Eigen behaviour of Random matrices with Heavy tailed independent columns

Authors*

Abstract

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1 The semi-metric and Lipschitz mapping

We introduce the semi-metric d_s on $\mathcal{D}_n(\mathbb{H}) = \{D \in \mathcal{D}_n, \forall i \in [n], \Im D_i > 0\}$:

$$d_s(\Delta, \Delta') = \sup_{1 \leq i \leq n} \frac{|\Delta - \Delta'|}{\sqrt{\Im(\Delta)\Im(\Delta')}}$$

The distance d_s is not a metric because it does not satisfy the triangular inequality, see the following counter-example:

$$d_s(4i, i) = \frac{3}{2} > \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = d_s(4i, 2i) + d_s(2i, 1i)$$

Indeed, one has the counter-triangular inequality when certain conditions are met:

Lemma 1.1. *Given $x, y, z \in \mathbb{R}$, $x < y < z$ implies that:*

$$d_s^2(a + xi, a + zi) > d_s^2(a + xi, a + yi) + d_s^2(a + yi, a + zi)$$

Proof. Here we construct the function

$$g : y \to \frac{(y-x)^2}{xy} + \frac{(z-y)^2}{yz}$$

and we differentiate it twice to get:

$$g'(y) = \frac{y^2 - x^2}{xy^2} + \frac{y^2 - z^2}{y^2 z} = \frac{1}{x} - \frac{x}{y^2} + \frac{1}{z} - \frac{z}{y^2}$$

$$g"(y) = \frac{3y}{x^3} + \frac{3z}{x^3} > 0$$

This shows that g is strictly convex on $[x, z]$, and the statement follows from the fact that $g(x) = g(z) = d_s^2(a + xi, a + yi)$ and that $g(y) = d_s^2(a + xi, a + yi) + d_s^2(a + yi, a + zi)$ □

*School of Data Science, The Chinese University of Hong Kong (Shenzhen), Shenzhen, China

Lemma 1.2. *Given* $\Delta, \Delta' \in \mathcal{D}_n(\mathbb{H})$: *and* $\Lambda \in \mathcal{D}_n^+$

$$d_s(\Lambda\Delta, \Lambda\Delta') = d_s(\Delta, \Delta')$$

$$d_s(-\Delta^{-1}, -\Delta'^{-1}) = d_s(\Delta, \Delta')$$

Lemma 1.3. *Given four diagonal matrices* $\Delta, \Delta', D, D' \in \mathcal{D}_n(\mathbb{H})$:

$$d_s(\Delta + D, \Delta' + D') \leq \max(d_s(\Delta, \Delta'), d_s(D, D'))$$

Proof. For any $\Delta, \Delta', D, D' \in \mathcal{D}_n(\mathbb{H})$:, there exist $i_0 \in [n]$ such that:

$$\begin{aligned} d_s(\Delta + D, \Delta' + D') &= \frac{|\lambda_{i_0} - \Lambda'_{i_0} + D_{i_0} - D'_{i_0}|}{\sqrt{\Im(\Delta_{i_0} + D_{i_0})\Im(\Delta'_{i_0} + D'_{i_0})}} \\ &\leq \frac{|\lambda_{i_0} - \Lambda'_{i_0}| + |D_{i_0} - D'_{i_0}|^2}{\sqrt{\Im(\Delta_{i_0})\Im(\Delta'_{i_0})} + \sqrt{\Im(D_{i_0})\Im(D'_{i_0})}} \\ &\leq \max\left(\frac{|\lambda_{i_0} - \Lambda'_{i_0}|}{\sqrt{\Im(\Delta_{i_0})\Im(\Delta'_{i_0})}}, \frac{|D_{i_0} - D'_{i_0}|}{\sqrt{\Im(D_{i_0})\Im(D'_{i_0})}}\right) \end{aligned}$$

□

In proving this property we have used the following elementary inequality results.

Lemma 1.4. *Given four positive real numbers* a, b, α, β*:*

$$\sqrt{ab} + \sqrt{\alpha\beta} \leq \sqrt{(a+\alpha)(b+\beta)}$$

$$\frac{a+\alpha}{b+\beta} \leq max(\frac{a}{b}, \frac{\alpha}{\beta})$$

Proof. For the first result, we deduce from the inequality $2\sqrt{ab\alpha\beta} \leq a\beta + b\alpha$:

$$(\sqrt{ab} + \sqrt{\alpha\beta})^2 = ab + \alpha\beta + 2\sqrt{ab\alpha\beta} \leq ab + \alpha\beta + a\beta + b\alpha$$

For the second result, we simply bound:

$$\frac{a+\alpha}{b+\beta} = \frac{a}{b}\frac{b}{b+\beta} + \frac{\alpha}{\beta}\frac{\beta}{b+\beta} \leq max\left(\frac{a}{b}, \frac{\alpha}{\beta}\right).$$

