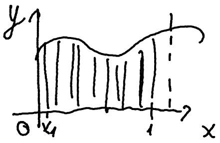
Seminar 6

Monday, November 21, 2022 6:15 PM

ex. 1.

a) $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}\right)=?$ using Riemann integral



$\Delta=(x_0=0,\ x_1,\dots,x_n=1)$

$\xi=(\xi_1,\xi_2,\dots,\xi_n)$

$\|\Delta\|=\max_{i\in\{1,\dots,n\}}|x_i-x_{i-1}|$

$$\lim_{\|\Delta\|\to0}\sum_{i=1}^{n}(x_i-x_{i-1})f(\xi_i)=\int_a^b f(x)\,dx$$

$$\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}=\frac{1}{n}\left(\frac{1}{1+\underbrace{\frac{1}{n}}_{\to0}}+\frac{1}{1+\frac{2}{n}}+\dots+\frac{1}{1+\underbrace{\frac{n}{n}}_{1}}\right)=$$

$$=\frac{1}{n}\cdot\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}=\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{1+\frac{n}{k}}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{n}\cdot\frac{1}{1+\frac{k}{n}}\right)=\int_0^1\frac{1}{1+x}\,dx=\ln(1+x)\Big|_0^1=\ln 2$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\ln 2$$

c) $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}\to e^{\ln\dots}$ to transform a product into a sum

$\ln(x\cdot y)=\ln x+\ln y$ → isomorphism

$$\lim_{n\to\infty}e^{\ln\frac{\sqrt[n]{n!}}{n}}=e^{\lim_{n\to\infty}\ln\frac{\sqrt[n]{n!}}{n}}$$

$$\lim_{n\to\infty}\ln\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\left(\ln\sqrt[n]{n!}-\ln n\right)=\lim_{n\to\infty}\left(\frac{1}{n}\ln(1\cdot2\cdot\ldots\cdot n)-\ln n\right)=$$

$$=\lim_{n\to\infty}\left[\frac{1}{n}(\ln1+\ln2+\dots+\ln n)-\ln n\right]=$$

$$=\lim_{n\to\infty}\frac{1}{n}(\ln1+\ln2+\dots+\ln n-n\cdot\ln n)=$$

$$=\lim_{n\to\infty}\frac{1}{n}(\ln1-\ln n+\ln2-\ln n+\dots+\ln(n-1)-\ln n+\ln n-\ln n)=$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\ln\frac{1}{n}+\ln\frac{2}{n}+\dots+\ln\frac{n-1}{n}+\ln\frac{n}{n}\right)=$$

$$=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}\ln\frac{k}{n}=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{n}\cdot\ln\underbrace{\left(\frac{k}{n}\right)}_{\in(0,1]}\right)=$$

$$=\int_0^1\ln x\,dx=x\ln x\Big|_0^1-\int_0^1 1\,dx=x\ln x\Big|_0^1-1=-1$$

$$\lim_{x\to0}x\ln x=\lim_{x\to0}\frac{\ln x}{\frac{1}{x}}\overset{\frac{\infty}{\infty}}{\underset{H}{=}}\lim_{x\to0}\frac{\frac{1}{x}}{\frac{-1}{x^2}}=0$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=e^{-1}=\frac{1}{e}$$

2) Study the Riemann integrability of the function

When is a function Riemann integrable?

1. Must be continous

$f:[0,1]\to\mathbb{R}$

$f(x)=\begin{cases}0, & x\in\mathbb{Q}\\ 1, & x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$

prove it's not integrable

$\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{n}f(\xi_i)=0$ choose ξ_i ... ξ_n s.t. it's not integrable

$\xi=(\xi_1,\dots,\xi_n)$, $\xi_i\in\mathbb{Q}$

$\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{n}\cdot f(\underset{\mathbb{R}\setminus\mathbb{Q}}{\xi_i})=\infty\to f$ is not Riemann integrable

ex. 3.

a) $\int_1^2 \frac{1}{x(x-2)}\,dx = \lim\limits_{t\nearrow 2}\int_1^t \frac{dx}{x(x-2)}$

$\frac{1}{x(x-2)} = \frac{A}{x}+\frac{B}{x-2} = \frac{1}{2}\cdot\frac{x-(x-2)}{x(x-2)}$

$\int \frac{1}{x(x-2)}\,dx = \int\left(\frac{\frac12}{x-2}-\frac{\frac12}{x}\right)dx = \frac12\ln|x-2|-\frac12\ln|x|+c,\ c\in\mathbb{R}$

$\int_1^2 \frac{1}{x(x-2)}\,dx = \lim\limits_{\substack{t\to 2\\ t<2}}\int_1^t\frac{1}{x(x-2)}\,dx = \lim\limits_{\substack{t\to 2\\ t<2}}\left[\frac12\ln|t-2|-\frac12\ln|t|\right]-0=$

$=\frac12(-\infty)-\frac12\ln 2=-\infty$

$\int_1^2\frac{1}{x(x-2)}\,dx=-\infty$

b) $\int_0^\infty x\cdot e^{-x^2}\,dx = \frac12\int_0^\infty e^{-t}\,dt = -\frac12 e^{-t}\Big|_0^\infty$

