CosmoLattice

School 2022 - Valencia

Non-linear dynamics of Axion-inflation

Joanes Lizarraga & Ander Urio

Basics of the model

[K. Freese, J. A. Frieman, A. V. Olinto (PRL 65,3233 1990)]
...

$$\boxed{\text{Axion-inflation} \quad + \quad \frac{\phi}{\Lambda} F\tilde{F}}$$

Shift symmetry $\phi \to \phi + c$

Action:

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} m_{\rm pl}^2 R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\phi}{4\Lambda} F_{\mu\nu} \tilde{F}^{\mu\nu} \right)$$

$\phi \longrightarrow$ Pseudo-scalar axion field

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$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \tilde{F}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

$$A_\mu \to A_\mu + \partial_\mu \alpha(x)$$

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Simple derivative
Non covariant

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Interaction only through this vertex
Axion coupling!

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Λ Coupling constant $[m_{\rm pl}^{-1}]$

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[M. M. Anber, L. Sorbo (0908.4089)]
[J. Cook, L. Sorbo (1109.0022)]
[N. Barnaby, E. Pajer, M. Peloso (1110.3327)]

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Simple derivative
Non covariant

Generated from external mechanism breaks the shift symmetry explicitly

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Vectorial form

$$S = \int d^4x \left[\frac{1}{2} a^3 \pi_\phi^2 - \frac{1}{2} a \left(\vec{\nabla}\phi \right)^2 - \frac{1}{2} a^3 m^2 \phi^2 + \frac{1}{2} a \left(\vec{E}^2 - \frac{\vec{B}^2}{a^2} \right) + \frac{\phi}{\Lambda} \vec{E} \cdot \vec{B} \right]$$

@ FLRW:

$$ds^2 = -dt^2 + a^2 d\vec{x}^2$$

In cosmic time

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In cosmic time

Our choice of the potential, also:

$$V(\phi) = V_0\left(1 - e^{-\sqrt{\frac{2}{3}}\phi}\right)^2$$

$$V(\phi) = V_0\left(1 - \left(\frac{\phi}{\eta}\right)^4\right)^2$$

$$V(\phi) = ...$$

3 vertex

Basics of the model

Continuum equations of motion:

Temporal gauge

$$A_0 = 0$$

CL momentum variables

$$\tilde{\pi}_\phi = a^3 \pi_\phi, \quad \tilde{\vec{E}} = a\vec{E}$$

No Hubble friction term in the EoMs

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Dynamical equations:

$$\dot{\tilde{\pi}}_\phi = a\vec{\nabla}^2\phi - a^3 m^2 \phi + \frac{1}{a\Lambda}\tilde{\vec{E}}\cdot\vec{B} \ ,$$

$$\dot{\tilde{\vec{E}}} = -\frac{1}{a}\vec{\nabla}\times\vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B} + \frac{1}{a\Lambda}\vec{\nabla}\phi\times\tilde{\vec{E}}$$

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Constraint equation: Gauss's law

$$\vec{\nabla} \cdot \tilde{\vec{E}} + \frac{1}{\Lambda}\vec{\nabla}\phi \cdot \vec{B} = 0$$

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Non-linear terms

Constraint equation: Gauss's law

$$\vec{\nabla}\cdot\tilde{\vec{E}} + \frac{1}{\Lambda}\vec{\nabla}\phi\cdot\vec{B} = 0$$

Basics of the model: linear regime

Let's go now deep inside inflation

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Inflaton in slow-roll,
Homogeneous
+
dominated by the potential

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- No backreaction in inflation's dynamics
- No generation of inhomogeneities

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Gauge fields get excited by this term

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EoMs

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Constraint equation:
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Expansion equation:

$$\dot{\pi}_a = \frac{-a}{6m_{\rm pl}^2}(3p+\rho) = \frac{a}{3m_{\rm pl}^2}\langle -2K_\phi + V - K_A - G_A\rangle$$

$$\left(K_\phi \equiv \frac{1}{2}\pi_\phi^2 = \frac{1}{2a^6}\tilde{\pi}_\phi^2\,, \quad G_\phi \equiv \frac{1}{2a^2}\sum(\partial_i\phi)^2\,, \quad V \equiv \frac{1}{2}m^2\phi^2\,, \right.$$
$$\left. K_A \equiv \frac{1}{2}\sum_i \frac{E_i^2}{a^2} = \frac{1}{2}\sum_i\frac{\tilde{E}_i^2}{a^4}\,, \quad G_A \equiv \frac{1}{2}\sum_i\frac{B_i^2}{a^4} \right)$$

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EoMs

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EoMs

Gauge fields evolve in the background set by the axion field

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$$\vec{\nabla}\cdot\tilde{\vec{E}} + \frac{1}{\Lambda}\vec{\nabla}\phi\cdot\vec{B} = 0$$

Basics of the model: linear regime

An analitycal solution can be found

Changing to the helicity basis: + Conformal time

$$\vec{A}(\vec{x},\tau) = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^3} A^\lambda(k,\tau)\vec{\varepsilon}^\lambda(\hat{k}) e^{i\vec{k}\cdot\vec{x}}$$

Photon with 2 helicity states

$$k_i \varepsilon_i^\pm(\hat{k}) = 0, \quad \epsilon_{ijk} k_j \varepsilon_k^\pm(\hat{k}) = \mp i k \varepsilon_i^\pm(\hat{k})$$

$$\varepsilon_i^\pm(\hat{k})^* = \varepsilon_i^\pm(-\hat{k}), \quad \varepsilon_i^\lambda(\hat{k})\varepsilon_i^{\lambda'}(\hat{k}) = \delta_{\lambda,\lambda'}$$

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Two polarisations:
- One exponentially amplified
- The other not

Chiral instability

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Two polarisations:
- One exponentially amplified
- The other not

Chiral instability

$$|A^+| \gg |A^-|$$

Basics of the model: linear regime

Analytical solution: [M. M. Anber, L. Sorbo (0908.4089)]

$$\boxed{\xi \equiv \frac{\dot\phi}{2H\Lambda}}$$

$$\left(\frac{\partial^2}{\partial\tau^2} + k^2 \pm \frac{2k\xi}{\tau}\right) A_\pm(k,\tau) = 0$$

Instability controlled by ξ

Not accurate towards the end of inflation

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Deep inside inflation $\xi \to \text{const}$

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Not accurate towards the end of inflation

Topological term:

Appears in EoMs

$$\frac{1}{a^3}\langle \vec{E}\cdot\vec{B}\rangle \simeq 2.4\cdot 10^{-4} \frac{H^4}{|\xi|^4} e^{2\pi|\xi|}$$

EM energy:

$$\frac{1}{2a^4}\langle a^2E^2 + B^2\rangle \simeq 1.4\cdot 10^{-4} \frac{H^4}{|\xi|^3} e^{2\pi|\xi|}$$

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Amplified around the Hubble scale at each time

Topological term:

Appears in EoMs

$$\frac{1}{a^3}\langle \vec{E}\cdot\vec{B}\rangle \simeq 2.4\cdot 10^{-4}\frac{H^4}{|\xi|^4}e^{2\pi|\xi|}$$

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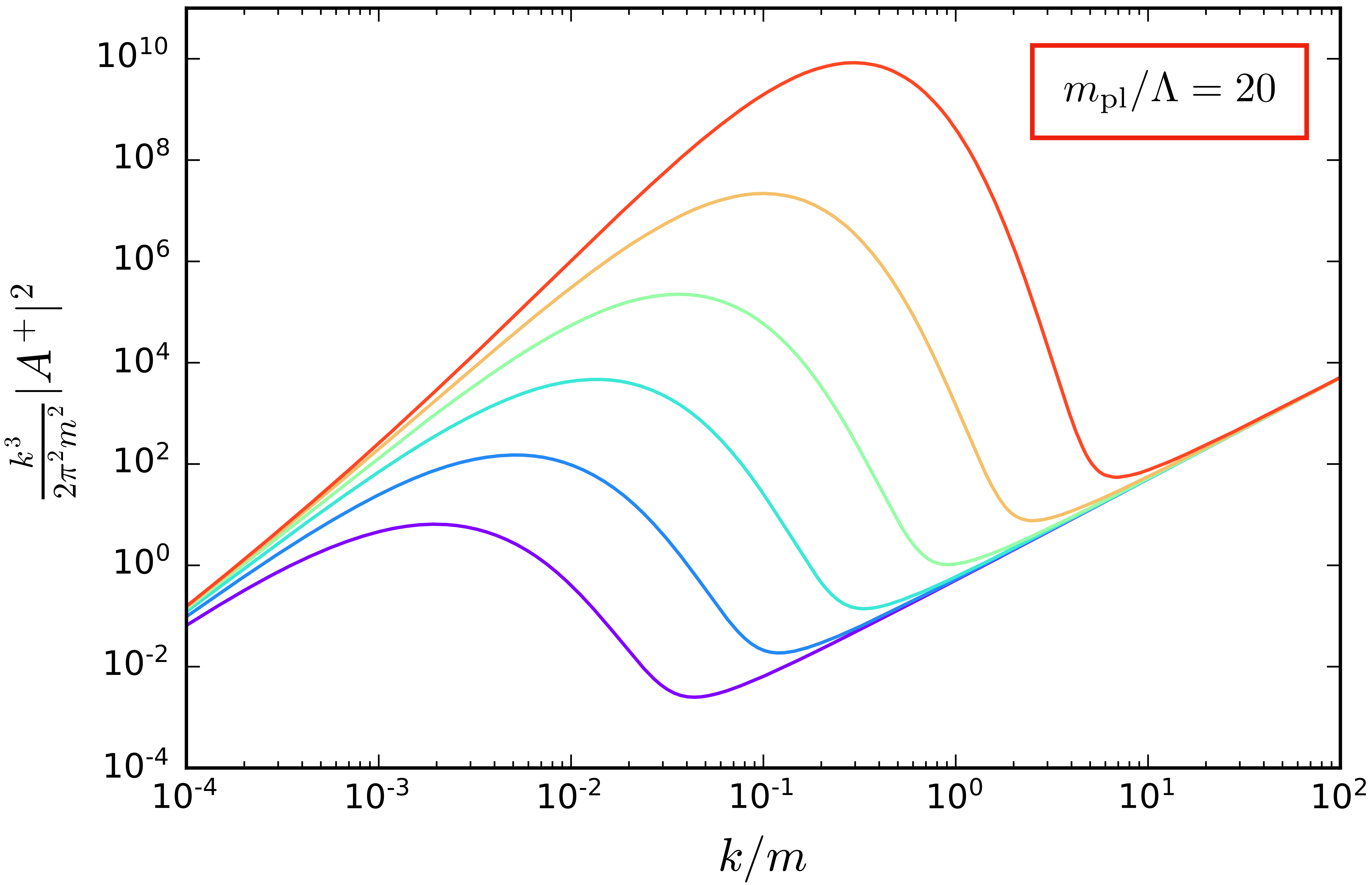
-6 eFolds

0 eFolds



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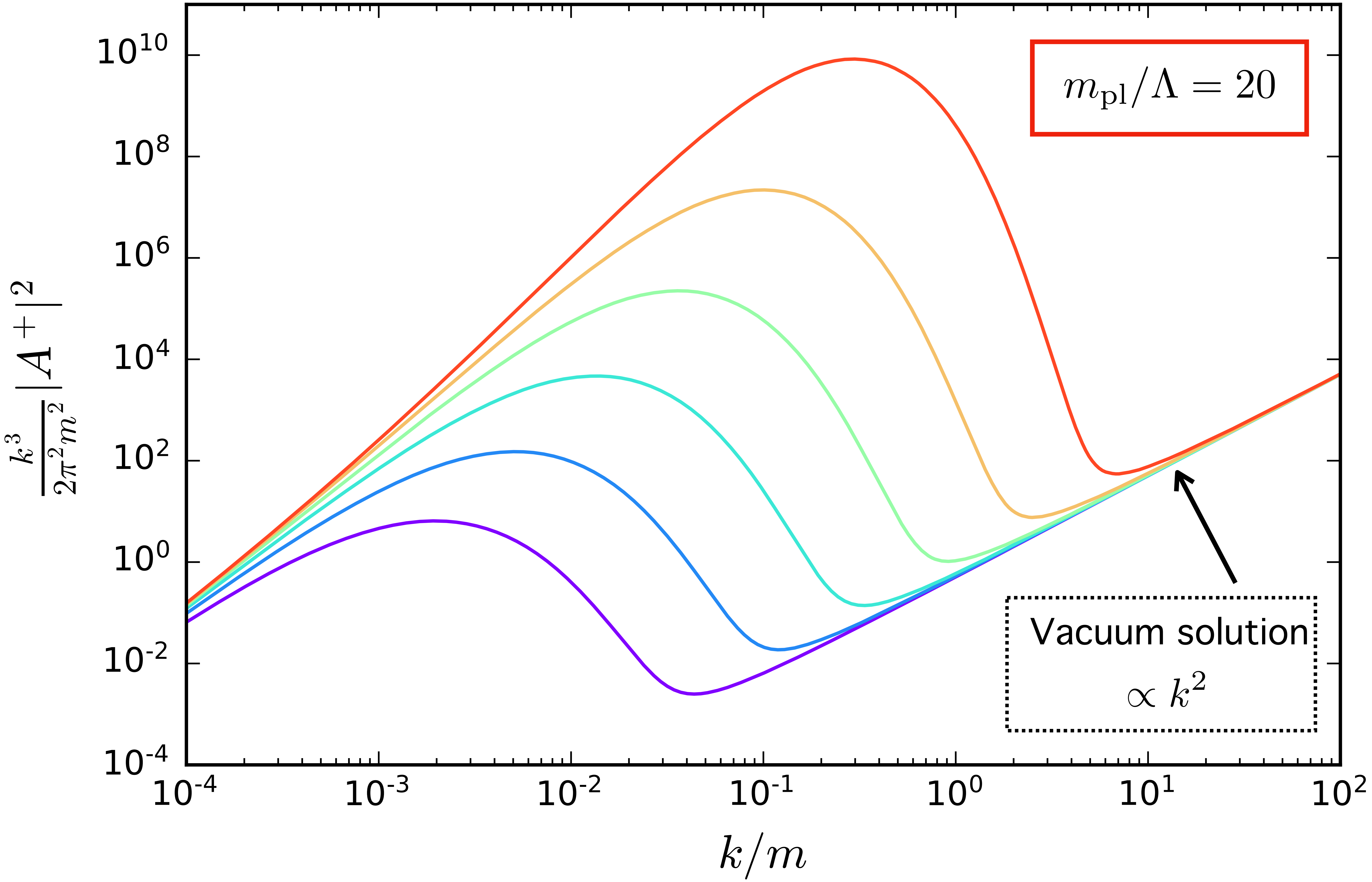

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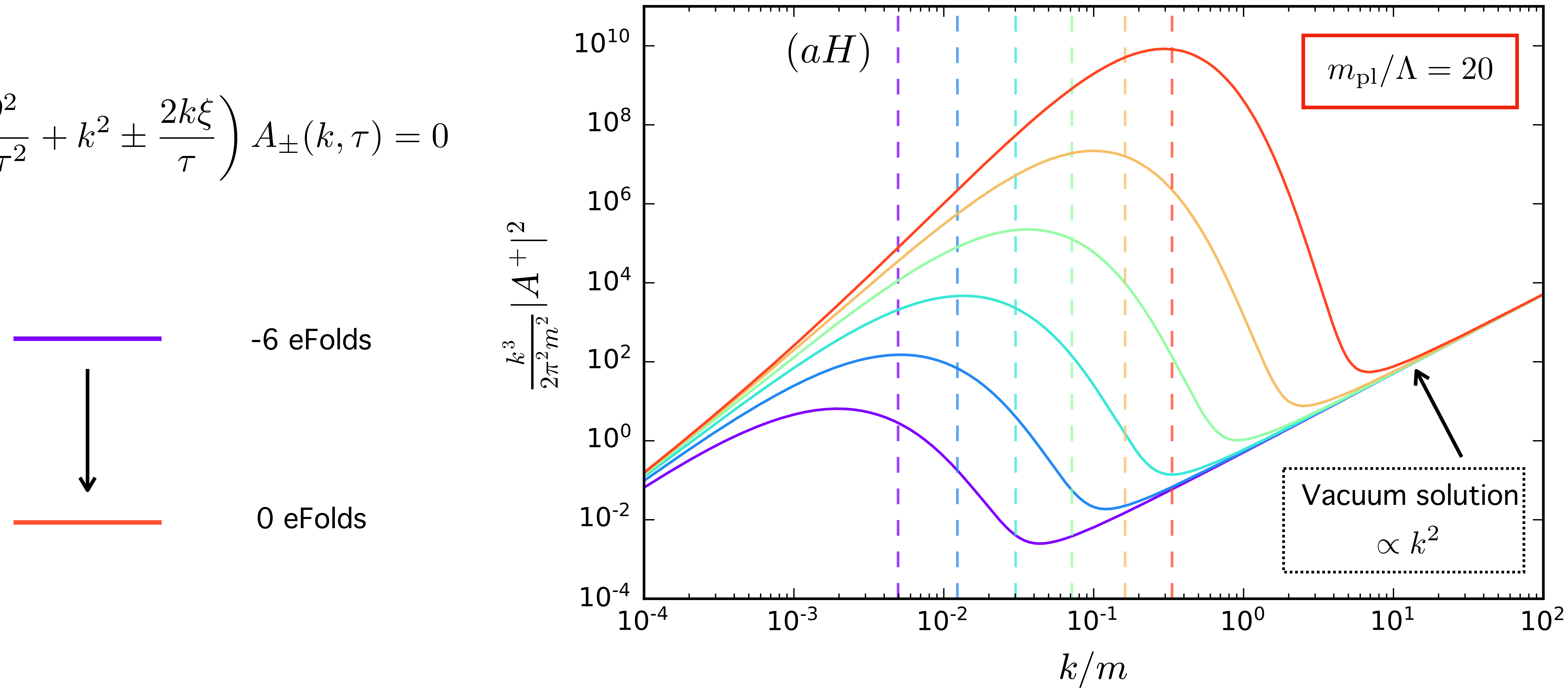
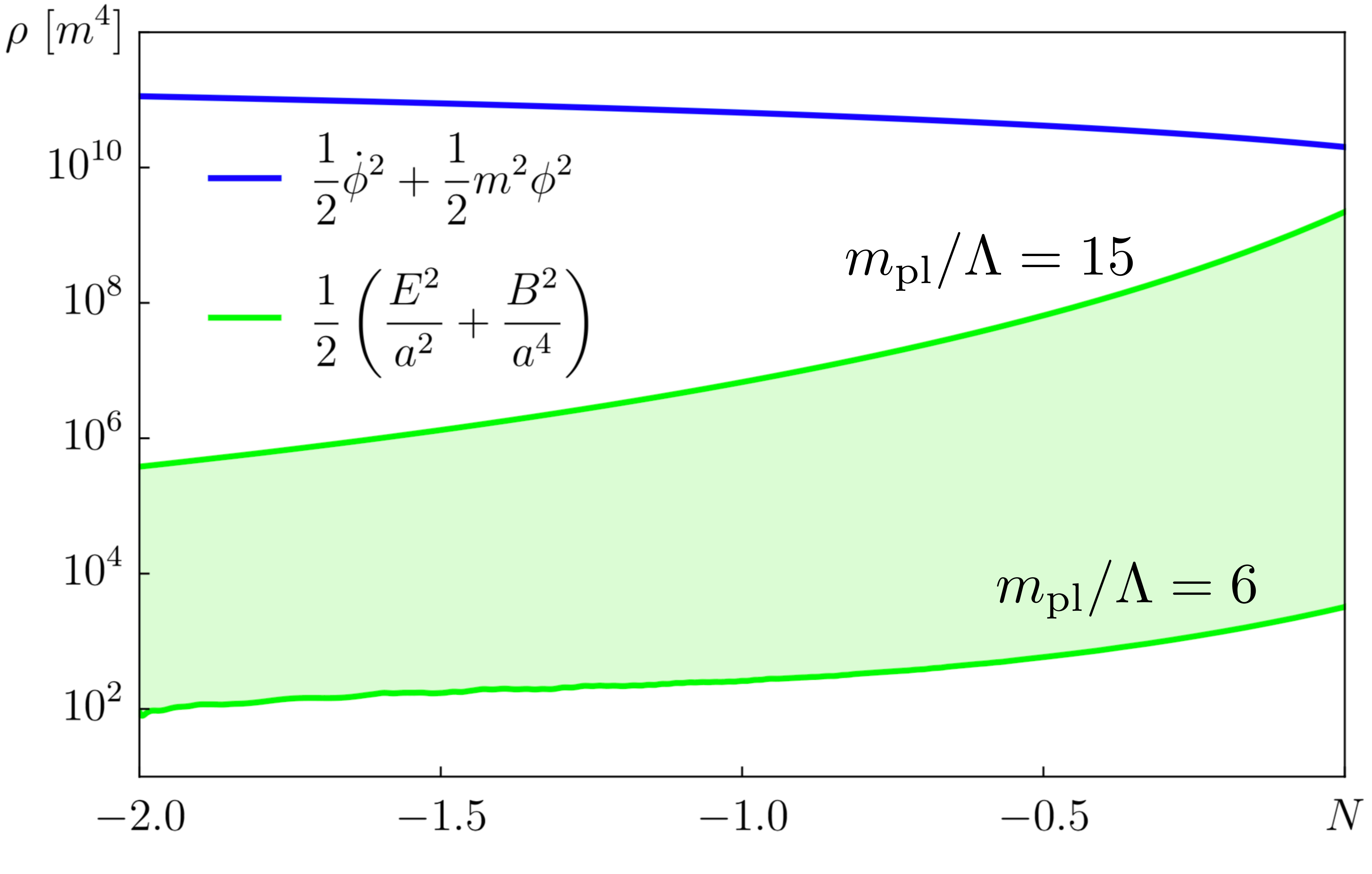


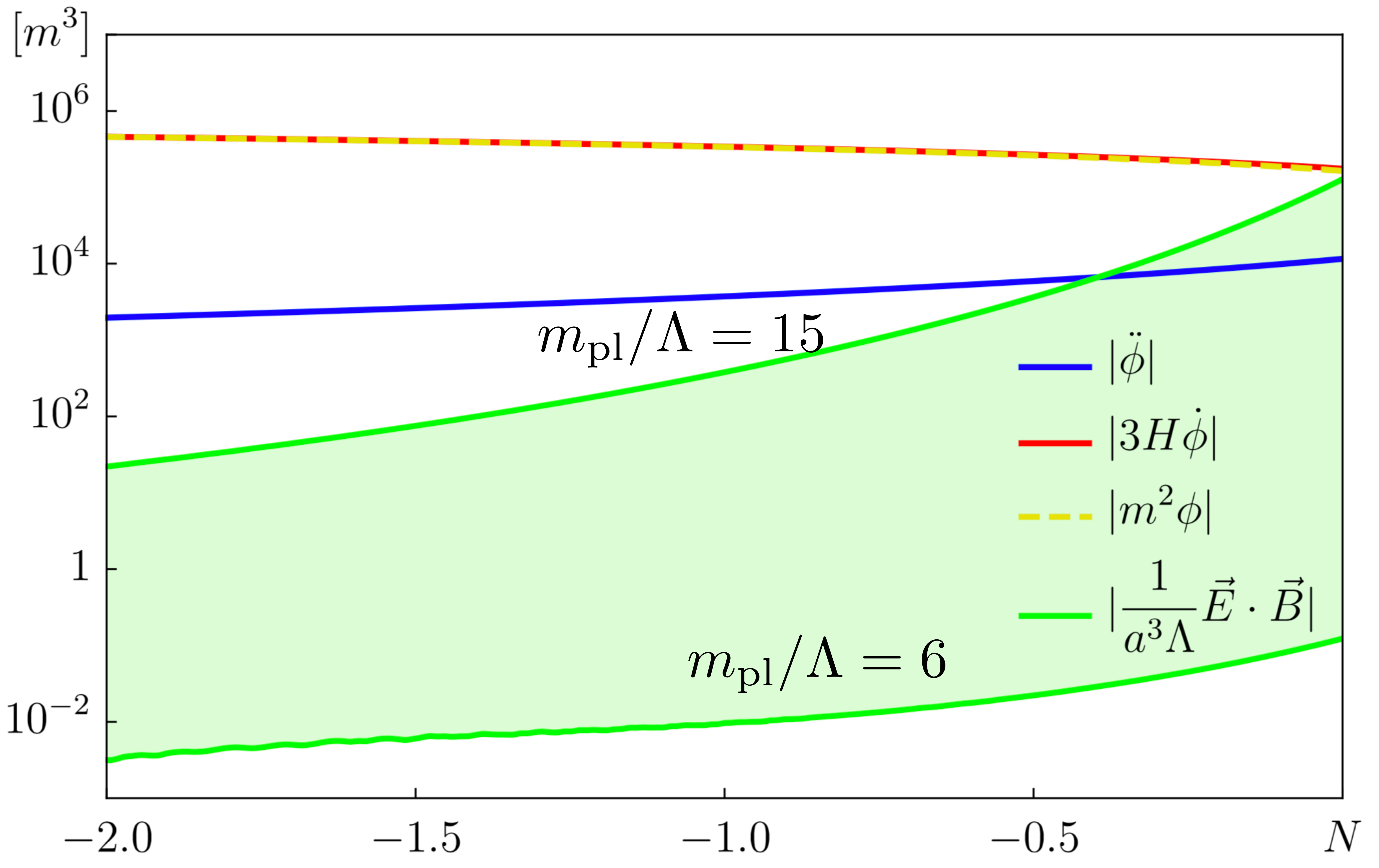
Basics of the model: linear regime

[J. R. C. Cuissa , D. G. Figueroa (1812.03132)]

Energy components



Topological term vs slow-roll



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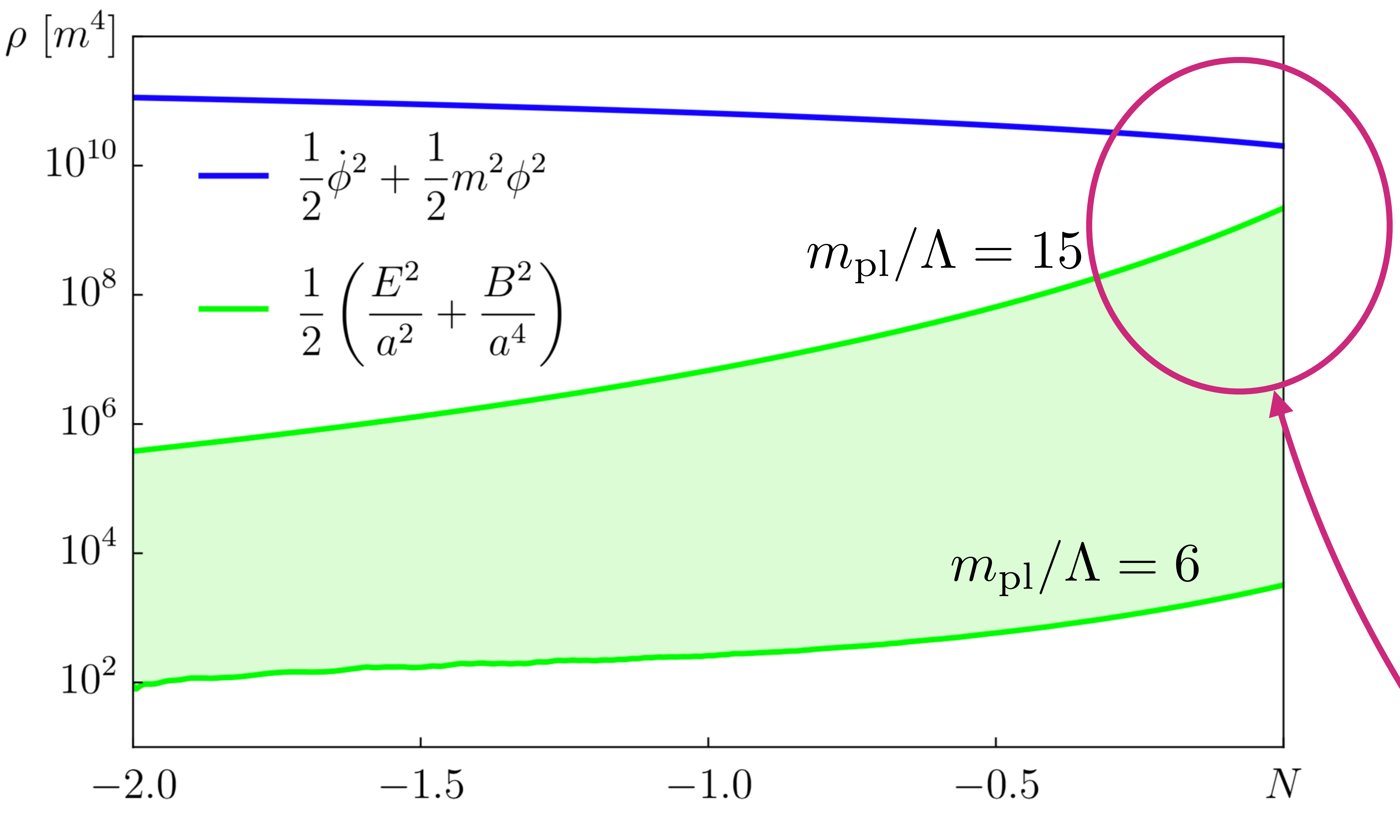

