Seminar 10

Monday, December 12, 2022 6:04 PM

1) $f:\mathbb{R}^2\to\mathbb{R}$ $f(x,y)=x\cdot y$

Prove that $\Delta f(x_0,y_0)(x,y)=y_0x+x_0y$

$\mathbb{R}^n\to\mathbb{R}^m$

$A=()\}m$

Let $A\subseteq\mathbb{R}^2$, $(x_0,y_0)\in A$

f is differentiable at (x_0,y_0) if $\exists\ \Delta f(x_0,y_0)\in \mathcal{M}_{1\times2}(\mathbb{R}) = \mathbb{R}^{1\times 2}$

$$\lim_{(x,y)\to(x_0,y_0)}\frac{\|f(x,y)-f(x_0,y_0)-\Delta f(x_0,y_0)(x-x_0,y-y_0)\|}{\|(x-x_0,y-y_0)\|}\overset{?}{=}0$$

→ you can get rid of the norm because we are in $\mathbb{R}$

$$\lim_{(x,y)\to(x_0,y_0)}\frac{|x\cdot y-x_0\cdot y_0-y_0(x-x_0)-x_0(y-y_0)|}{\|(x-x_0,y-y_0)\|}=$$

the norm is used for $\mathbb{R}^m$

$\|x\|=\sqrt{x^2}=|x|\to\mathbb{R}$

in $\mathbb{R}^m$: $\|x\|=\sqrt{x_1^2+x_2^2+\dots+x_m^2}$

$$=\lim_{(x,y)\to(x_0,y_0)}\frac{|xy-x_0y_0-xy_0+x_0y_0-x_0y+x_0y_0|}{\|(x-x_0,y-y_0)\|}=$$

$$=\lim_{(x,y)\to(x_0,y_0)}\frac{|x(y-y_0)-x_0(y-y_0)|}{\|(x-x_0,y-y_0)\|}=\lim_{(x,y)\to(x_0,y_0)}\frac{|\overbrace{(x-x_0)}^{a}\overbrace{(y-y_0)}^{b}|}{\|(x-x_0,y-y_0)\|}=$$

$$=\lim_{(a,b)\to(0,0)}\frac{|a\cdot b|}{\|(a,b)\|}=\lim_{(a,b)\to(0,0)}\frac{|a\cdot b|}{\sqrt{a^2+b^2}}=$$

$|a\cdot b|=|a|\cdot|b|=\sqrt{a^2}\cdot\sqrt{b^2}\le\sqrt{a^2+b^2}\cdot\sqrt{b^2+a^2}$

$$=\lim_{(a,b)\to(0,0)}\frac{|a|\cdot|b|}{\sqrt{a^2+b^2}}=\lim_{(a,b)\to(0,0)}\frac{\sqrt{a^2}\cdot\sqrt{b^2}}{\sqrt{a^2+b^2}}\le\lim_{(a,b)\to(0,0)}\frac{\sqrt{a^2+b^2}\cdot\cancel{\sqrt{a^2+b^2}}}{\cancel{\sqrt{a^2+b^2}}}=\sqrt{0}=0$$

$\Rightarrow \Delta f(x_0,y_0)(x,y)=y_0x+x_0y$

2) $f:\mathbb{R}^2\to\mathbb{R}$

$f(x,y)=x^2+xy$

a) $\nabla f(1,0)=?$

direction of steepest descent at $(1,0)=?$

$$=\frac{-\nabla f(1,0)}{\|\nabla f(1,0)\|}$$

↓ theoretically, it will be 0 at some point

$$\nabla f(x,y)=\left(\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial x}(x,y)\right)=(2x+y,x)$$

$\nabla f(1,0)=(2,1)$

$$\frac{-\nabla f(1,0)}{\|\nabla f(1,0)\|}=\frac{-(2,1)}{\sqrt{4+1}}=\frac{(-2,-1)}{\sqrt5}=\left(-\frac{2}{\sqrt5},-\frac{1}{\sqrt5}\right)$$

b) $\Delta_v(1,0)=?$

$v=\vec{i}+\vec{j}=(1,1)\in\mathbb{R}^2$

$\Delta_v(x,y)=\langle\Delta f(x,y),v\rangle$

$$\Delta_{(1,1)}(1,0)=\langle\nabla f(1,0),(1,1)\rangle=\langle(2,1),(1,1)\rangle=2+1=3$$

$2\cdot1+1\cdot1$

c) Find the equation of the tangent plane to the surface $z=f(x,y)$ at $(1,0,1)$

$$z-z_0=\frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0)+\frac{\partial f}{\partial x}(x_0,y_0)\cdot(y-y_0)$$

$(x_0,y_0,z_0)=(1,0,1)$

$$z-1=\frac{\partial f}{\partial x}(1,0)\cdot(x-1)+\frac{\partial f}{\partial y}(1,0)\cdot(y-0)\Leftrightarrow$$

