Seminar 1

10 October 2022 18:08

ex. 1) Find the lower and upper bound, sup, inf, min and max

a) $A=(-1,1]\cup(2,+\infty)$

$lb(A)=\{x\in\mathbb{R} / x\leq a\ \forall a\in A\}$ - lower bound of A

$lb(A)=(-\infty,-1]$

$ub(A)=\{x\in\mathbb{R} / x\geq a\ \forall a\in A\}$ - upper bound of A

$ub(A)=\emptyset$

We assume that $\alpha\in ub(A)$, $\alpha\geq 2$, but then $\alpha+1\in A$ and

$\alpha+1>\alpha$

$\Rightarrow \alpha\notin ub(A) \Rightarrow ub(A)=\emptyset$

sup = smallest value from the upper bound =

= $\min(ub(A)) \Rightarrow \nexists \sup(A) \Rightarrow \nexists \max A$

inf (infimum) = $\inf(A) = \max(lb(A)) = -1 \notin A \Rightarrow \nexists \min A$

$\max(A) = ub(A)\cap A$

$\min(A) = lb(A)\cap A$

b) $B=(-3,2)\cup\{3\}$

$lb(B)=(-\infty,-3]$

$ub(B)=[3,+\infty)$

$\sup(B)=3 \overset{\in B}{\Rightarrow} \max(B)=3$

$\inf(B)=-3\notin B \Rightarrow \nexists \min(B)$

c) $[-5,5]\cap\mathbb{Z}=\{-4,-3,\dots,3,4\}=C$

$lb(C)=(-\infty,-4)$

$ub(C)=[4,+\infty)$

$\sup(C)=4$

$\inf(C)=-4$

$\max(C)=4=C\cap ub(C)$

$\min(C)=C\cap lb(C)=-4$

2) sup, inf, min, max

a) $A=\{x\in\mathbb{Q} / x^2<2\}$

$x^2<2 \,/\sqrt{(\ )}$

$-\sqrt{2}<x<\sqrt{2} \simeq -1{,}4<x<1{,}4$

$x\in(-\sqrt{2},\sqrt{2})\cup\mathbb{Q} \Rightarrow A=(-\sqrt{2},\sqrt{2})\cap\mathbb{Q}$

$\sup(A)=$

We assume that $\alpha\in ub(A)$, $\alpha<\sqrt{2}$, $\alpha\in\mathbb{R}_+^*$

The density of $\mathbb{Q}$; $\forall a,b\in\mathbb{R}, a<b \Rightarrow \exists q\in\mathbb{Q}$:
$a<q<b$

Let $a=\alpha$ and $b=\sqrt{2}$
$a<q<b$ | $\xrightarrow[\text{dense in } \mathbb{R}]{\mathbb{Q}\text{ is}}$ $(\exists)\, q\in\mathbb{Q}$ so that $\alpha<q<\sqrt{2}$

$q\in A$
$\alpha<q$ | $\Rightarrow \alpha\notin ub(A) \Rightarrow \sup(A)=\sqrt{2}$

upper bound

$\inf(A) = -\sqrt{2}$
$\max(A) =$ doesn't $\exists$
$\min(A) =$ doesn't $\exists$

c) $C = \{ \frac{n}{n+1} \mid n \in \mathbb{N} \}$ $\mathbb{N} = \{1,2,3,\ldots\}$ $\mathbb{N}_0 = \{0,1,\ldots\}$

$C = \{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \}$

$\inf(C) = \frac{1}{2} \in C \Rightarrow \min(C) = \frac{1}{2}$

$\frac{n}{n+1} \geq \frac{1}{2} \; / \cdot 2(n+1)$

$2n \geq n+1 \Rightarrow n \geq 1$ true by definition

$\frac{n}{n+1} < 1$

$n < n+1$

$0 < 1$ True

Let $\alpha \in \mathbb{R}^*_{+}$, $\alpha < 1$ such that $\alpha \in ub(C) \Rightarrow \frac{n}{n+1} \leq \alpha, \forall n \in \mathbb{N}$

