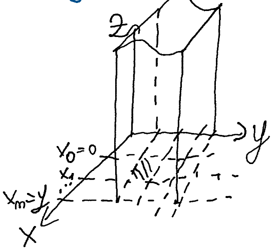
Course 13

Friday, January 13, 2023 10:38 AM

poză 13.01.2023

Def Riemann sum is $\sum_{i=1}^{m} \sum_{j=1}^{m} \underbrace{f(x_i^*, y_j^*)}_{\text{Height}} \underbrace{(x_i - x_{i-1})(y_j - y_{j-1})}_{\text{Base}} = \sigma(f, p)$



Let $P = \{A_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]\} \to$ partition of A

Norm $\|P\| = \max\{\max_i \{x_i - x_{i-1}\}, \max_j \{y_j - y_{j-1}\}\}$

Def Let $f: [a,b] \times [c,d] \to \mathbb{R}$, f is Riemann integrable if:
For any partition P of A with $\|P\| \to 0$
we have that $\sigma(f, p)$ converges to a limit I.
We write $I = \iint_A f(x,y)\,dx\,dy = \iint_A f(x,y)\,dA$

Remark • $\iint_A f(x,y)\,dx\,dy \to$ Volume below the surface $z = f(x,y)$

A • $\iint_A 1\,dx\,dy \to$ Area of A

Properties Let $f, g: A \to \mathbb{R}$ integrable

- $\iint_A [f(x,y) + g(x,y)]\,dx\,dy = \iint_A f(x,y)\,dx\,dy \pm \iint_A g(x,y)\,dx\,dy$
- $\iint_A \alpha f(x,y)\,dx\,dy = \alpha \iint_A f(x,y)\,dx\,dy$
- If $f(x,y) \le g(x,y)$, $\forall (x,y) \in A$

then $\iint_A f(x,y)\,dx\,dy \le \iint_A g(x,y)\,dx\,dy$

Property Let $A_1, A_2, A \subseteq \mathbb{R}^2$,
$A = A_1 \cup A_2$, $\operatorname{int} A_1 \cap \operatorname{int} A_2 = \emptyset$

$\underbrace{\boxed{A_1 \mid A_2}}_{A}$

$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Theorem (FUBINI): Let $A = [a,b] \times [c,d] \subseteq \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$

$\iint_A f(x,y)\,dx\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$

$= \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy$

Proof (Assume f is cont.) $= \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy$

Let $F(x) = \int_c^d f(x,y)\,dy = \sum_{j=1}^{m} \int_{y_{j-1}}^{y_j} f(x,y)\,dy$, for $c = y_0 < y_1 < \dots < y_m = d$

$\int_{y_{j-1}}^{y_j} f(x,y)\,dy \underset{\uparrow}{=} (y_j - y_{j-1}) f(x, y_j^*)$, $y_j^* \in [y_{j-1}, y_j]$
Mean value theorem

Recall that $\int_a^b f(x)\,dx = (b-a) f(c)$

$F(x) = \sum_{j=1}^{m} f(x, y_j^*)(y_j - y_{j-1})$ $(*)$

Take $\int_a^b F(x)\,dx = \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} F(x)\,dx$, with $a = x_0 < x_1 < \dots < x_m = b$

$\int_a^b F(x)\,dx \underset{\uparrow}{=} \sum_{i=1}^{m} (x_i - x_{i-1}) F(x_i^*)$, $x_i^* \in [x_{i-1}, x_i]$

Mean Value Theorem

$\int_a^b F(x)\,dx \overset{(*)}{=} \sum_{i=1}^{m} (x_i - x_{i-1}) \sum_{j=1}^{m} f(x_i^*, y_j^*)(y_j - y_{j-1})$

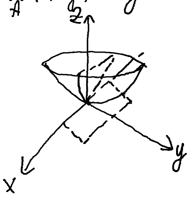
$$\Rightarrow \int_a^b F(x)\,dx = \underbrace{\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)(x_i - x_{i-1})(y_j - y_{j-1})}_{\text{Riemann Sum} \to \iint_A f(x,y)\,dx\,dy}$$

Hence $\iint_A f(x,y)\,dx\,dy = \int_a^b F(x)\,dx$
$= \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$

Example

$A = [-1,1] \times [0,1]$

$\iint_A (x^2+y^2)\,dx\,dy$



FUBINI:

$\iint_A (x^2+y^2)\,dx\,dy = \int_0^1 \left(\int_{-1}^1 (x^2+y^2)\,dx\right) dy = \int_0^1 \left(\frac{2}{3} + 2y^2\right) dy = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$

$\int_{-1}^1 (x^2+y^2)\,dx = \frac{x^3}{3}\Big|_{-1}^{1} + y^2 \cdot x\Big|_{-1}^{1} = \frac{2}{3} + 2y^2$

Hw: Check that $\frac{4}{3} = \int_{-1}^1 \left(\int_0^1 (x^2+y^2)\,dy\right) dx$

$\int_{-1}^1 \left(\frac{y^3}{3}\Big|_0^1 + x^2 \cdot y\Big|_0^1\right) dx = \int_{-1}^1 \left(\frac{1}{3} + x^2\right) dx = \frac{1}{3} \cdot x\Big|_{-1}^{1} + \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{4}{3}$

Double integral over a simple domain

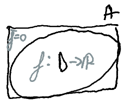
Def Let $\Delta \subseteq \mathbb{R}^2$ general domain.

We say that $f : \Delta \to \mathbb{R}$ is integrable on Δ

if $\exists A \subseteq \mathbb{R}^2$ rectangle, such that $\Delta \subseteq A$

and the function $\tilde{f} : A \to \mathbb{R}$, $\tilde{f}(x) = \begin{cases} f(x), & x \in \Delta \\ 0, & x \in A \setminus \Delta \end{cases}$

is integrable and $\iint_\Delta f(x,y)\,dx\,dy = \iint_A \tilde{f}(x,y)\,dx\,dy$

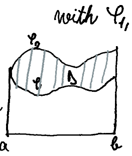


Def A set $\Delta \subseteq \mathbb{R}^2$ is called

- simple with respect to the y-axis if:

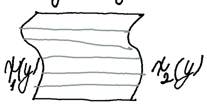
  $\Delta = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}$

  with φ_1, φ_2 cont. (phi)

y-simple


- simple w.r.t. the x-axis if

  $\Delta = \{(x,y) \mid c \le y \le d,\ x_1(y) \le x \le x_2(y)\}$



X-simple

Theorem $f : \Delta \to \mathbb{R}$

If Δ is simple $\begin{pmatrix} x\text{-simple} \\ y\text{-simple} \end{pmatrix}$ then $\iint f(x,y)\,dx\,dy = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right) dx = \int_c^d \left(\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\right) dy$