□

Definition 1.5. *Given* $\lambda > 0$*, we denote* $\mathcal{C}_s^\lambda(\mathcal{D}_n(\mathbb{H}))$*, the class of functions* $f : \mathcal{D}_n(\mathbb{H}) \to \mathcal{D}_n(\mathbb{H})$*,* λ*-Lipschitz for the semi-metric* d_s*; i.e. satisfying for all* $D, D' \in \mathcal{D}_n(\mathbb{H})$:

$$d_s(f(D), f(D')) \leq \lambda d_s(D, D').$$

When $\lambda < 1$*, we say that* f *is contracting for the semi-metric* d_s*.*

Proposition 1.6. *Given three parameters* $\alpha, \lambda, \theta > 0$ *and two mappings* $f \in \mathcal{C}_s^\lambda$ *and* $g \in \mathcal{C}_s^\theta$*,*

$$\frac{-1}{f} \in \mathcal{C}_s^\lambda, \quad \alpha f \in \mathcal{C}_s^\lambda, \quad f \circ g \in \mathcal{C}_s^{\lambda\theta}, \quad f + g \in \mathcal{C}_s^{\max(\lambda,\theta)}$$

2 Fixed point theorem for contracting mapping

The Banach fixed point theorem states that a contracting function on a complete space admits a unique fixed point. The extension of this result to contracting mappings on $\mathcal{D}_n(\mathbb{H})$, for the semi-metric d_s, is not obvious: first, because d_s does not verify the triangular inequality and second because the completeness needs to be proven. The completeness is guaranteed by a boundedness condition that we impose on the matrices.

Theorem 2.1. *Given a subset $\mathcal{D}_b$ of $\mathcal{D}_n(\mathbb{H})$ where each diagonal entry has an imaginary part bounded from above and below and a mapping $f : \mathcal{D}_b \to \mathcal{D}_b$, if it is furthermore contracting for the stable semi-metric d_s on $\mathcal{D}_b$, then there exists a unique fixed point $\Delta^* \in \mathcal{D}_b$ satisfying $\Delta^* = f(\Delta^*)$.*

Proof. Noting $\lambda \in (0,1)$ the Lipschitz constant such that $\forall \Delta, \Delta' \in \mathcal{D}_n(\mathbb{H}), d_s(f(\Delta), f(\Delta')) \leq \lambda d_s(\Delta, \Delta')$, we show that the sequence $(\Delta^{(k)})_{k\geq 0}$ satisfying:

$$\Delta^{(0)} = I_n, \quad \forall k \geq 1, \Delta^{(k)} = f(\Delta^{(k-1)})$$

is a Cauchy sequence in $\bar{\mathcal{D}}_n(\mathbb{H})$, where $\bar{\mathcal{D}}_n(\mathbb{H}) \equiv \mathcal{D}_n(\mathbb{H} \bigcup \mathbb{R})$. $\forall p \in \mathbb{N}$, $\Delta^{(p)} \in \mathcal{D}_b$, i.e. there exists $\delta > 0$, such that $|\Im \Delta^{(p)}| \leq \delta$. We can then bound for any $p \in \mathbb{N}$:

$$\|\Delta^{(p+1)} - \Delta^{(p)}\| \leq \delta d_s(\Delta^{(p+1)}, \Delta^{(p)}) \leq \lambda^p \delta d_s(\Delta^{(1)}, \Delta^{(0)}).$$

Therefore, thanks to the triangular inequality in $(\mathcal{D}_n(\mathbb{H}), \|\cdot\|)$, for any $n \in \mathbb{N}$:

$$\begin{aligned}\|\Delta^{(p+n)} - \Delta^{(p)}\| &\leq \|\Delta^{(p+n)} - \Delta^{(p+n-1)}\| + \cdots + \|\Delta^{(p+1)} - \Delta^{(p)}\| \\ &\leq \frac{\delta d_s(\Delta^{(1)}, \Delta^{(0)})}{1-\lambda} \lambda^p \to 0.\end{aligned}$$

This allows us to conclude that $(\Delta^{(p)})_{p\in\mathbb{N}}$ is a Cauchy sequence, and therefore it converges to a diagonal matrix $\Delta^* \equiv \lim_{p\to\infty} \Delta^{(p)} \in \bar{\mathcal{D}}_n(\mathbb{H})$ which is a closed thus complete set. But since $\Delta^{(p)}$ has diagonal entries which are bounded from below, we know that $\Delta^* \in \mathcal{D}_b$. By contractivity of f, it is clearly unique. □

3 Stability of the stable semi-metric towards perturbations

We have first of all the following elementary inequality result.

