CosmoLattice
— School 2022 —

Gravitational Wave module

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Lecture is based on
Technical Note II: Gravitational Waves

Available on: https://cosmolattice.net/technicalnotes/

CosmoLattice

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CosmoLattice

A modern code for lattice simulations of scalar and gauge field dynamics in an expanding universe

What is CosmoLattice?

CosmoLattice is a modern package for **lattice simulations of field dynamics in an expanding universe**. We have developed CosmoLattice to provide a new up-to-date, relevant numerical tool for the scientific community working in the **physics of the early universe**.

CosmoLattice can simulate the dynamics of i) interacting scalar field theories, ii) Abelian U(1) gauge theories, and iii) non-Abelian SU(2) gauge theories, either in flat spacetime or an expanding FLRW background, including the case of self-consistent expansion sourced by the fields themselves. CosmoLattice is ready to simulate the dynamics of field theories described by an action and a background metric of the type:

$$S = -\int d^4x \left\{ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + (D_\mu^A\varphi)^*(D_A^\mu\varphi) + (D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathrm{Tr}\{G_{\mu\nu}G^{\mu\nu}\} + V(\phi,|\varphi|,|\Phi|) \right\}$$

DOCUMENTATION

IN A NUTSHELL

USER MANUAL

MONOGRAPHIC REVIEW

TECHNICAL NOTES

Technical notes

Here are some technical notes that expand some of the contents of the user manual, and/or explain new functionalities incorporated to the code in successive releases.

- **Technical Note I:** **Power spectra**
  Daniel G. Figueroa and Adrien Florio. Released on 06.05.2022.

- **Technical Note II:** **Gravitational Waves**
  Jorge Baeza-Ballesteros, Daniel G. Figueroa, Adrien Florio, Nicolas Loayza. Released on 06.05.2022.

Part I: Theoretical basis of Gravitational Waves

Gravitational Wave equation of motion

FLRW background + tensor perturbations

$$ds^2 = -dt^2 + a(t)^2(\delta_{ij} + h_{ij})dx_i dx_j \qquad h_{ii} = \partial_i h_{ij} = 0$$

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\{\Pi_{ij}\}^{TT}$$

Gravitational Wave equation of motion

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$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\{\Pi_{ij}\}^{TT}$$

Latin indices $\{i, j, ..\}$ spatial components

Summation over repeated indices

Gravitational Wave equation of motion

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\{\Pi_{ij}\}^{TT}$$

Gravitational Wave equation of motion

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\underbrace{\{\Pi_{ij}\}^{TT}}_{}$$

TT part of Anisotropic stress tensor

Gravitational Wave equation of motion

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\underbrace{\{\Pi_{ij}\}^{TT}}_{\text{TT part of Anisotropic stress tensor}}$$

$$\partial_i \Pi_{ij}^{TT} = \Pi_{ii}^{TT} = 0 \;\; \text{hold} \;\; \forall \mathbf{x}, \forall \mathbf{t}$$

Gravitational Wave equation of motion

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\underbrace{\{\Pi_{ij}\}^{TT}}$$

TT part of Anisotropic stress tensor

$$\partial_i \Pi_{ij}^{TT} = \Pi_{ii}^{TT} = 0 \ \text{ hold } \ \forall \mathbf{x}, \forall \mathbf{t}$$

To obtain TT part of a tensor in real space is a **non-local operation!**

TT projection in Fourier Space

$$f(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \; e^{i\mathbf{x}\cdot\mathbf{k}} \; \tilde{f}(\mathbf{k}, t)$$

Continuum Fourier Transform

TT projection in Fourier Space

$$f(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \; e^{i\mathbf{x}\cdot\mathbf{k}} \; \tilde{f}(\mathbf{k}, t)$$

Continuum Fourier Transform

$$\Pi^{TT}_{ij}(\mathbf{k}) = \Lambda_{ijkl}(\hat{\mathbf{k}})\Pi_{kl}(\mathbf{k}, t)$$

$$\Lambda_{ij,lm}(\hat{\mathbf{k}}) \equiv P_{il}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}}) - \frac{1}{2}P_{ij}(\hat{\mathbf{k}})P_{lm}(\hat{\mathbf{k}}) \quad \text{with} \quad P_{ij} = \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j \, , \, \hat{\mathbf{k}}_i = \mathbf{k}_i/k$$

TT projection in Fourier Space

$$f(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\; e^{i\mathbf{x}\cdot\mathbf{k}}\; \tilde{f}(\mathbf{k}, t)$$

Continuum Fourier Transform

$$\Pi^{TT}_{ij}(\mathbf{k}) = \Lambda_{ij,kl}(\hat{\mathbf{k}})\Pi_{kl}(\mathbf{k}, t)$$

$$\boxed{\text{check} \quad \Pi^{TT}_{ij}(\mathbf{k}) \quad \text{satisfies} \quad k_i\Pi^{TT}_{ij} = \Pi^{TT}_{ii} = 0}$$

$$\Lambda_{ij,lm}(\mathbf{k}) \equiv P_{il}(\mathbf{k})P_{jm}(\mathbf{k}) - \frac{1}{2}P_{ij}(\mathbf{k})P_{lm}(\mathbf{k}) \quad \text{with} \quad P_{ij} = \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j\,, \hat{\mathbf{k}}_i = \mathbf{k}_i/k$$

TT projection in Fourier Space

$$f(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\; e^{i\mathbf{x}\cdot\mathbf{k}}\; \tilde{f}(\mathbf{k},t)$$

Continuum Fourier Transform

$$\Pi^{TT}_{ij}(\mathbf{k}) = \Lambda_{ijkl}(\hat{\mathbf{k}})\Pi_{kl}(\mathbf{k},t)$$

$$\Lambda_{ij,lm}(\hat{\mathbf{k}}) \equiv P_{il}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}}) - \frac{1}{2}P_{ij}(\hat{\mathbf{k}})P_{lm}(\hat{\mathbf{k}}) \quad \text{with} \quad P_{ij} = \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j\,,\; \hat{\mathbf{k}}_i = \mathbf{k}_i/k$$

$$\boxed{\Pi^{TT}_{ij}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{x}\cdot\mathbf{k}}\Lambda_{ijkl}(\hat{\mathbf{k}})\int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\,\Pi_{kl}(\mathbf{x},t)}$$

TT projection in Fourier Space

$$f(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\; e^{i\mathbf{x}\cdot\mathbf{k}}\; \tilde{f}(\mathbf{k},t)$$

Continuum Fourier Transform

$$\Pi^{TT}_{ij}(\mathbf{k}) = \Lambda_{ijkl}(\hat{\mathbf{k}})\Pi_{kl}(\mathbf{k},t)$$

$$\Lambda_{ij,lm}(\hat{\mathbf{k}}) \equiv P_{il}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}}) - \frac{1}{2}P_{ij}(\hat{\mathbf{k}})P_{lm}(\hat{\mathbf{k}}) \quad \text{with} \quad P_{ij} = \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j\,,\; \hat{\mathbf{k}}_i = \mathbf{k}_i/k$$

Problem!

$$\boxed{\Pi^{TT}_{ij}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{x}\cdot\mathbf{k}}\Lambda_{ijkl}(\hat{\mathbf{k}})\int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\,\Pi_{kl}(\mathbf{x},t)}$$

Anisotropic stress tensor

$$\Pi_{\mu\nu} \equiv T_{\mu\nu} - (T_{\mu\nu})_{perfect\,fluid}$$

Spatial Component $\quad \Pi_{ij} \equiv T_{ij} - p g_{ij}$

p = homogeneous background pressure

$$g_{ij} = a^2(t)(\delta_{ij} + h_{ij})$$

Anisotropic stress tensor

$$\Pi_{\mu\nu} \equiv T_{\mu\nu} - (T_{\mu\nu})_{perfect\,fluid}$$

Spatial Component $\quad \Pi_{ij} \equiv T_{ij} - p g_{ij}$

p = homogeneous background pressure

$$g_{ij} = a^2(t)(\delta_{ij} + h_{ij})$$

Example

Scalar fields $\quad \Pi_{ij}^{TT} = \left\{ \sum_a \partial_i \phi_a \partial_j \phi_a \right\}^{TT}$

Energy density of Stochastic Gravitational Wave Background (SGWB)

$$\rho_{\mathrm{GW}}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x},t)\dot{h}_{ij}(\mathbf{x},t)\rangle_V$$

$\langle \ldots \rangle_V$ Volume average to encompass all relevant wavelengths

Energy density of Stochastic Gravitational Wave Background (SGWB)

$$\rho_{\mathrm{GW}}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x},t)\dot{h}_{ij}(\mathbf{x},t)\rangle_V$$

$$\approx \frac{1}{32\pi G V}\int_V \frac{d^3\mathbf{k}}{(2\pi)^3}\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t)$$

Fourier transform of $h_{ij}(\mathbf{x},t)$,
Approximation valid for
$kV^{1/3} \gg 1$

Energy density of Stochastic Gravitational Wave Background (SGWB)

$$\rho_{\mathrm{GW}}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x}, t)\dot{h}_{ij}(\mathbf{x}, t)\rangle_V$$

$$\approx \frac{1}{32\pi G V}\int_V \frac{d^3\mathbf{k}}{(2\pi)^3}\dot{h}_{ij}(\mathbf{k}, t)\dot{h}^*_{ij}(\mathbf{k}, t)$$

$$\rho_{\mathrm{GW}}(t) = \int \frac{d\rho_{\mathrm{GW}}(k, t)}{d\log k}\frac{dk}{k}$$

Energy density of Stochastic Gravitational Wave Background (SGWB)

$$\rho_{\mathrm{GW}}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x}, t)\dot{h}_{ij}(\mathbf{x}, t)\rangle_V$$

$$\approx \frac{1}{32\pi G V}\int_V \frac{d^3\mathbf{k}}{(2\pi)^3}\dot{h}_{ij}(\mathbf{k}, t)\dot{h}^*_{ij}(\mathbf{k}, t)$$

Energy density per logarithmic interval

$$\frac{d\rho_{\mathrm{GW}}}{d\log k} = \frac{k^3}{(4\pi)^3 G V}\int \frac{d\Omega_k}{4\pi}\dot{h}_{ij}(\mathbf{k}, t)\dot{h}^*_{ij}(\mathbf{k}, t)$$

Energy density of Stochastic Gravitational Wave Background (SGWB)

$$\rho_{\mathrm{GW}}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x}, t)\dot{h}^{*}_{ij}(\mathbf{x}, t)\rangle$$