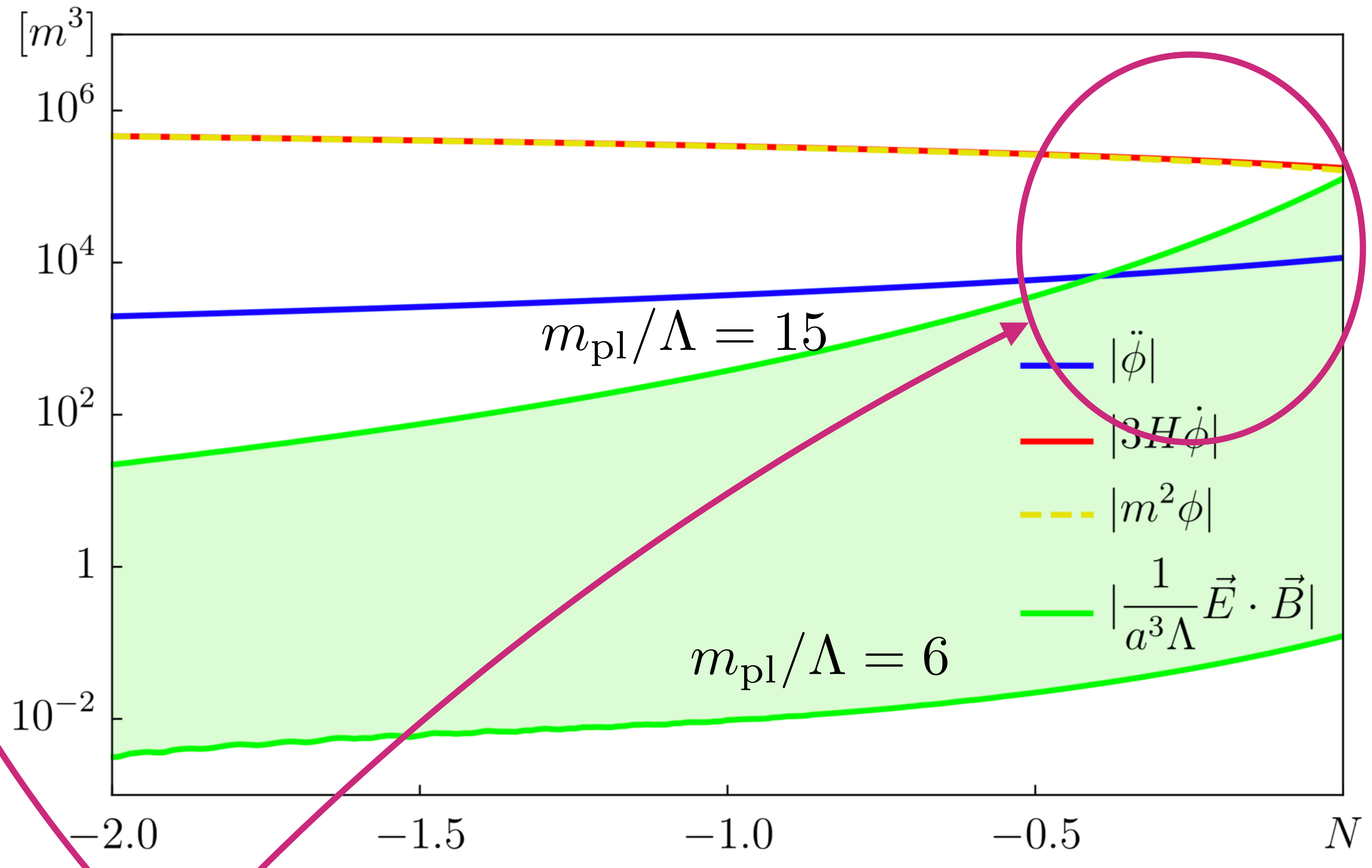

Backreaction effect cannot be neglected

Linear regime

Dynamical equations:

$$\dot{\tilde{\pi}}_\phi = a\vec{\nabla}^2\phi - a^3 m^2 \phi + \frac{1}{a\Lambda}\vec{\tilde{E}}\cdot\vec{B}\ ,$$

$$\dot{\vec{\tilde{E}}} = -\frac{1}{a}\vec{\nabla}\times\vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B} + \frac{1}{a\Lambda}\vec{\nabla}\phi\times\vec{\tilde{E}}$$

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$$\dot{\pi}_a = \frac{-a}{6m_{\rm pl}^2}(3p+\rho) = \frac{a}{3m_{\rm pl}^2}\langle -2K_\phi + V - K_A - G_A\rangle$$

CL

Basics of the model: linear regime

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Neglected

Basics of the model

Beyond the linear regime: backreaction

Dynamical equations:

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Backreaction terms

Basics of the model

Beyond the linear regime: backreaction

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Basics of the model

Beyond the linear regime: backreaction

1st approximation: homogeneous

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Due to backreaction not only slow-roll

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Basics of the model

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Why in the lattice?

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No gradient terms induced (yet)

Expansion equation:

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Lattice provides the solution: Real time evolution of fields

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Inhomogeneous source:
Induce gradients not
tractable analytically

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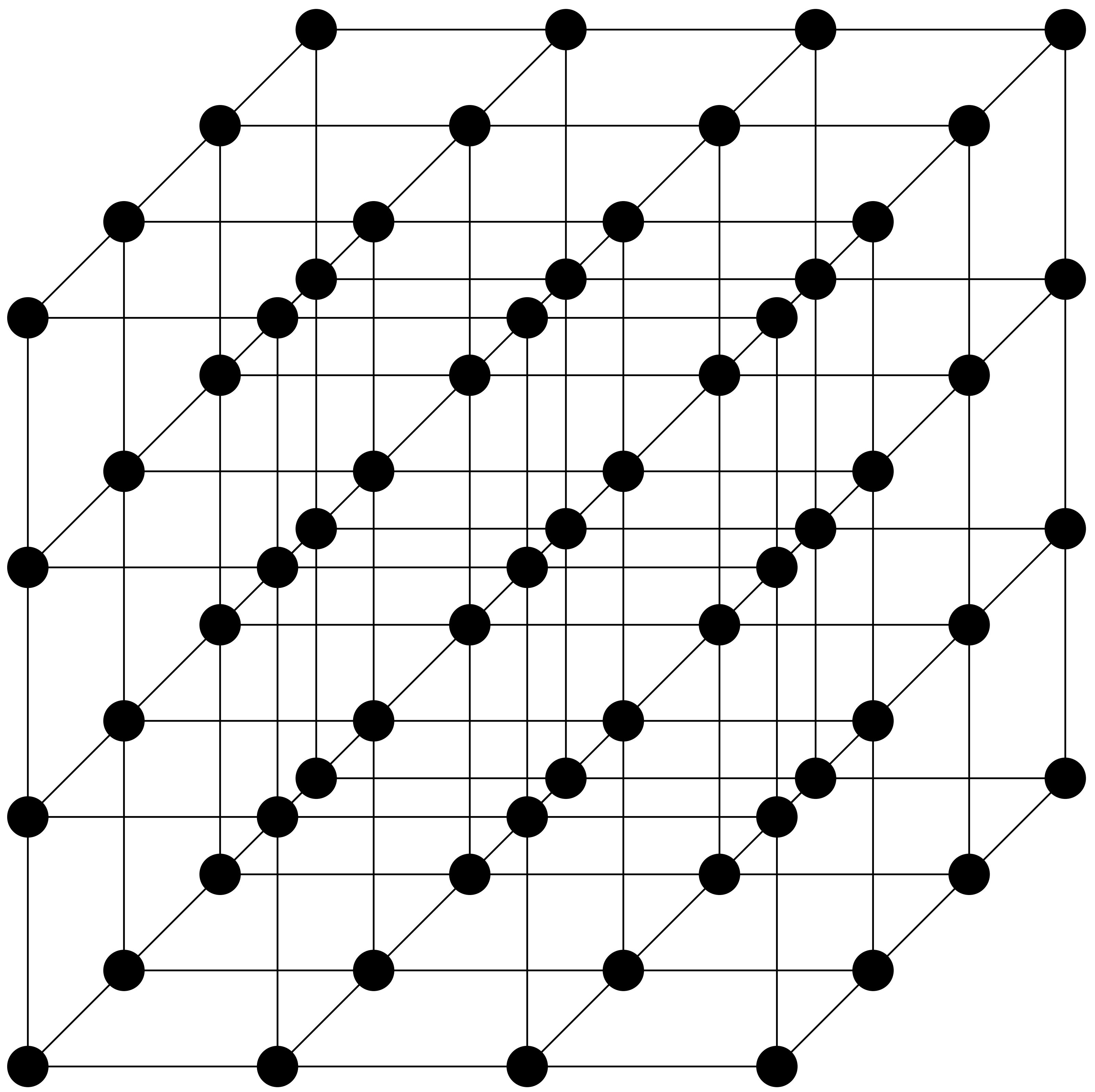
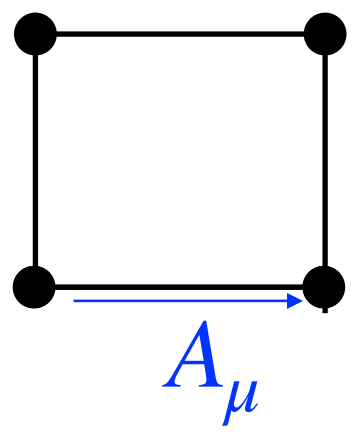
$\forall$ D.o.F will be simultaneously evolved

- Track of all inhomogeneities
- Full backreaction effects included

Discretisation procedure





ϕ @ sites

A_μ @ links

[D. G. Figueroa & M. Shaposhnikov (1705.09629)]

Want to preserve:

1- Gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \alpha(x)$$

2- Bianchi identities

$$\vec{\nabla} \times \vec{E} = \dot{\vec{B}} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

3- Topological nature of the coupling

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu K^\mu$$

Exact lattice shift symmetry

4- Continuum limit to $\mathcal{O}(dx^2)$

Discretisation procedure

[D. G. Figueroa, M. Shaposhnikov (1705.09629)]

Continuum

$$\frac{\phi}{\Lambda}\vec{E}\cdot\vec{B}$$

The only one that satisfies all conditions

Lattice

$$\boxed{\sum_i \frac{\phi}{\Lambda} E_i^{(2)} B_i^{(4)}}$$

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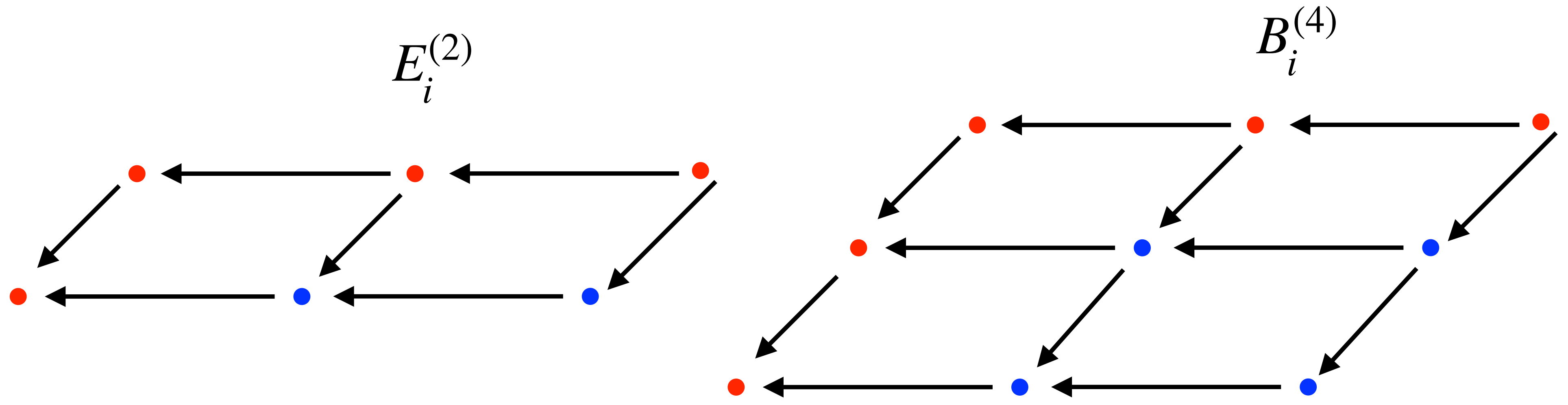

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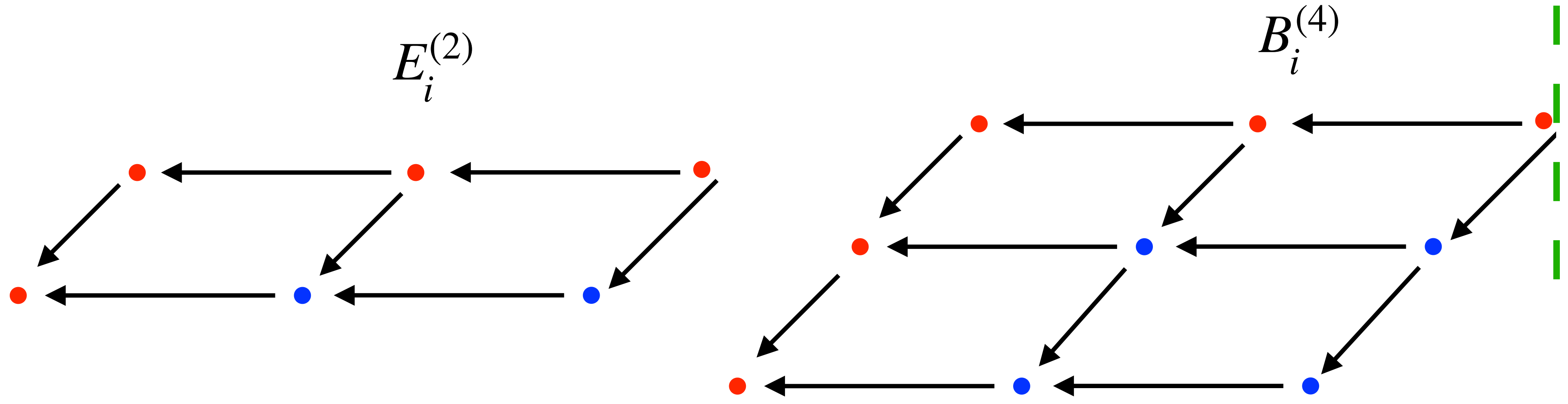

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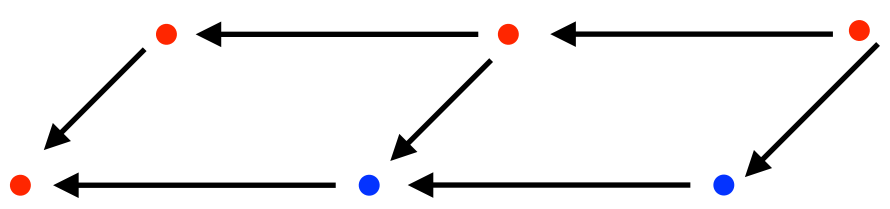

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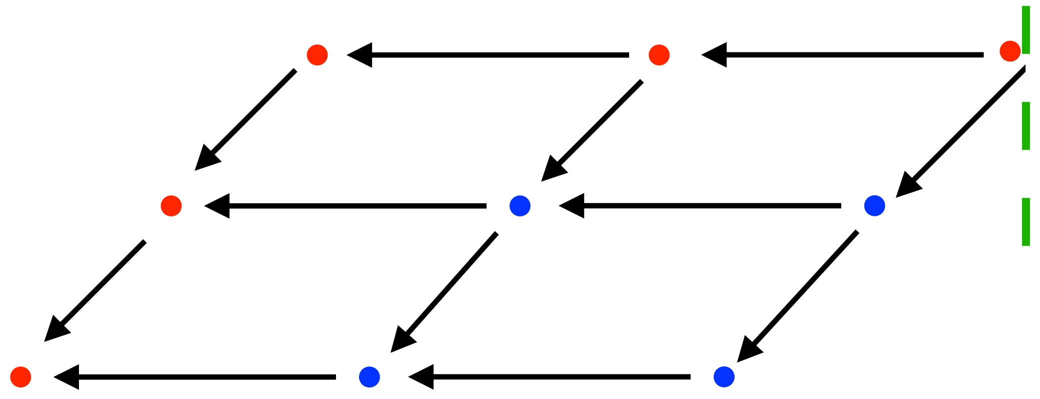

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Discretisation procedure

[D. G. Figueroa, M. Shaposhnikov (1705.09629)]
[J. R. C. Cuissa , D. G. Figueroa (1812.03132)]

$$S = \int d^4x \left[\frac{1}{2} a^3 \pi_\phi^2 - \frac{1}{2} a \left(\vec{\nabla} \phi \right)^2 - \frac{1}{2} a^3 m^2 \phi^2 + \frac{1}{2} a \left(\vec{E}^2 - \frac{\vec{B}^2}{a^2} \right) + \frac{\phi}{\Lambda} \vec{E} \cdot \vec{B} \right]$$

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Full discretised action:

$$\begin{aligned} S = \Delta t \Delta x^3 \sum_{t,\vec{n}} \Bigg[& \frac{1}{2a^3} \left(\tilde{\pi}_\phi \right)^2 - \frac{1}{2} a_{+\frac{\hat{0}}{2}} \left(\Delta_i^+ \phi_{+\frac{\hat{0}}{2}} \right)^2 - \frac{1}{2} a^3_{+\frac{\hat{0}}{2}} m^2 \phi_{+\frac{\hat{0}}{2}} \\ & + \frac{1}{2} a_{+\frac{\hat{0}}{2}} \sum_i \left(\Delta_0^+ A_i - \Delta_i^+ A_0 \right)^2 - \frac{1}{4a} \sum_{i,j} \left(\Delta_i^+ A_j - \Delta_j^+ A_i \right)^2 \\ & + \frac{\phi}{a\Lambda} \sum_i \frac{1}{2} \tilde{E}_i^{(2)} \left(B_i^{(4)} + B_{i,+\hat{0}}^{(4)} \right) \Bigg] \end{aligned}$$

Discretisation procedure

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Ready for variational principle

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$$+\frac{1}{2}a_{+\frac{\hat{0}}{2}}\sum_i\left(\Delta_0^+A_i - \Delta_i^+A_0\right)^2 - \frac{1}{4a}\sum_{i,j}\left(\Delta_i^+A_j - \Delta_j^+A_i\right)^2$$

$$\left. + \frac{\phi}{a\Lambda}\sum_i\frac{1}{2}\tilde{E}_i^{(2)}\left(B_i^{(4)} + B_{i,+\hat{0}}^{(4)}\right)\right]$$

Common kick-drift scheme

Evolution Kernels

$$\tilde{\pi}_{\phi,+\hat{0}} = \tilde{\pi}_{\phi} + dt\mathcal{K}_{\phi}^{L}\ , \quad \tilde{E}_{i,+\frac{\hat{0}}{2}} = \tilde{E}_{i,-\frac{\hat{0}}{2}} + dt\mathcal{K}_{i,A}^{L}$$

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Evolution Kernels

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$$\mathcal{K}_\phi^L = a_{+\frac{\hat{0}}{2}} \sum_i \Delta_i^- \Delta_i^+ \phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}} m^2 \phi_{+\frac{\hat{0}}{2}} + \frac{1}{2a\Lambda} \sum_i \tilde{E}^{(2)}_{i,+\frac{\hat{0}}{2}} \left(B_i^{(4)} + B_{i,+\hat{0}}^{(4)} \right)$$

$$\mathcal{K}_{i,A}^L = -\frac{1}{a} \sum_{j,k} \epsilon_{ijk} \Delta_j^- B_k - \frac{1}{2a^3\Lambda} \left(\tilde{\pi}_\phi B_i^{(4)} + \tilde{\pi}_{\phi,+i} B_{i,+i}^{(4)} \right)$$
$$+ \frac{1}{8a\Lambda}(2 + dx\Delta_i^+) \sum_\pm \sum_{j,k} \left\{ \epsilon_{ijk} [(\Delta_j^\pm \phi)\tilde{E}^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm \phi)\tilde{E}^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}} \right\}$$

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Gauss's constraint:

$$\sum_i \Delta^-_i \tilde{E}_{i,+\frac{\hat{0}}{2}} = -\frac{1}{4\Lambda} \sum_{\pm} \sum_i \left(\Delta^{\pm}_i \phi_{+\frac{\hat{0}}{2}} \right) \left(B^{(4)}_i + B^{(4)}_{i,+\hat{0}} \right)_{\pm i}$$

Discretisation procedure

Common kick-drift scheme

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Alternative approach in:
[A. Caravano, E. Komatsu, K. D. Lozanov, J. Weller (2204.12874)]

Discretisation procedure

Common kick-drift scheme

Evolution Kernels

$$b_{+\hat{0}} = b + dt\mathcal{K}_a^L$$

$$a_{+\frac{\hat{0}}{2}} = a_{-\frac{\hat{0}}{2}} + dtb$$

Expansion equations:

$$\mathcal{K}_a^L = -\frac{a_{+\hat{0}/2}}{6m_{\rm pl}^2}(\rho_L + 3p_L)_{+\hat{0}/2}$$

Hubble's constraint:

$$\pi_a^2 = \frac{a^2}{3m_{pl}^2}\rho_L$$

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$$(\rho_L + 3p_L)_{+\hat{0}/2} = 2(\bar{K}_\phi^L + \bar{K}_{\phi,+\hat{0}}^L) - 2\bar{V}_{\phi,+\hat{0}/2}^L + 2K_A^L + (G_A^L + G_{A,+\hat{0}}^L)$$

$$\rho_L = \bar{K}_\phi^L + \frac{1}{2}(\bar{G}_{\phi,-\hat{0}/2}^L + \bar{G}_{\phi,+\hat{0}/2}^L) + \frac{1}{2}(\bar{V}_{\phi,-\hat{0}/2}^L + \bar{V}_{\phi,+\hat{0}/2}^L) + \frac{1}{2}(\bar{K}_{A,-\hat{0}/2}^L + \bar{K}_{A,+\hat{0}/2}^L) + \bar{G}_A^L$$

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$$\boxed{\pi_a^2 = \frac{a^2}{3m_{pl}^2}\rho_L}$$

$$\bar{K}_\phi^L = \frac{1}{N^3}\sum_{\vec{n}}\frac{1}{2a^6}\tilde{\pi}_\phi^2, \quad \bar{G}_\phi^L = \frac{1}{N^3}\sum_{\vec{n}}\frac{1}{2}\sum_i(\Delta_i^+\phi_{+\hat{0}/2})^2, \quad \bar{V}_\phi^L = \frac{1}{N^3}\sum_{\vec{n}}\frac{1}{2}m^2\phi_{+\hat{0}/2}^2$$

$$\bar{K}_A^L = \frac{1}{N^3}\sum_{\vec{n}}\sum_i\frac{1}{2a_{+\hat{0}/2}^4}\tilde{E}_{i,+\hat{0}/2}^2, \quad \bar{G}_A^L = \frac{1}{N^3}\sum_{\vec{n}}\sum_i\frac{1}{4a^4}(\Delta_i^+A_j - \Delta_j^+A_i)^2$$

Evolution Kernels

Dynamical equations:

$$\mathcal{K}_\phi^L = a_{+\frac{\hat{0}}{2}} \sum_i \Delta_i^- \Delta_i^+ \phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}} m^2 \phi_{+\frac{\hat{0}}{2}} + \frac{1}{2a\Lambda} \sum_i \tilde{E}^{(2)}_{i,+\frac{\hat{0}}{2}} \left(B_i^{(4)} + B_{i,+\hat{0}}^{(4)} \right)$$

$$\mathcal{K}_{i,A}^L = -\frac{1}{a} \sum_{j,k} \epsilon_{ijk} \Delta_j^- B_k - \frac{1}{2a^3\Lambda} \left(\tilde{\pi}_\phi B_i^{(4)} + \tilde{\pi}_{\phi,+i} B_{i,+i}^{(4)} \right)$$

$$+ \frac{1}{8a\Lambda} (2 + dx\Delta_i^+) \sum_\pm \sum_{j,k} \left\{ \epsilon_{ijk} [(\Delta_j^\pm \phi) \tilde{E}^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm \phi) \tilde{E}^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}} \right\}$$

Evolution Kernels

Dynamical equations:

$$\mathcal{K}^L_\phi = a_{+\frac{\hat{0}}{2}} \sum_i \Delta_i^- \Delta_i^+ \phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}} m^2 \phi_{+\frac{\hat{0}}{2}} + \frac{1}{2a\Lambda} \sum_i \tilde{E}^{(2)}_{i,+\frac{\hat{0}}{2}} \left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}} \right)$$

$$\mathcal{K}^L_{i,A} = -\frac{1}{a} \sum_{j,k} \epsilon_{ijk} \Delta_j^- B_k - \frac{1}{2a^3\Lambda} \left(\tilde{\pi}_\phi B_i^{(4)} + \tilde{\pi}_{\phi,+i} B^{(4)}_{i,+i} \right)$$
$$+ \frac{1}{8a\Lambda} (2 + dx \Delta_i^+) \sum_{\pm} \sum_{j,k} \left\{ \epsilon_{ijk} [(\Delta_j^\pm \phi) \tilde{E}^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm \phi) \tilde{E}^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}} \right\}$$

Non-symplectic,
Conjugate momenta in kernels!
[Remember 4th lecture & 1st Topical]

$$\dot{\tilde{E}}_i = \mathcal{K}^L_{i,A} \left[a, b, \phi, A_i, \tilde{E}_i \right]$$

Evolution Kernels

Dynamical equations:

$$\mathcal{K}_\phi^L = a_{+\frac{\hat{0}}{2}} \sum_i \Delta_i^- \Delta_i^+ \phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}} m^2 \phi_{+\frac{\hat{0}}{2}} + \frac{1}{2a\Lambda} \sum_i \tilde{E}^{(2)}_{i,+\frac{\hat{0}}{2}} \left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}} \right)$$

$$\mathcal{K}_{i,A}^L = -\frac{1}{a} \sum_{j,k} \epsilon_{ijk} \Delta_j^- B_k - \frac{1}{2a^3\Lambda} \left(\tilde{\pi}_\phi B_i^{(4)} + \tilde{\pi}_{\phi,+i} B^{(4)}_{i,+i} \right)$$
$$+ \frac{1}{8a\Lambda} (2 + dx\Delta_i^+) \sum_{\pm} \sum_{j,k} \left\{ \epsilon_{ijk} [(\Delta_j^\pm \phi) \tilde{E}^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm \phi) \tilde{E}^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}} \right\}$$

Non-symplectic,
Conjugate momenta in kernels!
[Remember 4th lecture & 1st Topical]

$$\dot{\tilde{E}}_i = \mathcal{K}_{i,A}^L \left[a, b, \phi, A_i, \tilde{E}_i \right]$$

Discretisation procedure

Common kick-drift scheme

Evolution Kernels

$$\tilde{\pi}_{\phi,+\hat{0}} = \tilde{\pi}_{\phi} + dt\mathcal{K}^{L}_{\phi}\ , \quad \tilde{E}_{i,+\frac{\hat{0}}{2}} = \tilde{E}_{i,-\frac{\hat{0}}{2}} + dt\mathcal{K}^{L}_{i,A}$$

$$\phi_{+\frac{\hat{0}}{2}} = \phi_{-\frac{\hat{0}}{2}} + \frac{dt}{a^3}\tilde{\pi}_{\phi}\ , \quad A_{i,+\hat{0}} = A_i + \frac{dt}{a}\tilde{E}_{i,+\frac{\hat{0}}{2}}$$

Discretisation procedure

Common kick-drift scheme

Evolution Kernels

$$\tilde{\pi}_{\phi,+\hat{0}} = \tilde{\pi}_{\phi} + dt\mathcal{K}^L_{\phi} \ , \quad \tilde{E}_{i,+\frac{\hat{0}}{2}} = \tilde{E}_{i,-\frac{\hat{0}}{2}} + dt\mathcal{K}^L_{i,A}$$

$$\phi_{+\frac{\hat{0}}{2}} = \phi_{-\frac{\hat{0}}{2}} + \frac{dt}{a^3}\tilde{\pi}_{\phi} \ , \quad A_{i,+\hat{0}} = A_i + \frac{dt}{a}\tilde{E}_{i,+\frac{\hat{0}}{2}}$$

Dynamical equations:

$$\mathcal{K}^L_{\phi} = a_{+\frac{\hat{0}}{2}} \sum_i \Delta^-_i \Delta^+_i \phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}} m^2 \phi_{+\frac{\hat{0}}{2}} + \frac{1}{2a\Lambda} \sum_i \tilde{E}^{(2)}_{i,+\frac{\hat{0}}{2}} \left(B^{(4)}_i + B^{(4)}_{i,+\hat{0}} \right)$$

$$\mathcal{K}^L_{i,A} = -\frac{1}{a} \sum_{j,k} \epsilon_{ijk} \Delta^-_j B_k - \frac{1}{2a^3\Lambda} \left(\tilde{\pi}_{\phi} B^{(4)}_i + \tilde{\pi}_{\phi,+i} B^{(4)}_{i,+i} \right)$$
$$+ \frac{1}{8a\Lambda}(2 + dx\Delta^+_i) \sum_{\pm} \sum_{j,k} \left\{ \epsilon_{ijk} [(\Delta^{\pm}_j \phi) \tilde{E}^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta^{\pm}_j \phi) \tilde{E}^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}} \right\}$$

Common kick-drift scheme

Evolution Kernels

$$\tilde{\pi}_{\phi,+\hat{0}} = \tilde{\pi}_\phi + dt\mathcal{K}^L_\phi\,, \quad \tilde{E}_{i,+\frac{\hat{0}}{2}} = \tilde{E}_{i,-\frac{\hat{0}}{2}} + dt\mathcal{K}^L_{i,A}$$

$$\phi_{+\frac{\hat{0}}{2}} = \phi_{-\frac{\hat{0}}{2}} + \frac{dt}{a^3}\tilde{\pi}_\phi\,, \quad A_{i,+\hat{0}} = A_i + \frac{dt}{a}\tilde{E}_{i,+\frac{\hat{0}}{2}}$$

Dynamical equations:

$$\mathcal{K}^L_\phi = a_{+\frac{\hat{0}}{2}}\sum_i \Delta_i^-\Delta_i^+\phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}}m^2\phi_{+\frac{\hat{0}}{2}} + \frac{1}{2a\Lambda}\sum_i \tilde{E}^{(2)}_{i,+\frac{\hat{0}}{2}}\left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}}\right)$$

$$\mathcal{K}^L_{i,A} = -\frac{1}{a}\sum_{j,k}\epsilon_{ijk}\Delta_j^- B_k - \frac{1}{2a^3\Lambda}\left(\tilde{\pi}_\phi B_i^{(4)} + \tilde{\pi}_{\phi,+i}B^{(4)}_{i,+i}\right) + \frac{1}{8a\Lambda}(2 + dx\Delta_i^+)\sum_\pm\sum_{j,k}\left\{\epsilon_{ijk}[(\Delta_j^\pm\phi)\tilde{E}^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm\phi)\tilde{E}^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}}\right\}$$

Implicit scheme required

We need the value of the gauge's canonical momentum @ 0/2 for obtaining its value @ 0/2!

