Exercise 4

24. $\dot{x} = \frac{1}{2}(x-2x^2)$ $\quad$ f is a parameter

Find the flow: $\eta \in \mathbb{R}$, $\begin{cases} \dot{x} = \frac{1}{2}(x-2x^2) \\ x(0) = \eta \end{cases}$

you did ? $\quad \dot{x} = \frac{1}{2}(x-2x^2)$ $\quad$ $\frac{\dot{x}}{x(1-2x)} = \frac{1}{2}$ $\quad$ find order $L = \mathcal{O}(\mathbb{R})$

$\frac{1}{x} - \frac{2}{1-2x} = \frac{1}{2}$ $\quad$ separation

$\ln|x| + \ln|1-2x| = \frac{t}{2} + c$ $\quad$ $\Rightarrow x_0 = e^{-\frac{t}{2}}$

$x \in \mathbb{R} \Rightarrow$ two constant soln

you do: $x = ce^{\frac{t}{2}} + 21$, $c \in \mathbb{R}$ $\quad$ $x(0) = c + 21 = \eta \Rightarrow c = \eta - 21$

$\varphi(t,\eta) = (\eta-21)e^{\frac{t}{2}} + 21$, $\forall \eta \in \mathbb{R}$, $\forall t \in \mathbb{R}$.

Represent the phase portrait

$\dot{x} = \frac{1}{2}(x-2x^2)$

$f(x) = \frac{1}{2}(x-2x)$

Exponential: $\eta \cdot 10^{\frac{t}{10}} \geq 37^{\circ}$ $\quad$ $t = ?$

the initial state $\eta = 95^{\circ}$

the data after 10 when cooling at $\eta = 19^{\circ}$ is $\varphi(10, \eta) = 37^{\circ}$

$\Leftrightarrow (\eta - 21)e^{\frac{10}{2}} + 21 = 37 \Leftrightarrow 28e^{\frac{10}{2}} = 16 \Leftrightarrow$

$\Leftrightarrow e^{\frac{10}{2}} = \frac{16}{28}$ $\left(= \frac{4}{7}\right)$ $\Leftrightarrow -10 \cdot \ln\frac{4}{7} \Leftrightarrow \boxed{k = \frac{1}{10}\ln\frac{7}{4}}$

Pan problem: $\quad \eta \xrightarrow{20'} 37^{\circ}$ $\quad$ the above room

$\eta \geq 0$ $\quad$ $\eta = \frac{37}{e^{-\frac{20}{10}}}$

$\Leftrightarrow (\eta - 21) \cdot \left(\frac{4}{7}\right)^2 + 21 = 37 \Leftrightarrow \eta - 21 = 16 \cdot \frac{49}{16} \Leftrightarrow \boxed{\eta = 70^{\circ}}$

The linearization method: let $f \in C^1(\mathbb{R})$, $\eta^* \in \mathbb{R}$ be s.t. $f(\eta^*) = 0$.

If $f'(\eta^*) < 0$ then η^* is an attractive equilibrium of $\dot{x} = f(x)$.

If $f'(\eta^*) > 0$ then η^* is a repeller eq. of $\dot{x} = f(x)$.

Proof. $\quad f'(\eta^*) < 0$

$f'(\eta^*) > 0$

2. $0 < x < 1$ $\quad$ $\dot{x} = x(1-x) - cx$

Find the equilibria and study their stability using the $\frac{1}{2}x$ method

$f(x) = x(1-x) - cx = x(1-x-c) = x(1-c-x) = x(1-c-x)^2$

the equilibria: $\quad f(x) = 0 \Leftrightarrow \eta_1^* = 0$, $\eta_2^* = 1-c > 0$ for $\mathbb{Z}$

study the stability: $\quad f(x) = x \cdot 1-c-2x$

$f'(\eta_1^*) = f'(0) = 1-c > 0 \xrightarrow{\text{LHS}} \eta_1^* = 0$ is a repeller

$f'(\eta_2^*) = f'(1-c) = -(1-c) < 0 \xrightarrow{\text{LHS}} \eta_2^* = 1-c$ is an attractor.

Represent the phase portrait:

$-1 < -c < 0$

$0 < c < 1$

$x(t)$ is density of the fish

N the optimal density $\Rightarrow 1-c = N \Rightarrow \boxed{c = 1-N}$

initial density: $\frac{N}{2} = \eta_0$ $\quad$ rate of fishing

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$\dot{x} = x - x^2$ $\quad$ Represent the phase portrait. Find $\varphi(t,-1)$, $\varphi(t,0)$.