Energy density of Stochastic Gravitational Wave Background (SGWB)

$$\rho_{\rm GW}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x}, t)\dot{h}^*_{ij}(\mathbf{x}, t)\rangle$$

For stochastic sources use ensemble average $\langle \ldots \rangle$

Energy density of Stochastic Gravitational Wave Background (SGWB)

$$\rho_{\mathrm{GW}}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x}, t)\dot{h}_{ij}^{*}(\mathbf{x}, t)\rangle$$

$$\langle \dot{h}_{ij}(\mathbf{k}, t)\dot{h}_{ij}^{*}(\mathbf{k}', t)\rangle = (2\pi)^3 P_{\dot{h}}(k, t)\delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

$$\rho_{\mathrm{GW}}(t) \equiv \frac{1}{(4\pi)^3 G}\int \frac{dk}{k} k^3 P_{\dot{h}}(k, t)$$

$P_{\dot{h}}$ power spectrum of $\dot{h}$

Energy density of Stochastic Gravitational Wave Background (SGWB)

$$\rho_{\rm GW}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(\mathbf{x},t)\dot{h}^*_{ij}(\mathbf{x},t)\rangle$$

$$\langle \dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k}',t)\rangle = (2\pi)^3 P_{\dot{h}}(k,t)\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

$$\rho_{\rm GW}(t) \equiv \frac{1}{(4\pi)^3 G}\int \frac{dk}{k} k^3 P_{\dot{h}}(k,t)$$

Energy density per logarithmic interval

$$\frac{d\rho_{\rm GW}}{d\log k} = \frac{k^3}{(4\pi)^3 G}P_{\dot{h}}(k,t)$$

Energy density of Stochastic Gravitational Wave Background (SGWB)

Energy density per logarithmic interval

$$\frac{d\rho_{\rm GW}}{d\log k} = \frac{k^3}{(4\pi)^3 GV}\int \frac{d\Omega_k}{4\pi}\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t)$$

GW energy density per critical energy density

$$\Omega_{\rm GW} = \frac{1}{\rho_{\rm c}}\frac{d\rho_{\rm GW}}{d\log k}$$

Critical energy density

$$\rho_{\rm c} \equiv 3H^2/8\pi G$$

Gravitational Waves equation of motion

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} + \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\underbrace{\{\Pi_{ij}\}^{TT}}_{\text{Transverse-Traceless Tensor}}$$

$\partial_i \Pi_{ij}^{TT}$ **Problem!** $\mathbf{x}, \forall \mathbf{t}$

To obtain TT part of a tensor in real space is a **non-local operation!**

$$\boxed{\Pi_{ij}^{TT}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{x}\cdot\mathbf{k}} \Lambda_{ijkl}(\hat{\mathbf{k}}) \int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, \Pi_{kl}(\mathbf{x}, t)}$$

Evolution of GW modes

non-local operations are computationally expensive!

Every Time-step!

$$\boxed{\Pi_{ij}^{TT}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{x}\cdot\mathbf{k}}\Lambda_{ijkl}(\hat{\mathbf{k}}) \int d^3\mathbf{x}\; e^{-i\mathbf{k}\cdot\mathbf{x}}\,\Pi_{kl}(\mathbf{x},t)}$$

Evolution of GWs modes

non-local operations are computationally expensive!

Solution: we define a set of unphysical tensor modes u's

1) Evolve equation of motion of u's

$$\ddot{u}_{ij} + 3H\dot{u}_{ij} - \frac{\nabla^2}{a^2}u_{ij} = \frac{2}{m_p^2 a^2}\Pi_{ij}$$

Garcia-Bellido, Figueroa & Sastre '07

Evolution of GWs modes

non-local operations are computationally expensive!

Solution: we define a set of unphysical tensor modes u's

1) Evolve equation of motion of u's

$$\ddot{u}_{ij} + 3H\dot{u}_{ij} - \frac{\nabla^2}{a^2}u_{ij} = \frac{2}{m_p^2 a^2}\Pi_{ij}$$

2) When needed, (compute power spectrum energy density) we apply transformation

$$h_{ij}(k,t) = \Lambda_{ij,kl}(k)u_{kl}(k,t)$$

Garcia-Bellido, Figueroa & Sastre '07

Why it works?

$$h_{ij}(k,t) = \Lambda_{ij,kl}(k) u_{kl}(k,t)$$

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2} h_{ij} = \frac{2}{m_p^2 a^2} \{\Pi_{ij}\}^{TT}$$

Linear in Fourier Space
=> commute

Why it works?

$$h_{ij}(k,t) = \Lambda_{ij,kl}(k)u_{kl}(k,t)$$

$$\left(\frac{d^2}{dt^2} + 3H\frac{d}{dt} + \frac{k^2}{a^2}\right)h_{ij}(k,t) = \frac{2}{m_p^2 a^2}\{\Pi_{ij}\}^{TT}(k,t)$$

Why it works?

$$h_{ij}(k,t) = \Lambda_{ij,kl}(k) u_{kl}(k,t)$$

$$\left(\frac{d^2}{dt^2} + 3H\frac{d}{dt} + \frac{k^2}{a^2}\right)\Lambda_{ijkl}u_{kl}\,(k,t) = \frac{2}{m_p^2 a^2}\Lambda_{ijkl}\Pi_{kl}(k,t)$$

Why it works?

$$\ddot{u}_{ij} + 3H\dot{u}_{ij} - \frac{\nabla^2}{a^2}u_{ij} = \frac{2}{m_p^2 a^2}\Pi_{ij}$$

Every Time-step!

$$h_{ij}(k,t) = \Lambda_{ij,kl}(k)u_{kl}(k,t)$$

Few Times!

Energy density of Stochastic Gravitational Wave Background (SGWB)

Energy density per logarithmic interval

$$\frac{d\rho_{\rm GW}}{d\log k} = \frac{k^3}{(4\pi)^3 G V}\int \frac{d\Omega_k}{4\pi}\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t)$$

Energy density of Stochastic Gravitational Wave Background (SGWB)

Energy density per logarithmic interval

$$\frac{d\rho_{\mathrm{GW}}}{d\log k} = \frac{k^3}{(4\pi)^3 GV}\int \frac{d\Omega_k}{4\pi}\ \boxed{\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t)}$$

$$\begin{aligned}\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t) &= (\Lambda_{ijkl}(\mathbf{k})u_{kl}(\mathbf{k},t))(\Lambda_{ijmn}(\mathbf{k})u^*_{mn}(\mathbf{k},t)) \\ &= \Lambda_{klmn}(\mathbf{k})u_{kl}(\mathbf{k},t)u^*_{mn}(\mathbf{k},t)\end{aligned}$$

Energy density of Stochastic Gravitational Wave Background (SGWB)

Energy density per logarithmic interval

$$\frac{d\rho_{\rm GW}}{d\log k} = \frac{k^3}{(4\pi)^3 GV}\int \frac{d\Omega_k}{4\pi} \boxed{\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t)}$$

$$\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t) = (\Lambda_{ijkl}(\mathbf{k})u_{kl}(\mathbf{k},t))(\Lambda_{ijmn}(\mathbf{k})u^*_{mn}(\mathbf{k},t))$$

$$= \Lambda_{klmn}(\mathbf{k})u_{kl}(\mathbf{k},t)u^*_{mn}(\mathbf{k},t)$$

$$\boxed{\Lambda_{klmn}(\mathbf{k}) = \Lambda_{klpq}(\mathbf{k})\Lambda_{pqmn}(\mathbf{k})}$$

Energy density of Stochastic Gravitational Wave Background (SGWB)

$$\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t) = \mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}\,\mathtt{P}\,\dot{\mathtt{u}}^*) - \frac{1}{2}\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}})\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}^*)$$

$$(\dot{\mathtt{u}})_{ij} = \dot{u}_{ij} \qquad (\mathtt{P})_{ij} = P_{ij}$$

$$\Lambda_{ij,lm}(\hat{\mathbf{k}}) \equiv P_{il}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}}) - \frac{1}{2}P_{ij}(\hat{\mathbf{k}})P_{lm}(\hat{\mathbf{k}}) \quad \text{with} \quad P_{ij} = \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j\,,\ \hat{\mathbf{k}}_i = \mathbf{k}_i/k$$

Energy density of Stochastic Gravitational Wave Background (SGWB)

$$\boxed{\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t) = \mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}\,\mathtt{P}\,\dot{\mathtt{u}}^*) - \frac{1}{2}\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}})\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}^*)}$$

$$v_{ij} \equiv P_{ik}\dot{u}_{kj} \qquad \tilde{v}_{ij} \equiv P_{ik}\dot{u}^*_{kj}$$

$$\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}\,\mathtt{P}\,\dot{\mathtt{u}}^*) = v_{11}\tilde{v}_{11} + v_{22}\tilde{v}_{22} + v_{33}\tilde{v}_{33} + v_{12}\tilde{v}_{21} + v_{21}\tilde{v}_{12} + v_{13}\tilde{v}_{31} + v_{31}\tilde{v}_{13} + v_{23}\tilde{v}_{32} + v_{32}\tilde{v}_{23}$$

$$\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}) = v_{11} + v_{22} + v_{33}$$

$$\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}^*) = \tilde{v}_{11} + \tilde{v}_{22} + \tilde{v}_{33}$$

Energy density of Stochastic Gravitational Wave Background (SGWB)

$$\boxed{\dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k},t) = \mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}\,\mathtt{P}\,\dot{\mathtt{u}}^*) - \frac{1}{2}\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}})\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}^*)}$$

$$v_{ij} \equiv P_{ik}\dot{u}_{kj} \qquad \tilde{v}_{ij} \equiv P_{ik}\dot{u}^*_{kj}$$

$$\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}\,\mathtt{P}\,\dot{\mathtt{u}}^*) = v_{11}\tilde{v}_{11} + v_{22}\tilde{v}_{22} + v_{33}\tilde{v}_{33} + v_{12}\tilde{v}_{21} + v_{21}\tilde{v}_{12} + v_{13}\tilde{v}_{31} + v_{31}\tilde{v}_{13} + v_{23}\tilde{v}_{32} + v_{32}\tilde{v}_{23}$$

$$\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}) = v_{11} + v_{22} + v_{33}$$

$$\mathrm{Tr}(\mathtt{P}\,\dot{\mathtt{u}}^*) = \tilde{v}_{11} + \tilde{v}_{22} + \tilde{v}_{33}$$

