Seminar 12

Monday, January 9, 2023 6:05 PM

1) Let A be symmetric matrix of size $m\times m$

$f:\mathbb{R}^m \to \mathbb{R}$ $\quad f(x)=\frac{1}{2}\cdot x^{t}\cdot Ax$ $\leftarrow$ transposed

Prove that $\nabla f(x) = Ax$ and $Hf(x)=A$

$$f(x_1,x_2,\dots,x_m)=\frac{1}{2}(x_1,x_2,\dots,x_m)\cdot\begin{pmatrix}a_{11}&\cdots&a_{1m}\\ \vdots&&\\ a_{m1}&\cdots&a_{mm}\end{pmatrix}\cdot\begin{pmatrix}x_1\\ \vdots\\ x_m\end{pmatrix}$$

$Q(h)=h^{t}\cdot Hf\cdot h=\sum_{i=1}^{m}\sum_{j=1}^{m}$

$$=\frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{m}x_i\cdot a_{ij}x_j\right)=\frac{1}{2}\sum_{i=1}^{m}\left(a_{ii}x_i^2+\sum_{\substack{j=1\\ i\neq j}}^{m}x_i\cdot a_{ij}\cdot x_j\right)$$

$$\frac{\partial f}{\partial x_k}(x)=\frac{\partial}{\partial x_k}\left(\frac{1}{2}\cdot\sum_{i=1}^{m}\left(a_{ii}x_i^2+\sum_{\substack{j=1\\ i\neq j}}^{m}x_i a_{ij}x_j\right)\right)=\frac{1}{2}\left(2a_{kk}\cdot x_k+\sum_{\substack{j=1\\ j\neq k}}^{m}a_{kj}\cdot x_j+\sum_{\substack{j=1\\ j\neq k}}^{m}x_j\cdot a_{jk}\right)=$$

$$=\frac{1}{2}\left(\sum_{j=1}^{m}a_{kj}\cdot x_j+\sum_{j=1}^{m}a_{jk}\cdot x_j\right)=\sum_{j=1}^{m}a_{kj}\cdot x_j$$

$=$ they are the same, A symmetric

$$A\cdot x=\begin{pmatrix}a_{11}&\cdots&a_{1m}\\ \vdots&&\vdots\\ a_{m1}&\cdots&a_{mm}\end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ x_m\end{pmatrix}=\begin{pmatrix}\sum_{j=1}^{m}a_{1j}x_j\\ \vdots\\ \sum_{j=1}^{m}a_{mj}x_j\end{pmatrix}=\begin{pmatrix}\frac{\partial f}{\partial x_1}(x)\\ \vdots\\ \frac{\partial f}{\partial x_m}(x)\end{pmatrix}=\nabla f(x)$$

$$\frac{\partial^2 f}{\partial x_\ell\partial x_k}=\frac{\partial}{\partial x_\ell}\left(\sum_{j=1}^{m}a_{kj}\cdot x_j\right)=a_{k\ell}=\frac{\partial^2 f}{\partial x_k\partial x_\ell}(x)$$

$$\Rightarrow Hf(x)=\begin{pmatrix}\frac{\partial^2 f}{\partial x_1^2}&\cdots&\frac{\partial^2 f}{\partial x_1\partial x_m}\\ \vdots&&\\ \frac{\partial^2 f}{\partial x_m\partial x_1}&\cdots&\frac{\partial^2 f}{\partial x_m\partial x_m}\end{pmatrix}=\begin{pmatrix}a_{11}&\cdots&a_{1m}\\ \vdots&&\vdots\\ a_{m1}&\cdots&a_{mm}\end{pmatrix}$$

2) A an $m\times m$ matrix and $b\in\mathbb{R}^m$

$$\min_{x\in\mathbb{R}^m}\|Ax-b\|^2$$

Prove that the solution x^* satisfies the following equality:

$$A^t\cdot A\cdot x^*=A^t\cdot b$$

let $f:\mathbb{R}^m\to\mathbb{R}$, $f(x)=\|Ax-b\|^2=\langle Ax-b, Ax-b\rangle=$

$=\langle Ax, Ax-b\rangle-\langle b, Ax-b\rangle=\langle Ax,Ax\rangle-\langle Ax,b\rangle-\langle b,Ax\rangle+\langle b,b\rangle$

