# Tidal field and initial metric setup for numerical relativity simulations of p-g instability in neutron stars

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## I. INTRODUCTION

We are attempting to simulate the deformation of a neutron star in an inspiraling binary system due to the tidal field of its companion. The idea is to solve for the perturbed metric inside and outside the star separately.

### A. The internal equilibrium solution

We consider our star to be Newtonian and nonrotating. The equilibrium (unperturbed) structure of such a star is defined by a metric of the form

$$G_{\alpha\beta}dX^{\alpha}dX^{\beta} = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega^2 \qquad (1)$$

where X=(t,r,$\theta$,$\phi$) and $d\Omega^2 = d\theta^2 + r^2 d\phi^2$. The radial dependent mass parameter is defined as

$$e^{\lambda(r)} = \left(1 - \frac{2m(r)}{r}\right)^{-1} \qquad (2)$$

We also assume a perfect-fluid energy-momentum tensor and the functions $\nu(r)$, $m(r)$ and $p(r)$ to be spherically symmetric. Then the TOV equations become

$$\frac{dm}{dr} = 4\pi r^2 e \qquad (3)$$

$$\frac{dp}{dr} = -(e+p)\frac{m + 4\pi r^3 p}{r^2 - 2mr} \qquad (4)$$

$$\frac{d\nu}{dr} = \frac{2(m + 4\pi r^3 p)}{r^2 - 2mr} \qquad (5)$$

These equations describe the equilibrium configuration of the star. Given a barotropic EOS, these equations can be integrated from the centre outward with the following initial conditions:

$$\lim_{r\to 0} m(r) = \frac{4}{3}\pi r^3 e_c \qquad (6)$$

$$\lim_{r\to 0} p(r) = p(e_c) \qquad (7)$$

where $e_c$ is the neutron star critical energy density.

### B. Perturbations of the internal metric

The metric can be broken down as

$$G_{\alpha\beta}(X) = G^0_{\alpha\beta}(X) + H_{\alpha\beta}(X) \qquad (8)$$

The perturbation metric $H_{\alpha\beta}(X)$ can be further broken down as

$$H_{\alpha\beta}(X) = H^{(e)}_{\alpha\beta}(X) + H^{(o)}_{\alpha\beta}(X) \qquad (9)$$

where the superscripts (e) and (o) denote expansions in terms of even and odd tensor spherical harmonics respectively.

We shall take the form of the metric perturbation given by Thorne and Campolattaro[1] in the Regge-Wheeler gauge.

$$H_{\alpha\beta}= \begin{bmatrix} H_0 e^{\nu} & H_1 & 0 & 0 \\ H_1 & H_2 e^{\lambda} & 0 & 0 \\ 0 & 0 & r^2 K & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta K \end{bmatrix} P_l(\cos\theta) \qquad (10)$$

We are concerned only with the even-parity barotropic perturbations as the odd-parity perturbations will not couple with the energy density and pressure inside the star as they are scalar fields and hence have even parity. The odd-parity perturbations consist only of metric fluctuations and are characterised by a stationary, differential rotation of the fluid inside the star and by gravitational waves which do not couple to the star at all.

For the even-parity stationary barotropic perturbations, we have the following simplifications:

1. $H_0 = H_2 = H$ (This can be verified from the perturbed Einstein equations by evaluating $\delta G^{\theta}_{\theta} - \delta G^{\phi}_{\phi} = 0$)
2. $H_1 = 0$
3. The fluid perturbations are described by the logarithmic enthalpy function h defined as

$$\delta h = \delta p/(e+p) \qquad (11)$$

In the absence of entropy perturbations, this is related to the metric perturbation H as

$$\delta h = -\frac{1}{2}H \qquad (12)$$

Finally, it is possible to find a single second-order equation in H as derived by Damour and Nagar[2].

$$H'' + C_1 H' + C_0 H = 0 \tag{13}$$

Using the procedure in appendix A and the equilibrium TOV equations, we get

$$C_1 = \frac{2}{r} + \frac{1}{2}(\nu' - \lambda') = \frac{2}{r} + e^{\lambda}\left[\frac{2m}{r^2} + 4\pi r(p-e)\right] \tag{14}$$

$$C_0 = e^{\lambda}\left[-\frac{l(l+1)}{r^2} + 4\pi(e+p)\frac{de}{dp} + 4\pi(e+p)\right] + \nu'' + (\nu')^2 + \frac{1}{2r}(2 - r\nu')(3\nu' + \lambda') \tag{15}$$

$$= e^{\lambda}\left[-\frac{l(l+1)}{r^2} + 4\pi(e+p)\frac{de}{dp} + 4\pi(5e+9p)\right] - (\nu')^2 \tag{16}$$

