Analysis midterm work

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Seminar 1

1. ub, lb, min, max, sup, inf

a) $A=(-1,1)\cup(2,\infty)$

$lb=\{x\in\mathbb{R} \mid x\leq a, \forall a\in A\}$
$lb=(-\infty,-1]$
$ub=\{x\in\mathbb{R} \mid x\geq a, \forall a\in A\}$
$ub=\emptyset$

we assume that $\alpha\in ub(A)$, $\alpha\geq 2$, but then $\alpha+1\in A$ and $\alpha+1>\alpha$ $\Big\}\Rightarrow ub(A)=\emptyset$

$sup=\min(ub(A))\Rightarrow \nexists\, sup(A)\Rightarrow \nexists\, max(A)$
$inf=\max(lb(A))=-1\notin A\Rightarrow \nexists\, min(A)$

b) $B=(-3,2)\cup\{3\}$
$lb(B)=(-\infty,-3]$
$ub(B)=[3,+\infty)$
$inf(B)=-3\notin B\Rightarrow \nexists\, min(B)$
$sup(B)=3\Rightarrow max(B)=3$

c) $C=(-5,5)\cap\mathbb{Z}=\{-4,-3,-2,-1,0,1,2,3,4\}$
$lb=(-\infty,-4]$
$ub=[4,+\infty)$
$inf(C)=-4$
$sup(C)=4$
$max(C)=4$
$min(C)=-4$

d) $D=\emptyset$
$lb(D)=\mathbb{R}^*$
$up(D)=\mathbb{R}^*$
$sup=+\infty\notin D\Rightarrow \nexists\, max(D)$
$inf=-\infty\notin D\Rightarrow \nexists\, min(D)$

2. ub, lb, sup, inf, min, max

a) $A=\{x\in\mathbb{Q} \mid x^2<2\}$
$x^2<2$
$-\sqrt{2}<x<\sqrt{2}\simeq -1{,}4<x<1{,}4$
$x\in(-\sqrt{2},2)\cup\mathbb{Q}\Rightarrow A=(-\sqrt{2},\sqrt{2})\cap\mathbb{Q}$

The density of $\mathbb{Q}$ ($\mathbb{Q}$ is dense in $\mathbb{R}$)
$\forall a,b\in\mathbb{R}, a<b\Rightarrow \exists\, q\in\mathbb{Q}:$
$a<q<b$

Let $a=\alpha$ and $b=\sqrt{2}$ $\Big\}\xrightarrow[\text{dense in } \mathbb{R}]{\mathbb{Q} \text{ is}}$

$\Rightarrow \exists\, q\in\mathbb{Q}$ so that $\alpha<q<\sqrt{2}$
$q\in A$
$\alpha<q$ $\Big\}\Rightarrow \alpha\notin ub(A)\Rightarrow$
$\Rightarrow sup(A)=\sqrt{2}$

$\Rightarrow \nexists \max(A)$

let $b = \beta,\ a = -\sqrt{2}$ $\Big|$ $\mathbb{Q}$ d $\Rightarrow \exists\, q \in \mathbb{Q}$ so that $-\sqrt{2} < q < \beta$ ($\mathbb{R}$)