Discretisation procedure

Common kick-drift scheme

Evolution Kernels

$$\tilde{\pi}_{\phi,+\hat{0}} = \tilde{\pi}_{\phi} + dt\mathcal{K}^L_{\phi}\ , \quad \tilde{E}_{i,+\frac{\hat{0}}{2}} = \tilde{E}_{i,-\frac{\hat{0}}{2}} + dt\mathcal{K}^L_{i,A}$$

$$\phi_{+\frac{\hat{0}}{2}} = \phi_{-\frac{\hat{0}}{2}} + \frac{dt}{\ }\ \quad A_{i,+\hat{0}} = A_i + \frac{dt}{a}\tilde{E}_{i,+\frac{\hat{0}}{2}}$$

Dynamical equations:

$$\mathcal{K}^L_{\phi} = a_{+\frac{\hat{0}}{2}} \sum_i \Delta_i^- \Delta_i^+ \phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}} m^2 \phi \quad + B^{(4)}_{i,+\hat{0}}\Big)$$

$$\mathcal{K}^L_{i,A} = -\frac{1}{a} \sum_{j,k} \epsilon_{ijk} \Delta^- \quad + \tilde{\pi}_{\phi,+i} B^{(4)}_{i,+i}\Big)$$

$$+ \frac{1}{8a} \sum_{\pm} \sum_{j,k} \Big\{ \epsilon_{ijk} [(\Delta_j^{\pm}\phi)\tilde{E}^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^{\pm}\phi)\tilde{E}^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}} \Big\}$$

Need for a non-symplectic integrator!
[Ander]

Implicit scheme required

We need the value of the gauge's canonical momentum @ 0/2 for obtaining its value @ 0/2!

ANDER URIO

Our Current Work

Explicit-in-time integrator

$\boxed{\text{Joanes}}$

$\boxed{\text{Ander}}$

Explicit-in-time integrator

Joanes

If momenta appear in kernels

Ander

Explicit-in-time integrator

Joanes

If momenta appear in kernels

Not solvable by explicit symplectic integrators

Ander

Explicit-in-time integrator

Joanes

If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Ander

Explicit-in-time integrator

Joanes

If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Why?

Ander

Joanes

If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Why?

$$\pi_{+\frac{\hat{0}}{2}} = \pi_{-\frac{\hat{0}}{2}} + \mathcal{K}[\pi_{+\frac{\hat{0}}{2}}]dt$$

Ander

Explicit-in-time integrator

Joanes

If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Why?

$$\pi_{+\frac{\hat{0}}{2}} = \pi_{-\frac{\hat{0}}{2}} + \mathcal{K}[\pi_{+\frac{\hat{0}}{2}}]dt$$

Solution

Ander

Explicit-in-time integrator

Joanes

If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Why?

$$\pi_{+\frac{\hat{0}}{2}} = \pi_{-\frac{\hat{0}}{2}} + \mathcal{K}[\pi_{+\frac{\hat{0}}{2}}]dt$$

Solution

Ander

Non-sympletic integrators

Explicit-in-time integrator

Joanes

If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Why?

$$\pi_{+\frac{\hat{0}}{2}} = \pi_{-\frac{\hat{0}}{2}} + \mathcal{K}[\pi_{+\frac{\hat{0}}{2}}]dt$$

Solution

Ander

Non-sympletic integrators

Runge-Kutta

Explicit-in-time integrator

Joanes

If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Why?

$$\pi_{+\frac{\hat{0}}{2}} = \pi_{-\frac{\hat{0}}{2}} + \mathcal{K}[\pi_{+\frac{\hat{0}}{2}}]dt$$

Solution

Ander

Non-sympletic integrators

Runge-Kutta

(explicit version)

Explicit-in-time integrator

Joanes

If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Why?

$$\pi_{+\frac{\hat{0}}{2}} = \pi_{-\frac{\hat{0}}{2}} + \mathcal{K}[\pi_{+\frac{\hat{0}}{2}}]dt$$

Solution

Ander

Non-sympletic integrators

(explicit version)

Runge-Kutta

No longer!

Explicit-in-time integrator

1 st order ODE

$$\frac{dx}{dt} = v(t, x)$$

Explicit-in-time integrator

$\boxed{\text{RK2 \& 2 stages}}$

1st order ODE

$$\frac{dx}{dt} = v(t, x)$$

Explicit-in-time integrator

RK2 & 2 stages

1st order ODE

$$\frac{dx}{dt} = v(t, x) \quad \Rightarrow \quad \left\{ \quad x(t_{n+1}) = x(t_n) + dt\,(b_1 G_1 + b_2 G_2) \quad \right\}$$

RK2 & 2 stages

1 st order ODE

$$\frac{dx}{dt} = v(t, x) \rightarrow \left\{ \begin{array}{c} x(t_{n+1}) = x(t_n) + dt\,(b_1 G_1 + b_2 G_2) \\ G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + c_2 dt, x(t_n) + a_{21} G_1 dt) \end{array} \right\}$$

Explicit-in-time integrator

RK2 & 2 stages

1st order ODE $$\frac{dx}{dt} = v(t,x)$$ $\Rightarrow$

$$\left\{ \begin{array}{c} x(t_{n+1}) = x(t_n) + dt\,(b_1 G_1 + b_2 G_2) \\ G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + c_2 dt, x(t_n) + a_{21} G_1 dt) \end{array} \right\}$$

$$x^{\rm Cont}(t_{n+1}) - x^{\rm RK2}(t_{n+1}) = \mathcal{O}(dt^3)$$

Explicit-in-time integrator

RK2 & 2 stages

1 st order ODE

$$\frac{dx}{dt} = v(t,x) \quad \Rightarrow \quad \left\{ \begin{aligned} & x(t_{n+1}) = x(t_n) + dt\,(b_1 G_1 + b_2 G_2) \\ & G_1 = v(t_n, x(t_n)) \\ & G_2 = v(t_n + c_2 dt, x(t_n) + a_{21} G_1 dt) \end{aligned} \right\}$$

$$x^{\mathrm{Cont}}(t_{n+1}) - x^{\mathrm{RK2}}(t_{n+1}) = \mathcal{O}(dt^3)$$

$$\text{Constraints} \rightarrow b_1 + b_2 = 1\ ,\ \ b_2 c_2 = 1/2\ ,\ \ b_2 a_{21} = 1/2$$

Explicit-in-time integrator

$\boxed{\text{RK2 \& 2 stages}}$

1st order ODE

$$\frac{dx}{dt} = v(t,x) \quad \Rightarrow \quad \begin{Bmatrix} x(t_{n+1}) = x(t_n) + dt\,(b_1 G_1 + b_2 G_2) \\ G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + c_2 dt, x(t_n) + a_{21} G_1 dt) \end{Bmatrix}$$

$$x^{\text{Cont}}(t_{n+1}) - x^{\text{RK2}}(t_{n+1}) = \mathcal{O}(dt^3)$$

$$\text{Constraints} \rightarrow b_1 + b_2 = 1\ ,\ \ b_2 c_2 = 1/2\ ,\ \ b_2 a_{21} = 1/2$$

Modified Euler

Explicit-in-time integrator

$\boxed{\text{RK2 \& 2 stages}}$

$\boxed{1^{\text{st}}\text{ order ODE}}$ $\quad \dfrac{dx}{dt} = v(t,x) \;\rightarrow$

$$\left\{ \begin{array}{c} x(t_{n+1}) = x(t_n) + dt\,(b_1 G_1 + b_2 G_2) \\ G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + c_2 dt, x(t_n) + a_{21} G_1 dt) \end{array} \right\}$$

$$x^{\rm Cont}(t_{n+1}) - x^{\rm RK2}(t_{n+1}) = \mathcal{O}(dt^3)$$

$$\text{Constraints} \rightarrow b_1 + b_2 = 1\ ,\ \ b_2 c_2 = 1/2\ ,\ \ b_2 a_{21} = 1/2$$

$\boxed{\text{Modified Euler}}$

$$b_1 = 1/2\ ,\ \ b_2 = 1/2\ ,\ \ c_2 = 1\ , a_{21} = 1$$

Explicit-in-time integrator

RK2 & 2 stages

1st order ODE

$$\frac{dx}{dt} = v(t,x) \rightarrow \left\{ \begin{array}{c} x(t_{n+1}) = x(t_n) + dt\,(b_1 G_1 + b_2 G_2) \\ G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + c_2 dt, x(t_n) + a_{21} G_1 dt) \end{array} \right\}$$

$$x^{\rm Cont}(t_{n+1}) - x^{\rm RK2}(t_{n+1}) = \mathcal{O}(dt^3)$$

$$\text{Constraints} \rightarrow b_1 + b_2 = 1\,,\ b_2 c_2 = 1/2\,,\ b_2 a_{21} = 1/2$$

Modified Euler

$$b_1 = 1/2\,,\ b_2 = 1/2\,,\ c_2 = 1\,, a_{21} = 1$$

$$\boxed{\begin{array}{c} x(t_{n+1}) = x(t_n) + \frac{dt}{2}(G_1 + G_2) \\ G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + dt, x(t_n) + G_1 dt) \end{array}}$$

Explicit-in-time integrator

Explicit-in-time integrator

$\boxed{2^{\text{nd}} \text{ order ODE}}$

$$\frac{d^2x}{dt^2} = \mathcal{K}\left(t, x, \frac{dx}{dt}\right)$$

Explicit-in-time integrator

$\boxed{\text{2}^{\text{nd}}\text{ order ODE}}$

$$\frac{d^2x}{dt^2} = \mathcal{K}\left(t, x, \frac{dx}{dt}\right)$$

1st order ODE RK2 ?

Explicit-in-time integrator

2nd order ODE

1st order ODE RK2 ?

$$\frac{d^2x}{dt^2} = \mathcal{K}\left(t, x, \frac{dx}{dt}\right)$$

Coupled 1st order ODEs

Explicit-in-time integrator

$\boxed{\text{2}^{\text{nd}}\text{ order ODE}}$

1st order ODE RK2 ?

$$\frac{d^2x}{dt^2} = \mathcal{K}\left(t, x, \frac{dx}{dt}\right)$$

$$\frac{dx}{dt} = v(t,x) \longleftarrow \boxed{\text{Coupled 1}^{\text{st}}\text{ order ODEs}} \longrightarrow \frac{dv}{dt} = \mathcal{K}(t,x,v)$$

Explicit-in-time integrator

2nd order ODE

$$\frac{d^2x}{dt^2} = \mathcal{K}\left(t, x, \frac{dx}{dt}\right)$$

1st order ODE RK2 ?

Coupled 1st order ODEs

$$\frac{dx}{dt} = v(t,x)$$

$$\frac{dv}{dt} = \mathcal{K}(t,x,v)$$

RK

$$\begin{gathered} G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + dt, x(t_n) + G_1 dt) \\ x(t_{n+1}) = x(t_n) + \frac{dt}{2}(G_1 + G_2) \end{gathered}$$

Explicit-in-time integrator

2nd order ODE

1st order ODE RK2 ?

$$\frac{d^2x}{dt^2} = \mathcal{K}\left(t, x, \frac{dx}{dt}\right)$$

Coupled 1st order ODEs

$$\frac{dx}{dt} = v(t,x)$$

$$\frac{dv}{dt} = \mathcal{K}(t,x,v)$$

RK

RK

$$G_1 = v(t_n, x(t_n))$$
$$G_2 = v(t_n + dt, x(t_n) + G_1 dt)$$
$$x(t_{n+1}) = x(t_n) + \frac{dt}{2}(G_1 + G_2)$$

$$G_1^* = \mathcal{K}(t_n, x(t_n), v(t_n))$$
$$G_2^* = \mathcal{K}(t_n + dt, x(t_n) + dtG_1, v(t_n) + dtG_1^*)$$
$$v(t_{n+1}) = v(t_n) + \frac{dt}{2}(G_1^* + G_2^*)$$

Explicit-in-time integrator

Axion-Inflation using RK2

Explicit-in-time integrator

Axion-Inflation using RK2

$t_n \equiv (1)$

Explicit-in-time integrator

Axion-Inflation using RK2

$t_n \equiv (1)$

$$\begin{cases} \phi^{(1)} = \phi(t_n) \ , \\ A_i^{(1)} = A_i(t_n) \ , \\ a^{(1)} = a(t_n) \ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n) \ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n) \ , \\ \pi_a^{(1)} = \pi_a(t_n) \ , \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_n \equiv (1)$$

$$\begin{cases} \phi^{(1)} = \phi(t_n)\ , \\ A_i^{(1)} = A_i(t_n)\ , \\ a^{(1)} = a(t_n)\ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n)\ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n)\ , \\ \pi_a^{(1)} = \pi_a(t_n)\ , \end{cases}$$

$$\begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_n \equiv (1)$$

$$\begin{cases} \phi^{(1)} = \phi(t_n)\ , \\ A_i^{(1)} = A_i(t_n)\ , \\ a^{(1)} = a(t_n)\ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n)\ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n)\ , \\ \pi_a^{(1)} = \pi_a(t_n)\ , \end{cases} \longrightarrow \begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases} \longrightarrow \begin{cases} \mathcal{K}_\phi^{(1)} = \mathcal{K}_\phi\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \tilde{\vec{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_A^{(1)} = \mathcal{K}_A\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \tilde{\vec{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_a^{(1)} = \mathcal{K}_a\left[a^{(1)}, K_\phi^{(1)}, V^{(1)}, K_A^{(1)}, G_A^{(1)}\right]\ . \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_n \equiv (1)$$

$$\begin{cases} \phi^{(1)} = \phi(t_n)\ , \\ A_i^{(1)} = A_i(t_n)\ , \\ a^{(1)} = a(t_n)\ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n)\ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n)\ , \\ \pi_a^{(1)} = \pi_a(t_n)\ , \end{cases}$$

$$\begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases}$$

$$\begin{cases} \mathcal{K}_\phi^{(1)} = \mathcal{K}_\phi\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_A^{(1)} = \mathcal{K}_A\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_a^{(1)} = \mathcal{K}_a\left[a^{(1)}, K_\phi^{(1)}, V^{(1)}, K_A^{(1)}, G_A^{(1)}\right]\ . \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$t_n \equiv (1)$

$$\begin{cases} \phi^{(1)} = \phi(t_n)\ , \\ A_i^{(1)} = A_i(t_n)\ , \\ a^{(1)} = a(t_n)\ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n)\ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n)\ , \\ \pi_a^{(1)} = \pi_a(t_n)\ , \end{cases}$$

$$\begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases}$$

$$\begin{cases} \mathcal{K}_\phi^{(1)} = \mathcal{K}_\phi\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_A^{(1)} = \mathcal{K}_A\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_a^{(1)} = \mathcal{K}_a\left[a^{(1)}, K_\phi^{(1)}, V^{(1)}, K_A^{(1)}, G_A^{(1)}\right]\ . \end{cases}$$

$t_{n+1} \equiv (2)$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t\ , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t\ , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t\ , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t\ , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t\ , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t\ . \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_n \equiv (1) \qquad \begin{cases} \phi^{(1)} = \phi(t_n)\ , \\ A_i^{(1)} = A_i(t_n)\ , \\ a^{(1)} = a(t_n)\ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n)\ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n)\ , \\ \pi_a^{(1)} = \pi_a(t_n)\ , \end{cases} \longrightarrow \begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases} \longrightarrow \begin{cases} \mathcal{K}_\phi^{(1)} = \mathcal{K}_\phi\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_A^{(1)} = \mathcal{K}_A\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_a^{(1)} = \mathcal{K}_a\left[a^{(1)}, K_\phi^{(1)}, V^{(1)}, K_A^{(1)}, G_A^{(1)}\right]\ . \end{cases}$$

$$t_{n+1} \equiv (2) \qquad \begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t\ , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t\ , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t\ , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t\ , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t\ , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t\ . \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$t_n \equiv (1)$

$$\begin{cases} \phi^{(1)} = \phi(t_n) \ , \\ A_i^{(1)} = A_i(t_n) \ , \\ a^{(1)} = a(t_n) \ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n) \ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n) \ , \\ \pi_a^{(1)} = \pi_a(t_n) \ , \end{cases} \quad \begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases} \quad \begin{cases} \mathcal{K}_\phi^{(1)} = \mathcal{K}_\phi\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right] \ , \\ \mathcal{K}_A^{(1)} = \mathcal{K}_A\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right] \ , \\ \mathcal{K}_a^{(1)} = \mathcal{K}_a\left[a^{(1)}, K_\phi^{(1)}, V^{(1)}, K_A^{(1)}, G_A^{(1)}\right] \ . \end{cases}$$

$t_{n+1} \equiv (2)$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t \ , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t \ , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t \ , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t \ , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t \ , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t \ . \end{cases} \quad \begin{cases} K_\phi^{(2)} = K_\phi(a^{(2)}, \tilde{\pi}_\phi^{(2)}) \\ V^{(2)} = V(\phi^{(2)}) \\ K_A^{(2)} = K_A(a^{(2)}, \tilde{E}_i^{(2)}) \\ G_A^{(2)} = G_A(a^{(2)}, A_i^{(2)}) \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_n \equiv (1)$$

$$\begin{cases} \phi^{(1)} = \phi(t_n) \,, \\ A_i^{(1)} = A_i(t_n) \,, \\ a^{(1)} = a(t_n) \,, \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n) \,, \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n) \,, \\ \pi_a^{(1)} = \pi_a(t_n) \,, \end{cases} \quad \begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases} \quad \begin{cases} \mathcal{K}_\phi^{(1)} = \mathcal{K}_\phi\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right] \,, \\ \mathcal{K}_A^{(1)} = \mathcal{K}_A\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right] \,, \\ \mathcal{K}_a^{(1)} = \mathcal{K}_a\left[a^{(1)}, K_\phi^{(1)}, V^{(1)}, K_A^{(1)}, G_A^{(1)}\right] \,. \end{cases}$$

$$t_{n+1} \equiv (2)$$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t \,, \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t \,, \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t \,, \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t \,, \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t \,, \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t \,. \end{cases} \quad \begin{cases} K_\phi^{(2)} = K_\phi(a^{(2)}, \tilde{\pi}_\phi^{(2)}) \\ V^{(2)} = V(\phi^{(2)}) \\ K_A^{(2)} = K_A(a^{(2)}, \tilde{E}_i^{(2)}) \\ G_A^{(2)} = G_A(a^{(2)}, A_i^{(2)}) \end{cases} \quad \begin{cases} \mathcal{K}_\phi^{(2)} = \mathcal{K}_\phi\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \vec{\tilde{E}}^{(2)}, \vec{B}^{(2)}\right] \,, \\ \mathcal{K}_A^{(2)} = \mathcal{K}_A\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \vec{\tilde{E}}^{(2)}, \vec{B}^{(2)}\right] \,, \\ \mathcal{K}_a^{(2)} = \mathcal{K}_a\left[a^{(2)}, K_\phi^{(2)}, V^{(2)}, K_A^{(2)}, G_A^{(2)}\right] \,. \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$t_n \equiv (1)$

$$\begin{cases} \phi^{(1)} = \phi(t_n)\ , \\ A_i^{(1)} = A_i(t_n)\ , \\ a^{(1)} = a(t_n)\ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n)\ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n)\ , \\ \pi_a^{(1)} = \pi_a(t_n)\ , \end{cases} \quad \begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases} \quad \begin{cases} \mathcal{K}_\phi^{(1)} = \mathcal{K}_\phi\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_A^{(1)} = \mathcal{K}_A\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_a^{(1)} = \mathcal{K}_a\left[a^{(1)}, K_\phi^{(1)}, V^{(1)}, K_A^{(1)}, G_A^{(1)}\right]\ . \end{cases}$$

$t_{n+1} \equiv (2)$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t\ , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t\ , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t\ , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t\ , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t\ , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t\ . \end{cases} \quad \begin{cases} K_\phi^{(2)} = K_\phi(a^{(2)}, \tilde{\pi}_\phi^{(2)}) \\ V^{(2)} = V(\phi^{(2)}) \\ K_A^{(2)} = K_A(a^{(2)}, \tilde{E}_i^{(2)}) \\ G_A^{(2)} = G_A(a^{(2)}, A_i^{(2)}) \end{cases} \quad \begin{cases} \mathcal{K}_\phi^{(2)} = \mathcal{K}_\phi\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \vec{\tilde{E}}^{(2)}, \vec{B}^{(2)}\right]\ , \\ \mathcal{K}_A^{(2)} = \mathcal{K}_A\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \vec{\tilde{E}}^{(2)}, \vec{B}^{(2)}\right]\ , \\ \mathcal{K}_a^{(2)} = \mathcal{K}_a\left[a^{(2)}, K_\phi^{(2)}, V^{(2)}, K_A^{(2)}, G_A^{(2)}\right]\ . \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$t_{n+1} \equiv (2)$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t \ , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t \ , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t \ , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t \ , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t \ , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t \ . \end{cases}$$

$$\begin{cases} K_\phi^{(2)} = K_\phi(a^{(2)}, \tilde{\pi}_\phi^{(2)}) \\ V^{(2)} = V(\phi^{(2)}) \\ K_A^{(2)} = K_A(a^{(2)}, \tilde{E}_i^{(2)}) \\ G_A^{(2)} = G_A(a^{(2)}, A_i^{(2)}) \end{cases}$$

$$\begin{cases} \mathcal{K}_\phi^{(2)} = \mathcal{K}_\phi\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \vec{\tilde{E}}^{(2)}, \vec{B}^{(2)}\right] \ , \\ \mathcal{K}_A^{(2)} = \mathcal{K}_A\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \vec{\tilde{E}}^{(2)}, \vec{B}^{(2)}\right] \ , \\ \mathcal{K}_a^{(2)} = \mathcal{K}_a\left[a^{(2)}, K_\phi^{(2)}, V^{(2)}, K_A^{(2)}, G_A^{(2)}\right] \ . \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_{n+1} \equiv (2)$$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t \ , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t \ , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t \ , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t \ , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t \ , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t \ . \end{cases}$$

$$\begin{cases} K_\phi^{(2)} = K_\phi(a^{(2)}, \tilde{\pi}_\phi^{(2)}) \\ V^{(2)} = V(\phi^{(2)}) \\ K_A^{(2)} = K_A(a^{(2)}, \tilde{E}_i^{(2)}) \\ G_A^{(2)} = G_A(a^{(2)}, A_i^{(2)}) \end{cases}$$

$$\begin{cases} \mathcal{K}_\phi^{(2)} = \mathcal{K}_\phi\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \tilde{\vec{E}}^{(2)}, \vec{B}^{(2)}\right] \ , \\ \mathcal{K}_A^{(2)} = \mathcal{K}_A\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \tilde{\vec{E}}^{(2)}, \vec{B}^{(2)}\right] \ , \\ \mathcal{K}_a^{(2)} = \mathcal{K}_a\left[a^{(2)}, K_\phi^{(2)}, V^{(2)}, K_A^{(2)}, G_A^{(2)}\right] \ . \end{cases}$$

$$\phi(t_{n+1}) = \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2$$

$$A_i(t_{n+1}) = A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2$$

$$a(t_{n+1}) = a(t_n) + (\pi_a^{(1)} + \pi_a^{(2)})\delta t/2$$

$$\tilde{\pi}_\phi(t_{n+1}) = \tilde{\pi}_\phi(t_n) + \frac{1}{2}(\mathcal{K}_\phi^{(1)} + \mathcal{K}_\phi^{(2)})\delta t$$

$$\tilde{E}_i(t_{n+1}) = \tilde{E}_i(t_n) + \frac{1}{2}(\mathcal{K}_A^{(1)} + \mathcal{K}_A^{(2)})\delta t$$

$$\pi_a(t_{n+1}) = \pi_a(t_n) + \frac{1}{2}(\mathcal{K}_a^{(1)} + \mathcal{K}_a^{(2)})\delta t$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_{n+1} \equiv (2)$$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t \,, \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t \,, \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t \,, \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t \,, \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t \,, \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t \,. \end{cases}$$

$$\begin{cases} K_\phi^{(2)} = K_\phi(a^{(2)}, \tilde{\pi}_\phi^{(2)}) \\ V^{(2)} = V(\phi^{(2)}) \\ K_A^{(2)} = K_A(a^{(2)}, \tilde{E}_i^{(2)}) \\ G_A^{(2)} = G_A(a^{(2)}, A_i^{(2)}) \end{cases}$$

$$\begin{cases} \mathcal{K}_\phi^{(2)} = \mathcal{K}_\phi\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \tilde{\vec{E}}^{(2)}, \vec{B}^{(2)}\right] \,, \\ \mathcal{K}_A^{(2)} = \mathcal{K}_A\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \tilde{\vec{E}}^{(2)}, \vec{B}^{(2)}\right] \,, \\ \mathcal{K}_a^{(2)} = \mathcal{K}_a\left[a^{(2)}, K_\phi^{(2)}, V^{(2)}, K_A^{(2)}, G_A^{(2)}\right] \,. \end{cases}$$

$$\boxed{\begin{aligned} \phi(t_{n+1}) &= \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2 \\ A_i(t_{n+1}) &= A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2 \\ a(t_{n+1}) &= a(t_n) + (\pi_a^{(1)} + \pi_a^{(2)})\delta t/2 \end{aligned}}$$

$$\boxed{\begin{aligned} \tilde{\pi}_\phi(t_{n+1}) &= \tilde{\pi}_\phi(t_n) + \frac{1}{2}(\mathcal{K}_\phi^{(1)} + \mathcal{K}_\phi^{(2)})\delta t \\ \tilde{E}_i(t_{n+1}) &= \tilde{E}_i(t_n) + \frac{1}{2}(\mathcal{K}_A^{(1)} + \mathcal{K}_A^{(2)})\delta t \\ \pi_a(t_{n+1}) &= \pi_a(t_n) + \frac{1}{2}(\mathcal{K}_a^{(1)} + \mathcal{K}_a^{(2)})\delta t \end{aligned}}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$t_{n+1} \equiv (2)$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t \; , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t \; , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t \; , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t \; , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t \; , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t \; . \end{cases}$$

$$\begin{cases} K_\phi^{(2)} = K_\phi(a^{(2)}, \tilde{\pi}_\phi^{(2)}) \\ V^{(2)} = V(\phi^{(2)}) \\ K_A^{(2)} = K_A(a^{(2)}, \tilde{E}_i^{(2)}) \\ G_A^{(2)} = G_A(a^{(2)}, A_i^{(2)}) \end{cases}$$

$$\begin{cases} \mathcal{K}_\phi^{(2)} = \mathcal{K}_\phi\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \vec{\tilde{E}}^{(2)}, \vec{B}^{(2)}\right] \; , \\ \mathcal{K}_A^{(2)} = \mathcal{K}_A\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \vec{\tilde{E}}^{(2)}, \vec{B}^{(2)}\right] \; , \\ \mathcal{K}_a^{(2)} = \mathcal{K}_a\left[a^{(2)}, K_\phi^{(2)}, V^{(2)}, K_A^{(2)}, G_A^{(2)}\right] \; . \end{cases}$$

$$\phi(t_{n+1}) = \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2$$
$$A_i(t_{n+1}) = A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2$$
$$a(t_{n+1}) = a(t_n) + (\pi_a^{(1)} + \pi_a^{(2)})\delta t/2$$

$$\tilde{\pi}_\phi(t_{n+1}) = \tilde{\pi}_\phi(t_n) + \frac{1}{2}(\mathcal{K}_\phi^{(1)} + \mathcal{K}_\phi^{(2)})\delta t$$
$$\tilde{E}_i(t_{n+1}) = \tilde{E}_i(t_n) + \frac{1}{2}(\mathcal{K}_A^{(1)} + \mathcal{K}_A^{(2)})\delta t$$
$$\pi_a(t_{n+1}) = \pi_a(t_n) + \frac{1}{2}(\mathcal{K}_a^{(1)} + \mathcal{K}_a^{(2)})\delta t$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$\phi(t_{n+1}) = \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2$$

$$A_i(t_{n+1}) = A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2$$

$$a(t_{n+1}) = a(t_n) + (\pi_a^{(1)} + \pi_a^{(2)})\delta t/2$$

$$\tilde{\pi}_\phi(t_{n+1}) = \tilde{\pi}_\phi(t_n) + \frac{1}{2}(\mathcal{K}_\phi^{(1)} + \mathcal{K}_\phi^{(2)})\delta t$$

$$\tilde{E}_i(t_{n+1}) = \tilde{E}_i(t_n) + \frac{1}{2}(\mathcal{K}_A^{(1)} + \mathcal{K}_A^{(2)})\delta t$$

$$\pi_a(t_{n+1}) = \pi_a(t_n) + \frac{1}{2}(\mathcal{K}_a^{(1)} + \mathcal{K}_a^{(2)})\delta t$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$\phi(t_{n+1}) = \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2$$

$$A_i(t_{n+1}) = A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2$$

$$a(t_{n+1}) = a(t_n) + (\pi_a^{(1)} + \pi_a^{(2)})\delta t/2$$

$$\tilde{\pi}_\phi(t_{n+1}) = \tilde{\pi}_\phi(t_n) + \frac{1}{2}(\mathcal{K}_\phi^{(1)} + \mathcal{K}_\phi^{(2)})\delta t$$

$$\tilde{E}_i(t_{n+1}) = \tilde{E}_i(t_n) + \frac{1}{2}(\mathcal{K}_A^{(1)} + \mathcal{K}_A^{(2)})\delta t$$

$$\pi_a(t_{n+1}) = \pi_a(t_n) + \frac{1}{2}(\mathcal{K}_a^{(1)} + \mathcal{K}_a^{(2)})\delta t$$

