Seminar 3

24 October 2022 18:03

1) Convergence + find the sum

a) $\sum_{n\geq 1} \frac{2}{3^n}$

$$\sum_{n\geq 1} x_n = \sum_{n=1}^{\infty} x_n = x_1 + x_2 + \dots + x_n + \dots = \lim_{k\to\infty} S_k$$

$$S_k = x_1 + \dots + x_k$$

* convergence without calculating *

$x_n = \frac{3}{2^n}, \ n \in \mathbb{N}$

$$l = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\frac{2}{3^{n+1}}}{\frac{2}{3^n}} = \frac{1}{3} < 1 \overset{R\ test}{\Rightarrow} \sum_{n\geq 1} \frac{2}{3^n} \text{ is convergent}$$

$$S_n = \sum_{k=1}^{n} \frac{2}{3^k} = 2\sum_{k=1}^{n} \frac{1}{3^k} = 2\left(\frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n}\right) = \frac{2}{3}\left(1 + \frac{1}{3} + \dots + \frac{1}{3^{n-1}}\right) = \frac{2}{\cancel{3}} \cdot \frac{1 - \frac{1}{3^n}}{\cancel{1-\frac{1}{3}}} = 1 - \frac{1}{3^n}$$

$$1 + r + r^2 + \dots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{3^n}\right) = 1 - 0 = 1$$

ex. 2) 1 c)

$$\sum_{n\geq 1} \frac{1}{4n^2 - 1}$$

Ratio Test $(x_n \geq 0)$

$$L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$$

if $L < 1 \Rightarrow \sum_{n\geq 1} x_n$ convergent

if $L > 1 \Rightarrow \sum_{n\geq 1} x_n$ divergent

if $L = 1 \Rightarrow$ inconclusive

$$L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\frac{1}{4(n+1)^2 - 1}}{\frac{1}{4n^2-1}} = \lim_{n\to\infty} \frac{4n^2-1}{4(n+1)^2-1} = \lim_{n\to\infty} \frac{4n^2-1}{4n^2+8n+3} = 1 \quad \text{(inconclusive)}$$

Let $y_n = \frac{1}{n^2} \Rightarrow \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n^2}{4n^2-1} = \frac{1}{4} \in \mathbb{R}$

The series x_n and y_n have the same nature $\Rightarrow$
$\Rightarrow$ they are convergent / divergent at the same time (1)

!! $\sum_{n\geq 1} \frac{1}{n^p}$ convergent $\Leftrightarrow p > 1$

$\sum_{n\geq 1} y_n$ convergent as $2 > 1$ (2)

from (1) and (2) $\Rightarrow \sum_{n\geq 1} x_n$ is convergent

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2-1} = \sum_{k=1}^{n} \frac{1}{(2n)^2 - 1^2} = \sum_{k=1}^{n} \frac{1}{(2n-1)(2n+1)} = \sum_{k=1}^{n} \frac{1}{2} \frac{2n+1-2n+1}{(2n-1)(2n+1)} =$$

$$= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = \frac{1}{2}\left(\sum_{k=1}^{n} \frac{1}{2k-1} - \sum_{k=1}^{n} \frac{1}{2k+1}\right) = \frac{1}{2}\left(\sum_{k=1}^{n} \frac{1}{2k-1} - \sum_{k=2}^{n+1} \frac{1}{2k-1}\right) =$$

$$=\frac{1}{2}\left(\frac{1}{2\cdot1-1}-\frac{1}{2(n+1)-1}\right)=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$$

$$\Rightarrow \lim_{n\to\infty} S_n=\frac{1}{2}(1-0)=\frac{1}{2}\in\mathbb{R}$$

$$\Rightarrow \sum_{n\geq1} x_n=\frac{1}{2}$$

ex.3) 1.e)

$$\sum_{n\geq1}\frac{n}{2^n}=S$$

$$L=\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}=\lim_{n\to\infty}\frac{n+1}{2n}=\frac{1}{2}<1\xrightarrow{\text{Ratio Test}}\sum_{n\geq1}x_n \text{ convergent}$$