$q \in A$, $q < \beta$ $\Big|$ $\Rightarrow \beta \notin lb(A) \Rightarrow$

$\Rightarrow \inf(A) = -\sqrt{2}$

$\Rightarrow \nexists \min(A)$

b) $B = \{x^2 - 4x + 3 \mid x \in \mathbb{R}\}$

$x^2 - 4x + 3 = (x-3)(x-1)$

$x_1 = 3,\ x_2 = 1 \Rightarrow$

x		1		3	
$f(x)$	+	0	− −	0	+

$y_{min} = -\dfrac{\Delta}{4a} = -\dfrac{16-12}{4} = -\dfrac{4}{4} = -1$

$x_{min} = -\dfrac{b}{2a} = \dfrac{4}{2} = 2$

$M(2, -1)$

$\nexists\, ub(B) \Rightarrow \nexists \sup(B) \Rightarrow \nexists \max(B)$

$lb(B) = (-\infty, -1] \Rightarrow \inf(B) = -1 \Rightarrow \min(B) = -1$

c) $C = \left\{ \dfrac{n}{n+1} \mid n \in \mathbb{N} \right\}$

$\dfrac{n}{n+1} < 1$

$\inf(C) = \dfrac{1}{2} \in C \Rightarrow \min(C) = \dfrac{1}{2}$

$\dfrac{n}{n+1} \leq \dfrac{1}{2}$

$2n \leq n+1 \Rightarrow n \geq 1$ true by definition

let $\sup(A) = 1$

1. 1 is an upper bound

   $\dfrac{n}{n+1} \leq 1$

   $n \leq n+1$ (true)

2. $\forall \varepsilon > 0,\ 1 - \varepsilon \notin ub(A)$

   $\dfrac{n}{n+1} > 1 - \varepsilon$

   $\varepsilon > 1 - \dfrac{n}{n+1}$

   $\varepsilon > 1 - \dfrac{n+1-1}{n+1}$

   $\varepsilon > \dfrac{1}{n+1}$

   $\varepsilon(n+1) > 1$

   $n + 1 > \dfrac{1}{\varepsilon}$

   $\forall \varepsilon \in \mathbb{R}, \exists n \in \mathbb{N} : n > \varepsilon$ $\Big|$ $\xrightarrow{\text{Archimedian Principle}}$ $\dfrac{n}{n+1} > 1 - \varepsilon \Rightarrow \sup(C) = 1 \notin C \Rightarrow \nexists \max(C)$

   $n > \dfrac{1}{\varepsilon} - 1$

3. $A = (0, 1) \cap \mathbb{Q}$.

$\inf A \stackrel{?}{=} 0$

$lb(A) = (-\infty, 0]$

let $a = 0$, $b = \beta$ $\left|\xrightarrow[\text{in } \mathbb{R}]{\mathbb{Q} \text{ dense}}\right.$ $\exists q \in \mathbb{Q}$ s.t. $0 < q < \beta$ $\left.\begin{matrix} q \in A \\ q < \beta \end{matrix}\right| \Rightarrow$

$\Rightarrow \beta \notin lb(A) \Rightarrow$

$\Rightarrow \inf(A) = 0 \Rightarrow$

4) $-S = \{-x \mid x \in S\}$, S nonempty, bounded from above

lb $\exists$?

$\inf(-S) = -\sup(S)$?

$\exists\, ub(S) \Rightarrow \exists \sup$

let $\alpha \in \mathbb{R}^*_+$, $\sup(S) = \alpha \Rightarrow \alpha \geq x$, $\forall x \in S$

$-\alpha \leq -x$, $\forall x \in S$

$-\alpha \leq y$, $\forall y \in -S \Rightarrow$

$\Rightarrow -S$ is bounded from below

let $\inf(-S) = \beta$, $\beta \overset{?}{=} -\alpha$

$\left.\begin{matrix} -\alpha \in lb(-S) \\ \beta = \max(lb(-S)) \end{matrix}\right| \Rightarrow -\alpha \leq \beta$ (1)

β

Seminar 2

1.

$\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

#

$\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ s.t. $\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$, $\forall n \geq N_\varepsilon \Leftrightarrow \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\left|\frac{1}{\sqrt{n}}\right| < \varepsilon$

$\frac{1}{\sqrt{n}} < \varepsilon \,|^2$

$\frac{1}{n} < \varepsilon^2$

$1 < n\varepsilon^2$

$n > \frac{1}{\varepsilon^2} \xRightarrow[\text{principle}]{\text{Archimedean}}$ True, $\forall \varepsilon > 0$

$\lim\limits_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$

$\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ s.t. $\left|\frac{n+1}{2n+3} - \frac{1}{2}\right| < \varepsilon$, $\forall n \in N_\varepsilon \Leftrightarrow \lim\limits_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$

