CL Tutorial 5

Exercise 1

- Reduce the sequent $(a \wedge \neg b) \vee (\neg a \wedge b) \vDash$ and use the result to derive the rule $(\oplus L)$.

$$\dfrac{\dfrac{a, \neg b \vDash}{a \wedge \neg b \vDash} \wedge L \qquad \dfrac{\neg a, b \vDash}{\neg a \wedge b \vDash} \wedge L}{(a \wedge \neg b) \vee (\neg a \wedge b) \vDash} \vee L$$

Right-hand rule:

$$\dfrac{\Gamma, a, \neg b \vDash \Delta \qquad \Gamma, \neg a, b \vDash \Delta}{\Gamma, a \oplus b \vDash \Delta} \oplus L$$

Exercise 2

- Reduce the sequents $\vDash \neg a \vee b$ and $\neg a \vee b \vDash$ and use the results to write rules $(\rightarrow R)$ and $(\rightarrow L)$, respectively.

1.

$$\dfrac{\dfrac{a \vDash b}{\vDash \neg a, b} \neg R}{\vDash \neg a \vee b} \vee R$$

Right-hand rule:

$$\dfrac{\Gamma, \mathrm{a} \vDash b, \Delta}{\Gamma \vDash a \rightarrow b, \Delta} \rightarrow R$$

2.

$$\dfrac{\dfrac{b \vDash a}{\neg a, b \vDash} \neg L}{\neg a \vee b \vDash} \vee L$$

Left-hand rule:

$$\dfrac{\Gamma, b \vDash a, \Delta}{\Gamma, a \rightarrow b \vDash \Delta} \rightarrow L$$

Exercise 3

- Reduce the sequents $\vDash (a \wedge b) \vee (\neg a \wedge \neg b)$ and $(a \wedge b) \vee (\neg a \wedge \neg b) \vDash$ and use the results to write rules $(\leftrightarrow R)$ and $(\leftrightarrow L)$, respectively.

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{a \vDash a}\, I}{\vDash a, \neg a} \neg R \qquad \dfrac{b \vDash a}{\vDash a, \neg b} \neg R}{\vDash a, (\neg a \wedge \neg b)} \wedge R \qquad \dfrac{\dfrac{a \vDash b}{\vDash b, \neg a} \neg R \qquad \dfrac{\overline{b \vDash b}\, I}{\vDash b, \neg b} \neg R}{\vDash b, (\neg a \wedge \neg b)} \wedge R}{\vDash (a \wedge b), (\neg a \wedge \neg b)} \wedge R}{\vDash (a \wedge b) \vee (\neg a \wedge \neg b)} \vee R$$

Right-hand rule:

$$\frac{\Gamma, \mathrm{a} \vDash b, \Delta \qquad \Gamma, b \vDash a, \Delta}{\Gamma \vDash a \leftrightarrow b, \Delta} \leftrightarrow R$$

2.

$$\frac{\dfrac{\dfrac{\dfrac{\overline{a \vDash a}\, I}{a, \neg a \vDash}\neg L \qquad \dfrac{a \vDash b}{a, \neg b \vDash}\neg L}{a, (\neg a \wedge \neg b) \vDash}\wedge L \qquad \dfrac{\dfrac{b \vDash a}{b, \neg a \vDash}\neg L \qquad \dfrac{\overline{b \vDash b}\, I}{b, \neg b \vDash}\neg L}{b, (\neg a \wedge \neg b) \vDash}\wedge L}{(a \wedge b), (\neg a \wedge \neg b) \vDash}\wedge L}{(a \wedge b) \vee (\neg a \wedge \neg b) \vDash}\vee L$$

Left-hand rule:

$$\frac{\Gamma, \mathrm{a} \vDash b, \Delta \qquad \Gamma, b \vDash a, \Delta}{\Gamma, a \leftrightarrow b \vDash \Delta} \leftrightarrow L$$

Exercise 4

1.

$$\frac{a \vDash b \qquad b \vDash a}{\vDash a \leftrightarrow b} \leftrightarrow R$$

$$\frac{\dfrac{a \vDash b}{\vDash (a \to b)} \to R \qquad \dfrac{b \vDash a}{\vDash (b \to a)} \to R}{\vDash (a \to b) \wedge (b \to a)} \wedge R$$

$a \leftrightarrow b = (a \to b) \wedge (b \to a)$ is universally valid as both sides reduce to $a \vDash b, b \vDash a$.