The metric variable $K$ can be expressed as a linear combination of $H$ and $H'$. Equation 13 can be rewritten in terms of the logarithmic derivative of $H$ which is defined as

$$y = \frac{rH'}{H} \tag{17}$$

In terms of this variable, the second-order differential equation is reduced to a first-order one (the Riccati equation).

$$ry' + y(y-1) + rC_1 y + r^2 C_0 = 0 \tag{18}$$

This can be solved numerically along with the TOV equations to give the value of $y$ at different points inside the star. The initial value of $y$ is to be taken equal to $l$ (the azimuthal mode no.).

Outside the star, with the substitutions $e = p = 0$ and $m(r) = M$ and by substituting $x = r/M - 1$, equation 13 becomes

$$(x^2 - 1)H'' + 2xH' - \left(l(l+1) + \frac{4}{x^2-1}\right)H = 0 \tag{19}$$

where the prime stands for differentiation with respect to $x$. Its general solution is

$$H = a_P \hat{P}_{l2}(x) + a_Q \hat{Q}_{l2}(x) \tag{20}$$

where $\hat{P}_{l2}$ and $\hat{Q}_{l2}$ are normalized associated Legendre polynomials of the first and second kind respectively. For a particular value of $l$, the exterior logarithmic derivative is

$$y_l^{\text{ext}} = (1+x)\frac{\hat{P}'_{l2}(x) + a_l \hat{Q}'_{l2}(x)}{\hat{P}_{l2}(x) + a_l \hat{Q}_{l2}(x)} \tag{21}$$

where $a_l \equiv a_Q/a_P$. The value of $a_l$ must be determined by matching the solutions at $r = R$. This gives us

$$a_l = -\left.\frac{\hat{P}'_{l2}(x) - cy_l \hat{P}_{l2}(x)}{\hat{Q}_{l2}(x) - cy_l \hat{Q}_{l2}(x)}\right|_{x=1/c-1} \tag{22}$$

The Love no. $k_l$ which is a measure of the tidal deformability of the neutron star is given by

$$k_l = \frac{1}{2}c^{2l+1} a_l \tag{23}$$

$$= -\frac{1}{2}c^{2l+1}\left.\frac{\hat{P}'_{l2}(x) - cy_l \hat{P}_{l2}(x)}{\hat{Q}_{l2}(x) - cy_l \hat{Q}_{l2}(x)}\right|_{x=1/c-1} \tag{24}$$

The explicit expressions for $k_l$ for $2 \leq l \leq 4$ have been listed by Damour and Nagar[2].

## APPENDIX A: DERIVATION OF EQUATION (13)

In Appendix A of Lindblom, Mendell, Ipser[3], we take the derivative of equation (A1) on both sides and equate its right-hand side to that of (A2). $\delta U$ and its derivatives can be ignored under the barotropic assumption. Simplifying the expression in the adiabatic limit ($\omega \to 0$), we obtain the given expressions for $C_0$ and $C_1$.

## APPENDIX B: ALTERNATIVE EQUATION SYSTEM FOR COMPUTING H

Due to the nonlinearity of the Riccati equation, it might blow up in numerical computations. It may be simpler to use the following system given in[4].

$$\frac{dH}{dr} = \beta \tag{B1}$$

$$\frac{d\beta}{dr} = 2\left(1-2\frac{m}{r}\right)^{-1} H\left[(-2\pi\left(5e+9p+\frac{de}{dp}(e+p)\right)\right.$$
$$\left. + \frac{3}{r^2} + 2\left(1-2\frac{m}{r}\right)^{-1}\left(\frac{m}{r^2}+4\pi rp\right)^2\right]$$
$$+\frac{2\beta}{r}\left(1-2\frac{m}{r}\right)^{-1}\left(-1+\frac{m}{r}+2\pi r^2(e-p)\right) \tag{B2}$$

The system is to be integrated outwards with the following initial conditions:

$$\lim_{r\to 0} H(r) = a_0 r^2 \tag{B3}$$

$$\lim_{r\to 0} \beta(r) = 2a_0 r \tag{B4}$$

The value $a_0$ determines the extent of deformation and can be chosen arbitrarily. The Love no. is independent of this parameter.


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