More intermediate kernels

Explicit-in-time integrator

Axion-Inflation using RK2

$$\phi(t_{n+1}) = \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2$$

$$A_i(t_{n+1}) = A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2$$

$$a(t_{n+1}) = a(t_n) + (\pi_a^{(1)} + \pi_a^{(2)})\delta t/2$$

$$\tilde{\pi}_\phi(t_{n+1}) = \tilde{\pi}_\phi(t_n) + \frac{1}{2}(\mathcal{K}_\phi^{(1)} + \mathcal{K}_\phi^{(2)})\delta t$$

$$\tilde{E}_i(t_{n+1}) = \tilde{E}_i(t_n) + \frac{1}{2}(\mathcal{K}_A^{(1)} + \mathcal{K}_A^{(2)})\delta t$$

$$\pi_a(t_{n+1}) = \pi_a(t_n) + \frac{1}{2}(\mathcal{K}_a^{(1)} + \mathcal{K}_a^{(2)})\delta t$$

Aux fields

More intermediate kernels

Explicit-in-time integrator

Axion-Inflation using RK2

$$\phi(t_{n+1}) = \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2$$
$$A_i(t_{n+1}) = A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2$$
$$a(t_{n+1}) = a(t_n) + (\pi_a^{(1)} + \pi_a^{(2)})\delta t/2$$

$$\tilde{\pi}_\phi(t_{n+1}) = \tilde{\pi}_\phi(t_n) + \frac{1}{2}(\mathcal{K}_\phi^{(1)} + \mathcal{K}_\phi^{(2)})\delta t$$
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More intermediate kernels $\rightarrow$ Aux fields $\rightarrow$ Memory disadvantages

Explicit-in-time integrator

Axion-Inflation using RK2

$$\phi(t_{n+1}) = \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2$$
$$A_i(t_{n+1}) = A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2$$
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Aux fields → Memory disadvantages

More intermediate kernels

More steps

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More intermediate kernels → Aux fields → Memory disadvantages

More intermediate kernels → More steps → Time disadvantages

→ We use low storage RK methods

[D. G. Figueroa, A. Florio, T. Opferkuch, B. Stefanek (2112.08388)]

Axion-Inflation using RK2

Explicit-in-time integrator

Axion-Inflation using RK2

Dynamical equations

$$\mathcal{K}_\phi^L = a\sum_i \Delta_i^-\Delta_i^+\phi - a^3m^2\phi + \frac{1}{2a\Lambda}\sum_i \tilde{E}_i^{(2)}B_i^{(4)}$$

$$\mathcal{K}_A^L = -\frac{1}{a}\sum_{j,k}\epsilon_{ijk}\Delta_j^- B_k - \frac{1}{2a^3\Lambda}\left(\tilde{\pi}_\phi B_i^{(4)} + \tilde{\pi}_{\phi,+i}B_{i,+i}^{(4)}\right)$$

$$+\frac{1}{4a\Lambda}\sum_\pm\sum_{j,k}\epsilon_{ijk}\left\{\left[(\Delta_j^\pm\phi)\tilde{E}_{k,\pm j}^{(2)}\right]_{+i} + \left[(\Delta_j^\pm\phi)\tilde{E}_{k,\pm j}^{(2)}\right]\right\}$$

$$\mathcal{K}_a^L = -\frac{a}{6m_{pl}^2}(\rho_L + 3p_L)$$

Axion-Inflation using RK2

Dynamical equations

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$$\mathcal{K}_a^L = -\frac{a}{6m_{pl}^2}(\rho_L + 3p_L)$$

Constraints

$$\sum_i \Delta_i^- \tilde{E}_i = -\frac{1}{2\Lambda}\sum_\pm\sum_i \left(\Delta_i^\pm\phi\right) B_{i,\pm i}^{(4)}$$

$$\pi_a^2 = \frac{a^2}{3m_{pl}^2}\rho_L$$

Explicit-in-time integrator

Axion-Inflation using RK2

No temporal semi-sums

Dynamical equations

$$\mathcal{K}_\phi^L = a\sum_i \Delta_i^- \Delta_i^+ \phi - a^3 m^2 \phi + \frac{1}{2a\Lambda}\sum_i \tilde{E}_i^{(2)} B_i^{(4)}$$

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Axion-Inflation using RK2

Dynamical equations

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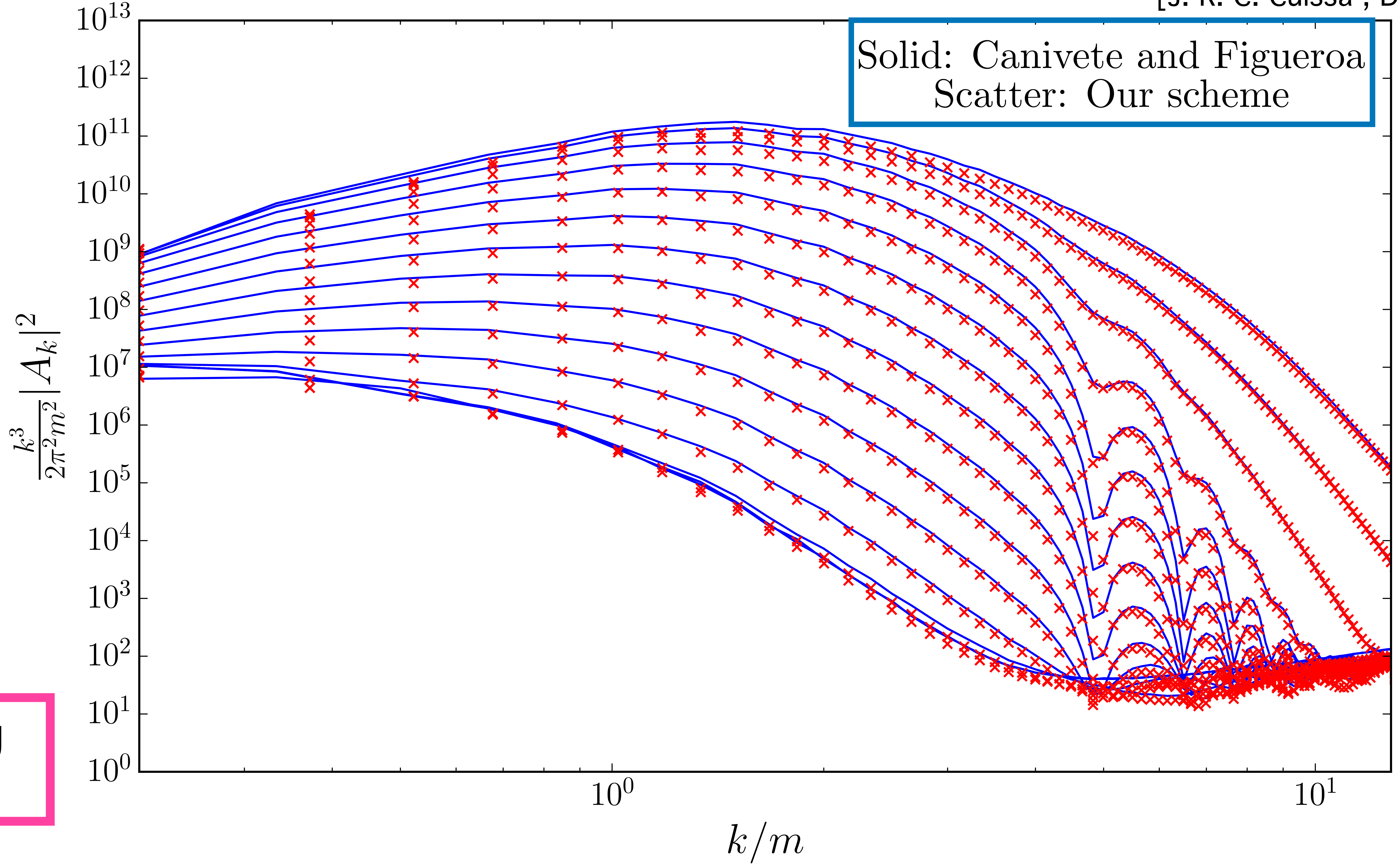
Explicit-in-time integrator

Axion-Inflation using RK2: validation of the scheme

[J. R. C. Cuissa , D. .G. Figueroa (1812.03132)]

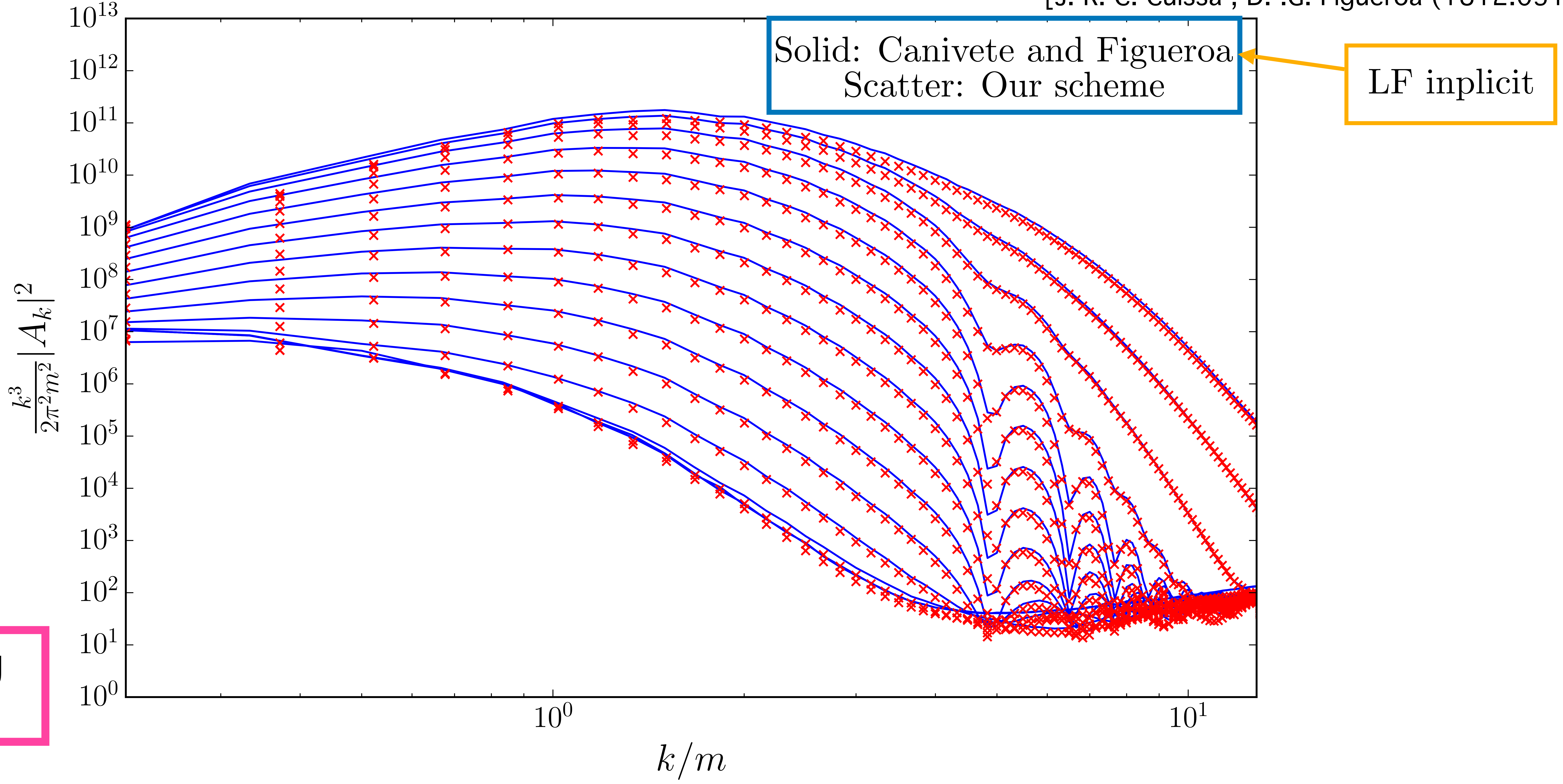


Preheating example

Explicit-in-time integrator

Axion-Inflation using RK2: validation of the scheme

[J. R. C. Cuissa , D. .G. Figueroa (1812.03132)]



Preheating example

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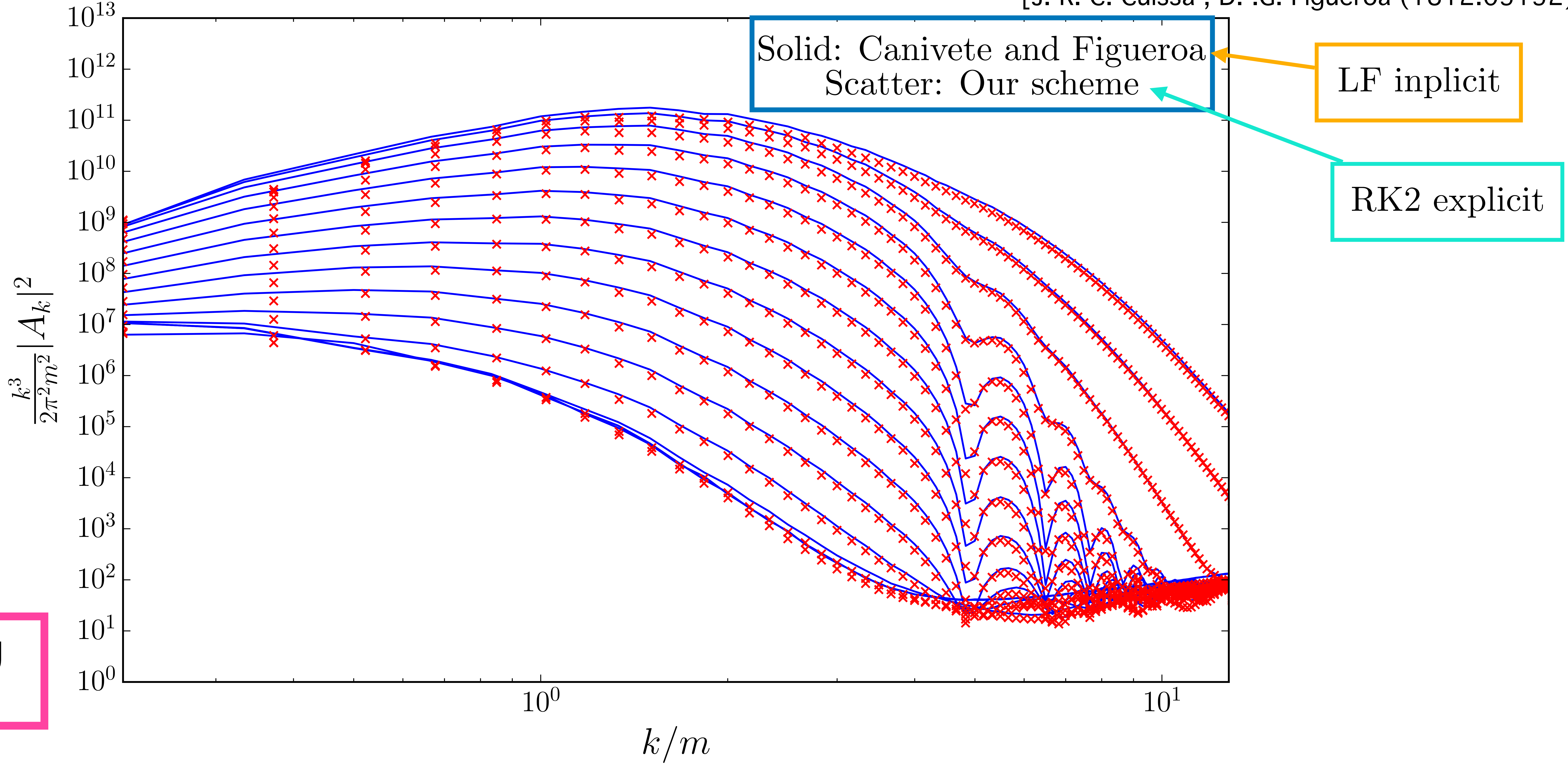

Preheating example

Explicit-in-time integrator

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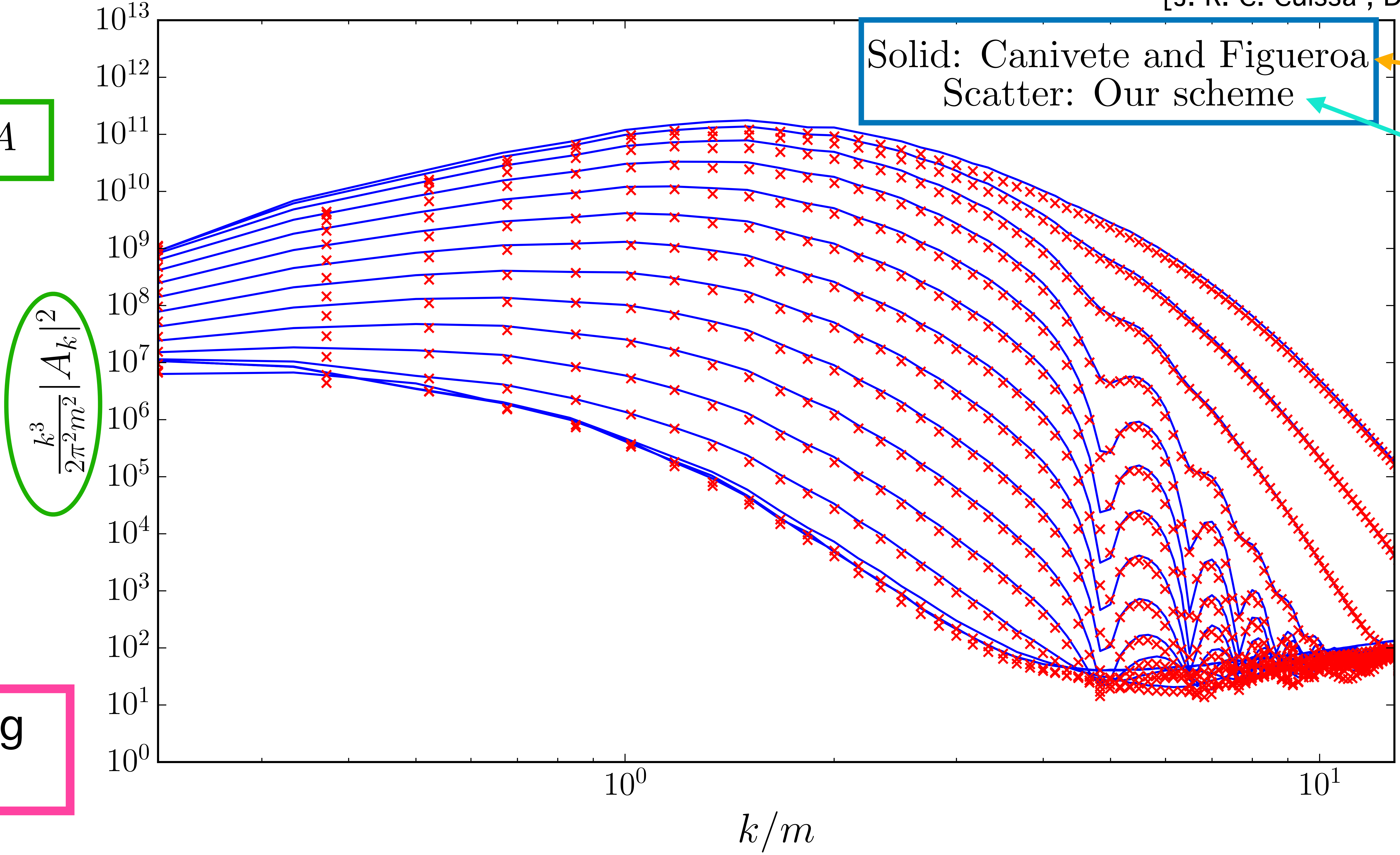
Spectrum of A



LF inplicit

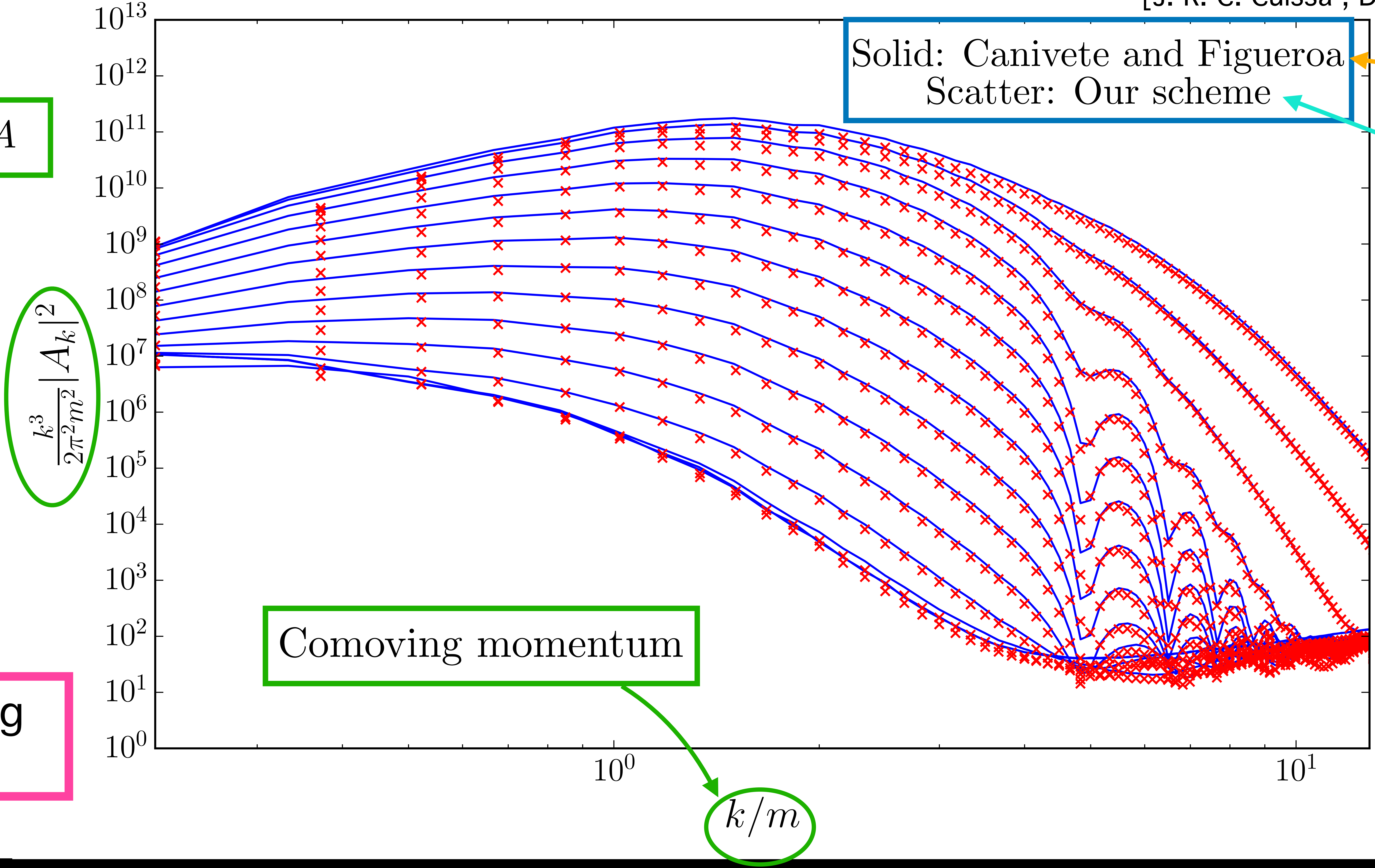
RK2 explicit

Preheating example

Axion-Inflation using RK2: validation of the scheme

[J. R. C. Cuissa , D. .G. Figueroa (1812.03132)]

Spectrum of A



LF inplicit

RK2 explicit

Comoving momentum

Preheating example

Explicit-in-time integrator

Axion-Inflation using RK2: validation of the scheme

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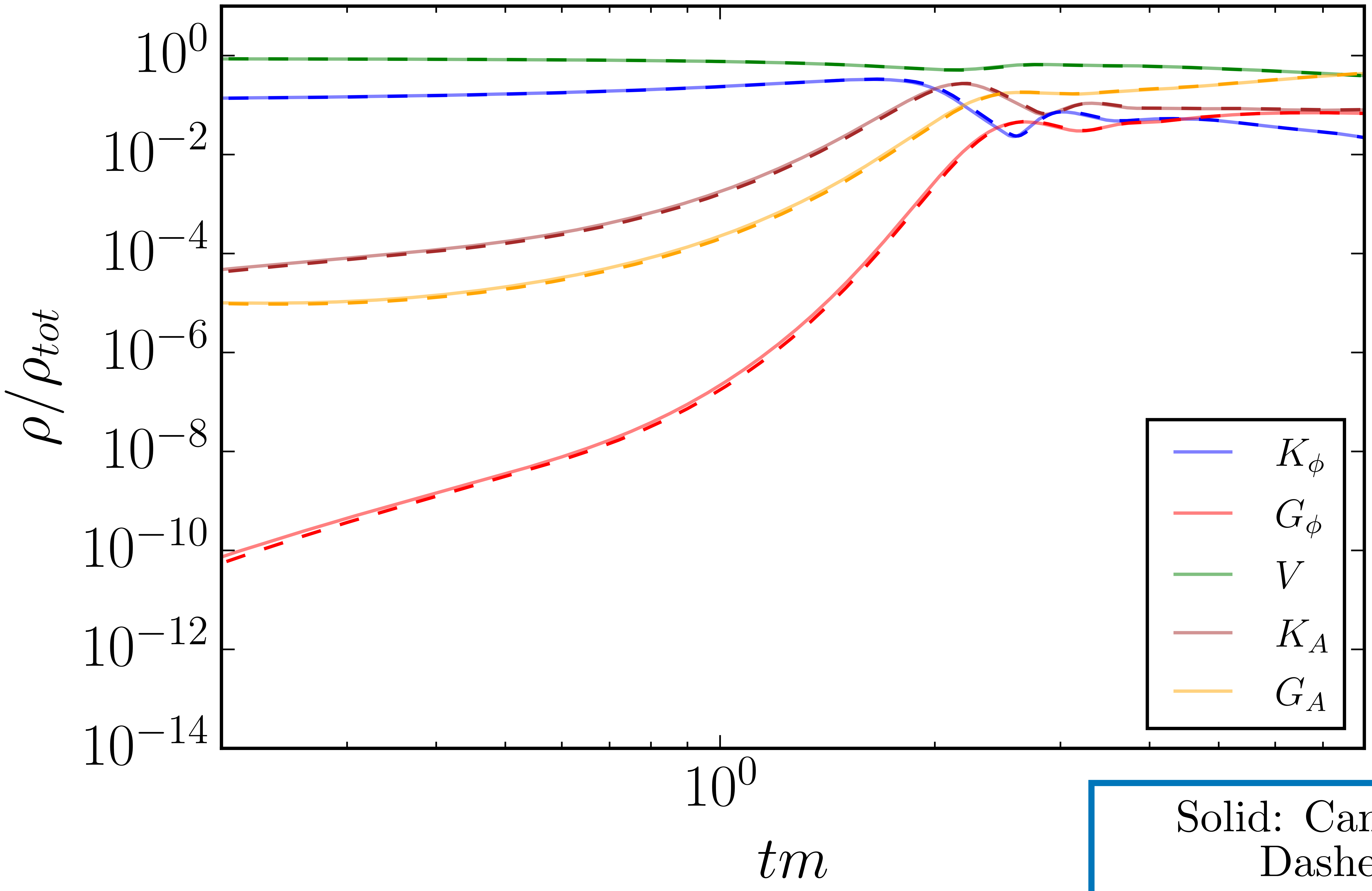