$t=x^2$

$dt=2x\,dx$

$=\lim\limits_{s\to+\infty}\left(-\frac12\cdot e^{-s}+\frac12\right)$

$=\lim\limits_{s\to\infty}\underbrace{\left(-\frac12\cdot\frac{1}{e^s}\right)}_{\to 0}+\frac12=\frac12$

c) $\int_0^1\frac{\ln x}{\sqrt{x}}\,dx = \lim\limits_{\substack{t\to 0\\ t>0}}\int_t^1\frac{\ln x}{\sqrt{x}}\,dx = \lim\limits_{t\searrow 0}\int \ln x\cdot(2\sqrt{x})'\,dx=$

$\frac{1}{\sqrt{x}}=(2\sqrt{x})'$ $\quad=\lim\limits_{t\searrow 0}\ln x\cdot 2\sqrt{x}\Big|_t^1-\int_t^1\frac{2\sqrt{x}}{x}\,dx$

$=\lim\limits_{t\searrow 0}\left(\underbrace{\ln 1\cdot 2}_{0}-\ln t\cdot(\underset{?}{2\sqrt{t}})-2\cdot 2\sqrt{x}\Big|_t^1\right)$

$=\lim\limits_{t\searrow 0}-\ln t\cdot 2\sqrt{t}-2(2-2\sqrt{t})=$

$=\lim\limits_{t\searrow 0}-\ln t\cdot 2\sqrt{t}-4+\underbrace{4\sqrt{t}}_{\to 0}$

$=-4-2\lim\limits_{t\searrow 0}\frac{\ln t}{\frac{1}{\sqrt{t}}}\overset{\frac{\infty}{\infty}}{\underset{?}{=}}-4-2\lim\limits_{t\searrow 0}\frac{\frac{1}{t}}{-\frac12\sqrt{t}}=-4$

4) Study the convergence

a) $\int_1^\infty\frac{dx}{x\sqrt{1+x^2}}\le\int_1^\infty\frac{dx}{x\sqrt{x^2}}=\int_1^\infty\frac{dx}{x^2}$

$\int_1^\infty x^{-2}=-\frac{1}{x}\Big|_1^\infty=\lim\limits_{t\to\infty}\frac{-1}{t}+1=1\Rightarrow\int_1^\infty\frac{dx}{x^2}$ convergent

Comparison $\int^\infty dx$ ... convergent

$\overset{\text{}}{\underset{\text{test}}{\Rightarrow}}$ $\int_1 \frac{1}{x\sqrt{1+x^2}} \in [0,1] \Rightarrow$ convergent

b) $\int_0^{\frac{\pi}{2}} \frac{1}{\cos x} dx$

Comparison test 2

$\lim_{x \nearrow \frac{\pi}{2}} \frac{f(x)}{g(x)} = L \in \mathbb{R}^* \Rightarrow \int f(x)dx \quad \int g(x)dx$ have the same nature

$\left. \begin{array}{l} f(x) = tg\, x \\ g(x) = \frac{1}{\cos x} \end{array} \right| \Rightarrow \lim_{x \nearrow \frac{\pi}{2}} \frac{tg\, x}{\frac{1}{\cos x}} = \lim_{x \nearrow \frac{\pi}{2}} \frac{\sin x}{\cos x} \cdot \cos x =$

$= \lim_{x \nearrow \frac{\pi}{2}} \sin x = 1 \in \mathbb{R}, \quad 1 \neq 0$

$\int_0^{\frac{\pi}{2}} tg\, x\, dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x} dx = -\ln|\cos x| \Big|_0^{\frac{\pi}{2}} =$

$= -\ln(\cos x)\Big|_0^{\frac{\pi}{2}} = \lim_{t \nearrow \frac{\pi}{2}} -\ln(\cos t) + 0 = +\infty \Rightarrow$

$\Rightarrow \int_0^{\frac{\pi}{2}} tg\, x\, dx$ is divergent $\overset{\text{comp.}}{\underset{\text{test 2}}{\Longrightarrow}} \int_0^{\frac{\pi}{2}} \frac{1}{\cos x} dx$ - is divergent

ex. 5

a) Using integral test, study the convergence

$\sum_{n \geq 1} \frac{1}{n^p}, p > 0$ conv. $\Leftrightarrow \int_1^{\infty} \frac{1}{x^p} dx$ convergent

$\int_1^{\infty} \frac{dx}{x^p} = \int_1^{\infty} x^{-p} dx \overset{p \neq 1}{=} \frac{x^{1-p}}{1-p} \Big|_1^{\infty} = \lim_{t \to \infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p} =$

$= \frac{1}{p-1} + \frac{1}{1-p} \left(\lim_{t \to \infty} t \right)^{1-p} = \begin{cases} +\infty, & 1-p > 0, \ p < 1 \\ \frac{1}{p-1}, & 1-p < 0, \ p > 1 \quad (1) \end{cases}$

$p = 1$

$\int_1^{\infty} \frac{dx}{x} = \ln x \Big|_1^{\infty} = \lim_{t \to \infty} \ln t = \infty$

$\Rightarrow p = 1 \Rightarrow \int_1^{\infty} \frac{1}{x^p} dx$ divergent (2)

$(1), (2) \Rightarrow \int_1^{\infty} \frac{dx}{x^p}$ conv. $\Leftrightarrow p > 1 \Rightarrow$

$\overset{\text{Int Test}}{\Longrightarrow} \sum_{n \geq 1} \frac{1}{n^p}$ conv. $\Leftrightarrow p > 1$