$\boxed{\text{Notice in Continuum } \tilde{v}_{ij} = v^*_{ij}}$

Working Examples

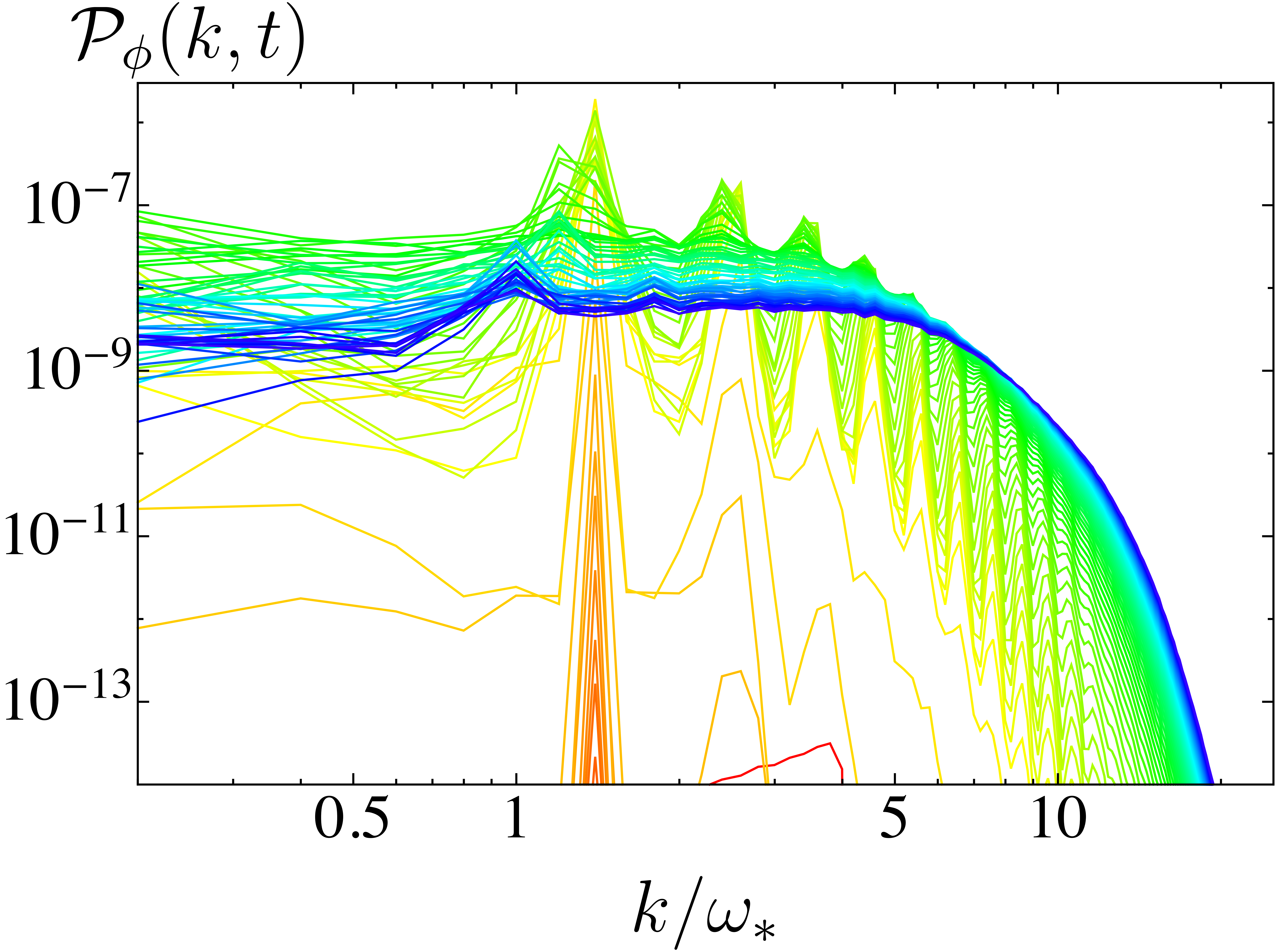
$\lambda\phi^4$ Self resonance

```
#Evolution
expansion = true
evolver = LF

#Lattice
N = 64
dt = 0.05
kIR = 0.5

#Times
tOutputFreq = 5
tOutputInfreq = 5
tMax = 1500

#Power spectrum options
PS_type = 1
PS_version = 1

#GWs
GWprojectorType = 1
withGWs=true

#IC
kCutOff = 4
initial_amplitudes = 5.6964e18  # homogeneous amplitudes in GeV
initial_momenta = -4.86735e30  # homogeneous amplitudes in GeV2

#Model Parameters
lambda = 9e-14
```