Preheating example

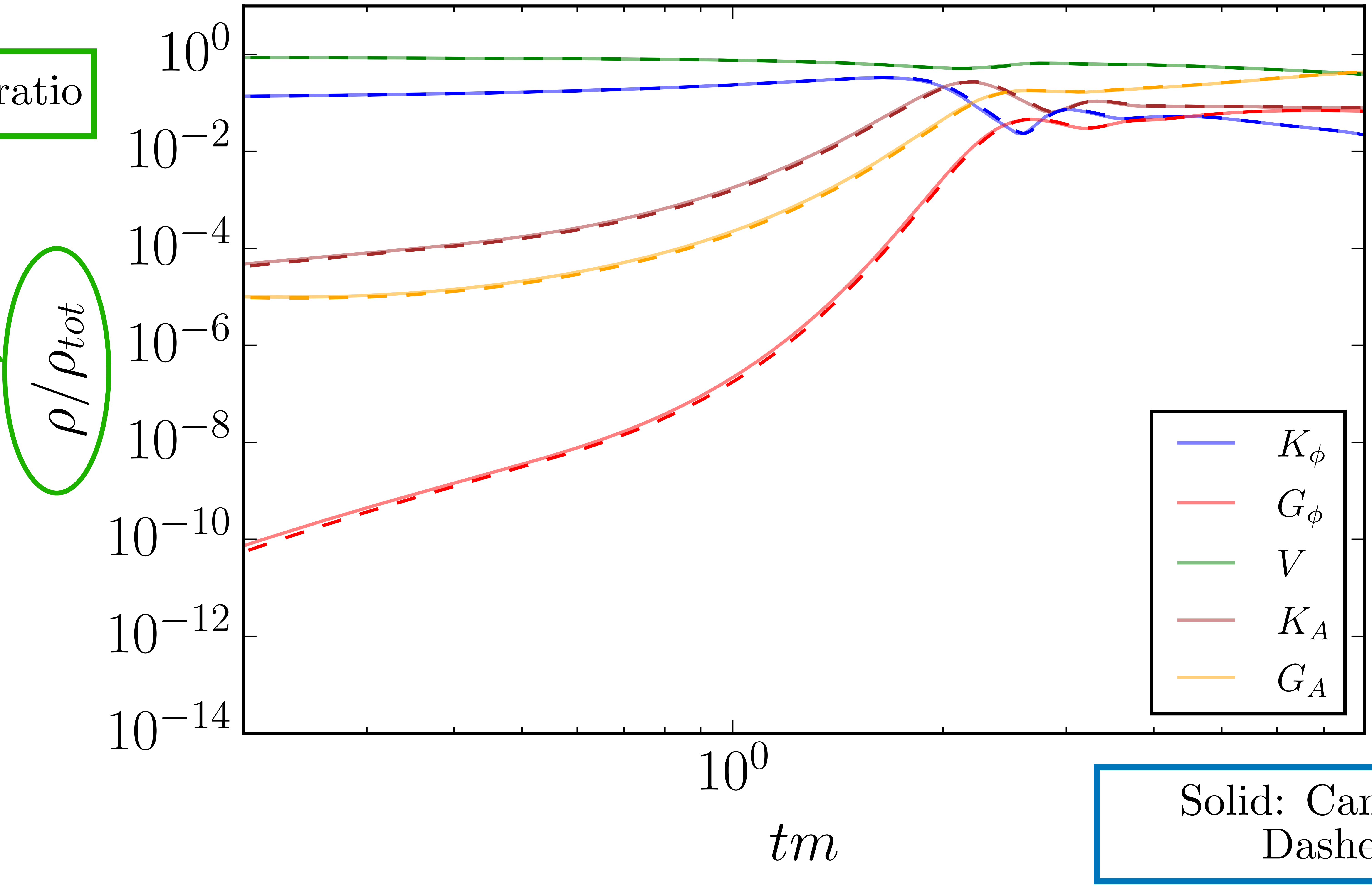
Solid: Canivete and Figueroa
Dashed: Our scheme

Explicit-in-time integrator

Axion-Inflation using RK2: validation of the scheme

[J. R. C. Cuissa , D. .G. Figueroa (1812.03132)]

Energy density ratio



Preheating example

Solid: Canivete and Figueroa
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Explicit-in-time integrator

Axion-Inflation using RK2: validation of the scheme

[J. R. C. Cuissa , D. .G. Figueroa (1812.03132)]

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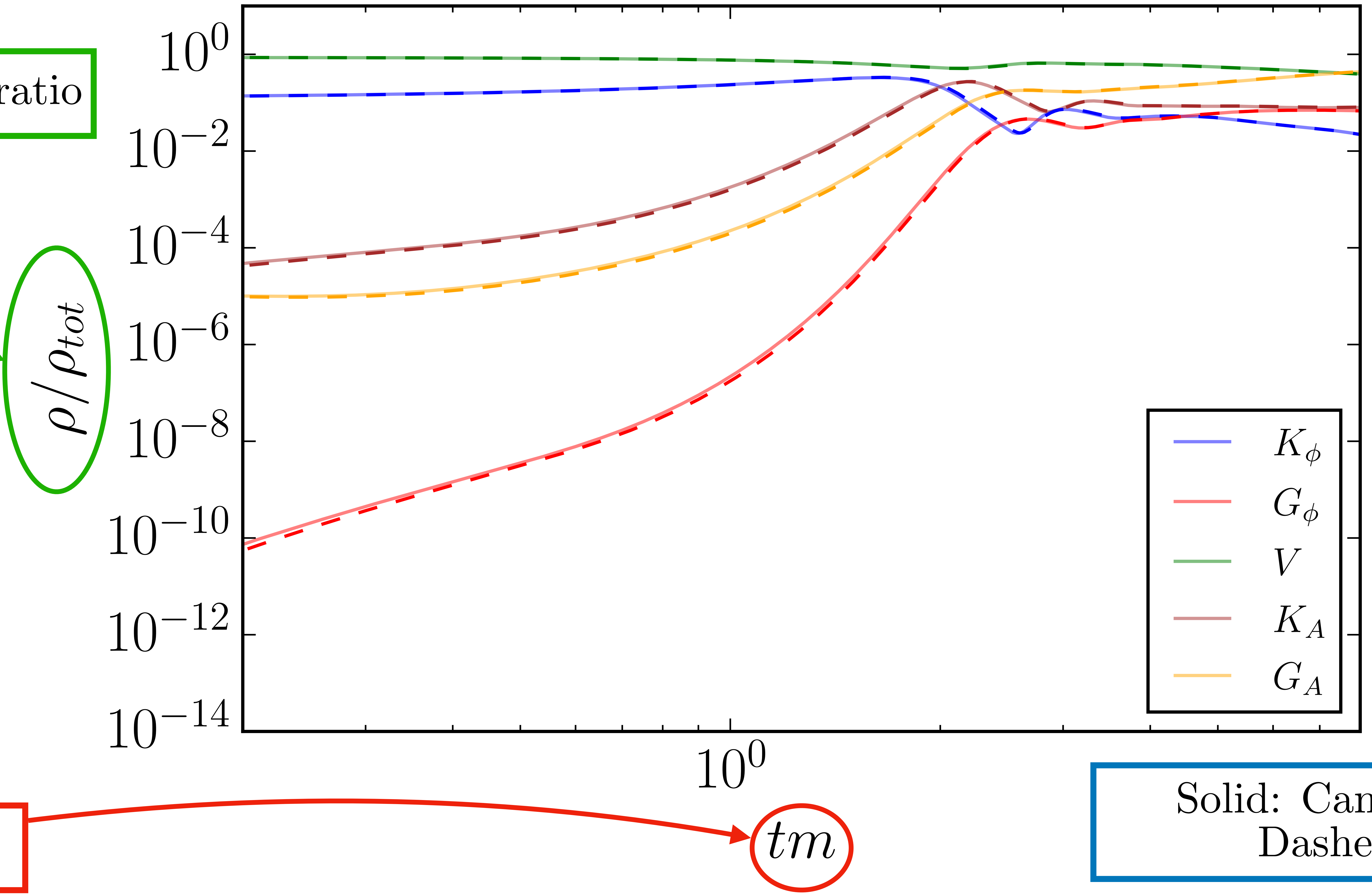


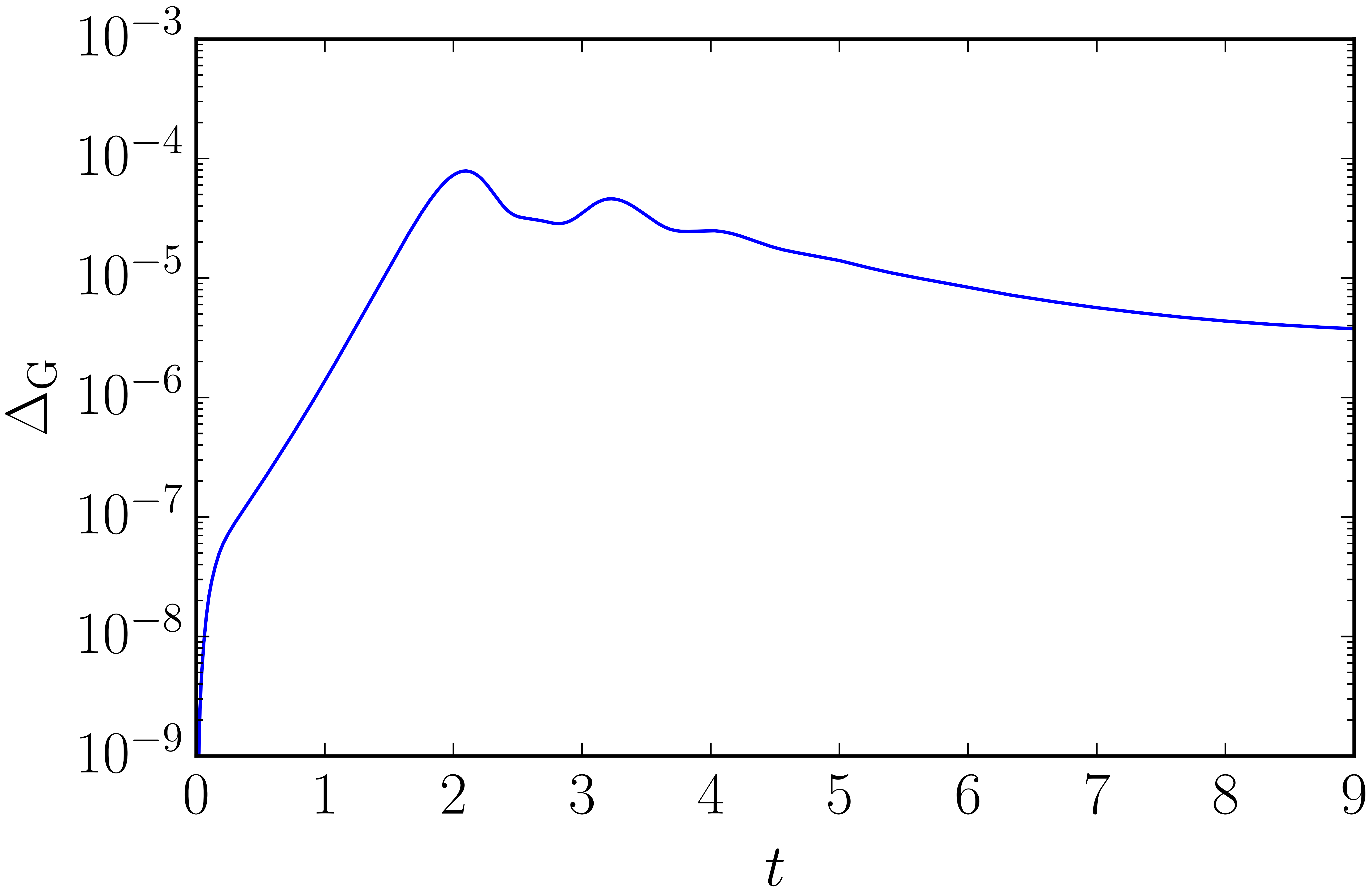
Preheating example

Cosmic time

Solid: Canivete and Figueroa
Dashed: Our scheme

Explicit-in-time integrator

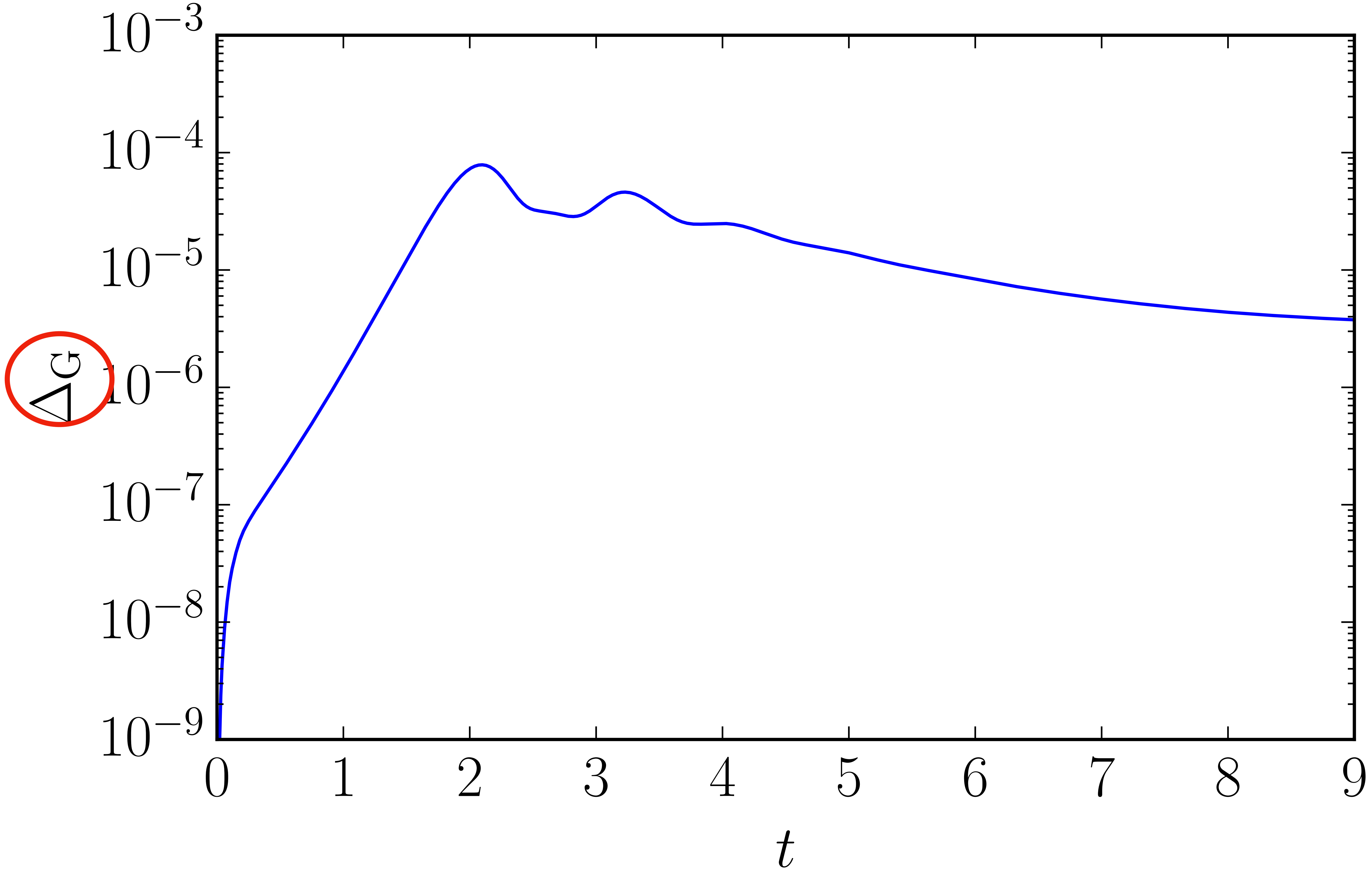
Axion-Inflation using RK2: validation of the scheme



Explicit-in-time integrator

Axion-Inflation using RK2: validation of the scheme

Gauss constraint observable



Initial Conditions in the Linear Regime

$$\dot{\tilde{\vec{E}}} = -\frac{1}{a}\vec{\nabla} \times \vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi \vec{B}$$

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Helicity basis

Conformal time

Initial Conditions in the Linear Regime

$$\dot{\vec{\tilde{E}}} = -\frac{1}{a}\vec{\nabla}\times\vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B} \quad \xrightarrow[\text{Conformal time}]{\text{Helicity basis}} \quad \left(\frac{\partial^2}{\partial\tau^2} + k^2 \pm \frac{2k\xi}{\tau}\right)A_\pm(k,\tau) = 0$$

Initial Conditions in the Linear Regime

$$\xi \equiv \frac{\dot{\phi}}{2H\Lambda}$$

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Deep inside Hubble $k \gg aH$

$$\left(\frac{\partial^2}{\partial\tau^2} + k^2\right) A_+(k,\tau) \simeq 0$$

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Bunch-Davies solution

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Cosmic time

$\tau \simeq -1/aH$

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Time derivative

Initial Conditions in the Linear Regime

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Helicity basis

Conformal time

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Time derivative

$$E_+(k,t) \simeq \frac{i}{a}\sqrt{\frac{k}{2}}e^{ik/a(t)H(t)}$$

Initial Conditions in the Linear Regime

$$\dot{\tilde{\vec{E}}} = -\frac{1}{a}\vec{\nabla}\times\vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B}$$

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Initial Conditions in the Linear Regime

Helicity basis

BD solutions for $A^{\pm}\,, E^{\pm}$

Initial Conditions in the Linear Regime

Helicity basis

BD solutions for $A^{\pm}\,,E^{\pm}$ $\longrightarrow$ CosmoLattice $A\,,E$

Cartesian basis

Initial Conditions in the Linear Regime

Helicity basis

BD solutions for $A^{\pm}$, $E^{\pm}$

Cartesian basis

CosmoLattice A , E

Objective

BD solution for each helicity starting from cartesian basis

Initial Conditions in the Linear Regime

Helicity basis

BD solutions for $A^{\pm}\ , E^{\pm}$

Cartesian basis

CosmoLattice $A\ , E$

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BD solution for each helicity

starting from cartesian basis

2 options

....

Initial Conditions in the Linear Regime

Helicity basis

BD solutions for $A^{\pm}$, $E^{\pm}$

Cartesian basis

CosmoLattice A , E

Objective

BD solution for each helicity

starting from cartesian basis

2 options

Helicity vectors ....

$$k_i \varepsilon_i^{\pm}(\hat{k}) = 0\ ,\ \ \epsilon_{ijk} k_j \varepsilon^{\pm}(\hat{k}) = \mp i k \varepsilon_i^{\pm}(\hat{k})\ ,$$

$$\varepsilon_i^{\pm}(\hat{k})^* = \varepsilon_i^{\pm}(-\hat{k})\ ,\ \ \varepsilon_i^{\lambda}(\hat{k}) \varepsilon_i^{\lambda'}(-\hat{k}) = \delta_{\lambda\lambda'}$$

Helicity basis

BD solutions for $A^{\pm}$, $E^{\pm}$

$\longrightarrow$

Cartesian basis

CosmoLattice A , E

Objective

BD solution for each helicity starting from cartesian basis

2 options

Helicity vectors

$$k_i \varepsilon_i^{\pm}(\hat{k}) = 0 \; , \;\; \epsilon_{ijk} k_j \varepsilon^{\pm}(\hat{k}) = \mp i k \varepsilon_i^{\pm}(\hat{k}) \; ,$$

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Projectors ✓

Use helicity projectors

$$A_i^{\pm} = P_{ij}^{\pm} A_j$$

Initial Conditions in the Linear Regime

How to build the Helicity Projector?

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Helicity operator

$$\Sigma^{\pm}_{ij}(k) = \pm\frac{i}{k}\epsilon_{ijk}k_k$$

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Transverse projector

$$(\Sigma^{\pm})^2_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{k^2}$$

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How to build the Helicity Projector?

2 eigenvalues: 1 and -1

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Helicity projector

$$P^{\pm}_{ij} = \frac{1}{2}\left(\Sigma^{\pm}_{ij} + (\Sigma^{\pm})^2_{ij}\right)$$

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$$\boxed{\begin{aligned} A^{\pm}_i &= P^{\pm}_{ij}A_j \\ E^{\pm}_i &= P^{\pm}_{ij}E_j \end{aligned}}$$

Gauss

$$\vec{\nabla}\cdot\vec{E} = 0$$

Two IC options

Initial Conditions in the Linear Regime

Two IC options

One option

Only project the $+$ helicity states

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Why?

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A^{+} is exponentially enhanced

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Loss of information!

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Our option

Project both states + and -

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Initial Conditions in the Linear Regime

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Loss of information!

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Main objective: non-linear regime

Can A^- become relevant?

Initial Conditions in the Linear Regime

Gauge initialialization process

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1 Assign BD + solution to A_i CL variables

$$A_i \rightarrow \frac{1}{\sqrt{2k}} e^{ik/aH}$$

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$$A_i \rightarrow A^+ = P^+_{ij} A_j$$

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Fourier

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5. Go back to real space via FFT

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EOMs evolved in real space!

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5 Go back to real space via FFT

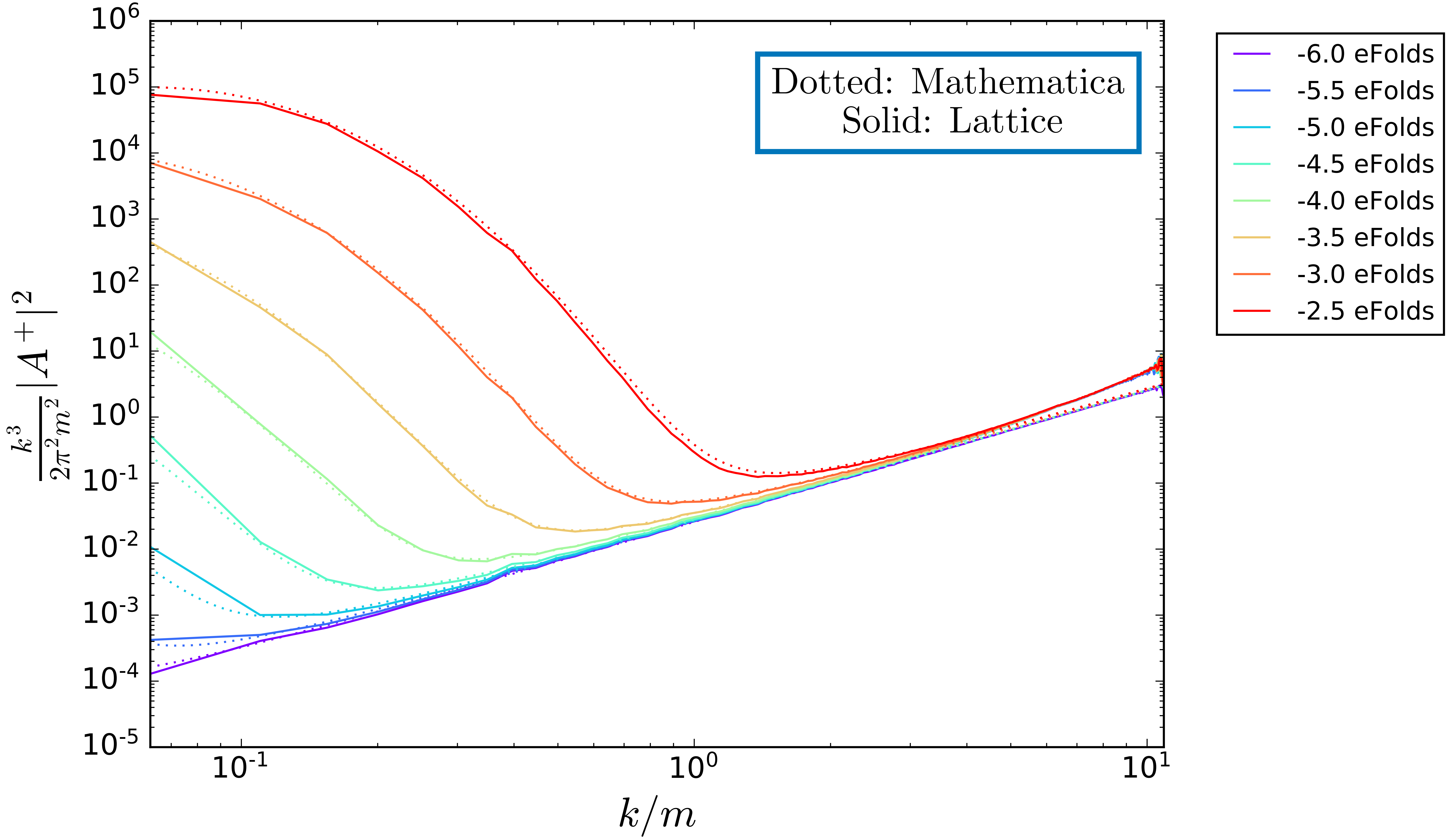
EOMs evolved in real space!

