Seminar 6

Monday, November 14, 2022 6:07 PM

ex.

1 a) Prove that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Taylor Series: $f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \dots$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \cdot (x-x_0)^n = f(x) \text{ around } x_0 \in \mathbb{R}$$

$(\sin x)' = \cos x$

$(\sin x)'' = -\sin x$

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$(\sin x)'''' = \sin x = f(x)$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{\sin^{(n)}(0)}{n!} x^n = \frac{\sin 0}{1} \cdot x^0 + \frac{\cos x}{1} \cdot x^1 - \frac{\sin 0}{2} \cdot x^2 - \frac{\cos x}{3} \cdot x^3 + \dots =$$

$$= 0 + 1 \cdot x - 0 - 1 \cdot \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \dots$$

$$\sin^{(n)} 0 = \begin{cases} 0, & n = 2k \quad k \in \mathbb{N} \cup \{0\} \\ 1, & n = 4k+1 \quad k \in \mathbb{N} \cup \{0\} \\ -1, & n = 4k+3 \quad k \in \mathbb{N} \cup \{0\} \end{cases}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k-1}$$

Ratio test: $l = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{x^{2(n+1)+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} =$

$$= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0, \ \forall x \in \mathbb{R}$$

$\Rightarrow$ Taylor series is abs. conv. $\Rightarrow$ it is convergent $\forall x \in \mathbb{R}$

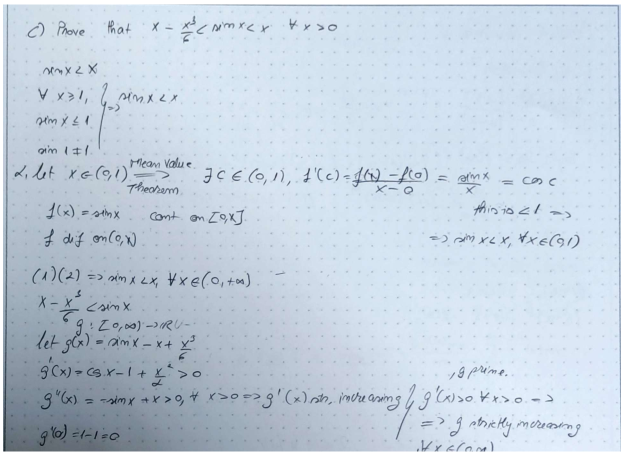
b) Prove that $(\sin x)' = \cos x$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n}$$

$$(\sin x)' = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}\right]' = \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1} + \dots\right]' =$$

$$(\sin x)' = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot (2n+1) \cdot x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n} = \cos x \Rightarrow (\sin x)' = \cos x, \ \forall x \in \mathbb{R}$$

c) Prove that $x - \frac{x^3}{6} < \sin x < x, \ \forall x > 0$



c) Prove that $x - \frac{x^3}{6} < \sin x < x \quad \forall x > 0$

$\sin x < x$

$\forall x \geq 1,$ $\sin x \leq 1$ $\sin 1 \neq 1$ $\Rightarrow \sin x < x$

2, let $x \in (0,1) \xrightarrow[\text{Theorem}]{\text{Mean Value}} \exists c \in (0,1), \ f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{\sin x}{x} = \cos c$

$f(x) = \sin x$ cont on $[0,x]$

f dif on $(0,x)$

this is $< 1 \Rightarrow$

$\Rightarrow \sin x < x, \ \forall x \in (0,1)$

$(1)(2) \Rightarrow \sin x < x, \ \forall x \in (0, +\infty)$

$x - \frac{x^3}{6} < \sin x$

$g: [0,\infty) \to \mathbb{R}$

let $g(x) = \sin x - x + \frac{x^3}{6}$

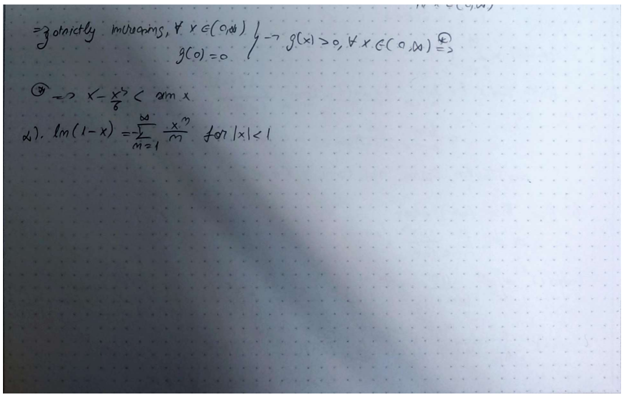
$g'(x) = \cos x - 1 + \frac{x^2}{2} > 0$, g prime.