$\left|\frac{2n+2-2n-3}{4n+6}\right| < \varepsilon$

$\frac{1}{4n+6} < \varepsilon$

$\frac{1}{\varepsilon} < 4n+6$

$n > \frac{1}{4\varepsilon} - 6 \xRightarrow[\text{property}]{\text{Archimedean}}$ True

2)

lim inf lim sup

$\lim\limits_{n\to\infty} \inf, \lim\limits_{n\to\infty} \sup$

$$\lim_{n\to\infty} \frac{(-1)^n \cdot n}{n+1}$$

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf_{m\geq n} \{x_m\}$$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{m\geq n} \{x_m\}$$

$$\frac{(-1)^n \cdot n}{n+1} = \begin{cases} n = 2k,\ k\in\mathbb{N}: \frac{n}{n+1} \\ n = 2k-1,\ k\in\mathbb{N}: -\frac{n}{n+1} \end{cases}$$

$$\sup_{m\geq n} \{x_m\} = 1 \Rightarrow \limsup_{n\to\infty} x_n = 1$$

$$\inf_{m\geq n} \{x_m\} = -1 \Rightarrow \liminf_{n\to\infty} x_n = -1$$

3) if (x_n) bounded, monotone, convergent

a) $x_n = \sqrt{n+1} - \sqrt{n}$

$$x_{n+1} - x_n = \sqrt{n+2} - \sqrt{n+1} - \sqrt{n+1} + \sqrt{n} = \sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n} \overset{?}{<} 0$$

$x_1 = \sqrt{2} - 1 \simeq 0{,}4$

$x_2 = \sqrt{3} - \sqrt{2} \simeq 0{,}3$

$$? \Rightarrow \sqrt{n+2} + \sqrt{n} < 2\sqrt{n+1}$$

$$n+2 + 2\sqrt{n^2+2n} + n < 4n+4$$

$$2\sqrt{n^2+2n} < 2n+2$$

$$\sqrt{n^2+2n} < n+1$$

$$n^2+2n < n^2+2n+1$$

$0<1$ (A) $\Rightarrow x_{n+1} < x_n \Rightarrow (x_n)$ strictly decreasing, monotone; $(x_n) > 0 \Rightarrow x_n$ bounded from above $\Big| \to x_n$ - Convergent

b) $x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}$

$$x_{n+1} - x_n = \cancel{\frac{1}{1\cdot 2}} + \ldots + \cancel{\frac{1}{n(n+1)}} + \frac{1}{(n+1)(n+2)} - \cancel{\frac{1}{1\cdot 2}} - \cancel{\frac{1}{2\cdot 3}} - \ldots - \cancel{\frac{1}{n(n+1)}} = \frac{1}{(n+1)(n+2)} > 0 \Rightarrow x_{n+1} > x_n \Rightarrow x_n \text{ str. inc.}$$

$$x_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \ldots + \cancel{\frac{1}{n}} - \frac{1}{n+1} = 1 - \frac{1}{n+1} < 1$$

$\Rightarrow (x_n)$ bounded from abo

4)

a) $\sqrt{n}(\sqrt{n+1} - \sqrt{n})$

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} \sqrt{n^2+n} - n = \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{1}{2}$$

b) $(a_1^n + a_2^n + \ldots + a_k^n)^{\frac{1}{n}}$, $a_i > 0$

we assume that $a_1 \leq a_2 \leq \ldots \leq a_k$

$$a_1^n + a_2^n + \ldots + a_k^n \leq k\cdot a_k^n$$

$$(a_1^n + a_2^n + \ldots + a_k^n)^{\frac{1}{n}} \leq (k\cdot a_k^n)^{\frac{1}{n}}$$

$$a_k = (a_k^n)^{\frac{1}{n}} \leq (a_1^n + a_2^n + \ldots + a_k^n)^{\frac{1}{n}} \leq (k\cdot a_k^n)^{\frac{1}{n}}$$

