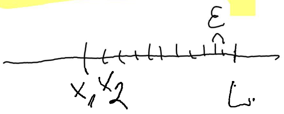
Seminar 2

17 October 2022 18:03

1) Prove using the ε-definition that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\lim_{n\to\infty} x_n = L \iff \forall\, \varepsilon > 0\ \exists\, N_\varepsilon \in \mathbb{N}: |x_n - L| < \varepsilon\ \forall\, n \geq N_\varepsilon$



$\forall\, \varepsilon > 0,\ \exists\, N_\varepsilon \in \mathbb{N}: \left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon,\ \forall\, n \geq N_\varepsilon \iff \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\left|\frac{1}{\sqrt{n}}\right| < \varepsilon \Rightarrow \frac{1}{\sqrt{n}} < \varepsilon \,|^2$

$\frac{1}{n} < \varepsilon^2$

$n \cdot \varepsilon^2 > 1$

$n > \frac{1}{\varepsilon^2} \xrightarrow[\text{principle}]{\text{Archimedian}}$ True, $\forall\, \varepsilon > 0$

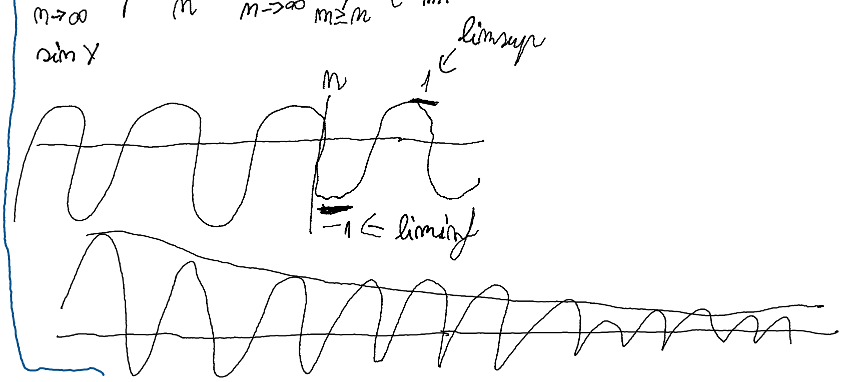
2) Find the $\liminf_{n\to\infty}$ and $\limsup_{n\to\infty}$

$\frac{(-1)^n \cdot n}{n+1}$

$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf_{m\geq n} \{x_m\}$

$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{m\geq n} \{x_m\}$

sin x



$\frac{(-1)^n \cdot n}{n+1} = \begin{cases} n = 2k,\ k\in\mathbb{N}: \frac{n}{n+1} \\ n = 2k-1,\ k\in\mathbb{N}: -\frac{n}{n+1} \end{cases}$

$\sup_{m\geq n} \{x_m\} = 1 \Rightarrow \limsup_{n\to\infty} x_n = 1$

$\inf_{m\geq n} \{x_m\} = -1 \Rightarrow \liminf_{n\to\infty} x_n = -1$

3) Study if the sequence (x_n) is bounded, monotone and convergent.

$x_n = \sqrt{n+1} - \sqrt{n}$

$x_{n+1} - x_n = \sqrt{n+2} - \sqrt{n+1} - \sqrt{n+1} + \sqrt{n} = \sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n} \overset{?}{<} 0$

$x_1 = \sqrt{2} - 1 \simeq 0,4$

$x_2 = \sqrt{3} - \sqrt{2} \simeq 0,3$

$? \Rightarrow \sqrt{n+2} + \sqrt{n} < 2\sqrt{n+1} \; /()^2$

$n+2+n+2\sqrt{n^2+2n} < 4(n+1)$

$2\sqrt{n^2+2n} < 2n+2 \; /:2$

$\sqrt{n^2+2n} < n+1 \; /()^2$

$n^2+2n < n^2+2n+1$

$0 < 1 \Rightarrow x_{n+1} - x_n < 0 \Rightarrow x_{n+1} < x_n \Rightarrow (x_n)$ is strictly monotone decreasing
$(x_n) > 0 \Rightarrow x_n$ is bounded from below
$\Rightarrow x_n$ - convergent

4) Find the limit

a) $\sqrt{n}(\sqrt{n+1} - \sqrt{n})$

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} \sqrt{n^2+n} - \lim_{n\to\infty} n = \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{n}{n\left(\sqrt{1+\underbrace{\frac{1}{n}}_{\to 0}}+1\right)}$$

b) $(a_1^n + a_2^n + \dots + a_k^n)^{\frac{1}{n}}, \; a_i > 0$

we assume that: $a_1 \le a_2 \le \dots \le a_k, \; k \in N$

$a_1^n + a_2^n + \dots + a_k^n \le k \cdot a_k^n$

$(a_1^n + a_2^n + \dots + a_k^n)^{\frac{1}{n}} \le (k \cdot a_k^n)^{\frac{1}{n}}$

