Seminar 5

Monday, November 7, 2022 6:13 PM

1) Find accumulation points of the following sets:

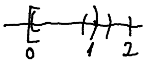
$A = [0,1) \cup \{2\}$

$c \in \overline{\mathbb{R}}$ is an accumulation point of $A \subseteq \mathbb{R} \iff \forall V \in \mathcal{V}(c) \quad V \cap (A - \{c\}) \neq \emptyset$ $A' \to$ set of acc. points

$\forall \varepsilon > 0 \quad (1-\varepsilon, 1+\varepsilon) \cap (A \setminus \{1\}) \neq \emptyset \Rightarrow 1 \in A'$

$A' = [0,1]$

Let $\varepsilon = \frac{1}{2}$ $\left(2-\frac{1}{2}, 2+\frac{1}{2}\right) \cap (A \setminus \{2\}) = \emptyset \Rightarrow 2 \notin A'$



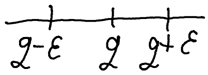
$A' = [0,1]$

$B = \mathbb{Z}$

Let $\varepsilon = \frac{1}{2}$ $\forall m \in \mathbb{Z}$ $\left(m-\frac{1}{2}, m+\frac{1}{2}\right) \cap (\mathbb{Z} \setminus \{m\}) = \emptyset$

$\Rightarrow B' = \emptyset$

$C = \mathbb{Q}$, $C' =$



$\forall \varepsilon > 0, \ \forall q \in \mathbb{Q} \quad (q-\varepsilon, q+\varepsilon) \cap (\mathbb{Q} \setminus \{q\}) \neq \emptyset \Rightarrow \mathbb{Q} \subseteq \mathbb{Q}'$

$\forall \varepsilon > 0, \ \forall r \in \mathbb{R} \setminus \mathbb{Q} \ \underbrace{(r-\varepsilon, r+\varepsilon)}_{\text{contains rational numbers}} \cap (\mathbb{Q} \setminus \{r\}) \neq \emptyset \Rightarrow \mathbb{R} - \mathbb{Q} \subset \mathbb{Q}'$

$\Rightarrow \mathbb{R} = \mathbb{Q}'$

2) Find a function $f: \mathbb{R} \to \mathbb{R}$ that is discontinous everywhere but $|f|$ is continous everywhere

$$f(x) = \begin{cases} -1, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

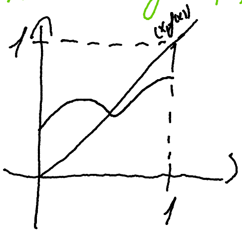
$\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$

$\underbrace{1, 1.4, 1.41, \ldots}_{\in \mathbb{Q}} \to \sqrt{2}$

f discontinous at every point and $|f| = 1$, which is continous at every point

f is continous at $x_0 \iff \lim_{x \to x_0} f(x) = f(x_0)$

3) If $f: [a,b] \to [a,b]$ is continous then it has at least one fixed point x^* with $f(x^*) = x^*$



$f(x) = x$

Let $g(x) = f(x) - x$ continuous

$g(a) = f(a) - a \geq 0$

$g(b) = f(b) - b \leq 0$

$\xrightarrow[\text{Value Theorem}]{\text{Intermediate}}$ Bolzano - Darboux $\exists c \in [a,b] : g(c) = 0$

$f(a) \geq a$

$f(b) \leq b$

$\Rightarrow g(c) = f(c) - c = 0 \Rightarrow f(c) = c \Rightarrow x^* = c$ is a fixed point

4) Study the continuity and the differentiability for f and f', where

$f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

f is differentiable at x_0 if $\exists \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \mathbb{R}$

for it to be differentiable it has to be finite

1. continuity of f

a) For $x \in \mathbb{R}$, $x \neq 0 \Rightarrow f(x) = x^2 \sin\frac{1}{x}$ (Continuous on $\mathbb{R}^*$)

b) For $x=0$, $f(0)=0$

$\lim\limits_{x\to 0} x^2 \underbrace{\sin\frac{1}{x}}_{\in[-1,1]} = 0$

$f(0) = \lim\limits_{x\to 0} f(x) = 0 \Rightarrow f$ is cont. in $x=0$ (2)

$\xRightarrow{(1),(2)} f$ cont. on $\mathbb{R}$

2. differentiability

f diff on $\mathbb{R}^*$ (because $f(x) = x^2 \sin\frac{1}{2}$ is diff) (1')

$\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x} = \lim\limits_{x\to 0} x \underbrace{\sin\frac{1}{x}}_{\in[-1,1]} = 0 \in \mathbb{R} \Rightarrow$

$\Rightarrow f$ is differentiable at $x=0$ (2')

$\xRightarrow{(1'),(2')} f$ is differentiable on $\mathbb{R} \Rightarrow f' : \mathbb{R} \to \mathbb{R}$, $f'(x) = \begin{cases} (x^2 \sin\frac{1}{x})', & x \neq 0 \\ 0', & x=0 \end{cases} = \begin{cases} 2x \sin\frac{1}{x} + x^2 \cos\frac{1}{x} \cdot \frac{-1}{x^2}, & x \neq 0 \\ 0, & x=0 \end{cases} = ($

continuity for f'

For $x \neq 0 \Rightarrow f'$ is continuous (operations with continuous functions)

For $x=0$:

$\left.\begin{array}{l} f'(0)=0 \\ \lim\limits_{x\to 0} 0 = 0 \end{array}\right| \Rightarrow$ For $x=0$, f' is continuous

$\lim\limits_{x\to 0} f'(x) = \lim\limits_{x\to 0} \left(2x \underbrace{\sin\frac{1}{x}}_{\in[-1,1]} - \cos\frac{1}{x}\right) = -\lim\limits_{x\to 0} \left(\cos\frac{1}{x}\right)$

$\left.\begin{array}{l} x_n = \frac{1}{2n\pi} \to 0;\ \cos\frac{1}{x_n} = \cos(2\pi n) = 1 \\ y_n = \frac{1}{2n\pi + \frac{\pi}{2}} \to 0;\ \cos\frac{1}{y_n} = \cos(2n\pi + \frac{\pi}{2}) = 0 \end{array}\right\} \Rightarrow \nexists \lim\limits_{x\to 0} \cos\frac{1}{x}$

$\lim\limits_{n\to\infty} \cos\frac{1}{x_n} \neq \lim\limits_{n\to\infty} \cos\frac{1}{y_n}$

6)

a) $\lim\limits_{x\to\infty} \frac{[x]}{x}$

We know $x \leq [x] \leq x+1 \,/\cdot \frac{1}{x}$

$1 \leq \frac{[x]}{x} \leq \frac{x+1}{x} \,/ \lim\limits_{x\to\infty}()$

$1 \leq \lim\limits_{x\to\infty} \frac{[x]}{x} \leq \lim\limits_{x\to+\infty} \frac{x+1}{x}$

$1 \leq \lim\limits_{x\to\infty} \frac{[x]}{x} \leq 1$

$1 \xRightarrow[\text{Theorem}]{\text{Squeeze}} \lim\limits_{x\to\infty} \frac{[x]}{x} = 1$

e) $\lim_{x\to\infty} x(\ln(x+2) - \ln(x+1))$

$= \lim_{x\to\infty} x \ln\frac{x+2}{x+1} = \lim_{x\to\infty} \ln\left(1+\frac{1}{x+1}\right)^x = \ln\left(e^{\lim_{x\to\infty}\frac{x}{x+1}}\right) = \ln e = 1$

d) $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} (e^{\ln x})^x = e^{\lim_{x\to 0^+} x\ln x} = e^{\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}} \overset{H}{\underset{\frac{\infty}{\infty}}{=}} e^{\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}} = e^{\lim_{x\to 0^+} -\frac{x}{1}} = e^0 = 1$

$x = e^{\ln x}$

f) $\lim_{x\to\infty} x\left(\left(1+\frac{1}{x}\right)^x - e\right) = \lim_{x\to\infty} \frac{\left(1+\frac{1}{x}\right)^x - e}{\frac{1}{x}}$

Let $y = \frac{1}{x}$

$\lim_{y\to 0^+} \frac{(1+y)^{\frac{1}{y}} - e}{y} \overset{\frac{0}{0}}{=} \lim_{y\to 0^+} \frac{e^{\ln(1+y)^{\frac{1}{y}}} - e}{y} \overset{\frac{0}{0}}{=} \lim_{y\to 0^+} e^{\frac{1}{y}\ln(1+y)} \cdot \frac{-1}{y^2}\ln(1+y) + \frac{1}{y}\cdot\frac{1}{1+y} - e$

$= \lim_{y\to 0^+} \underbrace{(1+y)^{\frac{1}{y}}}_{e} \cdot \frac{1}{y^2}\left(-\ln(1+y) + \frac{y}{1+y}\right) = e \cdot \lim_{y\to 0^+} \frac{-\ln(1+y) + \frac{y}{1+y}}{y^2} \overset{\frac{0}{0}}{=} e \cdot \lim_{y\to 0^+} \frac{-\frac{1}{1+y} + \frac{1+y-y}{(1+y)^2}}{2y} = e \cdot \lim_{y\to 0^+} \frac{-1-}{2y}$

ex. 8 Python → the minimum of f

7) Find the n^{th} derivative

a) $f: (-1, \infty) \to \mathbb{R}$

$f(x) = \ln(1+x)$

$P(1): f'(x) = \frac{1}{1+x}$

$P(2): f''(x) = \frac{-1}{(1+x)^2} = -1 \cdot (1+x)^{-2}$

$P(n): f^{(n)}(x) = (-1)^{n+1}(n-1)!\,(1+x)^{-n}$

$P(n+1): f^{(n+1)}(x) = (-1)^{n+2}\, n!\,(1+x)^{-n-1}$

Assume $P(n)$ true and prove $P(n+1)$

$(f^{(n)}(x))' = \left((-1)^{n+1}\cdot(n-1)!\,(1+x)^{-n}\right)' = (-1)^{n+1}\cdot(n-1)!\cdot(-n)(1+x)^{-n-1} =$

$= (-1)^{n+2}\cdot n!\cdot(1+x)^{-n-1} \Rightarrow P(n+1)$ true $\Rightarrow P(n)$ is true $\forall$

c) $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2 \cdot \sin x$

$f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} (x^2)^{(k)} \cdot (\sin x)^{(n-k)}$

$(f\cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} \cdot g^{(k)}$

$(\sin x)' = \cos x = \sin\left(x + \frac{\pi}{2}\right)$

$(\sin x)^{(n-k)} = \sin\left(x + \frac{(n-k)\pi}{2}\right)$