$\frac{n}{n+1} \leq \alpha < 1 \; / \cdot (n+1)$

$n \leq (n+1)\alpha < n+1 \; / - (n+1)$

$-1 \leq \underbrace{(n+1)(\alpha - 1)}_{\wedge \, 0} < 0 \; / \cdot (-1)$

$0 < (n+1)(1-\alpha) \leq 1 \; / : (1-\alpha), \; \alpha < 1$

$0 < n+1 \leq \frac{1}{1-\alpha} \; / -1$

$-1 < n \leq \underbrace{\frac{\alpha}{1-\alpha}}_{\in \mathbb{R}} \; \forall n \in \mathbb{N}$

$\Rightarrow$ False $\Rightarrow$

Archimedian Property

$\forall \alpha \in \mathbb{R} \; \exists n \in \mathbb{N} : n > \alpha$

$\Rightarrow \nexists \alpha \in 1$ so that $\alpha \in ub(C) \Rightarrow \sup(C) \neq 1$

3) Let $A = (0,1) \cap \mathbb{Q}$. Show that $\inf A = 0$, $\sup A = 1$

homework

4) S is non-empty and bounded from above
Show that the set $-S = \{-x, x \in S\}$ is bounded from below.
and $\inf(-S) = -\sup(S)$

$\underset{S}{(-\infty, 1]}$ $\underset{-S}{(-1, +\infty)}$

$-\sup(S) = \inf(-S) = -1$

bounded from above $\Rightarrow$ has a supremum

let $\alpha \in \mathbb{R}^*_+$, $\sup(S) = \alpha \Rightarrow \alpha \geq x, \forall x \in S$
$/ \cdot (-1)$

$\underbrace{-x}_{y} \geq -\alpha, \forall x \in S$

$-\alpha \leq y, \forall y \in -S \Rightarrow$

$\Rightarrow -S$ is bounded from below

let $\inf(-S) = \beta$, we need to prove that $\beta = -\alpha$

$A = B$ $\quad x = y$
$A \subseteq B$ $\quad x \leq y$
$B \subseteq A$ $\quad y \leq x$

$-\alpha \leq \beta$
$\beta \leq -\alpha$

$\left.\begin{array}{l} -\alpha \in lb(-S) \\ \beta = \max(lb(-S)) \end{array}\right| \Rightarrow -\alpha \leq \beta \ (1)$

$\beta \leq y \,/\!\cdot(-1)$, $\forall\ y \in -S$, (β is the infimum)

$-\beta \geq -y$
$-\beta \geq x$ (from above, $-y$ is x)
$-\beta \geq x$, $\forall x \in S$

$\left.\begin{array}{l} -\beta \text{ is } ub(S) \\ \alpha = \sup(S) \end{array}\right| \Rightarrow -\beta \geq \alpha \,/\!\cdot(-1)$

$\beta \leq -\alpha \ (2)$

from (1) and (2) $\Rightarrow \beta = -\alpha$

hw 2 lecture:

5) Let $a, b \in \mathbb{R}$ also S nonempty bounded from above.
Prove that $\sup_{x \in S}(ax+b) = a \sup_{x \in S}\{x\} + b$

7) Which of the following are neighborhoods of 0?

$U \subseteq \mathbb{R}$ is a neighborhood of x if $x \in U$ and $\exists\ \varepsilon > 0 : (x-\varepsilon, x+\varepsilon) \subseteq U$

ex: $A = [-1, 1] \cup \{2\}$ is a neighborhood because $(-1,1) \subseteq A$

$C = \bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right]$

$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$

$\forall\ \varepsilon > 0\ \exists\ n \in \mathbb{N} : n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon \Rightarrow (-\varepsilon, \varepsilon) \not\subseteq \left[-\frac{1}{n}, \frac{1}{n}\right] \subseteq C$

$\Rightarrow (-\varepsilon, \varepsilon) \not\subseteq C \Rightarrow C \notin V(0)$

it is not a neighborhood