$=(Ax)^t\cdot Ax-2(Ax)^t\cdot b+b^t\cdot b$

$\langle b,b\rangle=b^t\cdot b\ =(b_1,\dots,b_m)\begin{pmatrix}b_1\\ \vdots\\ b_m\end{pmatrix}$

$b\in\mathbb{R}^m$ row column $\qquad (AB)^t=B^t A^t$

$$\nabla f(x)=\nabla\Big((Ax)^t\cdot Ax-2(Ax)^t b+\underbrace{b^t\cdot b}_{\in\mathbb{R}}\Big)=\nabla\big((Ax)^t\cdot Ax\big)-2\nabla\big((Ax)^t\cdot b\big)=$$

$=\nabla\big(x^t\cdot(A^t\cdot A)\cdot x\big)-2\,\nabla(x^t\cdot A^t\cdot b)$

$A^t\cdot A$ symmetric $\overset{def}{\Rightarrow}(A^t\cdot A)^t=A^t\cdot A$

$(A^t\cdot A)^t=A^t\cdot(A^t)^t=A^t\cdot A\Rightarrow A^t\cdot A$ symmetric

$\overset{Ex.1}{=}2A^t\cdot A\cdot x-2\nabla(x^t\cdot A^t b)$

$x^t\cdot A^t\cdot b=(x_1,\dots,x_m)\cdot A^t\cdot b=$

$\quad=\big((x_1,0,\dots,0)+\dots+(0,0,\dots,x_m)\big)\cdot A^t b=$

$\quad=(x_1,\dots,0)\cdot A^t b+\dots+(0,\dots,0,x_m)\cdot A^t b$

$\dfrac{\partial(x^t\cdot A^t b)}{\partial x_k}=(0,\dots,1,\dots,0)\cdot A^t b\Rightarrow k^{th}$ row

$\qquad k^{th}$ place

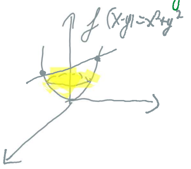
$\Rightarrow\nabla(x^t A^t b)=A^t b$

x^* is a minimum point of f $\Rightarrow$

$\nabla f(x^*)=0\Rightarrow\cancel{2}A^t\cdot A\cdot x-\cancel{2}\cdot A^t b=0\Rightarrow A^t A x=A^t b$

3) $f, g: \mathbb{R}^{(n)} \to \mathbb{R}$, $c \in \mathbb{R}$ - constant

minimize/maximize $f(x)$ subject to $g(x) = c$



$x^2+y^2=1$

a) x^2+y^2 subject to $x-y+1=0$

$f(x,y) = x^2+y^2$

$g(x,y) = x-y = \underbrace{-1}_{c}$

we define $L(x,y,\lambda) = f(x,y) - \lambda(g(x,y)-c)$ - Lagrange function

λ - Lagrange multiplier

$\nabla L(x,y,\lambda) = (2x-\lambda,\ 2y+\lambda,\ -x+y-1) = (0,0,0) \Rightarrow \lambda \neq 0$

$L(x,y,\lambda) = x^2+y^2-\lambda(x-y+1)$

$$\begin{cases} 2x-\lambda=0 \\ 2y+\lambda=0 \\ -x+y-1=0 \end{cases} \Rightarrow \begin{cases} x=\frac{\lambda}{2} \\ y=-\frac{\lambda}{2} \\ -\frac{\lambda}{2}-\frac{\lambda}{2}=1 \end{cases} \to \begin{cases} \lambda=-1 \\ x=\frac{1}{2} \\ y=\frac{1}{2} \end{cases}$$

$\Rightarrow$ the unique critical point of L is $(-\frac{1}{2}, \frac{1}{2}, -1)$

check if min/max

$f(-\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} \Rightarrow \frac{1}{2}$ is the min of f subject to $x-y=-1$