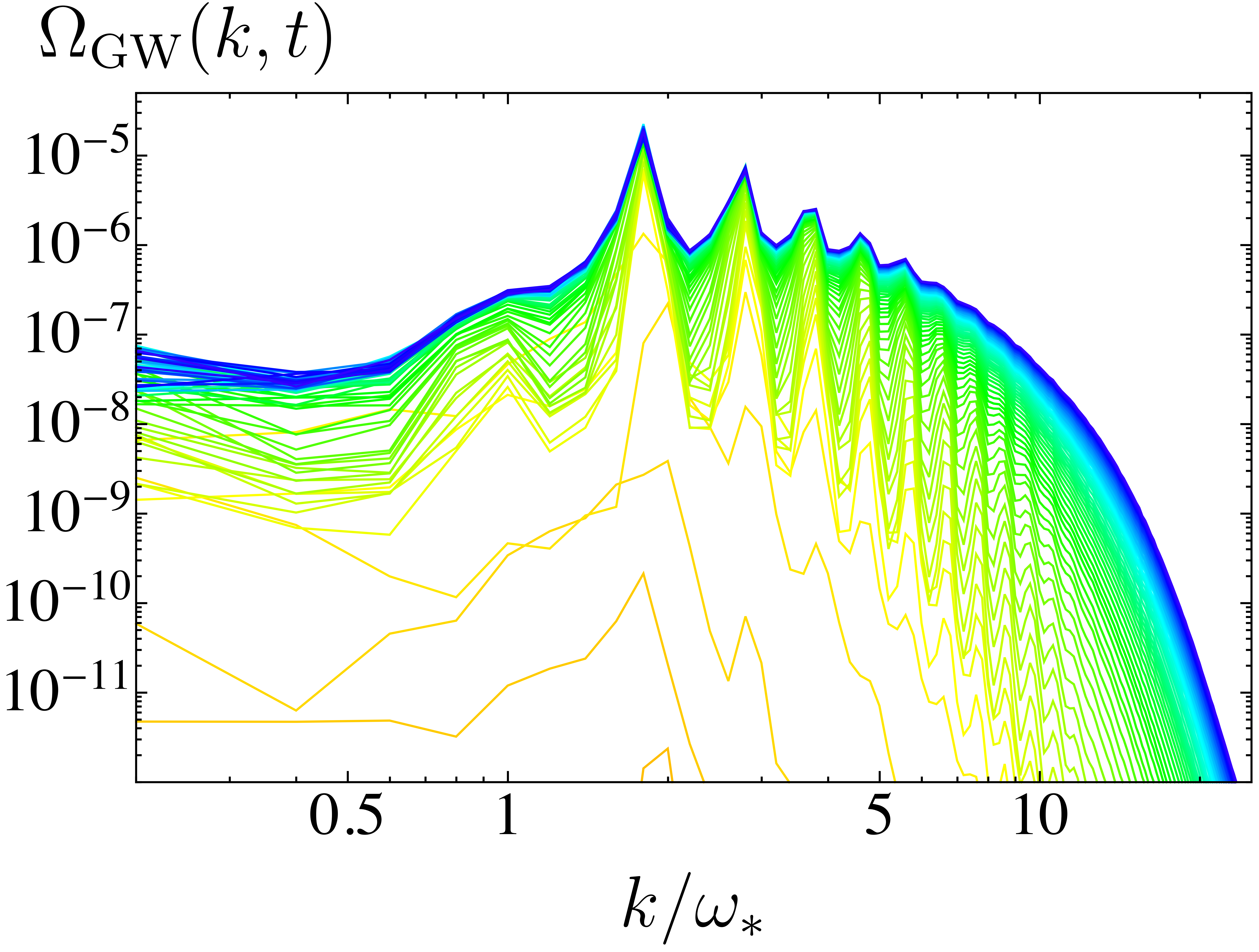
Write the model

$$V(\phi) = \frac{1}{4}\lambda\phi^4$$

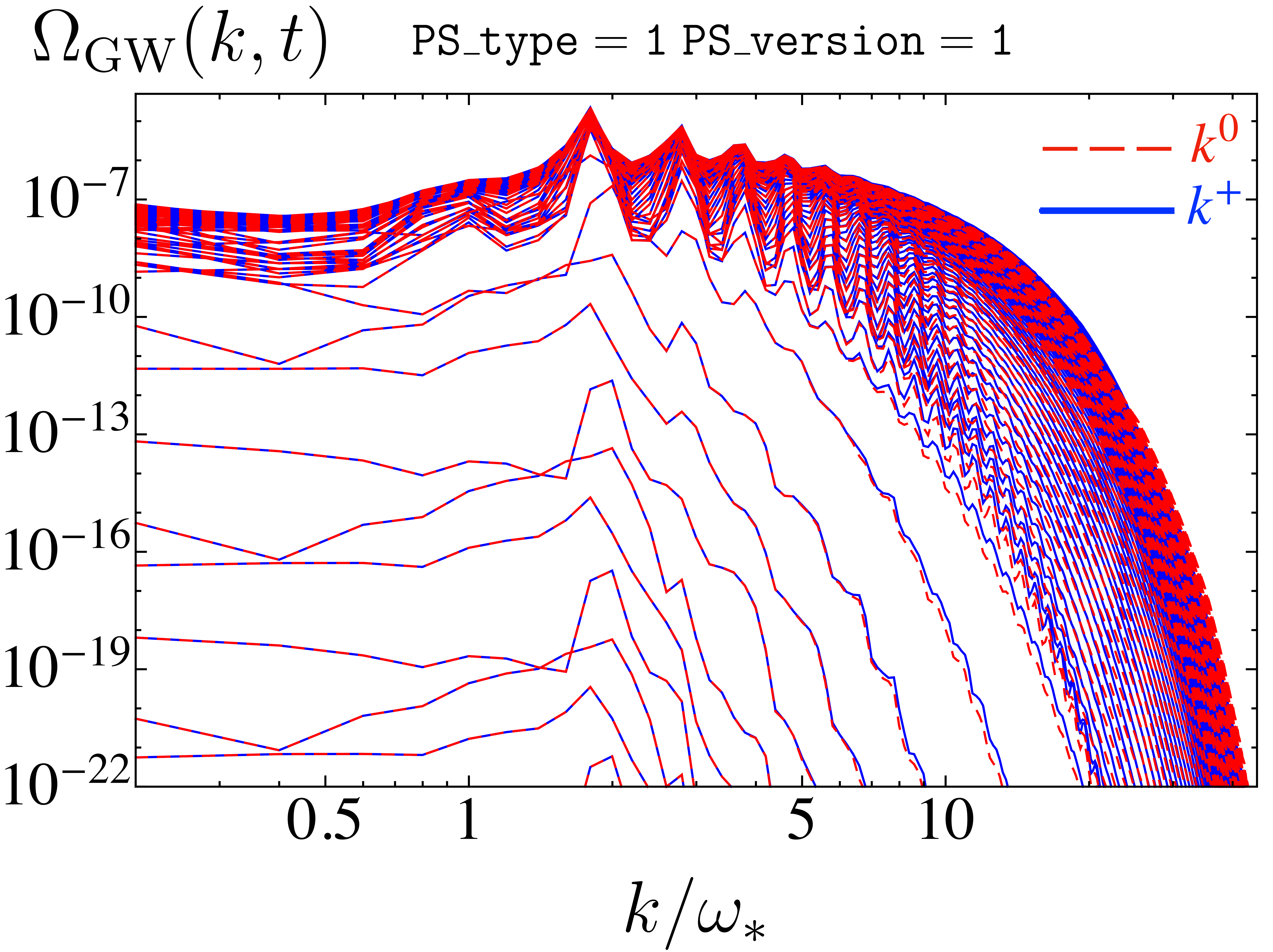
$$\ddot{\phi} + 3H\dot{\phi} + \lambda\phi^3 = 0$$



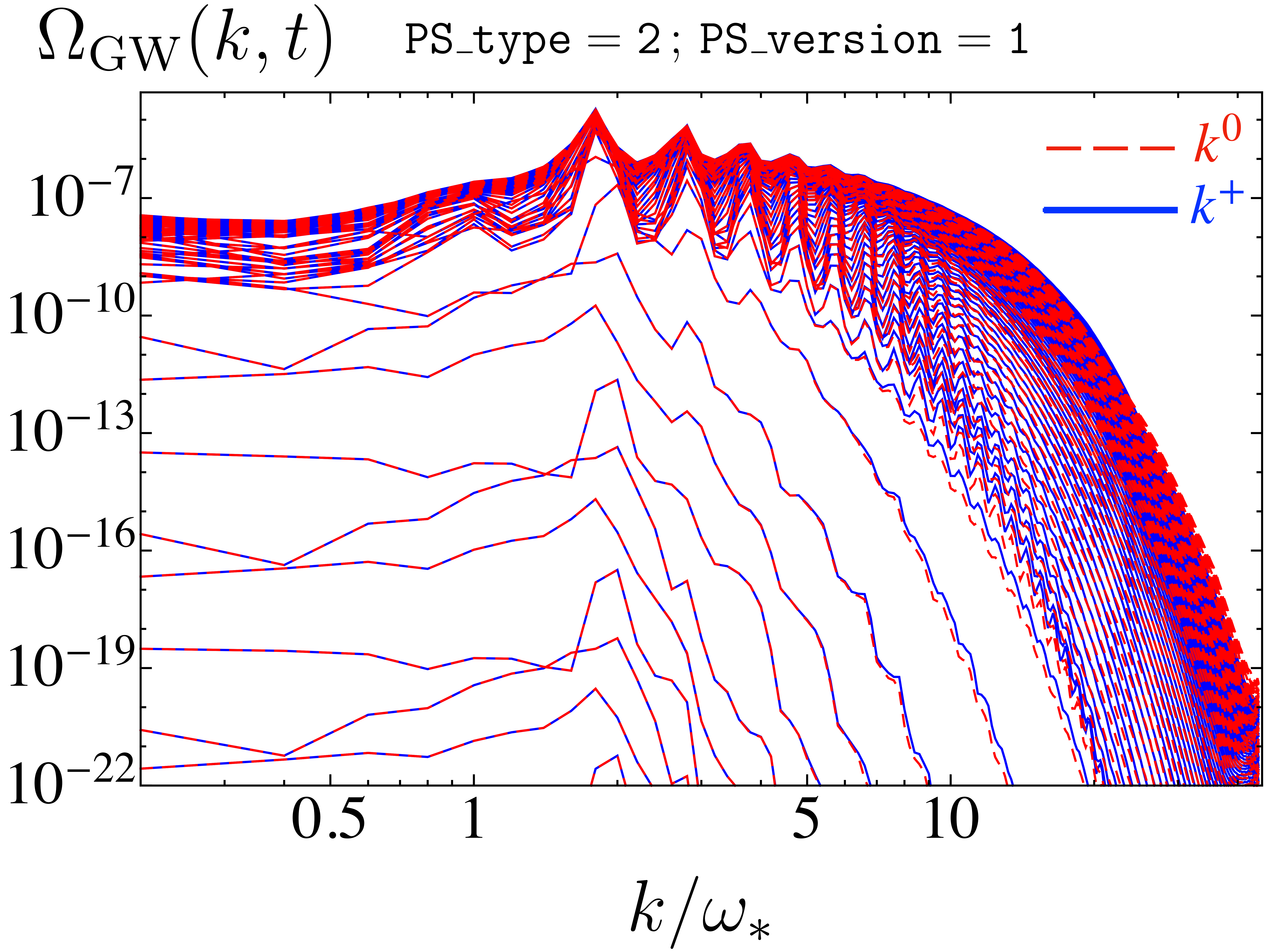
$\mathcal{P}_\phi(k,t)$
10^{-7}
10^{-9}
10^{-11}
10^{-13}
0.5
1
5
10
k/ω_*



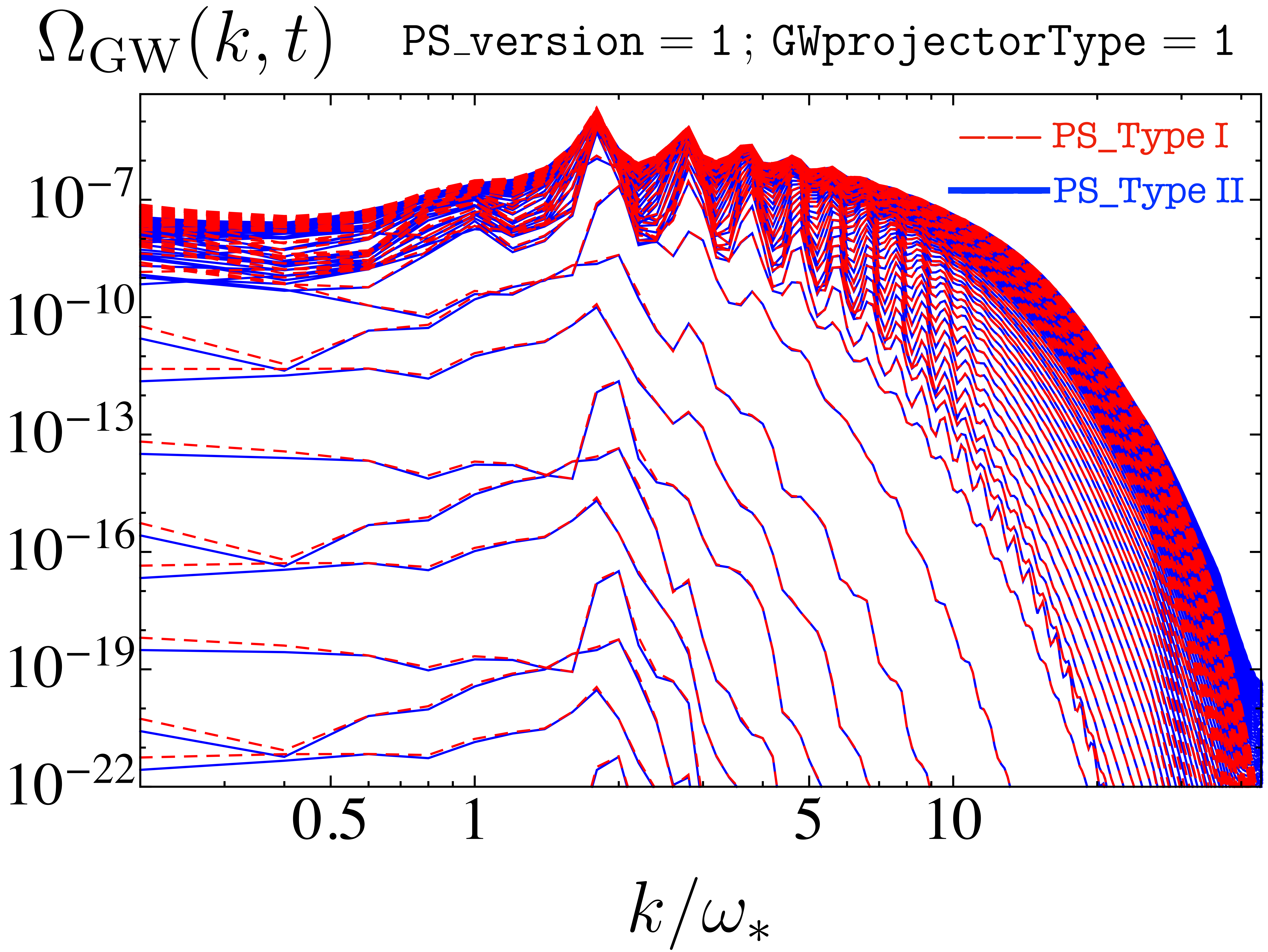
$\Omega_{\rm GW}(k,t)$
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
0.5
1
5
10
k/ω_*