$\searrow \swarrow$
a_k

$\lim_{n\to\infty}(a_1^n+a_2^n+\dots+a_k^n)^{\frac{1}{n}}=a_k$

c) $\lim_{n\to\infty}\sqrt[n]{n}$

$\sqrt[n]{n} > 1+\varepsilon_n$

$n=(1+\varepsilon_n)^n = 1+n\cdot\varepsilon_n+\binom{n}{2}\varepsilon_n^2+\dots+\varepsilon_n^n$

$\frac{n(n-1)}{2}\varepsilon_n^2 \overset{?}{<} (1+\varepsilon_n)^n \quad x>0,\ n\in\mathbb{Z}$

$\binom{n}{2}\varepsilon_n^2=\frac{n!}{2!(n-2)!}\varepsilon_n^2=\frac{(n-1)n}{2}\varepsilon_n^2<(1+\varepsilon_n)^n$

let $\varepsilon_n=\sqrt{\frac{2}{n-1}}$

$\frac{n\cancel{(n-1)}}{\cancel{2}}\left(\frac{\cancel{2}}{\cancel{n-1}}\right)$

$n<\left(1+\sqrt{\frac{2}{n-1}}\right)^n$

$1<\sqrt[n]{n}<1+\sqrt{\frac{2}{n-1}}$

Seminar 3

1.

a) $\sum_{n\geq 1}\frac{2}{3^n}$

$l=\lim_{n\to\infty}\frac{\frac{\cancel{2}}{3^{n+1}}}{\frac{\cancel{2}}{\cancel{3^n}}}=\frac{1}{3}<1 \Rightarrow \sum_{n=1}^{\infty}\frac{2}{3^n}$ is convergent

$S_n=\sum_{k=1}^{n}\frac{2}{3^k}=2\sum_{k=1}^{n}\frac{1}{3^k}=2\left(\frac{1}{3}+\frac{1}{3^2}+\dots+\frac{1}{3^n}\right)=\frac{2}{3}\left(1+\frac{1}{3}+\dots+\frac{1}{3^{n-1}}\right)=\frac{\cancel{2}}{\cancel{3}}\cdot\frac{1-\frac{1}{3^n}}{\cancel{1-\frac{1}{3}}}=1-\frac{1}{3^n}$

$\sum_{n=1}^{\infty}\frac{2}{3^n}=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\frac{1}{3^n}\right)>1-0=1$

c) $\sum_{n\geq 1}\frac{1}{4n^2-1}$

$l=\lim_{n\to\infty}\frac{\frac{1}{4(n+1)^2-1}}{\frac{1}{4n^2-1}}=\lim_{n\to\infty}\frac{4n^2-1}{4(n+1)^2-1}=1\Rightarrow$ inconclusive

let $y_n=\frac{1}{x_n}\Rightarrow \lim_{n\to\infty}\frac{x_n}{y_n}=\lim_{n\to\infty}\frac{n^2}{4n^2-1}=\frac{1}{4}\in\mathbb{R}$

The series x_n and y_n have the same nature $\Rightarrow$
$\Rightarrow$ they are convergent/divergent at the same time

$\frac{x_n}{y_n}$ Comp. test
$(0,+\infty) \to$ same nat.
0, y_n - conv. $\Rightarrow x_n$ conv.
∞, y_n - div $\Rightarrow x_n$ div.

Seminar 4

1.

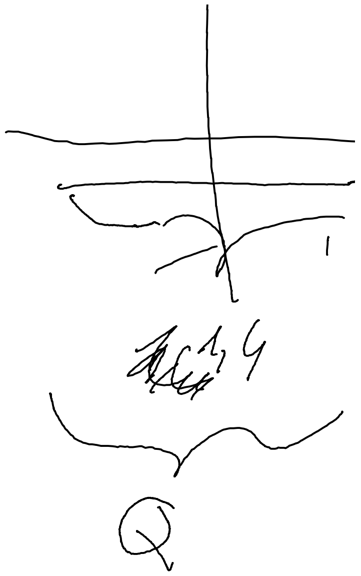
a) $\sum_{n\geq 2}\frac{1}{\ln n}$, $\quad x_n=\frac{1}{\ln n}$, $n\in\mathbb{N}^*$