$f(0,1) = 1$

has to $x-y=-1$

b) $(x+y)^2$ subject to $x^2+y^2=1$

$f(x,y) = (x+y)^2$

$g(x,y) = x^2+y^2 = 1$

$L(x,y,\lambda) = f(x,y) - \lambda(g(x,y)-c) =$

$= (x+y)^2 - \lambda(x^2+y^2-1)$

$\nabla L(x,y,\lambda) = (2x+2y-2\lambda x,\ 2x+2y-2\lambda y,\ 1-x^2-y^2) = (0,0,0)$

$$\begin{cases} 2x+2y-2\lambda x=0 \\ 2x+2y-2\lambda y=0 \\ 1-x^2-y^2=0 \end{cases} \Rightarrow \begin{cases} x+y=x\lambda \\ x+y=y\lambda \\ x^2+y^2=1 \end{cases} \Rightarrow \begin{cases} 2x+2y=\lambda(x+y) \\ x^2+y^2=1 \end{cases} \quad \lambda=2,\ x+y=0$$

1. $\lambda=2$

$x+y=2x \Rightarrow x=y$

$x^2+x^2=1 \Rightarrow x^2=\frac{1}{2} \Rightarrow x=\pm\frac{\sqrt{2}}{2} \Rightarrow y=\pm\frac{\sqrt{2}}{2}$

2. $x+y=0$, $x=-y$

$x^2+x^2=\frac{1}{2} \Rightarrow x=\pm\frac{1}{\sqrt{2}} \Rightarrow y=\mp\frac{1}{\sqrt{2}}$

2.1. $x=\frac{1}{\sqrt{2}} \Rightarrow y=-\frac{1}{\sqrt{2}} \Rightarrow \lambda \in \mathbb{R}$

2.2. $x=-\frac{1}{\sqrt{2}} \Rightarrow y=\frac{1}{\sqrt{2}} \Rightarrow \lambda \in \mathbb{R}$

$f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = 2$

$f(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = 0$ min=0

$f(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = 0$ max=0

$(x,y) \in \{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\}$ $f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = 2$

c) $f(x,y) = x^2-y^2$

$g(x,y) = x^2+y^2$

$c=1$

$L(x,y,\lambda) = f(x,y) - \lambda(g(x,y)-1) =$

$= x^2 - y^2 - \lambda(x^2+y^2-1)$

$\nabla L(x,y,\lambda) = (2x - 2\lambda x, -2y - 2\lambda y, 1-x^2-y^2) = (0,0,0)$

$2x - 2\lambda x = 0$

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d) $f(x,y,z) = x+2y+3z$

$g(x,y,z) = x^2+y^2+z^2 = 1$

$C = 1$

$L(x,y,z,\lambda) = f(x,y,z) - \lambda(g(x,y,z) - C)$

$L(x,y,z,\lambda) = x+2y+3z - \lambda(x^2+y^2+z^2-1)$

$\nabla L(x,y,z,\lambda) = (1-2\lambda x, 2-2\lambda y, 3-2\lambda z, -(x^2+y^2+z^2-1)) = (0,0,0,0)$

$$\begin{cases} 1-2\lambda x = 0 \Rightarrow x = \frac{1}{2\lambda} \\ 2-2\lambda y = 0 \Rightarrow y = \frac{1}{\lambda} \\ 3-2\lambda z = 0 \Rightarrow z = \frac{3}{2\lambda} \\ x^2+y^2+z^2 = 1 \end{cases}$$

$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{9}{4\lambda^2} = 1$

$\frac{10}{4\lambda^2} + \frac{4}{4\lambda^2} = 1$

$\frac{7}{2\lambda^2} = 1 \Rightarrow \lambda^2 = \frac{2}{7} \Rightarrow \lambda = \pm\sqrt{\frac{2}{7}}$

$$\begin{cases} x = \pm\frac{1}{2}\sqrt{\frac{7}{2}} \\ y = \pm\sqrt{\frac{7}{2}} \\ z = \pm\frac{3}{2}\sqrt{\frac{7}{2}} \end{cases}$$