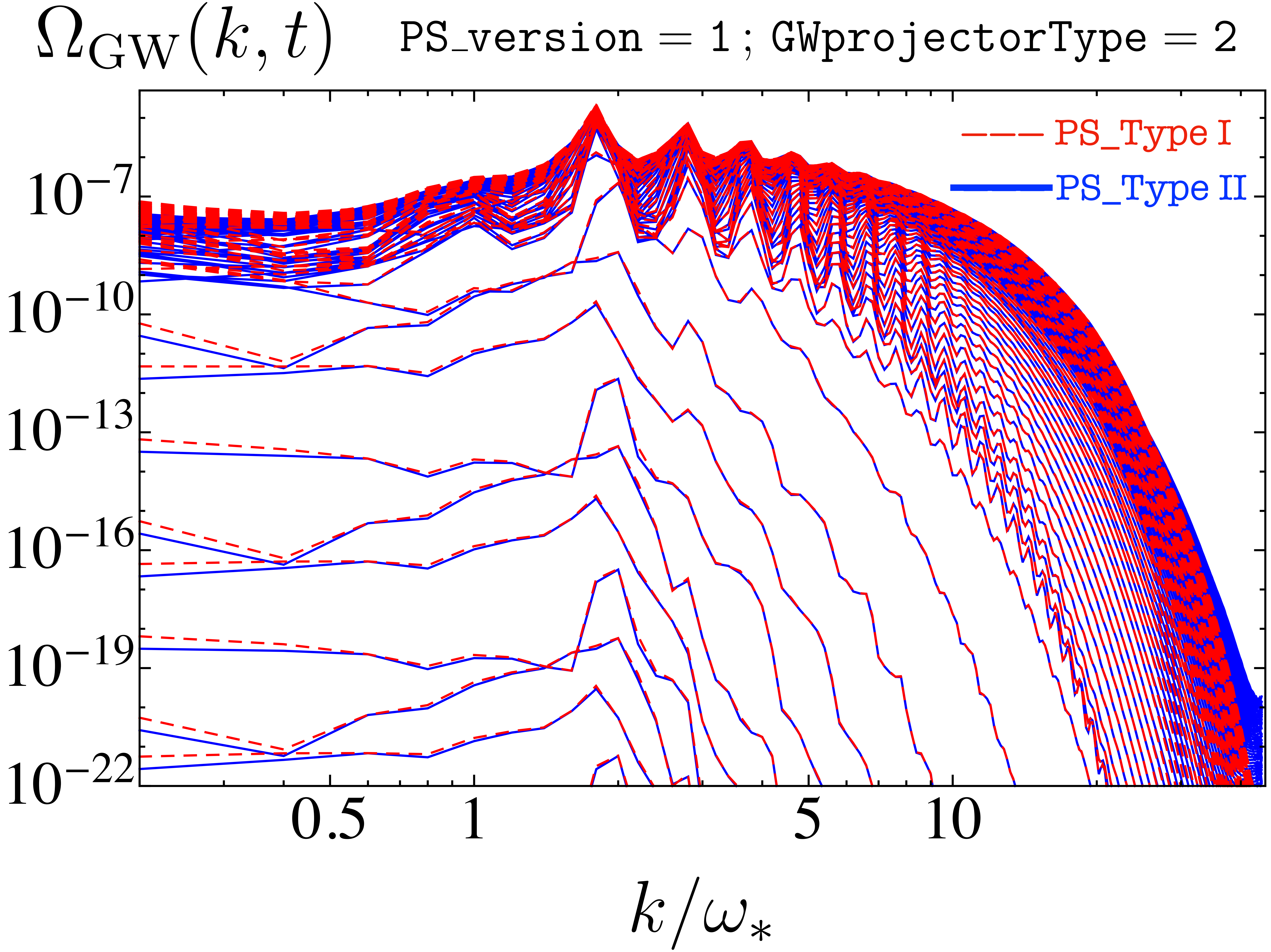


$\Omega_{\rm GW}(k,t)$
PS_type = 1 PS_version = 1
k^0
k^+
10^{-7}
10^{-10}
10^{-13}
10^{-16}
10^{-19}
10^{-22}
0.5
1
5
10
k/ω_*



$\Omega_{\rm GW}(k,t)$
PS_type = 2; PS_version = 1
k^0
k^+
10^{-7}
10^{-10}
10^{-13}
10^{-16}
10^{-19}
10^{-22}
0.5
1
5
10
k/ω_*

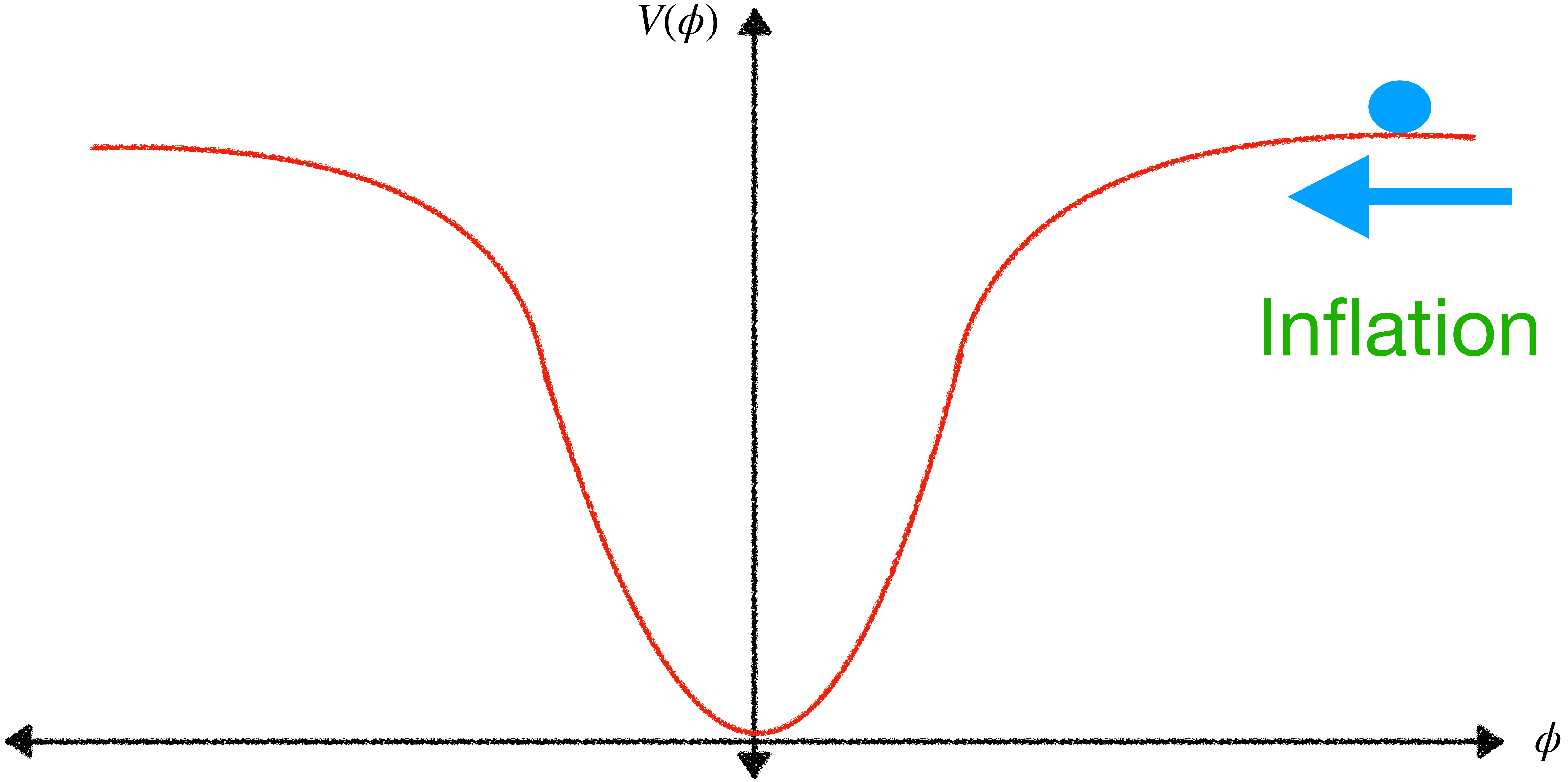

$\Omega_{\rm GW}(k,t)$ PS_version = 1 ; GWprojectorType = 1
PS_Type I
PS_Type II
10^{-7}
10^{-10}
10^{-13}
10^{-16}
10^{-19}
10^{-22}
0.5
1
5
10
k/ω_*



$\Omega_{\rm GW}(k,t)$ PS_version = 1 ; GWprojectorType = 2
PS_Type I
PS_Type II
10^{-7}
10^{-10}
10^{-13}
10^{-16}
10^{-19}
10^{-22}
0.5
1
5
10
k/ω_*

Preheating Scenario

$$V(\phi, \{\chi_j\}) = \frac{\Lambda^4}{2}\tanh^2\left(\frac{\phi}{M}\right) + \frac{1}{2}g_j^2\chi_j^2\phi^2$$