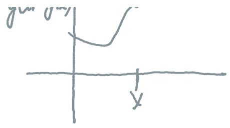
$\Leftrightarrow z-1=2(x-1)+1\cdot y\Rightarrow z=2x+y-1$ eq of the tangent plane

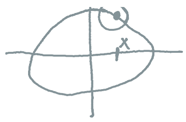
3) Find the equation of the tangent line to the ellipse.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(x_0,y_0)\in\mathbb{R}^2$

$y=ax+b$ eq. of a line

$y-y_0 = m\cdot(x-x_0)$
$m = y'(x_0)$





$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Big/ \cdot \frac{\partial}{\partial}$ $\quad y=y(x)$ around x

$\Rightarrow \frac{2x_0}{a^2}+\frac{2y_0\cdot y'(x_0)}{b^2}=0$

$\Rightarrow y'(x_0) = -\frac{x_0}{a^2}\cdot\frac{b^2}{y_0} = \frac{-b^2\cdot x_0}{a^2\cdot y_0}$

$\Rightarrow d:\ y-y_0 = \frac{-b^2 x_0}{a^2\cdot y_0}(x-x_0) \Big/ \cdot \frac{y_0}{b^2} \Rightarrow$

$\Rightarrow \frac{y_0\cdot y}{b^2} - \frac{y_0}{b^2} = \frac{-x_0 x}{a^2}+\frac{x_0^2}{a^2} \Rightarrow$

$\Rightarrow \frac{y_0\cdot y}{b^2}+\frac{x_0\cdot x}{a^2} = \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow d:\ \frac{x_0\cdot x}{a^2}+\frac{y_0\cdot y}{b^2}=1$

4) $f:\mathbb{R}^n\to\mathbb{R}$
$f(x)=\|x\|^2$
$D_v f$ in two ways (definition, gradient)
$D_n f < \nabla f(x), v>$
$D_n f = \lim_{t\to 0}\frac{f(x_0+tv)-f(x_0)}{t}$

$f = f(x_1,x_2,\dots,x_n) = \|(x_1,x_2,\dots,x_n)\|^2 = x_1^2+x_2^2+\dots+x_n^2$

$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x),\dots,\frac{\partial f}{\partial x_n}(x)\right) = (2x_1, 2x_2,\dots, 2x_n)$

$D_v f(x) = <\nabla f(x), v> = <(2x_1,\dots,2x_n),(v_1,v_2,\dots,v_n)> =$
$= 2x_1 v_1,\dots, 2x_n v_n =$
$= 2(x_1\cdot v_1+\dots+x_n v_n)$

$D_v f(x) = \lim_{t\to 0}\frac{f((x_1,\dots,x_n)+t\cdot(v_1,\dots,v_n))-f(x_1,\dots,x_n)}{t} =$

$= \lim_{t\to 0}\frac{f(x_1+tv_1,\dots,x_n+tv_n)-f(x_1,\dots,x_n)}{t} =$

$= \lim_{t\to 0}\frac{(x_1+tv_1)^2+\dots+(x_n+tv_n)^2-(x_1^2+\dots+x_n^2)}{t} =$

$= \lim_{t\to 0}\frac{\cancel{x_1^2}+2tx_1\cdot v_1+t^2v_1^2+\dots+\cancel{x_n^2}+2tx_nv_n+t^2v_n^2-\cancel{x_1^2}-\dots-\cancel{x_n^2}}{t} =$

$= \lim_{t\to 0}\frac{2\cancel{t}(x_1\cdot v_1+\dots+x_n\cdot v_n)+t^{\cancel{2}}(v_1^2+\dots+v_n^2)}{\cancel{t}} =$

$= \lim_{t\to 0}\left[2(x_1\cdot v_1+\dots+x_n\cdot v_n)+t\underbrace{(v_1^2+\dots+v_n^2)}_{\to 0}\right] =$

$= 2(x_1\cdot v_1+\dots+x_n\cdot v_n) = 2<x,v> = <\nabla f(x), v>$

5) $\frac{df}{dt} = ?\ f(t)$

a) $f(x,y)=\ln(x^2+y^2)$
$x=t,\ y=t^2$
$x(t)=t,\ y(t)=t^2$
pic from class

6) $f:\mathbb{R}^2\to\mathbb{R}$
$x=g_1(u,v)$
$y=g_2(u,v)$ $\quad f(x,y)=(f\circ g)(u,v)$
Prove that: $\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$$g = \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \end{pmatrix}$$

$$D(f \circ g)(u,v) = Df(g(u,v)) \cdot Dg(u,v)$$

$$Dg(u,v) = \begin{pmatrix} \nabla g_1 \\ \nabla g_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial u}(u,v) & \frac{\partial g_1(u,v)}{\partial v} \\ \frac{\partial g_2}{\partial u}(u,v) & \frac{\partial g_2(u,v)}{\partial v} \end{pmatrix}$$

$$Df = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$

$$D(f \circ g) = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$ and it checks out