$g''(x) = -\sin x + x > 0, \ \forall x > 0 \Rightarrow g'(x)$ str. increasing

$g'(0) = 1 - 1 = 0$

$g'(x) > 0 \ \forall x > 0 \Rightarrow$

$\Rightarrow g$ strictly increasing



$\Rightarrow g$ strictly increasing, $\forall x \in (0,\infty)$, $g(0)=0$ $\Big\} \to g(x) > 0, \forall x \in (0,\infty) \overset{*}{\Rightarrow}$

$* \Rightarrow x - \frac{x^3}{6} < \sin x$

2). $\ln(1-x) = -\sum_{m=1}^{\infty} \frac{x^m}{m}$ for $|x|<1$

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What is the convergence set?

Taylor Series: $f(x) = \sum_{m\geq 0} \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m$

$(\ln(1-x))' = \frac{-1}{1-x} = -(1-x)^{-1}$

$(\ln(1-x))'' = -(1-x)^{-2}$

$(\ln(1-x))^{(m)} = -(m-1)!(1-x)^{-m}$

$\ln(1-x) = 0 - \sum_{m\geq 1} \frac{(m-1)!(1-x_0)^m}{m!} \cdot x^m = -\sum_{m\geq 1} \frac{x^m}{m} = -\sum_{m\geq 1} \underbrace{\frac{1}{m}}_{a_m} x^m$

a_m, it is a power series

$\sum_{m=1}^{\infty} a_m (x-x_0)^m \quad R = \frac{1}{L}$

$L = \lim_{m\to\infty} \frac{a_{m+1}}{a_m}$

$L = \lim_{m\to\infty} \frac{a_{m+1}}{a_m} = \lim_{m\to\infty} \frac{1}{m+1} \cdot m = 1 > 0$

$R = \frac{1}{L} = 1$ is the radius of convergence

$\Rightarrow$ on $(-1,1)$ it is convergent

1. $x=-1$

$\ln 2 = \sum_{m\geq 1} \frac{1}{m}(-1)^m = 1 - \frac{1}{2} + \frac{1}{3} + \dots$ "True"

$\Rightarrow -1$ is from the convergent set C

2. $x = 1$

$\Rightarrow \ln 0$, doesn't exist

$\Rightarrow -\sum_{m\geq 1} \frac{1}{m}$ is divergent $\Rightarrow 1 \notin C$

$\Bigg\} \Rightarrow C = [-1,1)$

4) Prove that $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x>0 \\ 0, & x \leq 0 \end{cases}$$

infinitely differentiable at 0, but it's not expandable as a Taylor series around 0

$\lim_{x\searrow 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\searrow 0} \frac{e^{-\frac{1}{x}} - 0}{x} = \lim_{x\searrow 0} \frac{e^{-\frac{1}{x}}}{x} \overset{\frac{0}{0}}{=} \lim_{\substack{y=\frac{1}{x} \\ y\to\infty}} \frac{e^{-y}}{\frac{1}{y}} = \lim_{y\to\infty} y \cdot e^{-y} = 0 \in \mathbb{R} \Rightarrow$

$\Rightarrow f$ is diff at 0 with $f'(0) = 0$

$f'(x) = \{0,\ x \leq 0$

$$f'(x) = \begin{cases} 0, & x \le 0 \\ \frac{1}{x^2} \cdot e^{-\frac{1}{x}}, & x > 0 \end{cases}$$

$$f^{(n)}(x) = \begin{cases} 0, & x \le 0 \\ e^{-\frac{1}{x}}(\text{polynomial in } \tfrac{1}{x}), & x > 0 \end{cases} = \begin{cases} 0, & x \le 0 \\ e^{-y}(\text{polynomial in } y), & x > 0 \end{cases}$$

$$\lim_{y \to \infty} e^{-y}(\text{pol in } y) \cdot y = \lim_{y \to \infty} \frac{y}{e^y}(\text{pol in } y) = 0$$

$\Rightarrow f^{(n)}$ is dif in 0

$\Rightarrow f$ is infinitely dif. at 0

$$f^{(n)}(0) = 0$$

$$\left.\begin{array}{l} f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = 0, \ \forall x \in \mathbb{R} \\ f(1) = e^{-1} = \frac{1}{e} \neq 0 \end{array}\right| \Rightarrow \text{contradiction}$$

5) $x \in \mathbb{R}$

$|x| < 1$

$$(1+x)^{\alpha} = \underbrace{\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k}_{f(x)} \qquad \binom{\alpha}{k} = \frac{\alpha \cdot (\alpha-1) \cdot \ldots \cdot (\alpha-k+1)}{k!}$$

$$f'(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} k \cdot x^{k-1} = \sum \binom{\alpha}{k-1}(\alpha-k+1)x^{k-1} = \alpha + \alpha \cdot \sum_{k=1}^{\infty} \binom{\alpha}{k} x^k = \alpha \cdot f(x)$$

$$[f(x) \cdot (1+x)^{-\alpha}]' = 0$$

$$\Rightarrow f(x) \cdot (1+x)^{-\alpha} = c$$

$$f(0) = 1 \Rightarrow c = 1$$