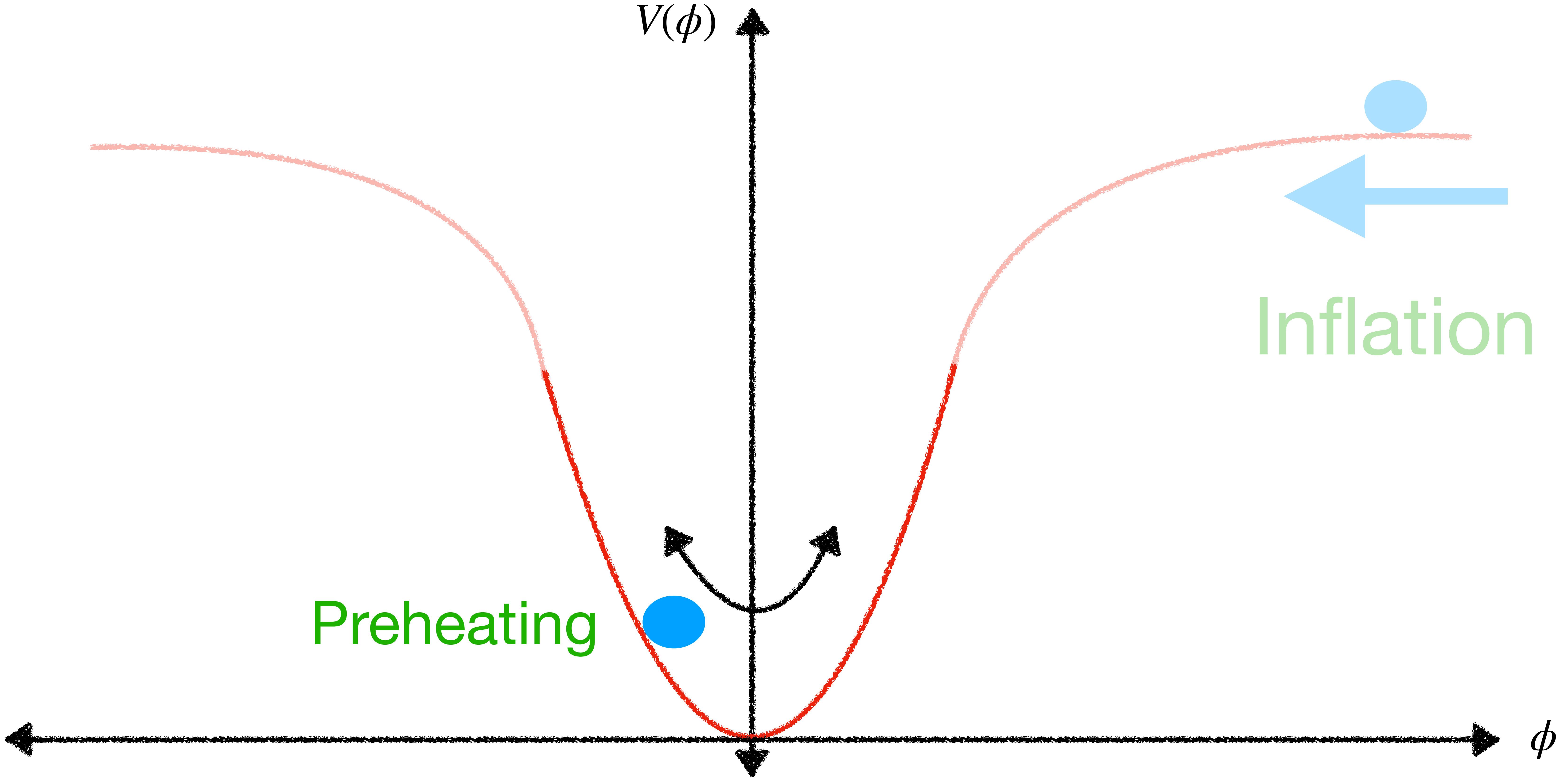


Inflationary model: *α-attractor*

Kallosh, Linde '13

Preheating Scenario

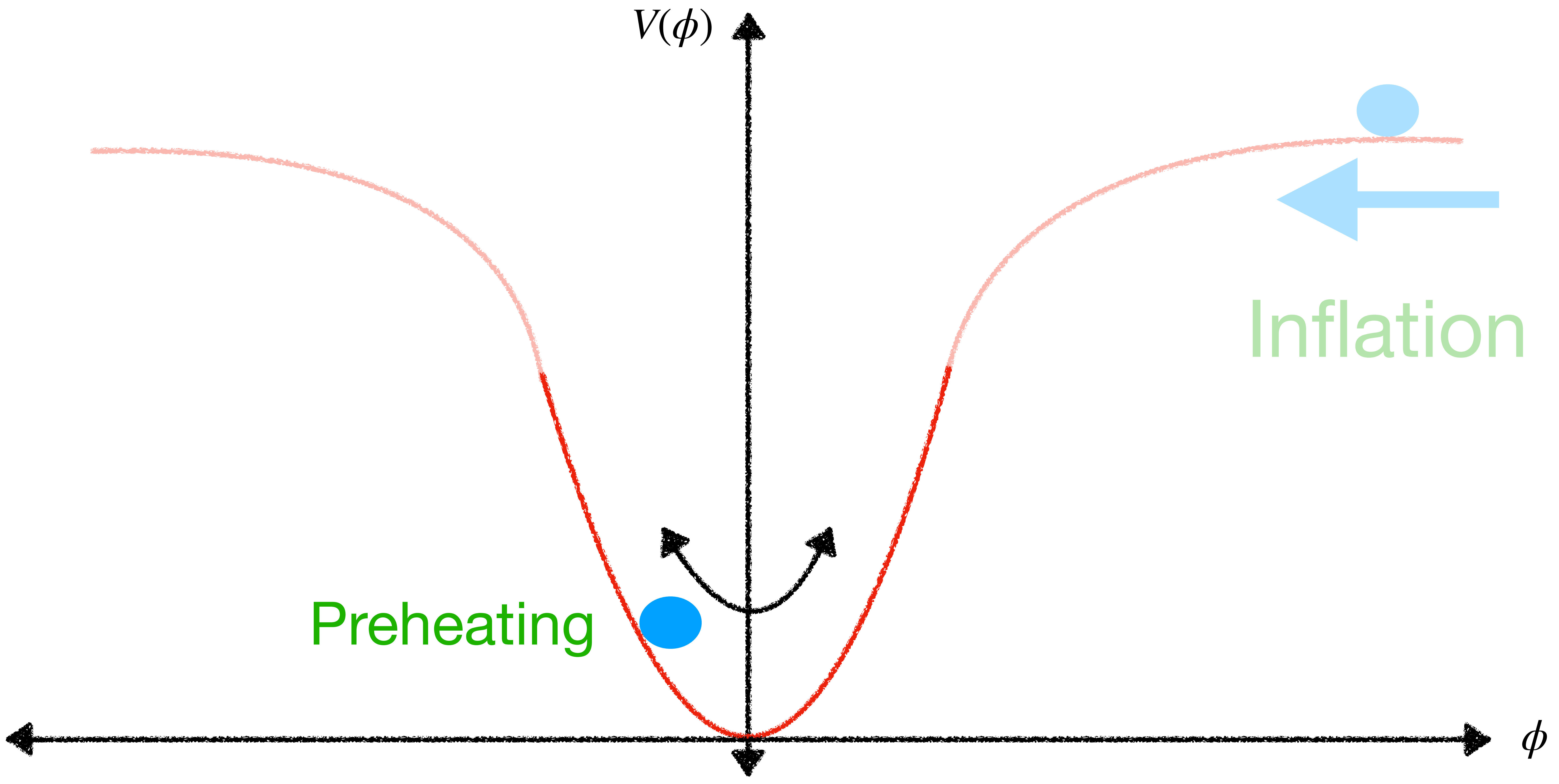
$$V(\phi, \{\chi_j\}) = \frac{\Lambda^4}{2}\tanh^2\left(\frac{\phi}{M}\right) + \frac{1}{2}g_j^2\chi_j^2\phi^2$$



Inflationary model: *α-attractor*

Preheating Scenario

$$V(\phi, \{\chi_j\}) = \frac{\Lambda^4}{2} \tanh^2\left(\frac{\phi}{M}\right) + \frac{1}{2}\boxed{g_j^2 \chi_j^2}\phi^2$$