Let $y_n=\frac{1}{n}$ - divergent

$l=\lim\frac{x_n}{y_n}=\lim\frac{\frac{1}{\ln n}}{\dots}=\lim\frac{n}{\dots}\overset{\frac{\infty}{\infty}}{=}\lim n=\infty$

Comp. Test $\Rightarrow (x_n)$ divergent

$\lim\limits_{n\to\infty} y_n \quad \lim\limits_{n\to\infty} \frac{1}{n} \quad \lim\limits_{n\to\infty} \ln n \quad \lim\limits_{n\to\infty}$



b) $\sum\limits_{n\geq 1} \frac{\ln(1+\frac{1}{n})}{n}$, $x_n = \frac{\ln(1+\frac{1}{n})}{n}$

let $y_n = \frac{1}{n^2}$, convergent

$\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \lim\limits_{n\to\infty} \frac{\frac{\ln(1+\frac{1}{n})}{n}}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \ln(1+\frac{1}{n})\, n = \lim\limits_{n\to\infty} \ln(1+\frac{1}{n})^n =$

$= \ln e^{\lim\limits_{n\to\infty}\frac{n}{n} = 1} = 1 \Rightarrow$ same nature $\Rightarrow (x_n)$ conv.

c) $\sum\limits_{n\geq 2} \frac{1}{n(\ln n)^p}$ $\quad x_n = \frac{1}{n(\ln n)^p}$

$\sum\limits_{n\geq 2} x_n \quad \sum\limits_{n\geq 2} 2^n x_{2^n}$ have the same nature

$\sum\limits_{n\geq 2} 2^n x_{2^n} = \sum\limits_{n\geq 2} \cancel{2^n} \frac{1}{\cancel{2^n}(\ln 2^n)^p} = \sum\limits_{n\geq 2} \frac{1}{(n\ln 2)^p} = \sum\limits_{n\geq 2} \frac{1}{n^p} \cdot \frac{1}{\ln^p 2} \overset{C.C.T}{\Rightarrow} \sum\limits_{n\geq 2} \frac{1}{n \cdot \ln n^p}$ converges

$p > 1 \Leftrightarrow$ converges

2) $\sum\limits_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$

check for absolute convergence:

$\sum\limits_{n\geq 1} |x_n|$ - conv. $\Rightarrow \sum\limits_{n\geq 1} x_n$ - conv.

$\sum\limits_{n\geq 1} \left|\frac{(-1)^{n+1}}{\sqrt{n(n+1)}}\right| = \sum\limits_{n\geq 1} \frac{1}{\sqrt{n(n+1)}} = \sum\limits_{n\geq 1} \frac{1}{\sqrt{n^2+n}}$

$y_n = \frac{1}{n}$

$\lim\limits_{n\to\infty} \frac{|x_n|}{y_n} = \lim\limits_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\sqrt{1+\frac{1}{n}}} = 1 \overset{C.T.2.}{\Rightarrow} |x_n|$ and y_n have the same nature $\Big|$ $\Rightarrow |x_n|$ abs. divergent

(y_n) is divergent

$\sum\limits_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$

$\left|\frac{(-1)^{n+1}}{\sqrt{n(n+1)}}\right|$ - decreasing?

$\lim\limits_{n\to\infty} \left|\frac{(-1)^{n+1}}{\sqrt{n(n+1)}}\right| = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n^2+n}} = 0$

$\frac{|x_{n+1}|}{|x_n|} = \frac{\sqrt{n(n+1)}}{\sqrt{(n+1)(n+2)}} \approx \frac{\sqrt{n}}{\sqrt{n+2}} < 1 \Rightarrow x_n$ decreasing

Leibniz $\Rightarrow \sum \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$

b) $\sum\limits_{n\geq 1} (-1)^n \sin\frac{1}{n}$

Absolute conv.

$|x_n|$ - conv?