6 Repeat initialization with $E^\pm$ BD solutions

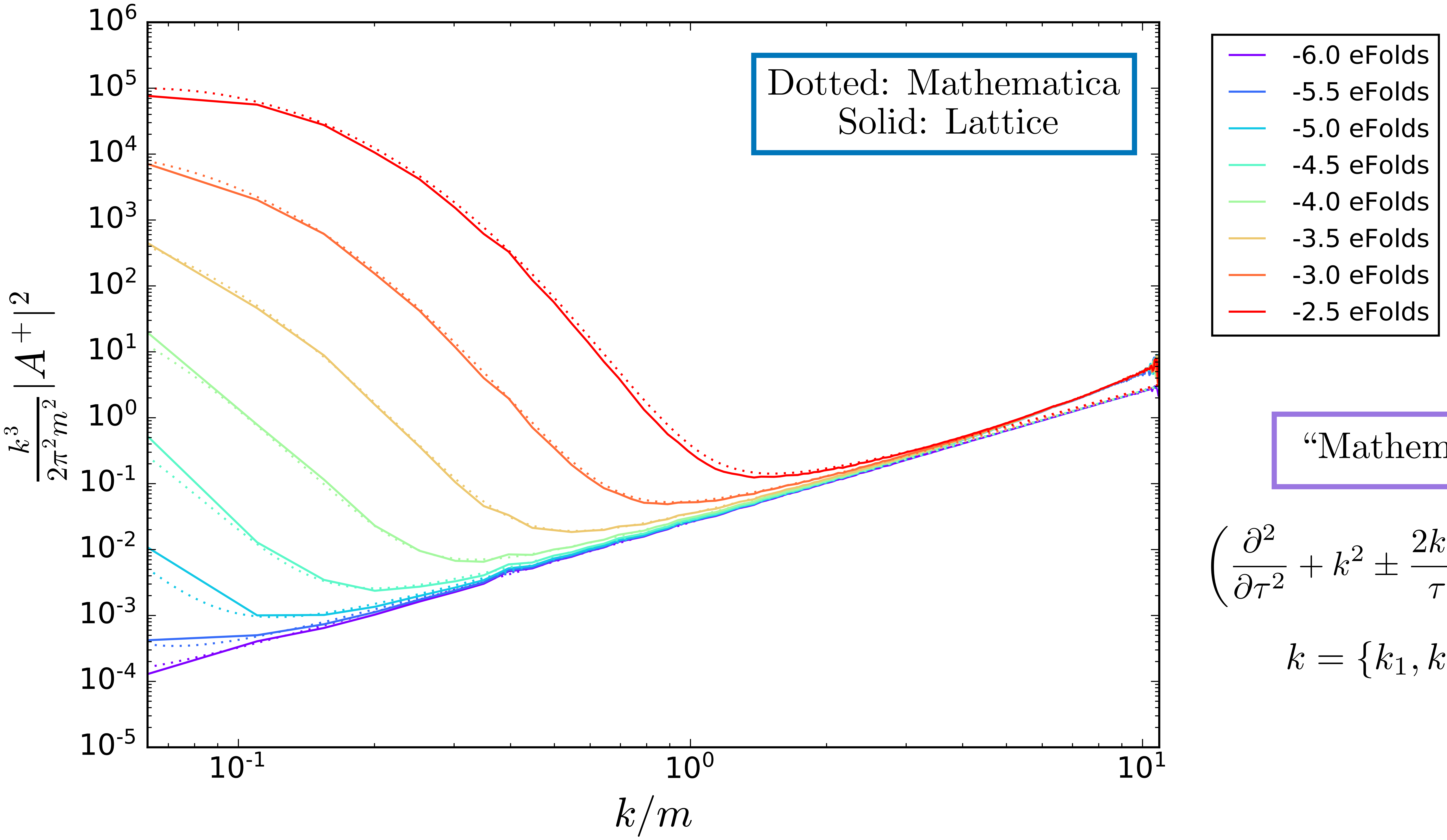
We will have an amplitude of 2 BD in A_i

$$\sum_i |A_i|^2 = |A_+|^2 + |A_-|^2$$

Simulations in the Linear Regime



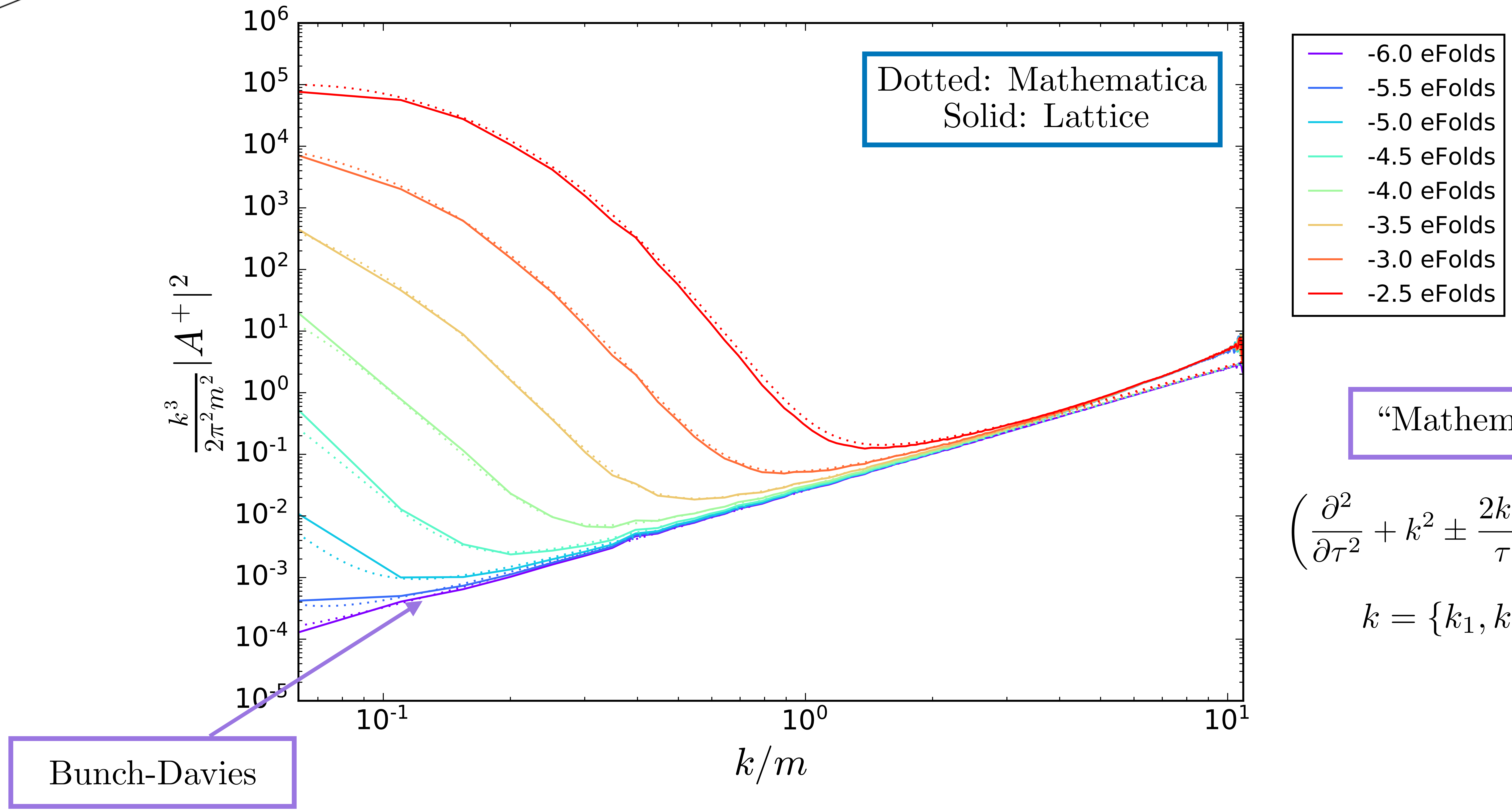
Simulations in the Linear Regime



"Mathematica"

$$\left(\frac{\partial^2}{\partial\tau^2} + k^2 \pm \frac{2k\xi}{\tau}\right) A_\pm(k,\tau) = 0$$

$$k = \{k_1, k_2, ..., k_N\}$$



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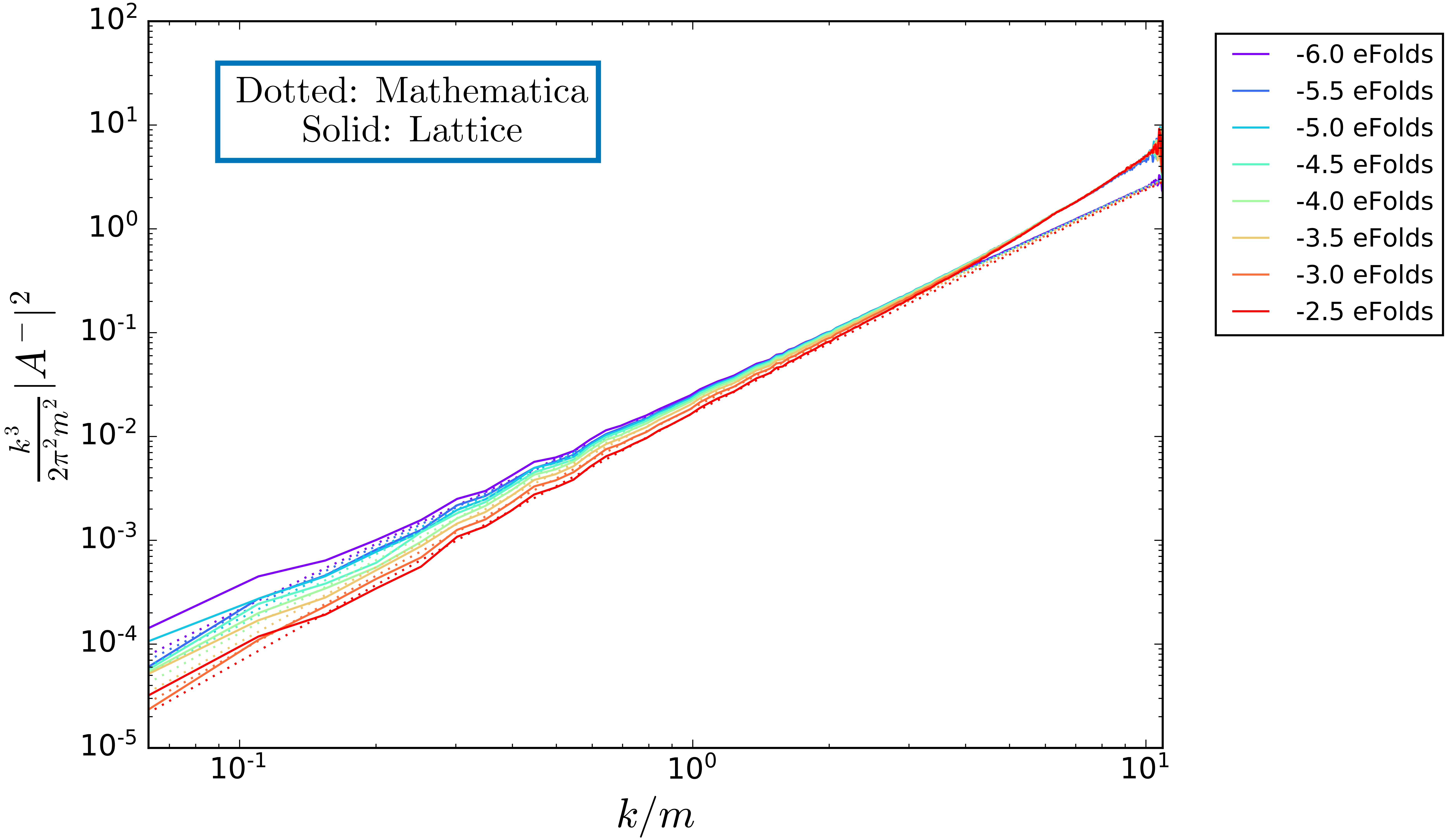

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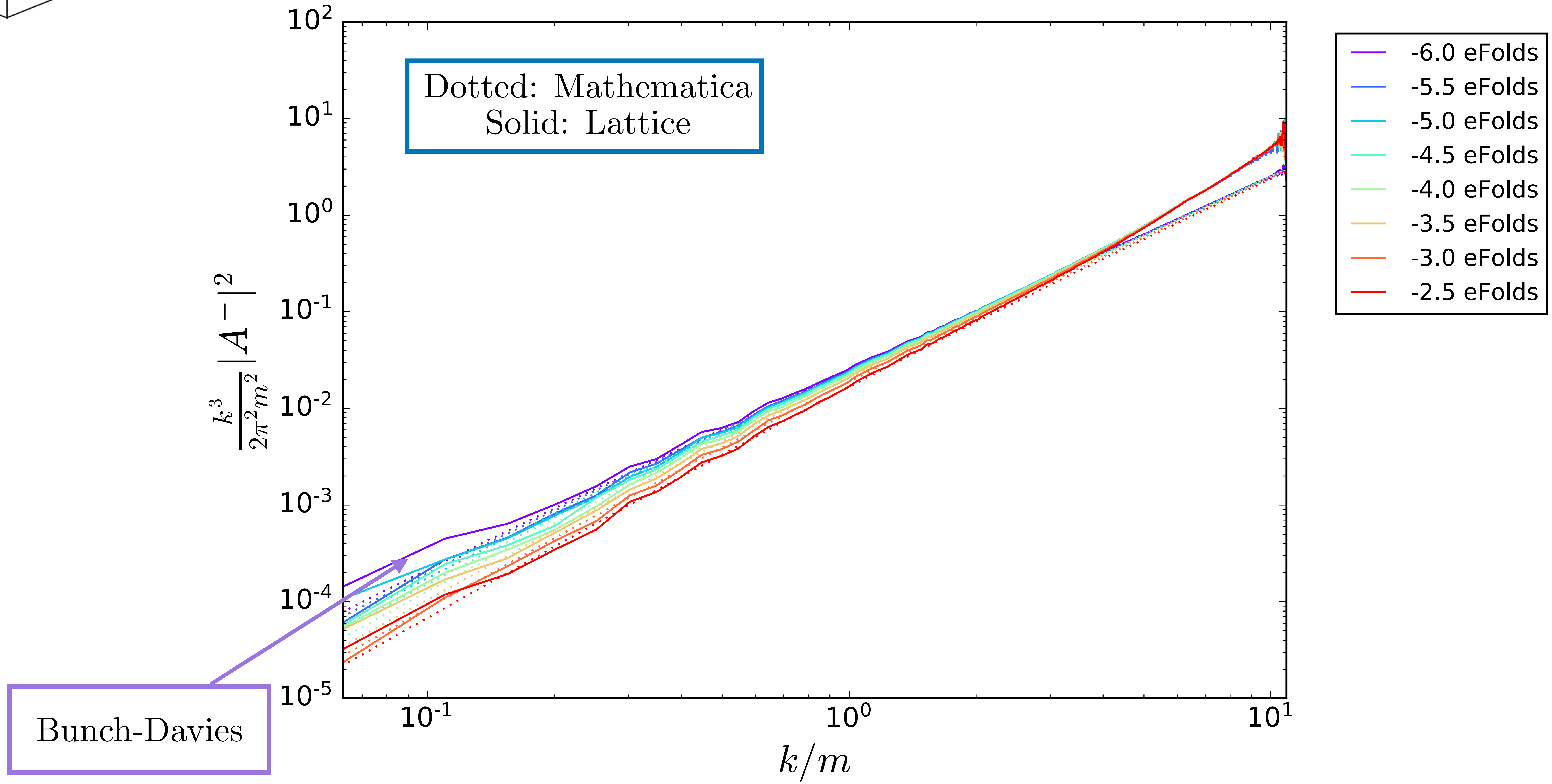

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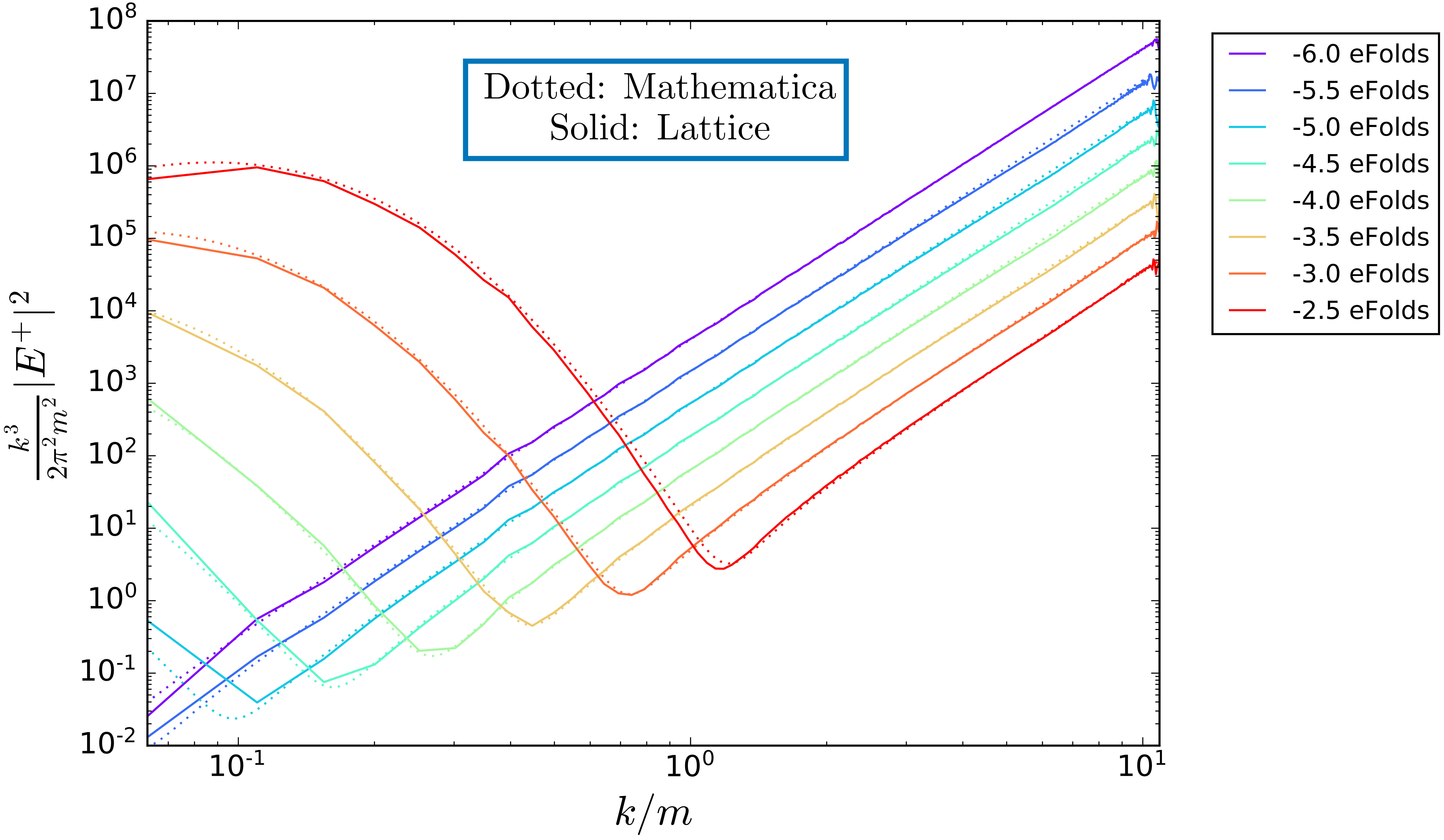

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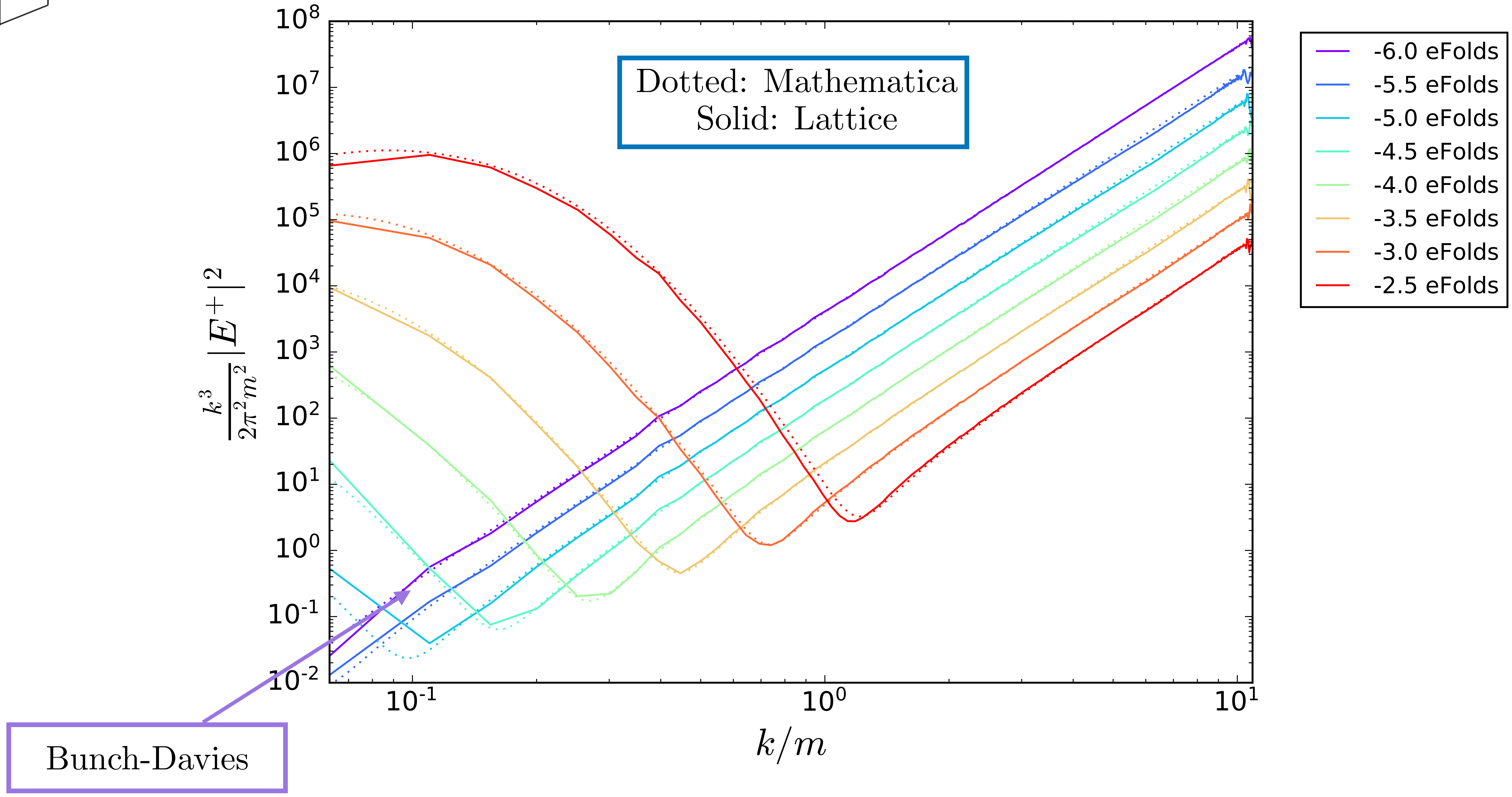

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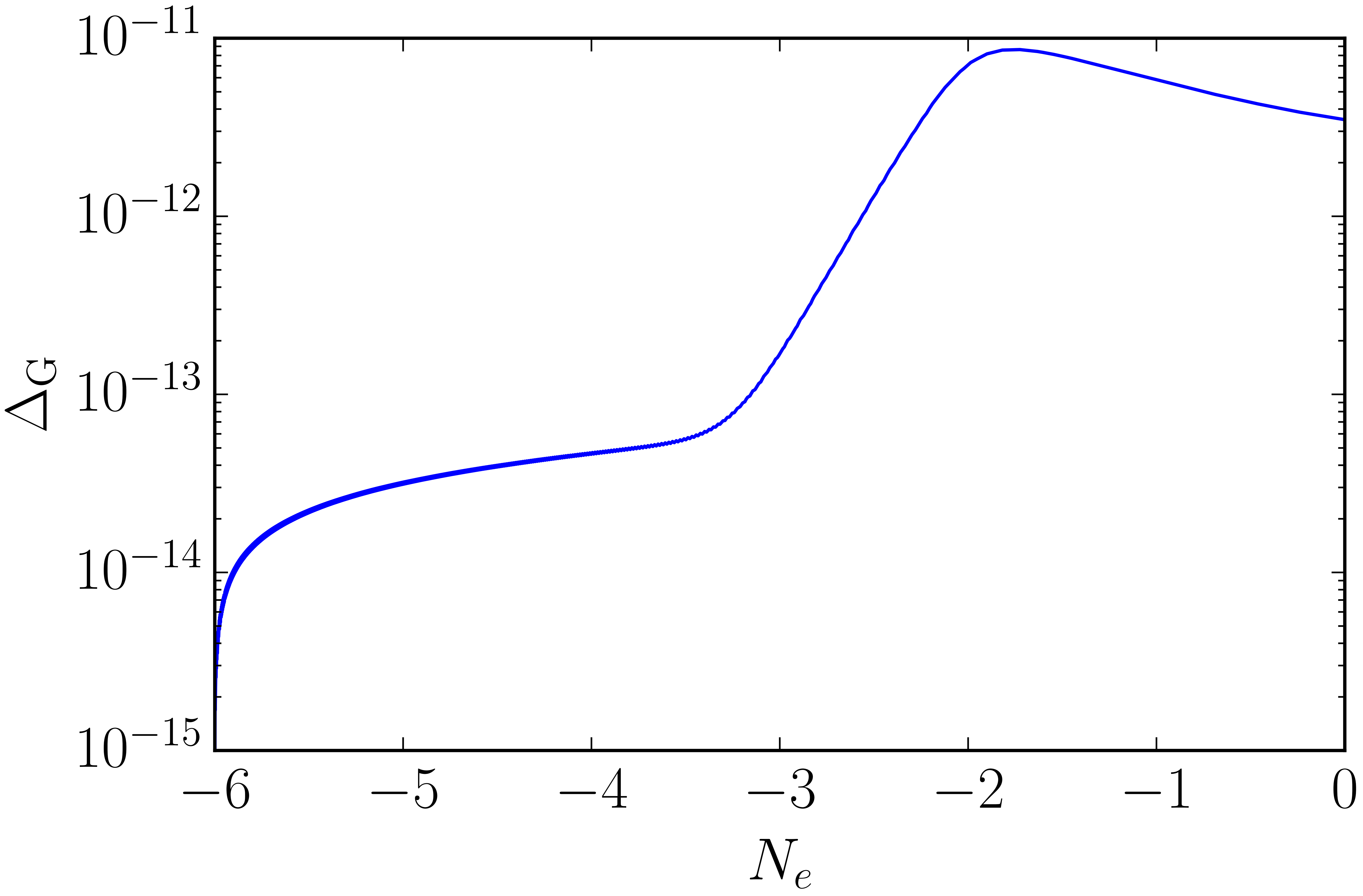

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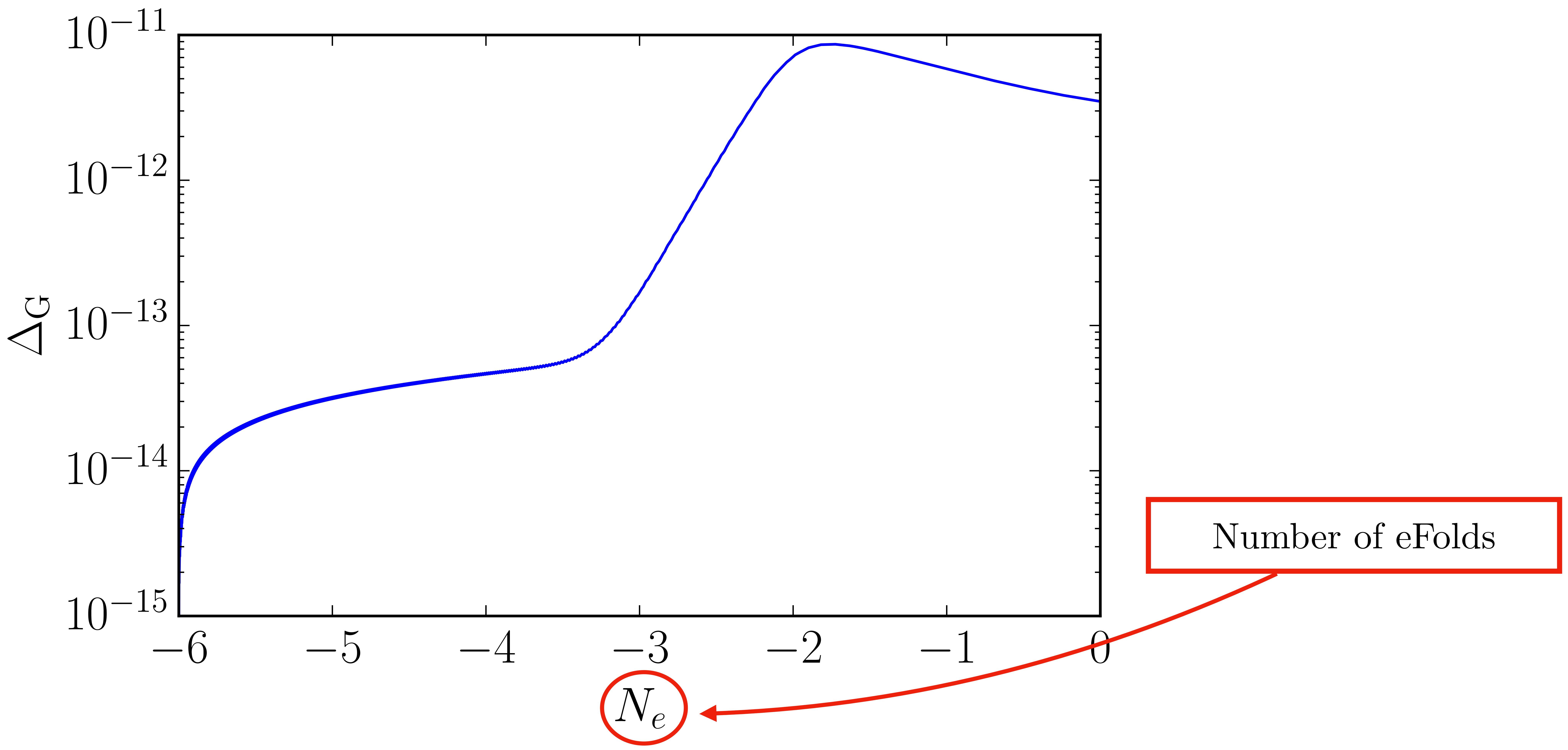

Non-linear evolution: example

Dynamical equations:

$$\dot{\tilde{\pi}}_\phi = a\vec{\nabla}^2\phi - a^3 m^2 \phi + \frac{1}{a\Lambda}\tilde{\vec{E}}\cdot\vec{B}\ ,$$

$$\dot{\tilde{\vec{E}}} = -\frac{1}{a}\vec{\nabla}\times\vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B} + \frac{1}{a\Lambda}\vec{\nabla}\phi\times\tilde{\vec{E}}$$