Lemma 3.1. *Given theree diagonal matrices $\Gamma^1, \Gamma^2, \Gamma^3 \in \mathcal{D}_n(\mathbb{H})$:*

$$\left\| \frac{\Gamma^3}{\sqrt{\Im(\Gamma^1)}} \right\| \leq \left\| \frac{\Gamma^3}{\sqrt{\Im(\Gamma^2)}} (1 + d_s(\Im(\Gamma^1), \Im(\Gamma^2))) \right\|$$

Proof. We simply bound:

$$\begin{aligned}\left\| \frac{\Gamma^3}{\sqrt{\Im(\Gamma^1)}} \right\| &\leq \left\| \frac{\Gamma^3}{\sqrt{\Im(\Gamma^2)}} \right\| + \left\| \frac{\Gamma^3 \left(\sqrt{\Im(\Gamma^2)} - \sqrt{\Im(\Gamma^1)} \right)}{\sqrt{\Im(\Gamma^2)\Im(\Gamma^1)}} \right\| \\ &\leq \left\| \frac{\Gamma^3}{\sqrt{\Im(\Gamma^2)}} \right\| + \left\| \frac{\Gamma^3}{\sqrt{\Im(\Gamma^2)}} \right\| \left\| \frac{\Im(\Gamma^2) - \Im(\Gamma^1)}{\sqrt{\Im(\Gamma^1)} \left(\sqrt{\Im(\Gamma^2)} + \sqrt{\Im(\Gamma^1)} \right)} \right\|\end{aligned}$$

□

Next we give the result to bound the distance between a diagonal matrix and the other one which is obtained as a fixed point.

Proposition 3.2. *Given a diagonal matrix $\Gamma \in \mathcal{D}_n(\mathbb{H})$, a mapping $f : \mathcal{D}_n(\mathbb{H}) \to \mathcal{D}_n(\mathbb{H})$ λ contractive for the semi-metric d_s with the Lipschitz coefficient $\lambda < 1$ and admitting the fixed point $\tilde{\Gamma} = f(\tilde{\Gamma})$, we have the bound:*

$$d_s(\Gamma, \tilde{\Gamma}) \leq \left\| \frac{f(\Gamma) - \Gamma}{\sqrt{\Im(\Gamma)\Im(\tilde{\Gamma})}} \right\| / (1 - \lambda - \lambda d_s(\Im(\Gamma)), \Im(f(\Gamma)))$$

Proof. Thanks to the above lemma, we can bound:

$$\begin{aligned} d_s(\Gamma, \tilde{\Gamma}) &\leq \left\| \frac{\tilde{\Gamma} - f(\Gamma)}{\sqrt{\Im(\Gamma)\Im(\tilde{\Gamma})}} \right\| + \left\| \frac{f(\Gamma) - \Gamma}{\sqrt{\Im(\tilde{\Gamma})\Im(\Gamma)}} \right\| \\ &\leq d_s(\tilde{\Gamma}, \Gamma)(1 + d_s(\Im(\Gamma)), \Im(f(\Gamma)) + \left\| \frac{f(\Gamma) - \Gamma}{\sqrt{\Im(\tilde{\Gamma})\Im(\Gamma)}} \right\| \\ &\leq \left\| \frac{f(\Gamma) - \Gamma}{\sqrt{\Im(\Gamma)\Im(\tilde{\Gamma})}} \right\| / (1 - \lambda - \lambda d_s(\Im(\Gamma)), \Im(f(\Gamma))) \end{aligned}$$

□

Proposition 3.3. *Let us consider a family of mappings$(f^m)_{m\in\mathbb{N}}$ of $\mathcal{D}_{n_m}(\mathbb{H})$, each f^m being λ-Lipschitz for the semi-metric d_s with $\lambda < 1$ and admitting the fixed point $\tilde{\Gamma}^m = f^m(\tilde{\Gamma}^m)$ and a family of diagonal matrices Γ^m. If one assume that $d_s(\Im(\Gamma^m), \Im(f^m(\Gamma^m))) \leq o_{m\to\infty}(1)$, then*

$$d_s(\Gamma^m, \tilde{\Gamma}^m) \leq O_{m\to\infty}\left(\left\| \frac{f^m(\Gamma^m)) - \Gamma^m}{\sqrt{\Im(\tilde{\Gamma^m})\Im(\Gamma^m)}} \right\| \right)$$

Proof. For m sufficiently big, we have $d_s(\Im(\Gamma^m), \Im(f^m(\Gamma^m))) \leq o(1) \leq \frac{1-\lambda}{2\lambda}$, so we have:

$$\begin{aligned} d_s(\Gamma^m, \tilde{\Gamma}^m) &\leq \left\| \frac{f(\Gamma^m) - \Gamma^m}{\sqrt{\Im(\Gamma^m)\Im(\tilde{\Gamma}^m)}} \right\| / (1 - \lambda - \lambda d_s(\Im(\Gamma^m)), \Im(f(\Gamma^m))) \\ &\leq \left(\left\| \frac{f^m(\Gamma^m)) - \Gamma^m}{\sqrt{\Im(\tilde{\Gamma^m})\Im(\Gamma^m)}} \right\| \right) \end{aligned}$$

□