\sqrt{x^2+y^2}) = \frac{\partial}{\partial x}((x^2+y^2)^{\frac{1}{2}}) = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x$

$\frac{\partial f}{\partial y}(x,y) \underset{x\text{-const.}}{\overset{y\text{-var}}{=\!=}} \frac{\partial}{\partial x}((x^2+y^2)^{\frac{1}{2}}) = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y$

b) $f(x,y) = \ln\sqrt{x^2+y^2}$

$\frac{\partial f}{\partial x}(x,y) \underset{y\text{-const.}}{\overset{x\text{-var}}{=\!=}} \frac{\partial}{\partial x}[\ln(x^2+y^2)^{\frac{1}{2}}] = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2}$

c) $f(x,y) = \cos x \cdot \cos y - \sin x \cdot \sin y = \cos(x+y)$

$\frac{\partial f}{\partial x}(x,y) = -\sin(x+y) \cdot \underbrace{\frac{\partial}{\partial x}(x+y)}_{=1}$

$\frac{\partial f}{\partial y}(x,y) = -\sin(x+y) \cdot 1$

5) Find the gradient of f at the point a:

$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)\right)$

$a = (a_1, a_2, a_3)$

$\nabla f(a_1, a_2, a_3) = \left(\frac{\partial f}{\partial x}(a), \frac{\partial f}{\partial y}(a), \frac{\partial f}{\partial z}(a)\right)$

a) $f(x,y) = e^{-x}\sin(x+2y)$, $a = (0, \frac{\pi}{4})$

$\frac{\partial f}{\partial x}(x,y) = -e^{-x} \cdot \sin(x+2y) + e^{-x} \cdot \cos(x+2y)$

$\frac{\partial f}{\partial y}(x,y) = 2 \cdot e^{-x} \cdot \cos(x+2y)$

$\nabla f(0, \frac{\pi}{4}) = \left(\frac{\partial f}{\partial x}(0, \frac{\pi}{4}), \frac{\partial f}{\partial y}(0, \frac{\pi}{4})\right) = (-1 \cdot 1 + 1 \cdot 0, 0) = (-1, 0) \in \mathbb{R}^2$

vector with partial derivatives

$x=0$
$y=\frac{\pi}{4}$

b) $f(x,y) = \arctan(\frac{y}{x})$, $a = (1,1)$

$\frac{\partial f}{\partial x}(x,y) = \frac{1}{1+(\frac{y}{x})^2} \cdot \left(-\frac{y}{x^2}\right)$

$\frac{\partial f}{\partial y}(x,y) = \frac{1}{\ldots} \cdot \frac{1}{\ldots}$

$$\frac{\partial f}{\partial x}(x,y)=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}$$

$$\nabla f(1,1)=\left(\frac{\partial f}{\partial x}(1,1),\ \frac{\partial f}{\partial x}(1,1)\right)=\left(-\frac{1}{2},\frac{1}{2}\right)$$

c) $f(x,y,z)=e^{xyz},\quad a=(0,0,0)$

$$\frac{\partial f}{\partial x}(x,y,z)=e^{xyz}\cdot yz$$

$$\frac{\partial f}{\partial y}(x,y,z)=e^{xyz}\cdot xz$$

$$\frac{\partial f}{\partial z}(x,y,z)=e^{xyz}$$

$$\nabla f(0,0,0)=(0,0,0)$$

d) $f(x,y,z)=\sqrt{x^2+y^2+z^2},\quad a=(1,1,1)$

$$\frac{\partial f}{\partial x}(x,y,z)=\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x$$

$$\frac{\partial f}{\partial y}(x,y,z)=\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2y$$

$$\frac{\partial f}{\partial z}(x,y,z)=\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2z$$

$$\nabla f(1,1,1)=\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)\in\mathbb{R}^3$$

6) Let $f:\mathbb{R}\to\mathbb{R}$ differentiable $g:\mathbb{R}^2\to\mathbb{R}$

$g(x,y)=f(x^2+y^2)$ we define $h(x,y)=x^2+y^2 \Rightarrow g=f\circ h$

Prove that $x\cdot\frac{\partial g}{\partial y}(x,y)=y\frac{\partial g}{\partial x}(x,y)$ $\quad g(x,y)=(f\circ h)(x,y)=f(h(x,y))=f(x^2+y^2)$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "h"] \arrow[rr, bend right, "g"'] & \mathbb{R} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$$\frac{\partial g}{\partial y}(x,y)=\frac{\partial(f\circ h)}{\partial y}(x,y)=f'(h(x,y))\cdot\frac{\partial h}{\partial y}(x,y)=2y\cdot f'(h(x,y))$$

$$\Rightarrow x\cdot\frac{\partial g}{\partial y}(x,y)=x\cdot 2y\cdot f'(h(x,y))\ (1)$$

$$\frac{\partial g}{\partial x}(x,y)=\frac{\partial(f\circ h)}{\partial x}(x,y)=f'(h(x,y))\cdot\underbrace{\frac{\partial h}{\partial x}(x,y)}_{2x}=2x\cdot f'(h(x,y))$$

$$\Rightarrow y\cdot\frac{\partial g}{\partial x}(x,y)=y\cdot 2x\cdot f'(h(x,y))\ (2)$$

$(1),(2)\Rightarrow(3)$