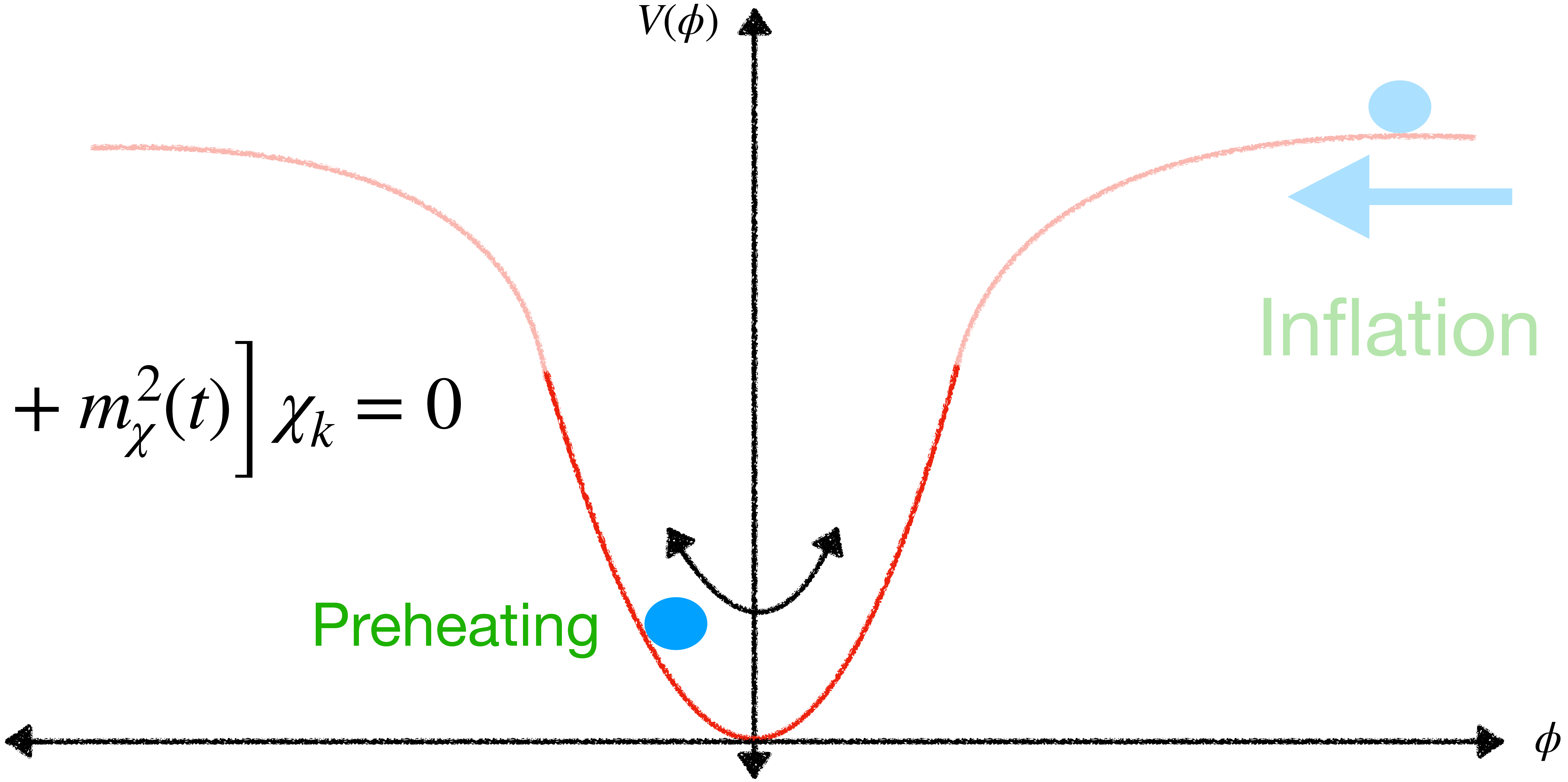


Inflationary model: *α-attractor*

Preheating Scenario

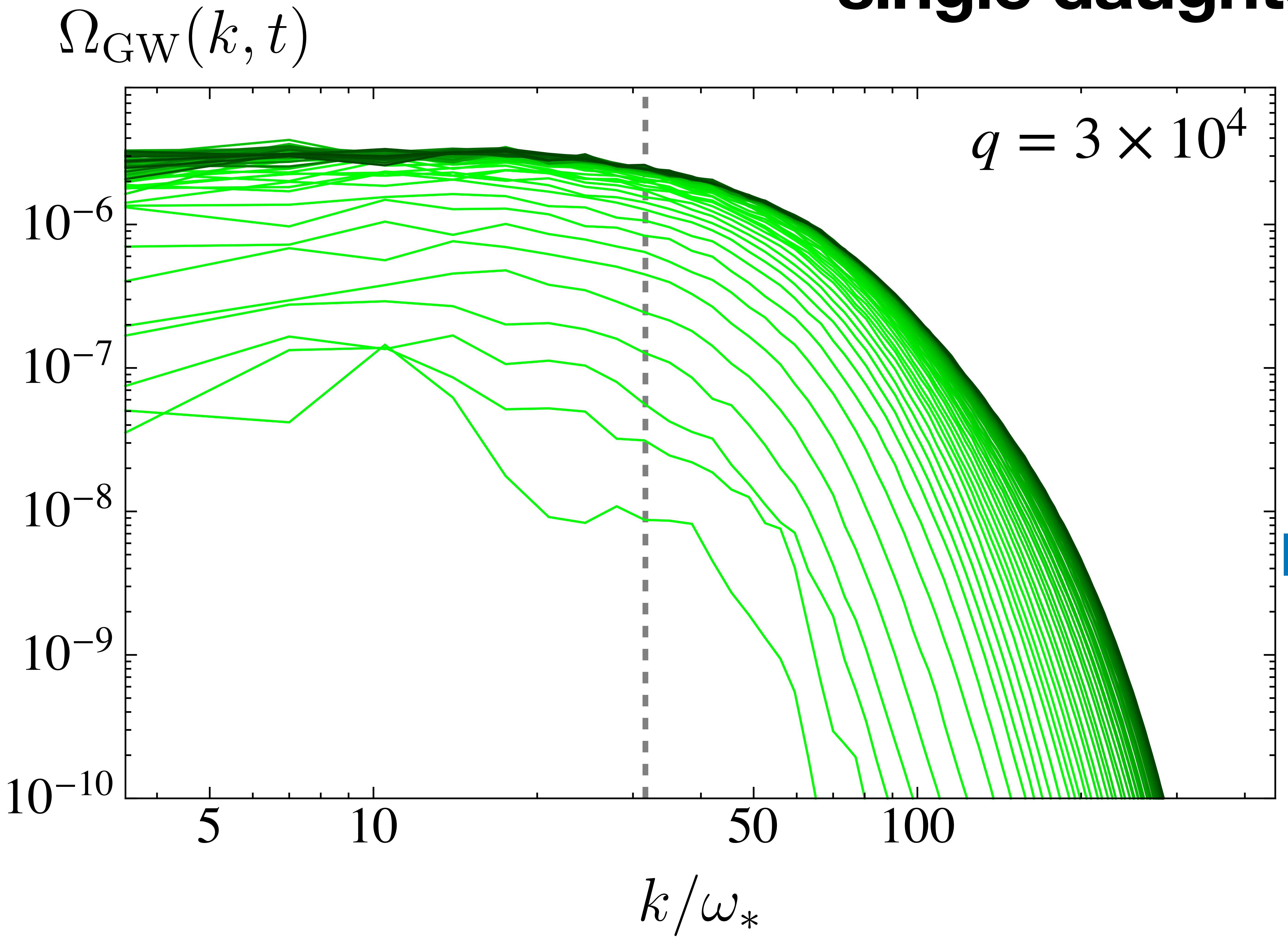
$$V(\phi, \{\chi_j\}) = \frac{\Lambda^4}{2} \tanh^2\left(\frac{\phi}{M}\right) + \frac{1}{2} g^2 \chi^2 \phi^2$$

$$\ddot{\chi}_k + \left[\kappa^2 + m_\chi^2(t)\right] \chi_k = 0$$



Inflationary model: α-*attractor*

GW spectrum parametrization of single daughter field



Peak position parameter dependence

$$\frac{k_p(q)}{\omega_*} = (21.22 \pm 7.64)\left(\frac{q}{10^4}\right)^{0.52\pm0.08}$$

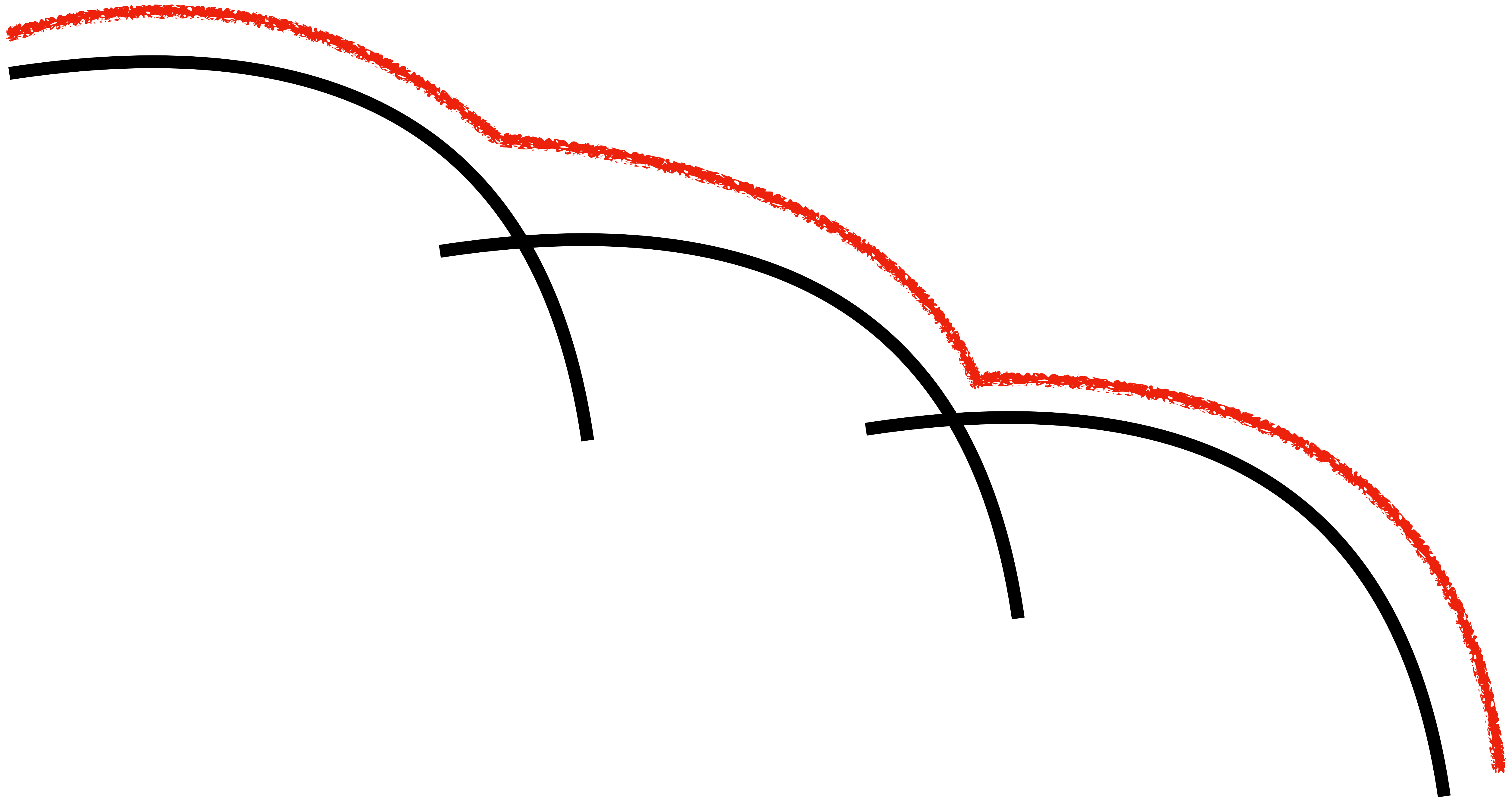
Peak amplitude parameter dependence

$$\Omega_{\rm GW}^{(p)}(q) = (1.2 \pm 0.7) \times 10^{-5}\left(\frac{q}{10^4}\right)^{-1.1\pm0.12}$$