$a_k = (a_k^n)^{\frac{1}{n}} \le (a_1^n + a_2^n + \dots + a_k^n)^{\frac{1}{n}} \le (k \cdot a_k^n)^{\frac{1}{n}}$

a_k

$\lim_{n\to\infty} (a_1^n + a_2^n + \dots + a_k^n)^{\frac{1}{n}} = a_k$

c) $\sqrt[n]{n}$

$n^{\frac{1}{n}} \quad e^{\frac{1}{n}\ln n}$

$\sqrt[n]{n} = 1 + \varepsilon_n \; |^n$

$n = (1+\varepsilon_n)^n = 1 + n \cdot \varepsilon_n + \binom{n}{2}\varepsilon_n^2 + \dots + \varepsilon_n^n$

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k \cdot b^{n-k}$

$C_n^k = \binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$

$\frac{\cancel{n}(n-1)}{2}\varepsilon_n^2 < \cancel{n} \Rightarrow \frac{n-1}{2}\varepsilon_n < 1 \Rightarrow \varepsilon_n^2 < \frac{2}{n-1}$

$$\lim_{n\to\infty} \varepsilon_n = 0$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

5) $e_n = \left(1+\frac{1}{n}\right)^n$

Prove that e_n is increasing and bounded above.

$\sqrt{ab} < \frac{a+b}{2}$ $\quad \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} \le \frac{x_1+x_2+\ldots+x_n}{n}$

$e_{n+1} \overset{?}{\geq} e_n$

$x_1 = 1, \quad x_2 = x_3 = \ldots = x_{n+1} = 1 + \frac{1}{n}$

$$\sqrt[n+1]{\left(1+\frac{1}{n}\right)^n} \le \frac{1+n\left(1+\frac{1}{n}\right)}{n+1}$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}} \le \frac{n+2}{n+1} \Rightarrow \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}} \le 1+\frac{1}{n+1} \Big|^{n+1} \Rightarrow$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^n \le \left(1+\frac{1}{n+1}\right)^{n+1}$$

$e_n \le e_{n+1} \Rightarrow e_n$ is increasing

$$e_n = \left(1+\frac{1}{n}\right)^n = \binom{n}{0} 1^n \cdot \frac{1}{n}^0 + \binom{n}{1} + \binom{n}{1} \cdot 1^{n-1} \cdot \left(\frac{1}{n}\right)^1 + \ldots + \binom{n}{n} \cdot 1^0 + \left(\frac{1}{n}\right)^n =$$

$$= 1 + n \cdot \frac{1}{n} + \sum_{k=3}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k$$

$$= 2 + \sum_{k=2}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k < 3 \Rightarrow \sum_{k=2}^{n} \binom{n}{k} \cdot \left(\frac{1}{n}\right)^k < 1$$

$$\binom{n}{k}\left(\frac{1}{n}\right)^k = \frac{n!}{(n-k)! \cdot k!} \cdot \frac{1}{n^k} = \frac{\overbrace{(n-k+1) \cdot \ldots \cdot n}^{k \text{ terms}}}{k} \cdot \frac{1}{n^k} = \frac{(n-k+1) \cdot \ldots \cdot n}{n^k} \cdot \frac{1}{k!} = \underbrace{\frac{n-k+1}{n}}_{<1} \cdot \underbrace{\frac{n-k+2}{n}}_{<1} \cdot \ldots \cdot \underbrace{\frac{n-1}{n}}_{<1} \cdot \underbrace{\frac{n}{n}}_{=1} \cdot \frac{1}{k!} < \frac{1}{k!}$$

$$\frac{1}{k \cdot (k-1) \cdot \ldots \cdot 1} \le \frac{1}{k(k-1)} = \frac{k-(k-1)}{k(k-1)} = \frac{k}{k(k-1)} - \frac{k-1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$$

$$\sum_{k=2}^{n} \binom{n}{k} \cdot \left(\frac{1}{n}\right)^k \le \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \ldots + \cancel{\frac{1}{n-1}} - \frac{1}{n} = 1 - \frac{1}{n} < 1 \Rightarrow$$

$\Rightarrow (e_n) < 3, \quad \forall n \in \mathbb{N} \Rightarrow$ (e_n) is upper bounded, (e_n) is increasing $\Big| \Rightarrow (e_n)$ is convergent

6) $\left(\frac{2n+1}{2n-1}\right)^n =$ $\quad e^{i\pi} = -1$

$$\lim_{n\to\infty} \left(1+\frac{2}{2n-1}\right)^n = e^{\lim_{n\to\infty} \frac{2n}{2n-1}} = e$$

Homework: two exercises 7, 11