Describe the properties of $\varphi(t,-2)$, $\varphi(t,-\frac{1}{2})$, $\varphi(t,3)$.

(Represent the phase portrait of $\dot{x} = x - x^2$.)

$f(x) = x - x^2$ $\quad$ $f \in C^1(\mathbb{R})$

$f(x) = 0 \Leftrightarrow 0, -1, 1$

$f'(x) = x(1-x)$

x	-1		0		1
$f(x)$	$+$	$-$	0	$+$	

The state are: $\{-\infty, -1\}$, $\{-1, 0\}$, $\{0, 1\}$, $\{1, +\infty\}$.

-1 is an attractor $\quad A_{-1} \xrightarrow{\text{def}} \{\eta \in \mathbb{R} : \lim_{t\to\infty} \varphi(t,\eta) = -1\}$

$A_{-1} = (-\infty, 0)$ $\quad$ 1 is an attractor $\quad A_1 = (0, +\infty)$

0 is a repeller

Find $\varphi(t,-1) = -1$, $\forall t \in \mathbb{R}$. since -1 is an equilibrium (definition)

$\varphi(t,0) = 0$, $\forall t \in \mathbb{R}$. since 0 ——— $\frac{1}{2}$ ———

$\varphi(t,-2)$ $\quad$ $-2 \in (-\infty, -1)$ an output so $x_0 = (-\infty, -1)$ so

the image of $\varphi(t,-2)$ is $(-\infty, -1)$

is strictly increasing. $\quad \lim_{t\to\infty} \varphi(t,-2) = -1$.

$\varphi(t,-\frac{1}{2})$ $\quad$ $-\frac{1}{2} \in (-1, 0)$ output so $x_0 = (-1, 0)$ so

the image of $\varphi(t,-\frac{1}{2})$ is $(-1, 0)$ so

the function is bounded & is defined on $\mathbb{R}$

is strictly increasing. $\quad \lim_{t\to\infty} \varphi(t,-\frac{1}{2}) = -1$. $\quad \lim_{t\to-\infty} \varphi(t,-\frac{1}{2}) = 0$.

$\varphi(t,3)$ $\quad$ HW.

6.4) Represent the phase portrait of $\dot{x} = x - x^3 - 1$.

(The phase portrait is, namely, qualitative, not necessarily quantitative)

$f(x) = x - x^3 - 1$, $f \in C^1(\mathbb{R})$

$f(x) = 0 \Leftrightarrow x - x^3 + 0 = -1$ $\quad$ the -1 order $\mathbb{Z}$ $\quad$ 1 has either 3 real roots or 1 real root.

x	$-\infty$		$-\frac{1}{\sqrt{3}}$		0		$\frac{1}{\sqrt{3}}$		$+\infty$
f		$+$	0	$+$	$+$	$-$	0	$-$	

$f'(x) = 1 - 3x^2 = 0$

$\Leftrightarrow x^2 = \frac{1}{3}$

$\Leftrightarrow x = \pm\frac{1}{\sqrt{3}}$

$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} - \left(\frac{1}{\sqrt{3}}\right)^3 + 1 = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 1 = \frac{2}{\sqrt{3}} + 1 > 0$

$f\left(-\frac{1}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} - \left(-\frac{1}{\sqrt{3}}\right)^3 + 1 = -\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 1 = 1 - \frac{2}{\sqrt{3}} > 0$

$\Rightarrow x_1^* = -\frac{1}{\sqrt{3}}$, $x_2^* = \frac{1}{\sqrt{3}}$

these roots a unique root of f. η^* and we have that $\eta^* \in (1, \frac{1}{\sqrt{3}}, +\infty)$. Then $f'(\eta^*) < 0$. Then η^* is attractor.

$f\left(\frac{1}{\sqrt{3}}\right) > 0$ $\quad$ $f(1) = 1 - 1 + 1 = 1 > 0$ $\quad$ $f(2) = 2 - 2^3 + 1 = -5 < 0$

$\Rightarrow \eta^* \in (1, 2)$ $\quad$ $f\left(\frac{1}{\sqrt{3}}\right) = \cdots$