Seminar 4

Monday, October 31, 2022 6:01 PM

1) a)

$\sum_{n\geq 1} \frac{1}{\ln n}$, $x_n = \frac{1}{\ln n}$, $n \in \mathbb{N}^*$

$n \leq e^n$ (we know it is true)

$\ln n \leq n \mid^{-1}$

$\frac{1}{\ln n} \geq \frac{1}{n} \Rightarrow \frac{1}{n} \leq x_n, \forall n \in \mathbb{N}^*, n \geq 2$

We know $\sum_{n\geq 2}^{\infty} \frac{1}{n}$ - divergent

$\Rightarrow \sum_{n\geq 2}^{\infty} x_n$ - divergent (comparison test 1)

b) $\sum_{n\geq 1} \frac{\ln(1+\frac{1}{n})}{n}$, $x_n = \frac{\ln(1+\frac{1}{n})}{n}$

$\lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{n} = 0$

$\lim_{n\to\infty} \frac{x_n}{y_n} > 0$

let $y_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{\frac{\ln(1+\frac{1}{n})}{n}}{\frac{1}{n^2}} = \lim_{n\to\infty} n \cdot \ln(1+\frac{1}{n}) = \lim_{n\to\infty} \underbrace{\ln(1+\frac{1}{n})^n}_{e} = 1 > 0 \xRightarrow{C.T.2} \sum x_n, \sum y_n$ have the same nature

$\sum_{n\geq 1} y_n = \sum_{n\geq 1} \frac{1}{n^2}$ which is convergent $(2>1)$ $\xRightarrow{C.T.2} \sum_{n\geq 1} x_n$

c) $\sum_{n\geq 2} \frac{1}{n(\ln n)^p}$ $x_n = \frac{1}{n(\ln n)^p}$

$\sum_{n\geq 2} x_n$ $\sum_{n\geq 2} 2^n \cdot x_{2^n}$ $\to$ Cauchy's condensation test

Ratio test

$\exists M \in \mathbb{N} : x_n \leq y_n \ \forall n \geq M$

$\sum_{n\geq 2} 2^n \cdot \frac{1}{2^n(\ln 2^n)^p} = \sum_{n\geq 2} \frac{1}{(n\ln 2)^p} = \sum_{n\geq 2} \frac{1}{n^p \ln^p 2} = \frac{1}{\ln^p 2} \cdot \sum_{n\geq 2} \frac{1}{n^p} \to$ converges $\Leftrightarrow p > 1 \xRightarrow{C.C.T.} \sum_{n\geq 2} \frac{1}{n(\ln n)^p} \Leftrightarrow p > 1$

2) Convergence and absolute convergence

$\sum_{n\geq 1} x_n$ absolute convergent $\Leftrightarrow \sum_{n\geq 1} |x_n|$ convergent

$\sum_{n\geq 1} |x_n|$ conv $\Rightarrow \sum_{n\geq 1} x_n$ converges

a) $\sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$

$\sum_{n\geq 1} \left|\frac{(-1)^{n+1}}{\sqrt{n(n+1)}}\right| = \sum_{n\geq 1} \frac{1}{\sqrt{n(n+1)}}$ $x_n = \frac{1}{\sqrt{n(n+1)}}$ $y_n = \frac{1}{n}$

$\lim_{x\to\infty} \frac{|x_n|}{y_n} = \lim_{x\to\infty} \frac{n}{\sqrt{n(n+1)}} = \lim_{x\to\infty} \frac{\cancel{n}}{\cancel{n}\sqrt{1+\underbrace{(\frac{1}{n})}_{\to 0}}} = 1 > 0 \xRightarrow{C.T.2} \sum x_n, \sum y_n$ are of the same nature

$\sum y_n = \sum \frac{1}{n}$ diverges

$\xRightarrow{C.T.2} \sum_{n\geq 1} |x_n|$ diverges $\Rightarrow$

$\sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$ alternating

$\left|\frac{(-1)^{n+1}}{\sqrt{n(n+1)}}\right|$ decreasing

$\xRightarrow{Leibniz} \sum x_n$ convergent

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{\sqrt{n(n+1)}}\right| = \lim_{n\to\infty}\frac{1}{\sqrt{n^2+n}} = \lim_{n\to\infty}\frac{1}{n\sqrt{1+\frac{1}{n}}} = 0$$

$$\frac{|x_{n+1}|}{|x_n|} = \frac{\frac{1}{\sqrt{(n+1)(n+2)}}}{\frac{1}{\sqrt{n(n+1)}}} = \frac{\sqrt{n}}{\sqrt{n+2}} = \sqrt{\frac{n}{n+2}} < 1 \Rightarrow x_n \text{ decreasing}$$

$n\geq 1$

b) $\sum_{n\geq 1}(-1)^n \cdot \sin\frac{1}{n}$ $\quad x_n = (-1)\cdot\sin\frac{1}{n},\ n\in\mathbb{N}^*$

