A.2 Geodesic Equation

In the absence of any non-gravitational forces, particles move along special paths in the curved spacetime. In this section, we will derive an equation that determines these trajectories.

A.2.1 Timelike Geodesics

Let $\mathcal{C}$ be a timelike curve connecting two points p and q. Use a parameter λ to label points along the curve, with $\mathcal{C}(0) = p$ and $\mathcal{C}(1) = q$. The total proper time between p and q, can then be written as

$$\Delta\tau = \int_0^1 \mathrm{d}\lambda \sqrt{g_{\mu\nu}\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}}\,. \tag{A.2.28}$$

A small deformation of a timelike curve remains timelike hence there exist infinitely many timelike curves connecting p and q. The path taken by a massive particle in general relativity is that which extremises the proper time (A.2.28). [Recall that the relativistic action of a massive particle is $S = -m\int \mathrm{d}\tau$.] This special curve is called a *geodesic*. Solving the Euler-Lagrange problem, we find an equation satisfied by particles moving along geodesics:

$$\frac{d^2X^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dX^\alpha}{d\tau}\frac{dX^\beta}{d\tau} = 0\,, \tag{A.2.29}$$

where $\Gamma^\mu_{\alpha\beta}$ are called *Christoffel symbols*, and are defined by

$$\boxed{\Gamma^\mu_{\alpha\beta} \equiv \frac{1}{2}g^{\mu\gamma}\left(\partial_\alpha g_{\gamma\beta} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}\right)}\,. \tag{A.2.30}$$

Derivation.—Let us write (A.2.28) as

$$\Delta\tau[X^\mu(\lambda)] = \int_0^1 \mathrm{d}\lambda \left(g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu\right)^{1/2} \equiv \int_0^1 \mathrm{d}\lambda\, L[X^\mu, \dot{X}^\mu]\,, \tag{A.2.31}$$

where $\dot{X}^\mu \equiv dX^\mu/d\lambda$. The geodesic equation for the path of extremal proper time follows from the Euler-Lagrange equation

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{X}^\mu}\right) - \frac{\partial L}{\partial X^\mu} = 0\,. \tag{A.2.32}$$

The derivatives in (A.2.32) are

$$\frac{\partial L}{\partial \dot{X}^\mu} = -\frac{1}{L}g_{\mu\nu}\dot{X}^\nu\,, \quad \frac{\partial L}{\partial X^\mu} = -\frac{1}{2L}\partial_\mu g_{\nu\rho}\dot{X}^\nu\dot{X}^\rho\,. \tag{A.2.33}$$

Before continuing, it is convenient to switch from the general parameterisation λ to the parameterisation using proper time τ. (We could not have used τ from the beginning since the value of τ at the final point q is different for different curves. The range of integration would then have been different for different curves.) Notice that

$$\left(\frac{d\tau}{d\lambda}\right)^2 = g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = L^2\,, \tag{A.2.34}$$

and hence $d\tau/d\lambda = L$. In the above equations, we can therefore replace $d/d\lambda$ with $Ld/d\tau$. The Euler-Lagrange equation then becomes

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dX^\nu}{d\tau}\right) - \frac{1}{2}\partial_\mu g_{\nu\rho}\frac{dX^\nu}{d\tau}\frac{dX^\rho}{d\tau} = 0\,. \tag{A.2.35}$$

Expanding the first term, we get

$$g_{\mu\nu}\frac{d^2X^\nu}{d\tau^2}+\partial_\rho g_{\mu\nu}\frac{dX^\rho}{d\tau}\frac{dX^\nu}{d\tau}-\frac{1}{2}\partial_\mu g_{\nu\rho}\frac{dX^\nu}{d\tau}\frac{dX^\rho}{d\tau}=0\,, \tag{A.2.36}$$

where ∂_ρ is shorthand for $\partial/\partial X^\rho$. In the second term, we can replace $\partial_\rho g_{\mu\nu}$ with $\frac{1}{2}(\partial_\rho g_{\mu\nu}+\partial_\nu g_{\mu\rho})$ because it is contracted with an object that is symmetric in ν and ρ. Contracting (A.2.36) with the inverse metric and relabelling indices, we find

$$\frac{d^2X^\mu}{d\tau^2}+\Gamma^\mu_{\alpha\beta}\frac{dX^\alpha}{d\tau}\frac{dX^\beta}{d\tau}=0\,. \tag{A.2.37}$$

This is the desired result (A.2.29).

