CL Tutorial 4

Exercise 1

Assuming $c \vDash \neg b,\ c \vDash \neg a$

By contraposition, $a \vDash \neg c,\ b \vDash \neg c$

$$\frac{a \vDash \neg c \qquad b \vDash \neg c}{a \wedge b \vDash \neg c}$$

$$\frac{a \wedge b \vDash \neg c}{c \vDash \neg(a \wedge b)}$$

Also,

$$\frac{c \vDash \neg a \qquad c \vDash \neg b}{c \vDash \neg a \vee \neg b}$$

Both of these proofs involve only equivalences, and both of them start from the same two premises. It follows that they can be combined to give the equivalence

$$\frac{c \vDash \neg(a \wedge b)}{c \vDash \neg a \vee \neg b}$$

Now, we can use this equivalence in the following proofs:

$$\frac{\overline{\neg a \vee \neg b \vDash \neg a \vee \neg b}}{\neg a \vee \neg b \vDash \neg(a \wedge b)}$$

$$\frac{\overline{\neg(a \wedge b) \vDash \neg(a \wedge b)}}{\neg(a \wedge b) \vDash \neg a \vee \neg b}$$

Interpreting $\vDash$ as set inclusion, this means that $\neg a \vee \neg b \subseteq \neg(a \wedge b)$ and $\neg(a \wedge b) \subseteq \neg a \vee \neg b$, that is,

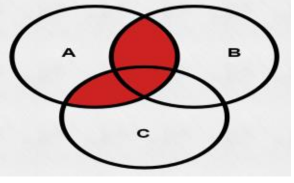
$$\neg(a \wedge b) = \neg a \vee \neg b$$

which is the second of De Morgan's laws.

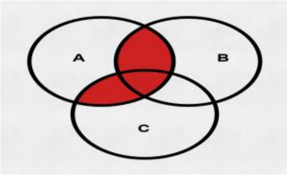
Exercise 2

Write a proof which reduces the conclusion $(x \wedge y) \vee (x \wedge z) \vDash x \wedge (y \vee z)$ to premises that can't be reduced further.

1. (the intersection of x and y) conjoined with (the intersection of x and z) looks like



where x = A, y = B, z = C.

2. The intersection of x and (the conjunction of y and z) looks like


where x = A, y = B, z = C.

A set of reduced premises would be $z \vDash x, y \vDash x$.

The set is not universally valid, as a universe with no y would not fill the intersection of x and y, so the existential assumption is required.

Exercise 3

Write a proof which reduces the conclusion $\vDash\ (x\ \land\ \neg y)\ \lor\ (\neg(x\ \lor\ z)\ \lor\ (y\ \lor\ z))$ to premises that can't be reduced further.

1. (X and not Y) or (not (X or Z) or (Y or Z))
2. If X or Z are false, truth propagates from $\neg(x\ \lor\ z)$ all the way out.
3. For the other two cases (X & Z are true, Y is true or false),
   a. if Y is false, $(x\ \land\ \neg y)$ makes the statement true.
   b. if Y is true, $(y\ \lor\ z)$ makes the statement true.
4. The statement is always true.

The statement is a tautology as it is true by necessity.

Exercise 4

The second cannot be reduced as it is invalid, $\neg(a\ \land\ b)$ is a superset of $\neg a\ \land\ \neg b$.

Neither one is universally valid, and the counterexample for the first is a universe with no examples of $\neg a$ and $\neg b$ and $\neg c$.

The counterexample for the second is an example within a but not within the intersection of a and b.