Let $|x_n| = \sin\frac{1}{n},\ n\in\mathbb{N}^*$

Let $y_n = \frac{1}{n},\ n\in\mathbb{N}^*$

$$\left.\begin{array}{l}\lim_{n\to\infty}\frac{|x_n|}{y_n} = \lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 > 0 \overset{CT2}{\Rightarrow} \sum_{n\geq 1}\frac{1}{n} \text{ and } \sum_{n\geq 1}|x_n| \text{ have the same nature} \\ \sum_{n\geq 1}\frac{1}{n} - \text{divergent}\end{array}\right| \Rightarrow$$

$\Rightarrow \sum_{n\geq 1}\sin\frac{1}{n}$ – divergent

$\sum_{n\geq 1}x_n$ is not absolutely convergent

$\sum_{n\geq 1}(-1)^n\cdot\sin\frac{1}{n}$

(1) x_n – is alternating

(2) $\lim_{n\to\infty}|x_n| = \lim_{n\to\infty}\sin\frac{1}{n} = 0$

(3) Show that $|x_n|$ – decreasing

$$|x_{n+1}| - |x_n| = \sin\frac{1}{n+1} - \sin\frac{1}{n} =$$

$$= 2\sin\frac{\frac{1}{n+1}-\frac{1}{n}}{2}\cos\frac{\frac{1}{n+1}-\frac{1}{n}}{2} =$$

a

$$= 2\sin\frac{-1}{2n(n+1)}\cos\frac{2n+1}{2n(n+1)} = -2\sin\underbrace{\frac{1}{2n(n+1)}}_{0<\ \ <1}\cos\underbrace{\frac{2n+1}{2n(n+1)}}_{0<\ \ <1} \Rightarrow |x_{n+1}| - |x_n| < 0$$

$|x_{n+1}| < |x_n| \Rightarrow |x_n|$ decreasing

$(1)(2)(3) \overset{Leibniz}{\Rightarrow} \sum_{n\geq 1}(-1)^n\sin\frac{1}{n}$ is convergent

c) $\sum_{n\geq 1}\frac{\sin n}{n^2}$

$$\left.\begin{array}{l}\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2} \\ \sum_{n\geq 1}\frac{1}{n^2} \text{ is convergent}\end{array}\right| \overset{CT1}{\Rightarrow} \sum_{n\geq 1}\left|\frac{\sin n}{n^2}\right| \text{ is convergent} \Rightarrow \sum_{n\geq 1}\frac{\sin n}{n^2} \text{ is absolutely convergent} \Rightarrow$$

$\Rightarrow \sum_{n\geq 1}\frac{\sin n}{n^2}$ is convergent

3)

a) $\sum_{n\geq 1}\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}$

$$x = \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}$$

$$\frac{x_{n+1}}{x_n} = \frac{\frac{1\cdot 3\cdot\ldots\cdot(2n+1)}{2\cdot 4\cdot\ldots\cdot(2n+2)}}{\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}} = \frac{2n+1}{2n+2}$$

* ratio test won't work lim = 1 *

$$n\left(\frac{2n+2}{2n+1} - 1\right) = n\left(\frac{2n+2-(2n+1)}{2n+1}\right) = n\frac{1}{2n+1} = \frac{n}{2n+1}$$

Raabe–Duhamel

$$\lim_{n\to\infty}\left[n\left(\frac{2n+2}{2n+1}\right)\right]=\lim_{n\to\infty}\frac{n}{2n}=\frac{1}{2}<1 \overset{R-D}{\Rightarrow} \text{series is divergent}$$

4) Prove that $\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}=\ln 2$

$$\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots$$

$$\gamma_n=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n \xrightarrow{n\to\infty}\gamma$$

$$S_{2n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{2n-1}-\frac{1}{2n}=\underbrace{\left[1+\frac{1}{2}+\dots+\frac{1}{2n}-\ln(2n)\right]}_{\gamma_{2n}}-2\left(\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2n}\right)+\ln 2n=$$

$$=\gamma_{2n}-\underbrace{\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\ln n\right)}_{\gamma_n}-\underbrace{\ln n+\ln 2n}_{\ln 2}=$$

$$=\gamma_{2n}-\gamma_n+\ln 2$$

$$\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}=\lim_{n\to\infty}S_{2n}=\lim_{n\to\infty}(\gamma_{2n}-\gamma_n+\ln 2)=\gamma-\gamma+\ln 2=\ln 2$$

– the order of summation in the series can lead to a different sum

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\dots=$$

$$=\underbrace{1-\frac{1}{2}}-\frac{1}{4}+\underbrace{\frac{1}{3}-\frac{1}{6}}_{\frac{1}{6}}-\frac{1}{8}+\underbrace{\frac{1}{5}-\frac{1}{10}}_{\frac{1}{10}}+\dots=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\dots\right)=\frac{1}{2}\ln 2$$

$$=\frac{1}{2}+$$