$\sum\limits_{n\geq 1} |x_n| = \sum\limits_{n\geq 1} \sin\frac{1}{n}$

let $y_n = \frac{1}{n}$, divergent

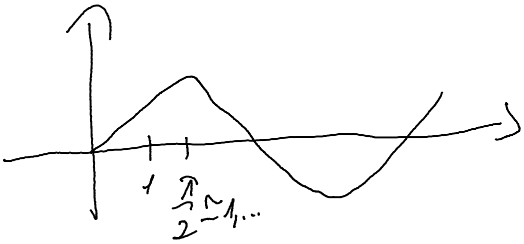
$\lim\limits_{n\to\infty} \frac{|x_n|}{y_n} = \lim\limits_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 \Rightarrow |x_n|$ and y_n have the same nature $\Big|$ $\Rightarrow |x_n|$ divergent

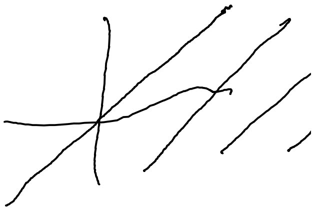
$\Rightarrow x_n$ is not abs. convergent

Convergence

$\sum_{n\geq 1} (-1)^n \cdot \sin\frac{1}{n}$ $\qquad$ $\sin a - \sin b = 2\sin\frac{a-b}{2}\cos\frac{a+b}{2}$

$0 \leq \frac{1}{n} \leq 1$ $\qquad$ $f(2) = 2$





$\sin(x)$ is increasing $\forall x \in [0,1]$
but $\frac{1}{n}$ is decreasing $\Big| \Rightarrow \sin(\frac{1}{n})$ is decreasing $\forall x \in [0,1]$

$\lim_{n\to\infty} \sin\frac{1}{n} = 0$

c) $\sum_{n\geq 1} \frac{\sin n}{n^2}$ $\quad x_n = \frac{\sin n}{n^2}$

absolute convergence

$\sum_{n\geq 1} |x_n| = \sum_{n\geq 1} \left|\frac{\sin n}{n^2}\right| = \sum_{n\geq 1} \frac{|\sin n|}{n^2}$

$\sin n \in [-1,1]$

$|\sin n| \in [0,1]$

$y_n = \frac{1}{n^2}$

$x_n \leq y_n$

$\frac{\sin n}{n^2} \leq \underbrace{\frac{1}{n^2}}_{\text{conv.}} \Rightarrow x_n$ abs. conv. $\Rightarrow x_n$ conv.

3) a)

$\sum_{n\geq 1} \frac{1\cdot 3 \cdot \ldots \cdot (2n-1)}{2\cdot 4 \cdot \ldots \cdot 2n}$

$R = \lim_{n\to\infty} n\left(\frac{\frac{\cancel{1\cdot 3\cdot\ldots\cdot(2n-1)}}{\cancel{2\cdot 4\cdot\ldots\cdot 2n}}}{\frac{\cancel{1\cdot 3\cdots(2n-1)}\cdot(2n+1)}{\cancel{2\cdot 4\cdots 2n}\cdot 2(n+1)}} - 1\right) = \lim_{n\to\infty}\left(n\frac{2n+2}{2n+1} - 1\right) = \lim_{n\to\infty} n\left(\cancel{1}+\frac{1}{2n+1} - \cancel{1}\right) = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2} < 1 \overset{RD}{\Rightarrow} \sum_{n=1}^{\infty} x_n$ divergent

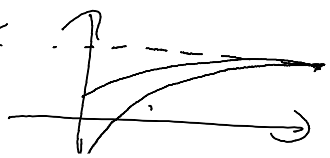
b) $\sum_{n\geq 1} \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\cdot\frac{1}{n^2}$

$R = \lim_{n\to\infty} n\left(\frac{2n+2}{2n+1}\cdot\frac{(n+1)^2}{n^2} - 1\right) = \lim_{n\to\infty}\left(\frac{(2n+2)(n^2+2n+1)}{2n^2+n} - n\right) = \lim_{n\to\infty}\left(\frac{2n^3+4n^2+2n+2n^2+4n+2}{2n^2+n} - n\right)$

$= \lim_{n\to\infty}\frac{\cancel{2n^3}+6n^2+6n+2-\cancel{2n^3}-n^2}{2n^2} = \frac{5}{2} > 1 \overset{RD}{\Rightarrow}$ conv.