$$S_n=\sum_{k=1}^{n}\frac{k}{2^k}=\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}$$

$$\sum_{n\geq1}x_n \text{ convergent}\overset{p\to q}{\Rightarrow}\lim_{n\to\infty}x_n=0$$

$$\sum\frac{1}{n}\Rightarrow\text{divergent}$$
$$\downarrow$$
$$0$$

$$S=\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\ldots+\frac{n}{2^n}+\ldots\ /\cdot2$$

$$2S=1+\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\ldots+\frac{n}{2^{n-1}}+\ldots$$

$$2S-S=1+\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\ldots+\frac{n}{2^{n-1}}+\ldots-\frac{1}{2}-\frac{2}{4}-\frac{3}{8}-\frac{4}{16}-\frac{5}{32}-\ldots-\frac{n}{2^n}-\ldots$$

$$S=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots$$

$$\sum_{n\geq1}x_n=\sum_{n\geq0}\frac{1}{2^n}=2$$

$$1+r+r^2+r^3+\ldots=\frac{1}{1-r}$$

ex.4) 1.f)

f) $\sum_{n\geq1}\frac{1}{\sqrt[3]{n}}=\sum_{n\geq1}\frac{1}{n^{\frac{1}{3}}}$ $\quad\frac{1}{3}<1\Rightarrow\sum_{n\geq1}x_n$ divergent

homework: ex.2

ex.5) 3. Convergent or not?

a) $\sum_{n\geq1}\left(\frac{n}{n+1}\right)^{n^2}=\sum_{n\geq1}\left(1-\frac{1}{n+1}\right)^{n^2}$

$$x_n=\left(\frac{n}{n+1}\right)^{n^2}$$

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n^2}=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n^2}=\lim_{n\to\infty}\left(1+\frac{-1}{n+1}\right)^{n^2}=$$

$$=\lim_{n\to\infty}\left[\left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-1}{n+1}}\right]^{n^2}=e^{\lim_{n\to\infty}\frac{-n^2}{n+1}}=e^{-\infty}=0\Rightarrow\text{inconclusive}$$

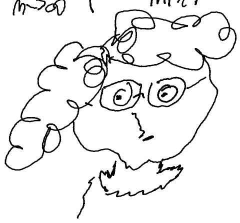
$$L=\lim_{n\to\infty}\sqrt[n]{x_n}$$

if $L<1 \Rightarrow \sum x_n$ convergent

if $L>1 \Rightarrow \sum x_n$ divergent

$$L=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n=e^{\lim_{n\to\infty}\frac{-n}{n+1}}=\frac{1}{e}<1 \Rightarrow \sum x_n \text{ convergent}$$

$$\sum_{n\geq 1}\frac{1}{n^2}=\frac{\pi^2}{6}$$



b) $\sum_{n\geq 1}\underbrace{\frac{x^n}{n^p}}_{a_n}$, $x>0$, $p\in\mathbb{R}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{x^{n+1}}{(n+1)^p}}{\frac{x^n}{n^p}}=\lim_{n\to\infty}\frac{x\cdot n^p}{(n+1)^p}=x\cdot\lim_{n\to\infty}\frac{n^p}{(n+1)^p}=x$$

if $x<1 \Rightarrow \sum_{n\geq 1} a_n$ is convergent

if $x>1 \Rightarrow \sum a_n$ divergent

if $x=1 \Rightarrow \sum_{n\geq 1} a_n=\sum_{n\geq 1}\frac{1}{n^p}$: if $p>1 \Rightarrow \sum a_n$ convergent

if $p\leq 1 \Rightarrow \sum a_n$ divergent

ex. 9 from seminar 2

$$\lim_{n\to\infty}\frac{1^p+2^p+3^p+\dots+n^p}{n^{p+1}},\ p\in\mathbb{N}^*$$

Cezaro Stolz $\Rightarrow \lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\lim_{n\to\infty}\frac{a_n}{b_n}$