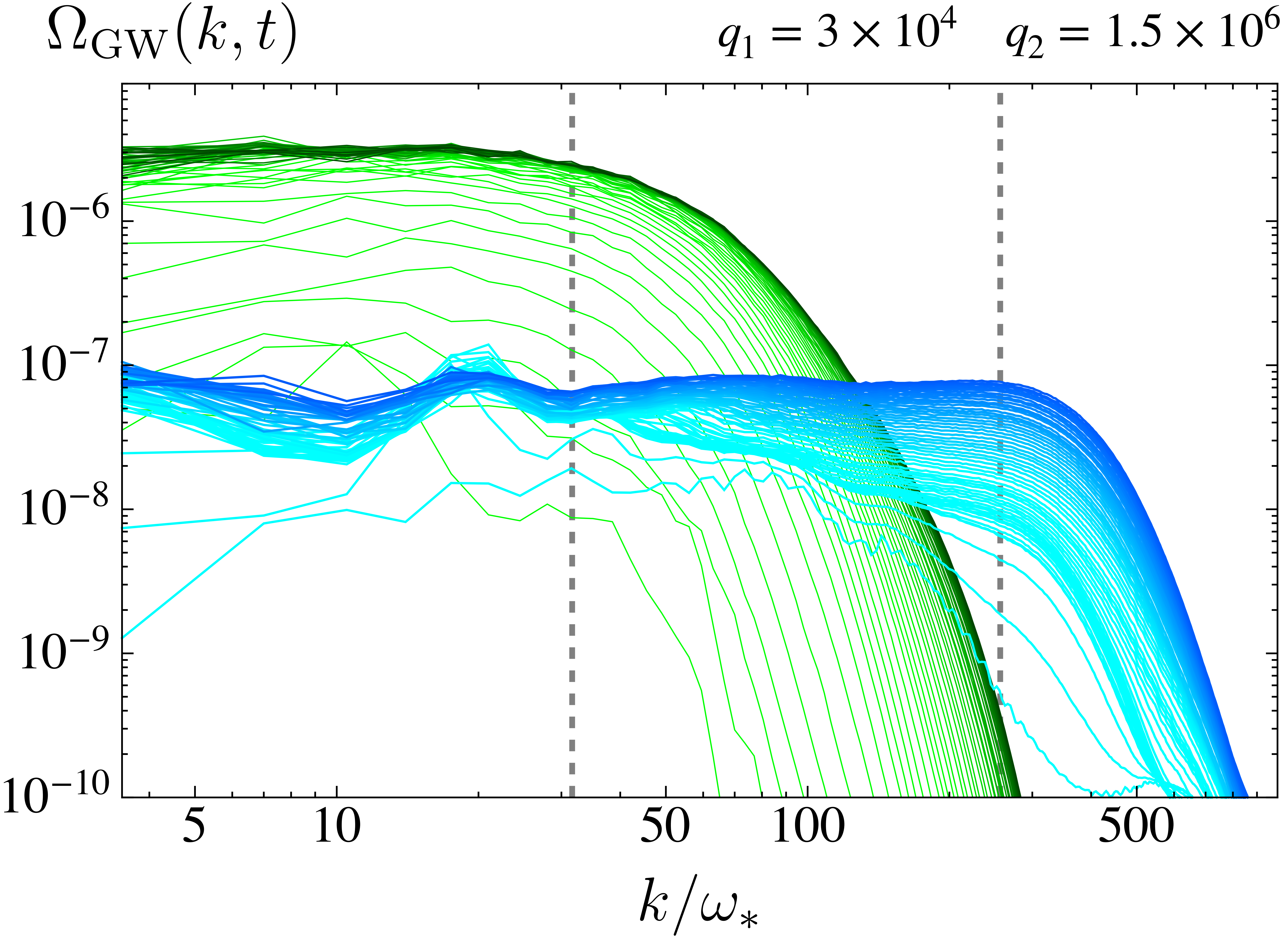
What if multiple daughter fields?

Multipeak Spectra?

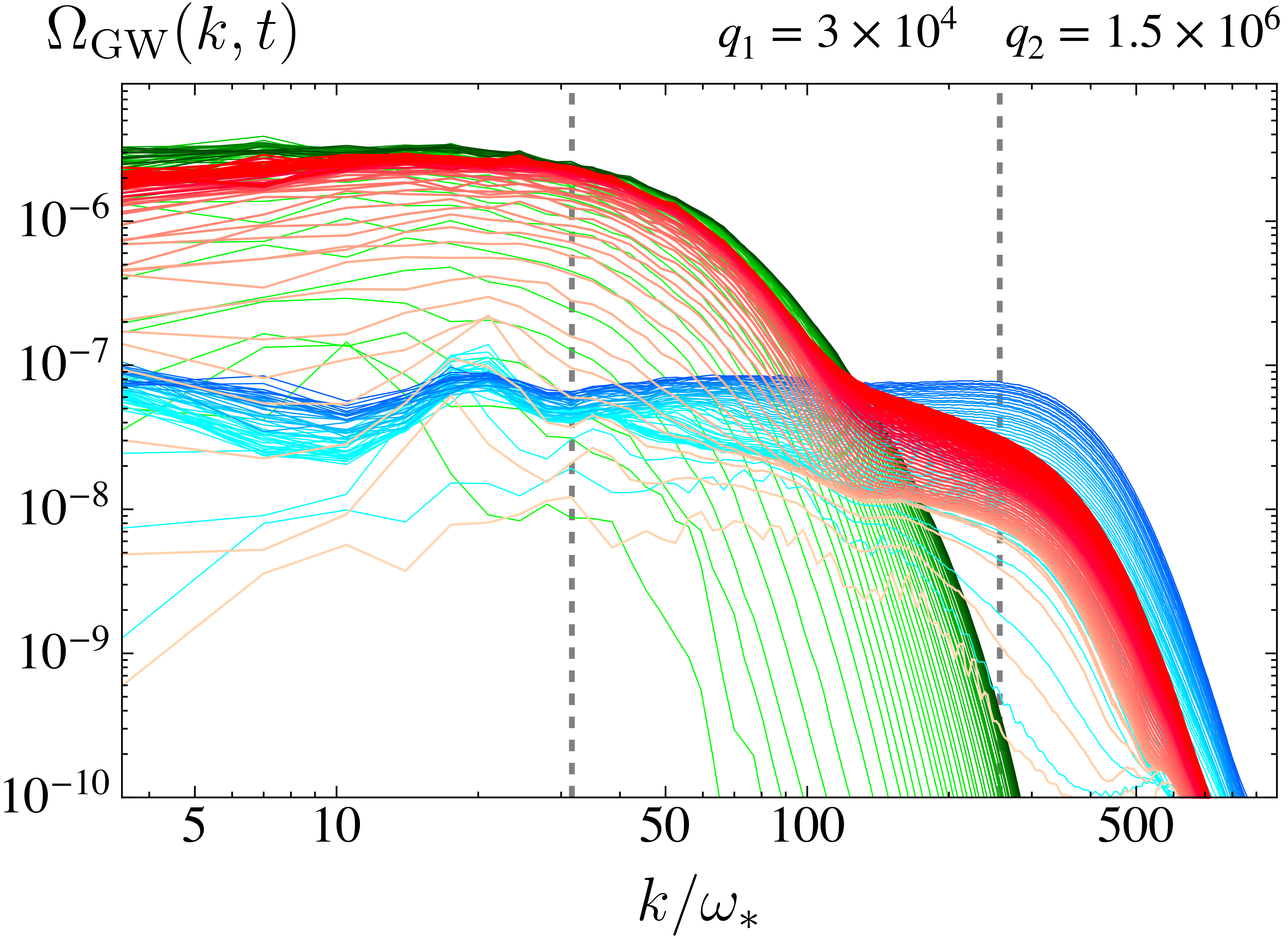
$$\ddot{h}_{ij} + 3H\dot{h}_{ij} + \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_p^2 a^2}\left\{\partial_i\phi\partial_j\phi + \sum_n (\partial_i\chi_n\partial_j\chi_n)\right\}^{TT}$$



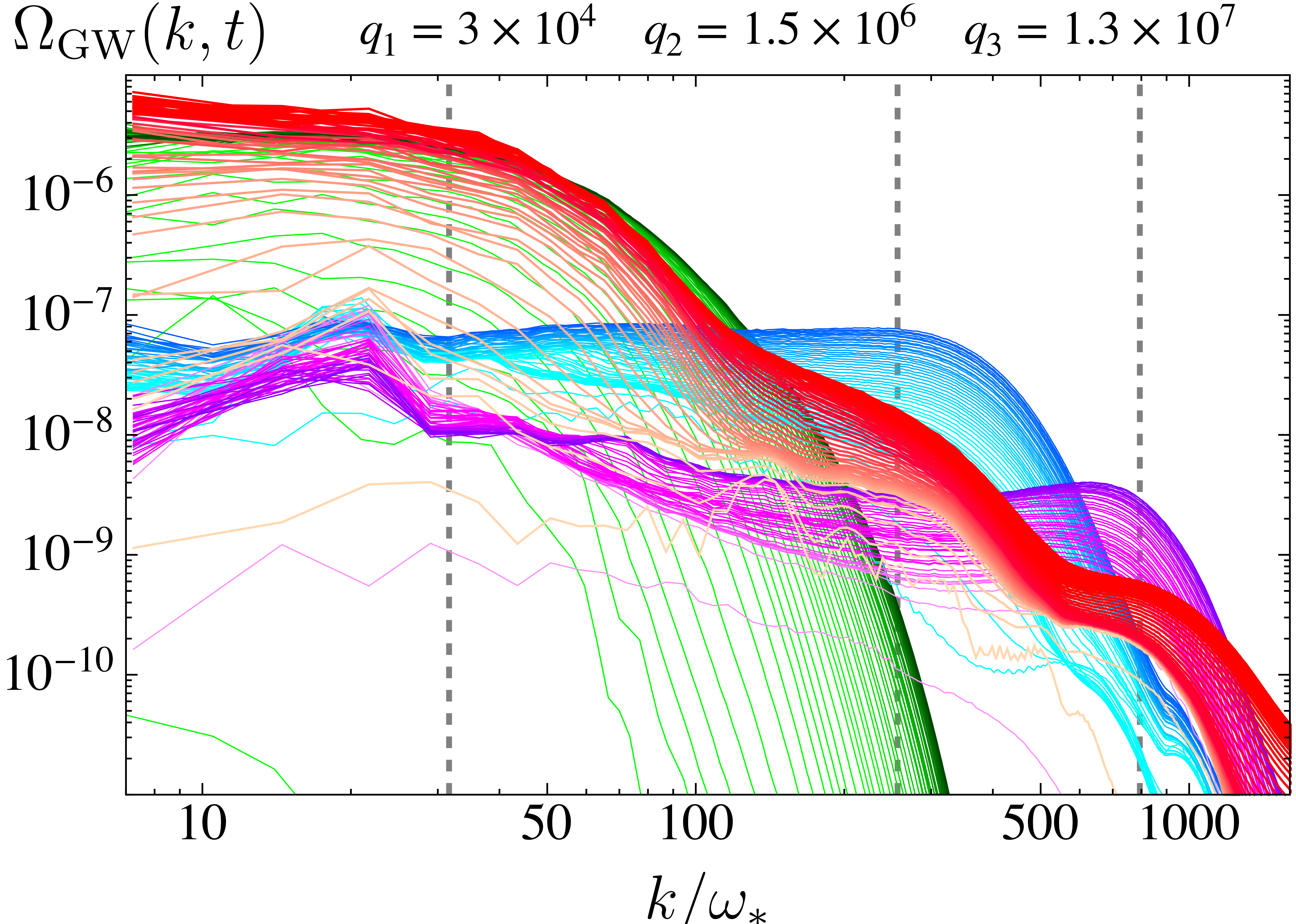
Preheating scenario with 2 daughter fields



Preheating scenario with 2 daughter fields



Preheating scenario with 3 daughter fields



Preheating scenario with 3 daughter fields