@

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Expansion equation:

$$\dot{\pi}_a = \frac{-a}{6m_{\rm pl}^2}(3p+\rho) =$$

$$\frac{a}{3m_{\rm pl}^2}\langle -2K_\phi + V - K_A - G_A\rangle$$

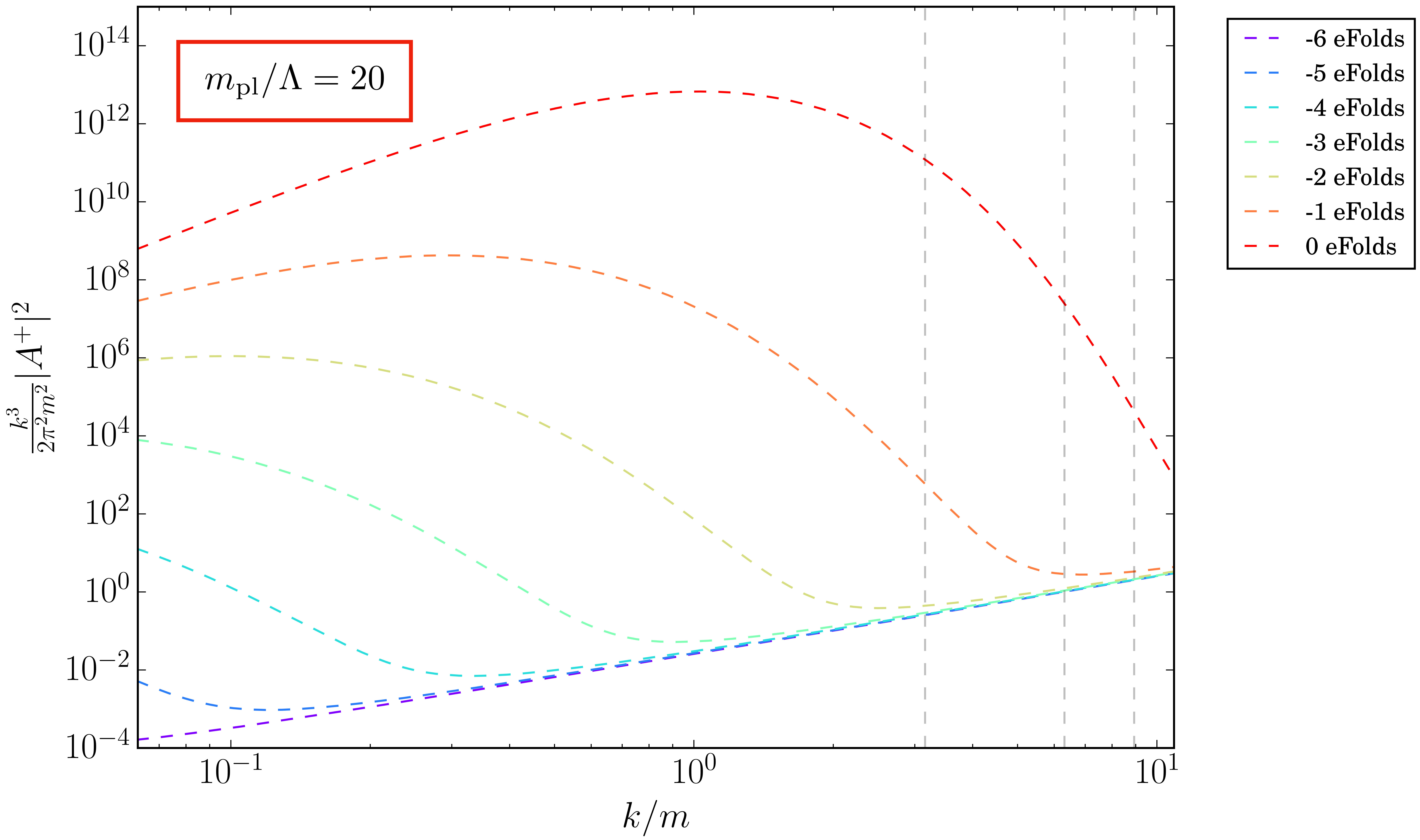
@

Non-linear evolution: example

Gauge PS

A^+

Linear



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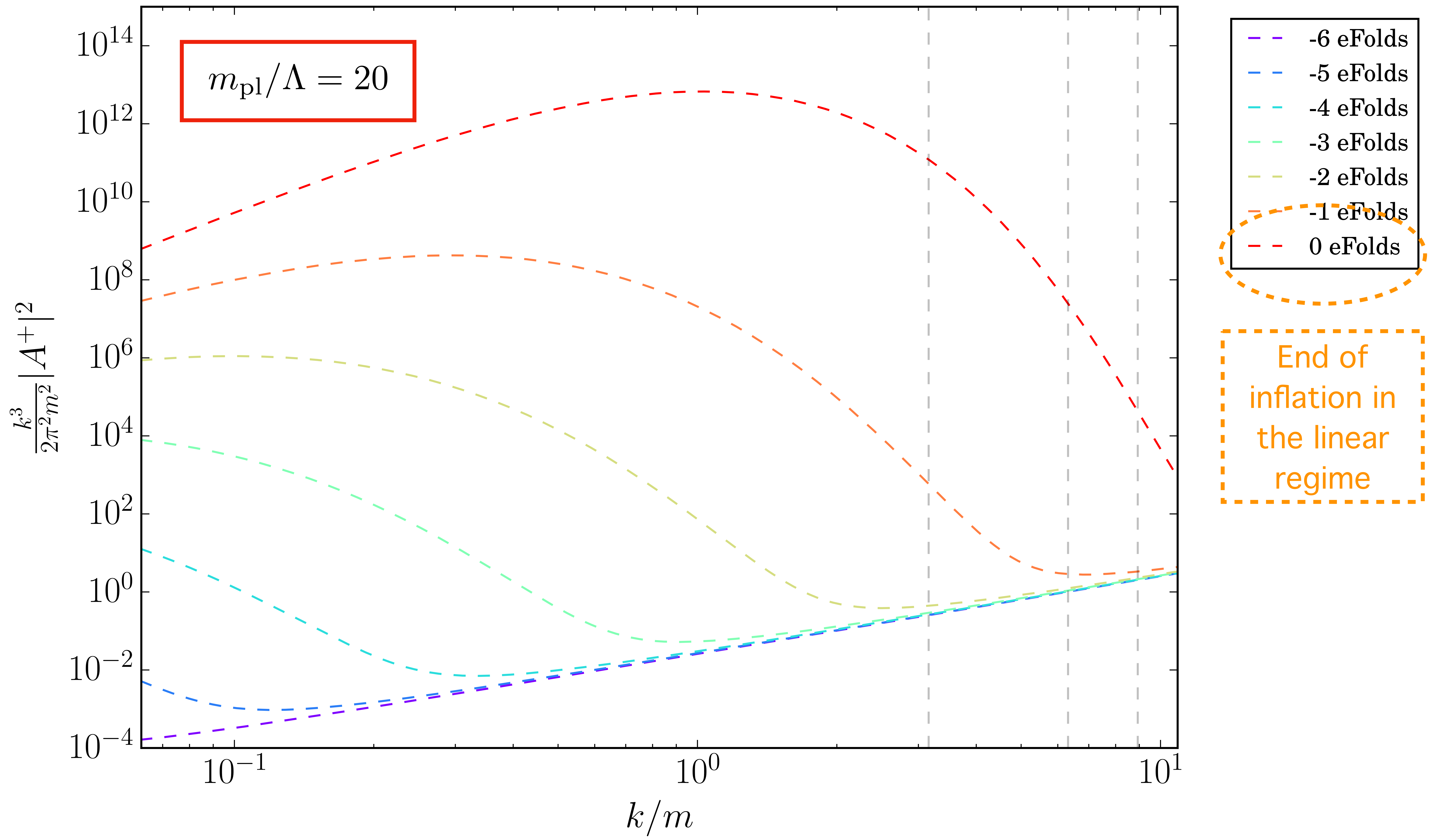

End of inflation in the linear regime

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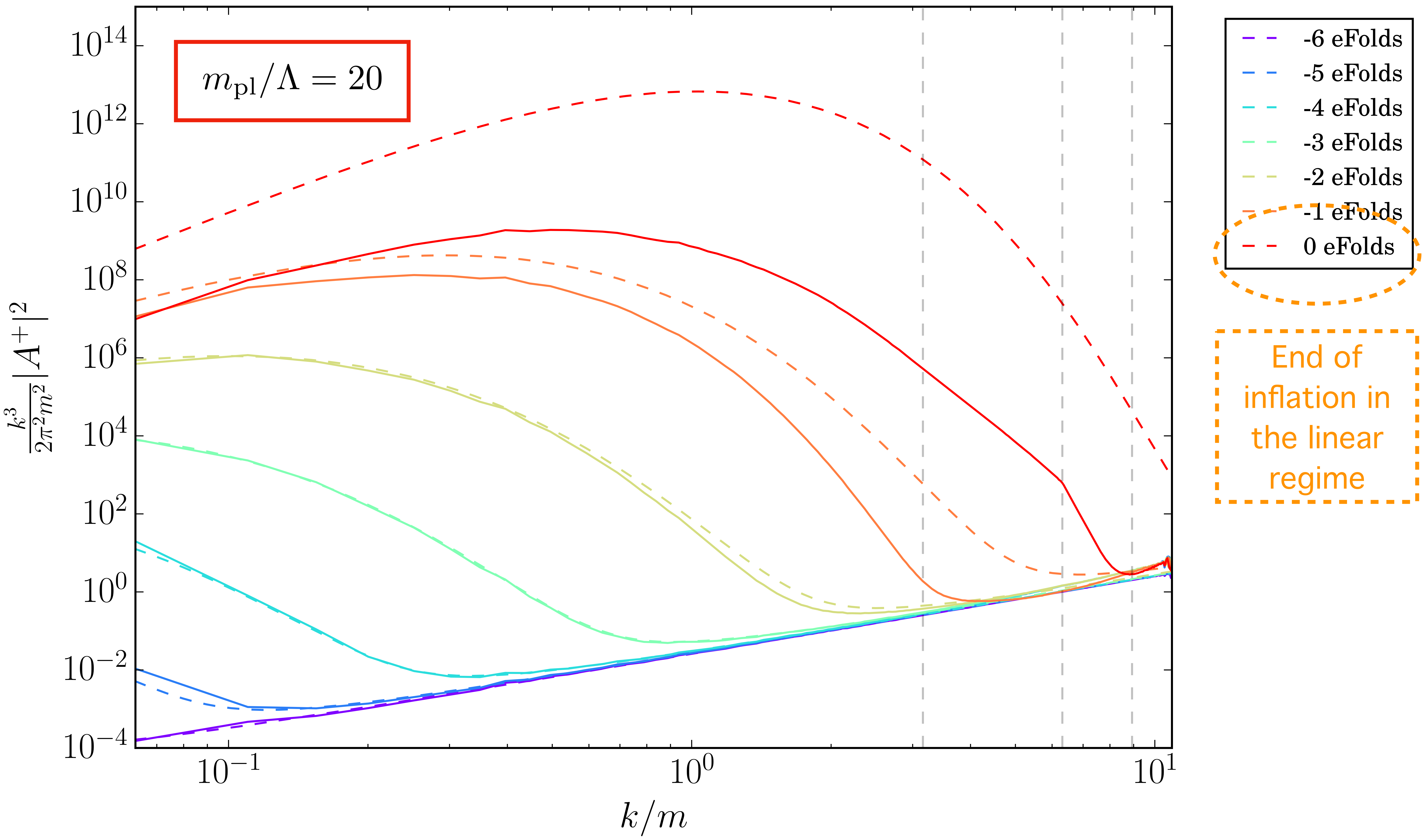

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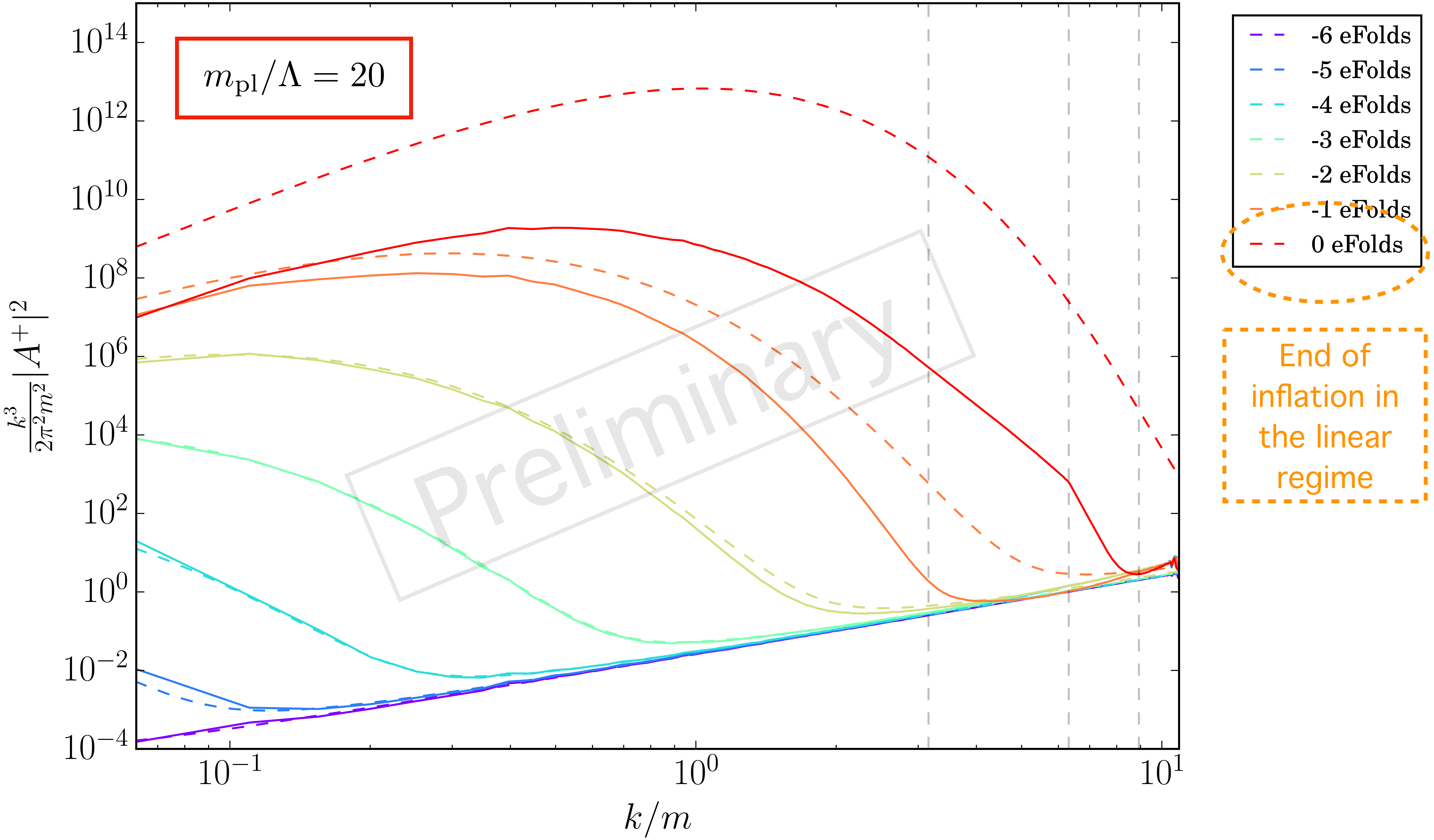

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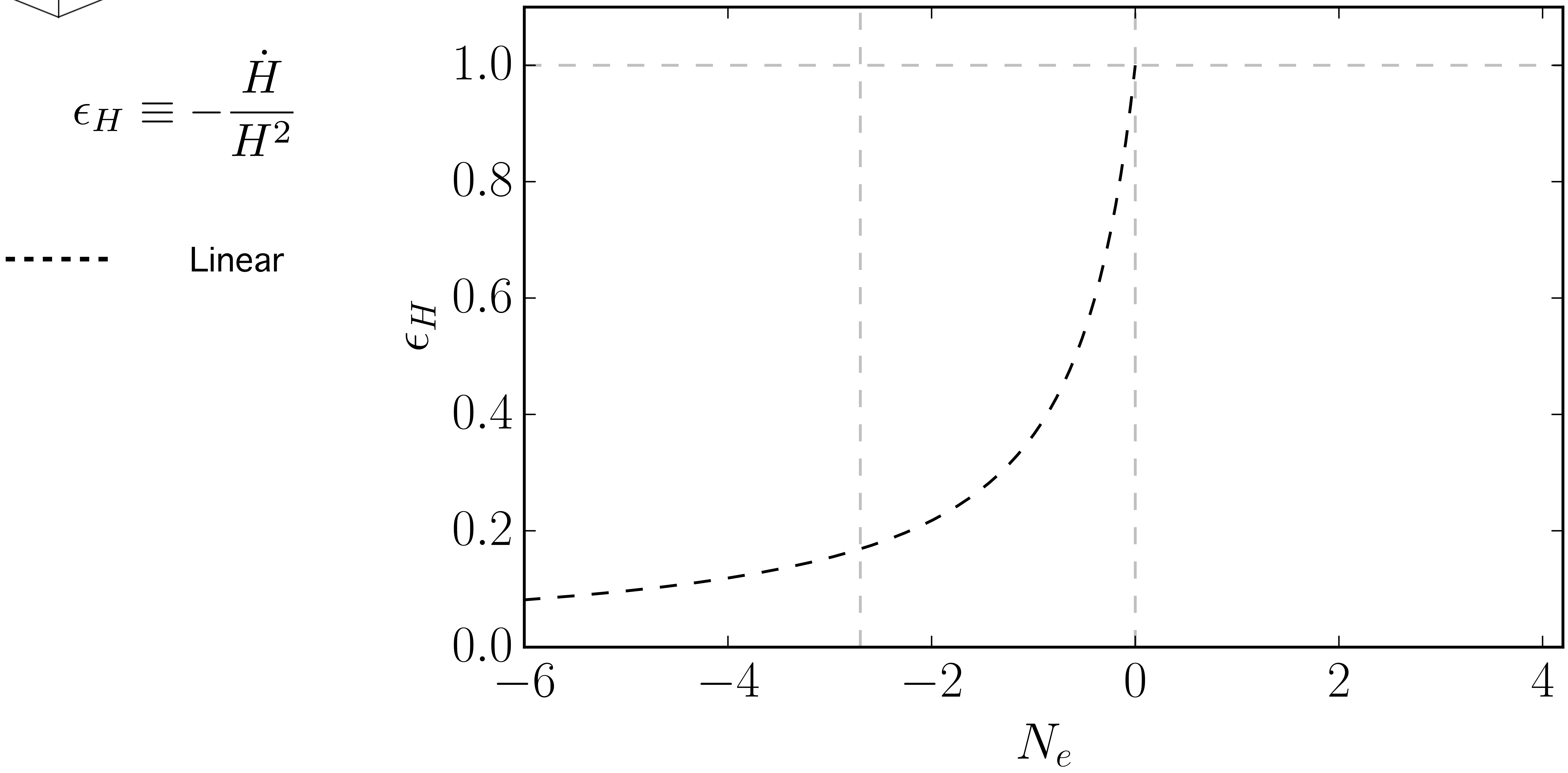
$$\epsilon_H \equiv -\frac{\dot{H}}{H^2}$$



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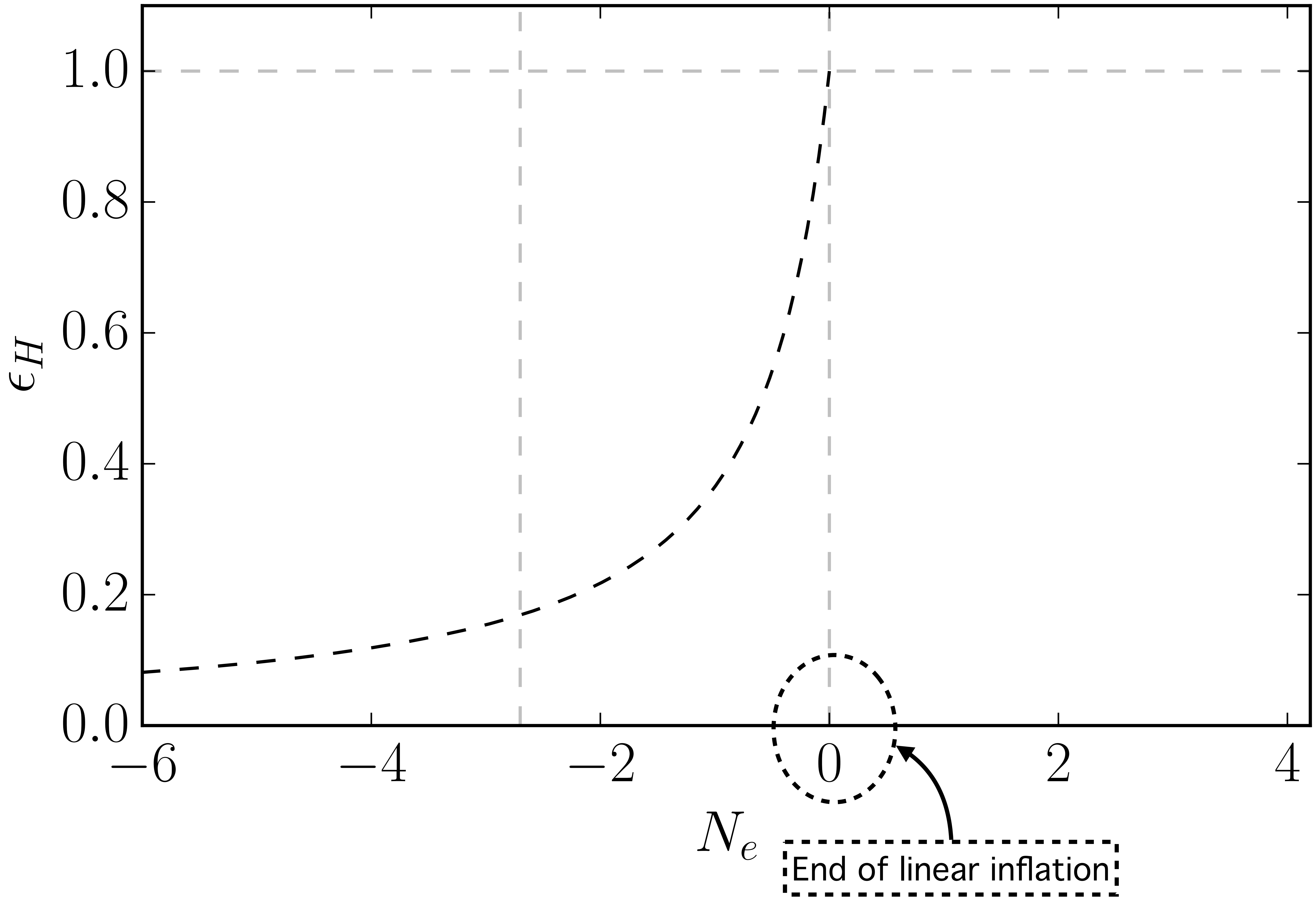

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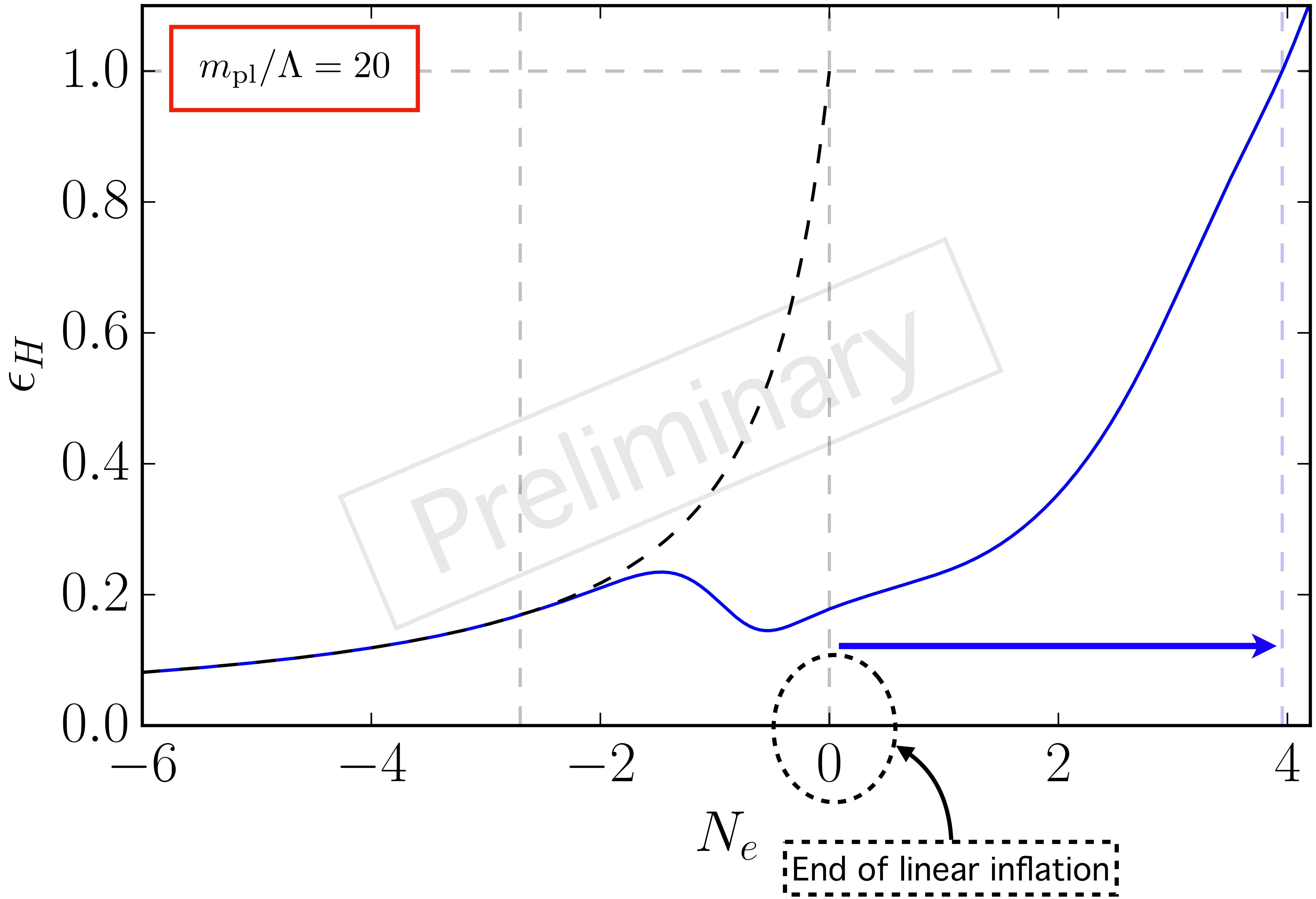

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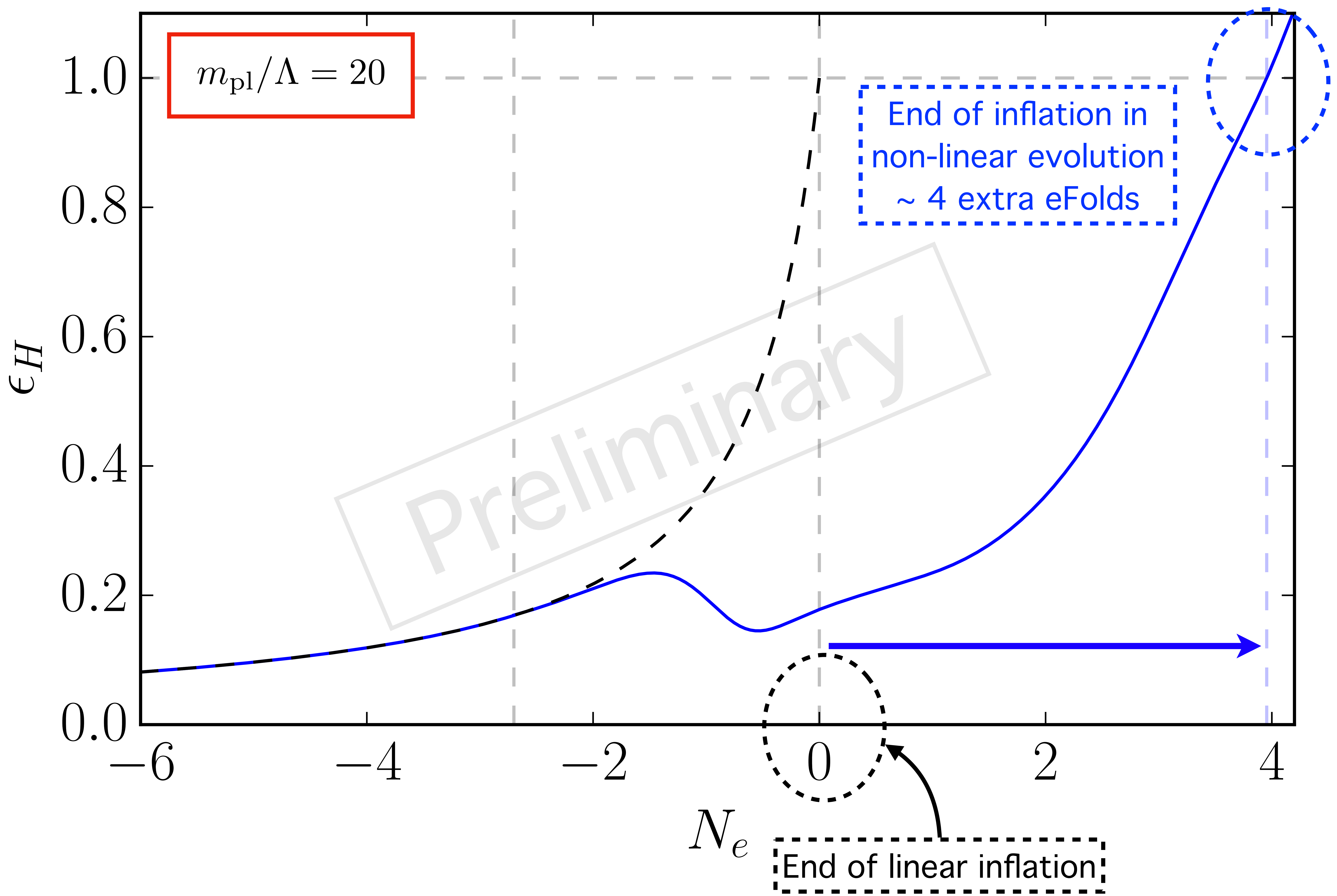
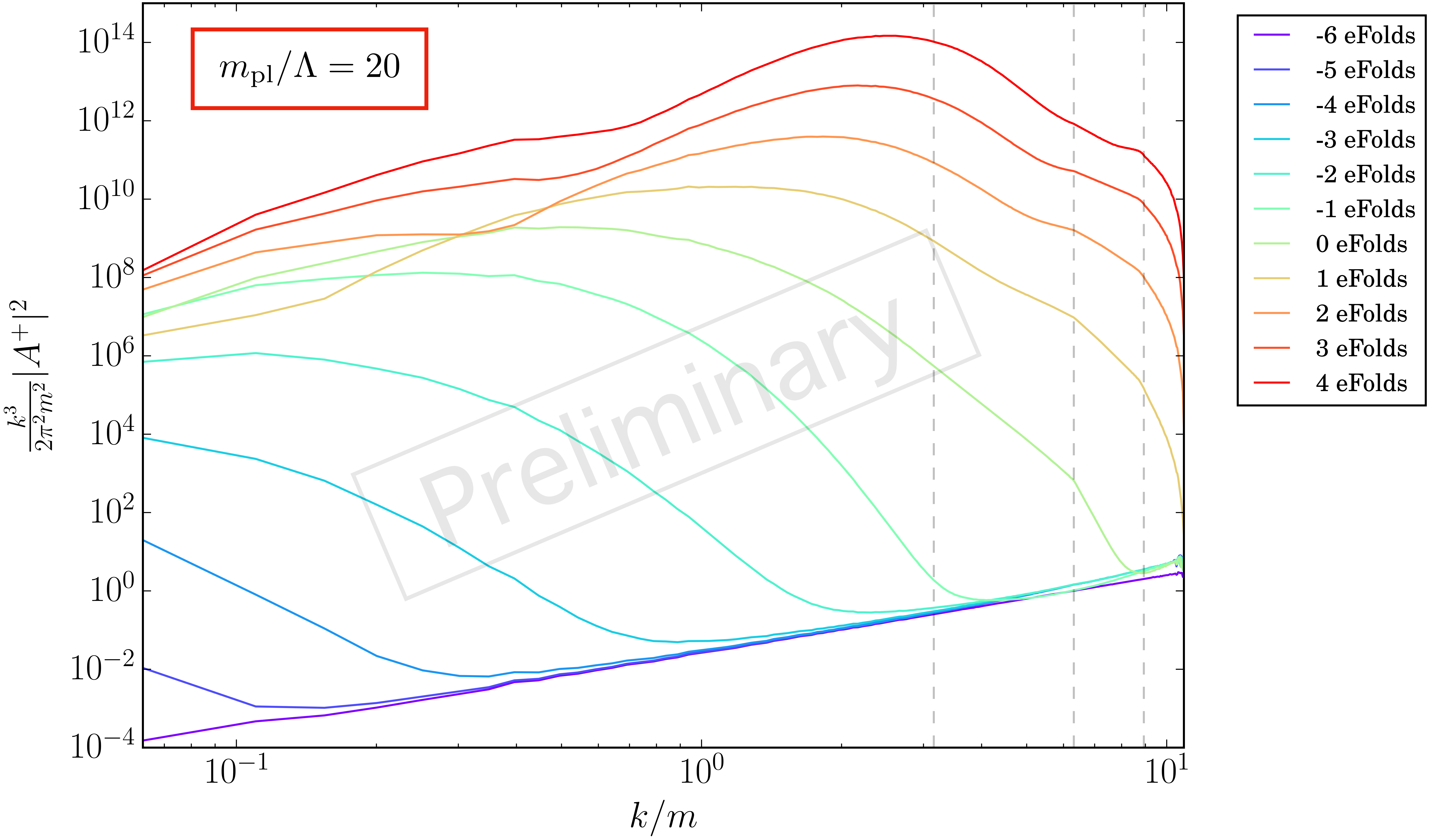


Non-linear evolution: example

What's happening in those extra folds?