4) $\sum_{n\geq 1}\frac{(-1)^{n+1}}{n} = \ln 2$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^n}{n-1} + \frac{(-1)^{n+1}}{n}$ γ -



$x_n = \frac{(-1)^{n+1}}{n}$

$$\gamma_n = \underbrace{1+\frac{1}{2}+\frac{1}{3}+\dots+n}-\ln n \xrightarrow{n\to\infty} \gamma$$

$$S_{2n} = 1-\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2n-1}-\frac{1}{2n} = \left[1+\frac{1}{2}+\dots+\frac{1}{2n}-\ln(2n)\right]$$

$$T_n(x) := f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2+\dots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^3$$

$$\lim \frac{x_{n+1}}{x_n}$$

4)

$f:\mathbb{R}\to\mathbb{R}$

$$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & x\neq 0,\ x\in(-\infty,0)\cup(0,+\infty) \\ 0, & x=0 \end{cases}$$

$$\lim_{\substack{x\to 0\\ x<0}} x^2\sin\frac{1}{x}=0$$

$$\lim_{\substack{x\to 0\\ x>0}} x^2\sin\frac{1}{x}=0$$

$f(0)=0$

$f(0-0)=f(0+0)=f(0) \Rightarrow f$ continous in $x_0=0$

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

$$\lim_{\substack{x\to 0\\ x>0}}\frac{f(x)-f(0)}{x-0}=\lim_{\substack{x\to 0\\ x>0}}\frac{x^{\cancel{2}}\sin\frac{1}{x}}{\cancel{x}}=0$$

$$\lim_{\substack{x\to 0\\ x<0}}\frac{f(x)-f(0)}{x-0}=\lim_{\substack{x\to 0\\ x<0}}\frac{x^2\sin\frac{1}{x}}{x}=0$$

$$f'(x)=2x\sin\frac{1}{x}+x^2\cos\frac{1}{x}\cdot\left(\frac{-1}{x^2}\right)$$

Cauchy sequence

$1+r+r^2+\dots+r^n$

$$S_n=\frac{b_1(q^n-1)}{q-1}=\frac{r^n-1}{r-1}$$

Comparison test 1

$x_n \le y_n$

$\sum y_n$ converges $\Rightarrow \sum x_n$ converges

$\sum x_n$ diverges $\Rightarrow \sum y_n$ diverges

Comparison test 2

$$\lim_{n\to\infty}\frac{x_n}{y_n}=l \Rightarrow$$

- $l\in(0,+\infty)$: $\sum x_n$ and $\sum y_n$ have the same nature
- $l=0$, if $\sum y_n$ converges $\Rightarrow$ $\Rightarrow \sum x_n$ converges

$l = \infty$, if $\sum y_n$ diverges $\Rightarrow$
$\Rightarrow \sum x_n$ diverges

Ratio test

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l$$

if $\begin{cases} l < 1, & \sum_{n=1}^{\infty} x_n \text{ is convergent} \\ l > 1, & \sum_{n=1}^{\infty} x_n \text{ is divergent} \\ l = 1 & \text{inconclusive} \end{cases}$

Root test

$$\lim_{n\to\infty} \sqrt[n]{x_n} = l$$

if $\begin{cases} l < 1 \Rightarrow \sum x_n \text{ convergent} \\ l > 1 \Rightarrow \sum x_n \text{ divergent} \end{cases}$

Kummer's test

$\lim_{n\to\infty} \left(c_n \frac{x_n}{x_{n+1}} - c_{n+1} \right)$: $\begin{cases} l > 0 \Rightarrow \sum x_n \text{ conv.} \\ l < 0 \Rightarrow \sum x_n \text{ div} \end{cases}$

Raabe-Duhamel test

$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1 \right)$: $\begin{cases} l > 1, & (x_n) \text{ convergent} \\ l < 1, & (x_n) \text{ divergent} \end{cases}$