2.

$$\frac{\dfrac{\dfrac{b \vDash a, c}{a \to b \vDash c} \to L}{\vDash (a \to b) \to c} \to R$$

$$\frac{\dfrac{\dfrac{a, b \vDash c}{a \vDash b \to c} \to R}{\vDash a \to (b \to c)} \to R$$

$(a \to b) \to c \ = a \to (b \to c)$ is not universally valid.

3.

$$\frac{\dfrac{a \vDash b, c \qquad b \vDash a, c}{a \leftrightarrow b \vDash c} \leftrightarrow L \qquad \dfrac{b, c \vDash a \qquad a, c \vDash b}{c \vDash a \leftrightarrow b} \leftrightarrow R}{\vDash (a \leftrightarrow b) \leftrightarrow c} \leftrightarrow R$$

$$\frac{\dfrac{\dfrac{a. b \vDash c \qquad b, c \vDash a}{a \vDash b \leftrightarrow c} \leftrightarrow R \qquad \dfrac{b \vDash a, c \qquad c \vDash a, b}{b \leftrightarrow c \vDash a} \leftrightarrow L}{\vDash a \leftrightarrow (b \leftrightarrow c)} \leftrightarrow R$$

$(a \leftrightarrow b) \leftrightarrow c \ = a \leftrightarrow (b \leftrightarrow c)$ is not universally valid.

$$\frac{\dfrac{a, \neg b \vDash c \qquad \neg a, b \vDash c}{(a \oplus b), c \vDash} \oplus L \qquad \dfrac{a, b \vDash c \qquad \vDash a, b, c}{\vDash (a \oplus b), c} \oplus R}{\vDash (a \oplus b) \oplus c} \oplus R$$

$$\dfrac{\dfrac{c, \neg b \vDash a \quad \neg c, b \vDash a}{a, (b \oplus c) \vDash} \oplus L \qquad \dfrac{b, c \vDash a \quad \vDash a, b, c}{\vDash a, (b \oplus c)} \oplus R}{\vDash a \oplus (b \oplus c)} \oplus R$$

$(a \oplus b) \oplus c \ = a \oplus (b \oplus c)$ is not universally valid.

Exercise 5

1.$r \leftrightarrow (a \wedge b)$

$$\dfrac{\dfrac{r \vDash a \quad r \vDash b}{r \vDash (a \wedge b)} \wedge R \qquad \dfrac{a, b \vDash r}{(a \wedge b) \vDash r} \wedge L}{\vDash r \leftrightarrow (a \wedge b)} \leftrightarrow R$$

Shifting everything to the right: $\vDash a, \neg r \quad \vDash b, \neg r \quad \vDash r, \neg a, \neg b$

CNF: $(a \vee \neg r) \wedge (b \vee \neg r) \wedge (r \vee \neg a \vee \neg b)$

2.$r \leftrightarrow (a \vee b)$

$$\dfrac{\dfrac{r \vDash a, b}{r \vDash (a \vee b)} \vee R \qquad \dfrac{a \vDash r \quad b \vDash r}{(a \vee b) \vDash r} \vee L}{\vDash r \leftrightarrow (a \vee b)} \leftrightarrow R$$

Shifting everything to the right: $\vDash a, b, \neg r \qquad \vDash r, \neg a \qquad \vDash r, \neg b$

CNF: $(a \vee b \vee \neg r) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$

3.$r \leftrightarrow (a \rightarrow b)$

$$\dfrac{\dfrac{r, a \vDash b}{r \vDash (a \rightarrow b)} \rightarrow R \qquad \dfrac{b \vDash r, a}{(a \rightarrow b) \vDash r} \rightarrow L}{\vDash r \leftrightarrow (a \rightarrow b)} \leftrightarrow R$$

Shifting everything to the right: $\vDash b, \neg a, \neg r \qquad \vDash r, a, \neg b$

CNF: $(b \vee \neg a \vee \neg r) \wedge (r \vee a \vee \neg b)$

4.$r \leftrightarrow (\neg a)$

$$\dfrac{r \vDash \neg a \qquad \neg a \vDash r}{\vDash r \leftrightarrow (\neg a)} \leftrightarrow R$$

Shifting everything to the right: $\vDash \neg a, \neg r \qquad \vDash a, r$

CNF: $(\neg a \vee \neg r) \wedge (a \vee r)$