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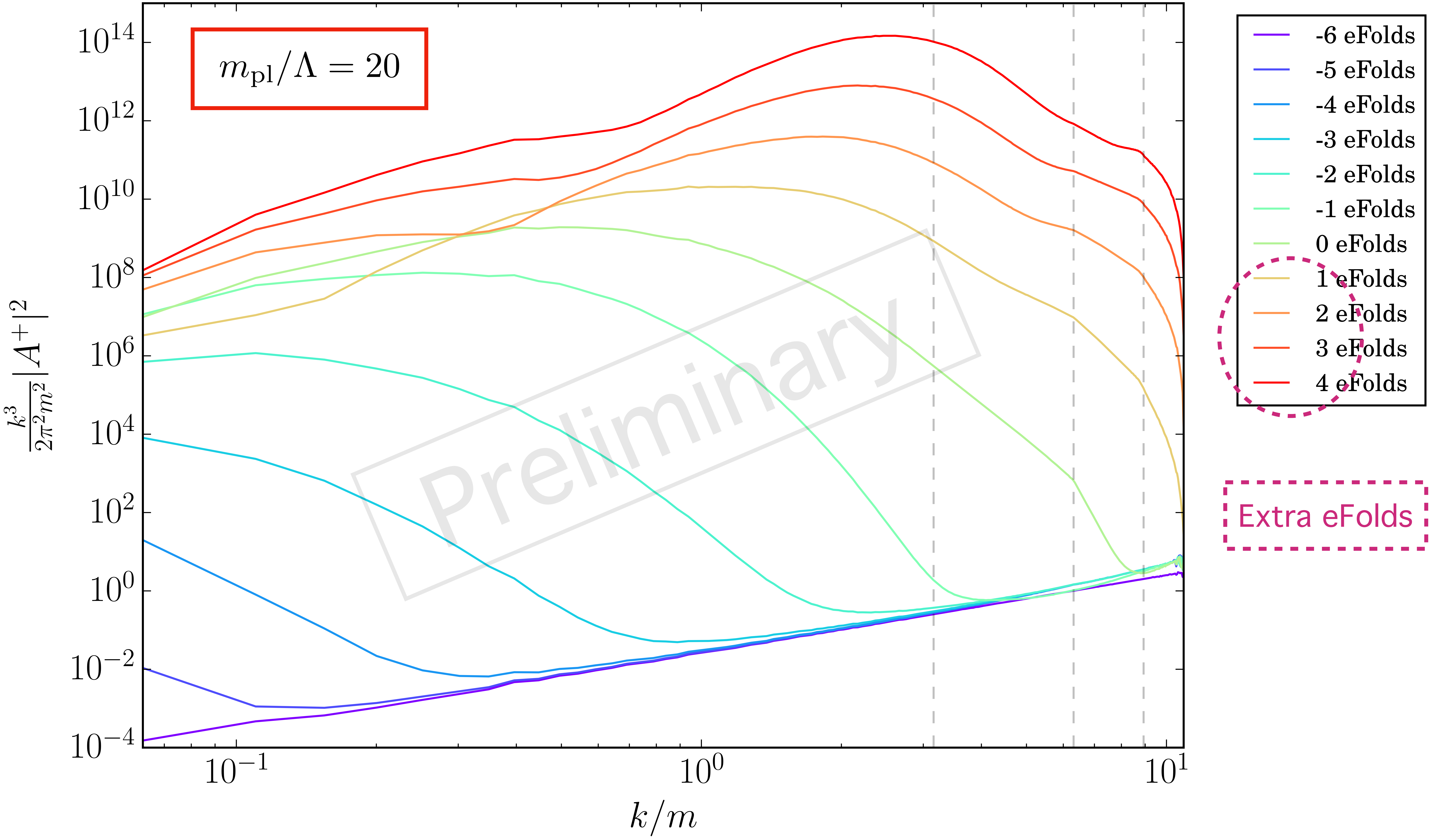

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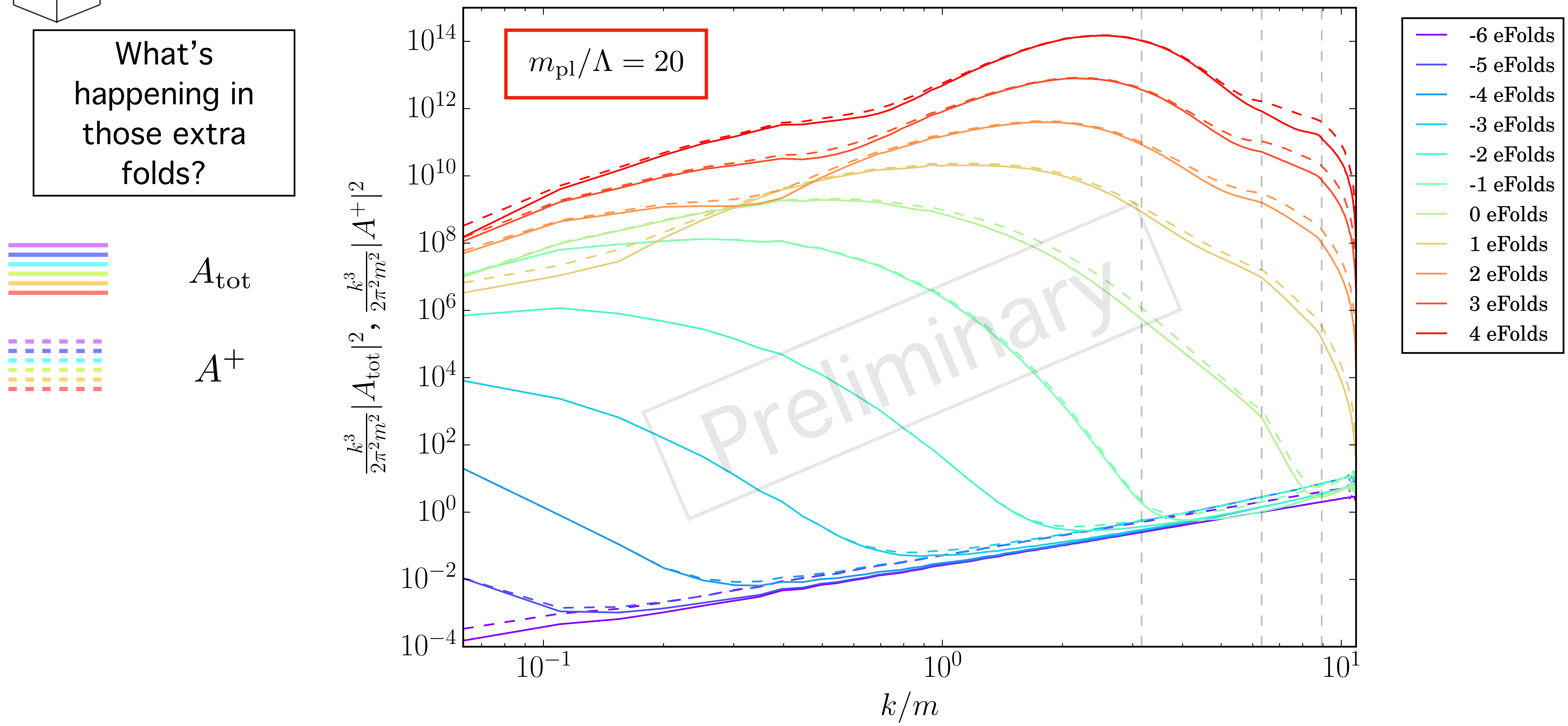

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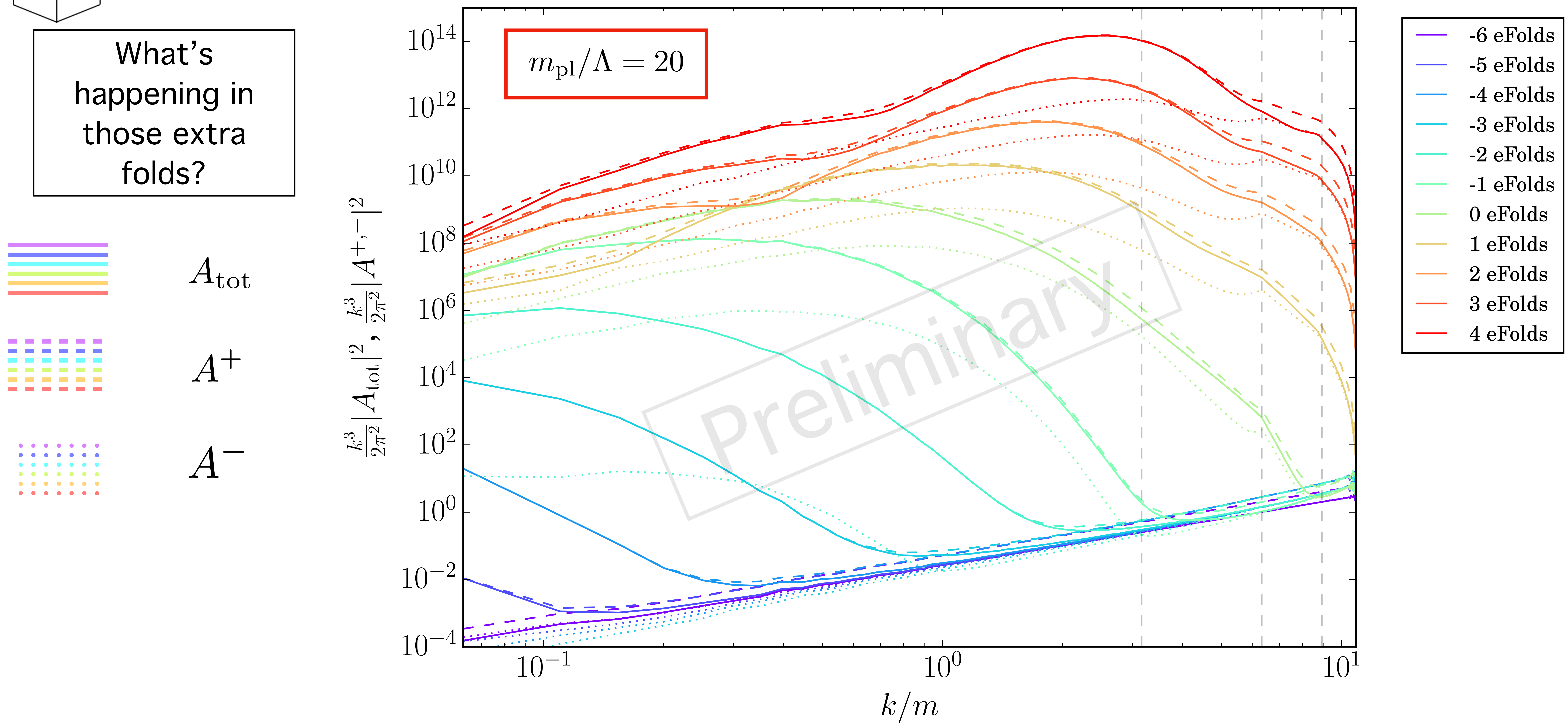
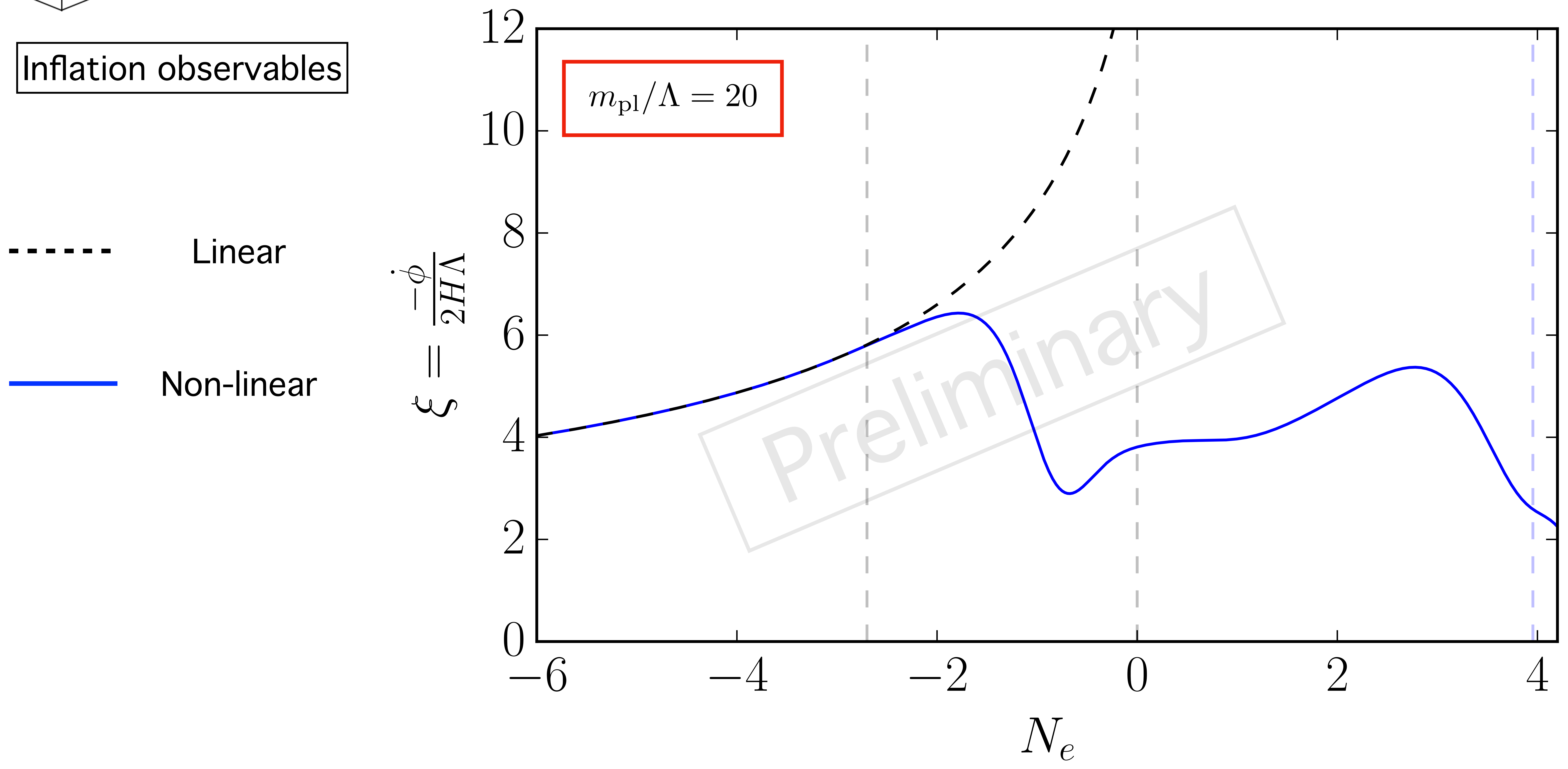


Non-linear evolution: example

Inflation observables

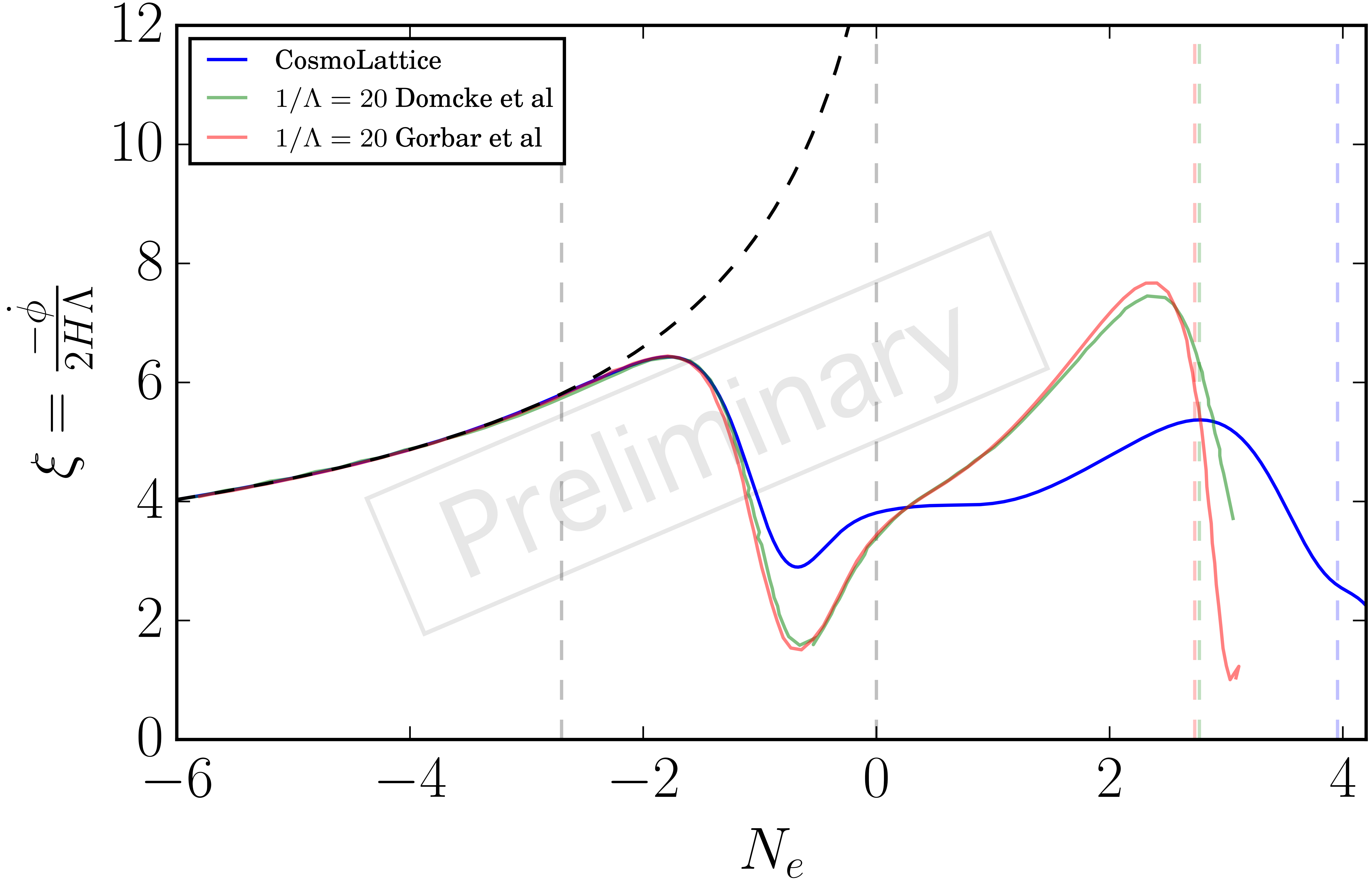


7- Non-linear evolution: example

Inflation observables

Linear

Non-linear

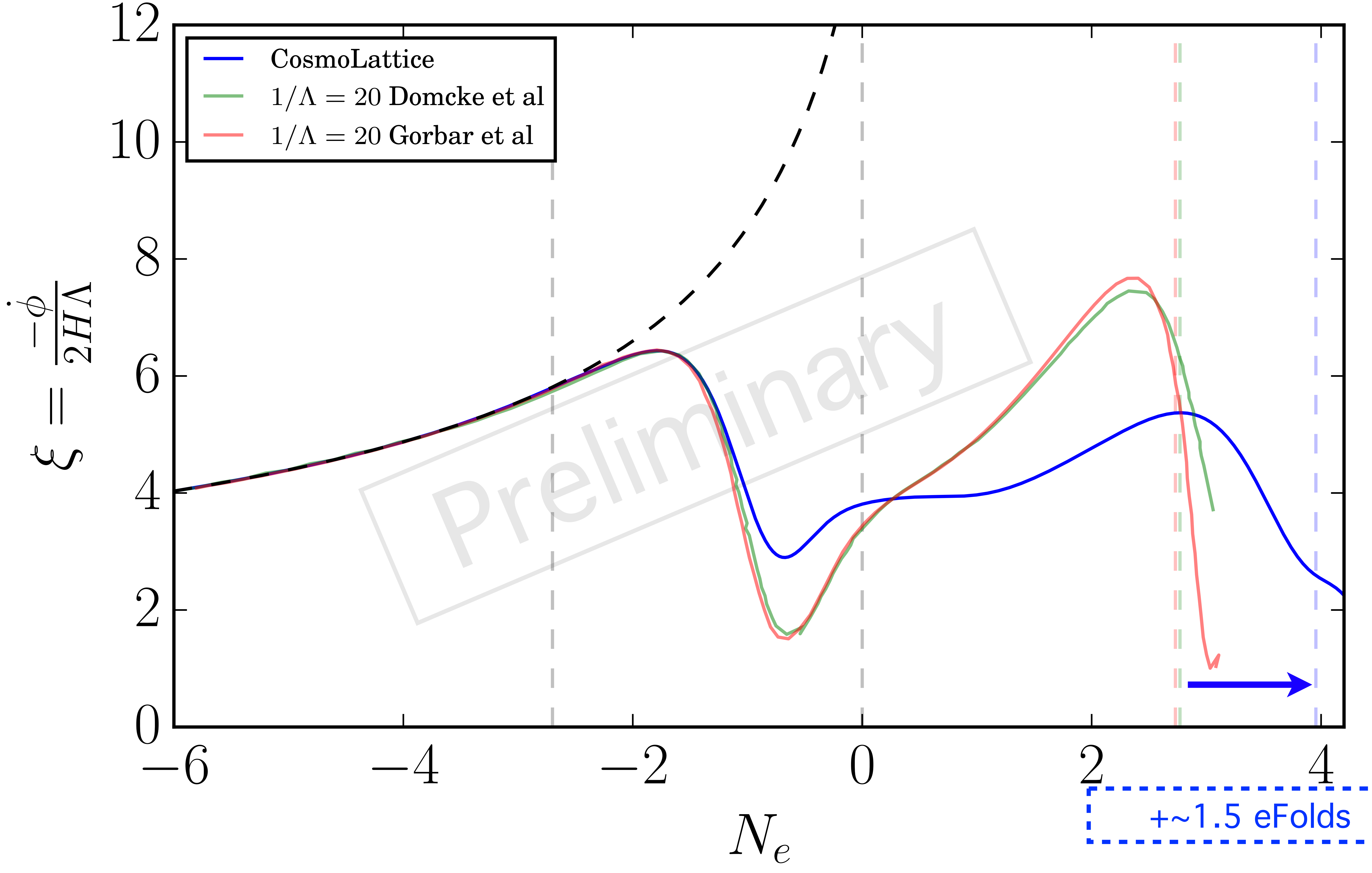


7- Non-linear evolution: example

Inflation observables

Linear

Non-linear



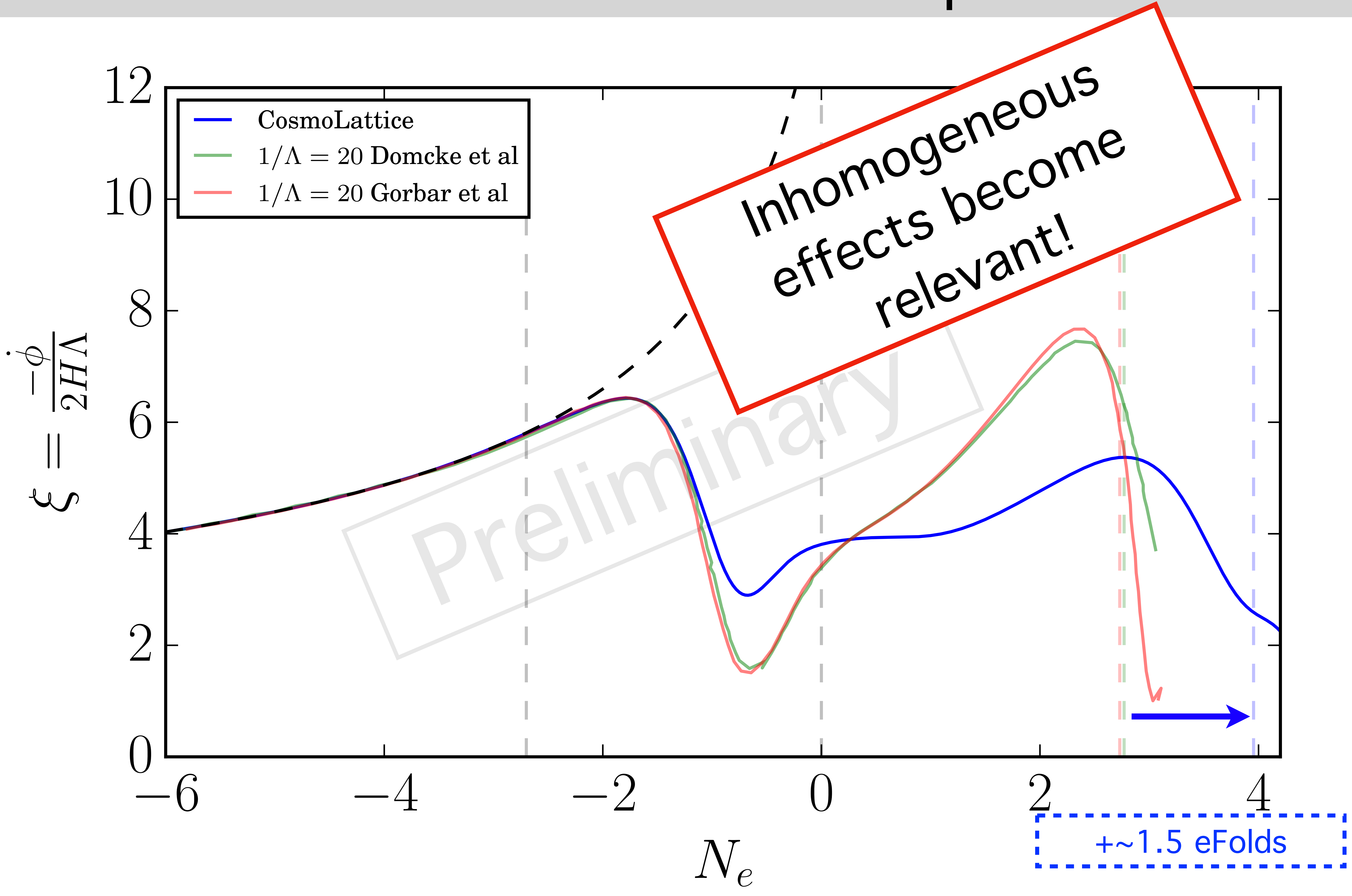
+~1.5 eFolds

7- Non-linear evolution: example

Inflation observables

Linear

Non-linear



Summary & Conclusions

- Successful implementation of axion-inflation in CL

$$\boxed{\frac{\phi}{\Lambda}F\tilde{F}}$$

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- Preliminar non-linear sims already show inhomogeneities non-negligible

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Ready to study non-linear phenomenology!

- Preliminar non-linear sims already show inhomogeneities non-negligible

Thank you!