Newtonian limit.—Let us use the geodesic equation (A.2.29) to study the dynamics of a massive test particle moving slowly in the spacetime (A.1.17). "Moving slowly" (with respect to the speed of light, $c\equiv 1$) means

$$\frac{dx^i}{dt}\ll 1 \quad\Rightarrow\quad \frac{dx^i}{d\tau}\ll\frac{dt}{d\tau}\,. \tag{A.2.38}$$

The geodesic equation then becomes

$$\frac{d^2X^\mu}{d\tau^2}\approx-\Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2\,. \tag{A.2.39}$$

The relevant component of the Christoffel symbol is

$$\Gamma^\mu_{00}=\frac{1}{2}g^{\mu\gamma}\left(\partial_0 g_{0\gamma}+\partial_0 g_{0\gamma}-\partial_\gamma g_{00}\right). \tag{A.2.40}$$

All time derivatives vanish since the metric (A.1.17) is static. Hence, we find

$$\Gamma^\mu_{00}=-g^{\mu j}\,\partial_j\Phi\,. \tag{A.2.41}$$

Since the metric is diagonal, $g^{0j}=0$, we get $\Gamma^0_{00}=0$. The $\mu=0$ component of (A.2.39) therefore becomes

$$\frac{d^2t}{d\tau^2}=0 \quad\Rightarrow\quad \frac{dt}{d\tau}=const. \tag{A.2.42}$$

The $\mu=i$ component of (A.2.39) then reads

$$\frac{d^2x^i}{d\tau^2}\approx-\Gamma^i_{00}\left(\frac{dt}{d\tau}\right)^2=-\partial^i\Phi\left(\frac{dt}{d\tau}\right)^2+\mathcal{O}(\Phi^2)\,. \tag{A.2.43}$$

Dividing by $(dt/d\tau)^2$ and using $dt/d\tau=const.$, we find

$$\frac{d^2x^i}{dt^2}\approx-\partial^i\Phi\,, \tag{A.2.44}$$

which is Newton's law if we identify Φ with the gravitational potential. Newtonian gravity has therefore been expressed geometrically as geodesic motion in the curved spacetime (A.1.17).

The geodesic equation (A.2.29) can be written in a more elegant form. First, recall that $dX^\mu/d\tau$ is the four-velocity U^μ of the particle, so that (A.2.29) reads

$$\frac{dU^\mu}{d\tau}+\Gamma^\mu_{\alpha\beta}U^\alpha U^\beta=0\,. \tag{A.2.45}$$

Using

$$\frac{dU^\mu}{d\tau} = \frac{dX^\alpha}{d\tau}\frac{dU^\mu}{dX^\alpha} = U^\alpha \partial_\alpha U^\mu \,, \tag{A.2.46}$$

we get

$$U^\alpha \left(\partial_\alpha U^\mu + \Gamma^\mu_{\alpha\beta} U^\beta\right) \equiv U^\alpha \nabla_\alpha U^\mu \,, \tag{A.2.47}$$

where we have introduced the *covariant derivative* of the four-vector, ∇_α. The geodesic equation therefore takes the following form

$$U^\alpha \nabla_\alpha U^\mu = 0 \,. \tag{A.2.48}$$

In a proper GR course, you would derive this form of the geodesic equation from the concept of "parallel transport". Finally, note that in terms of the four-momentum $P^\mu = mU^\mu$ the geodesic equation (A.2.48) becomes

$$\boxed{P^\alpha \nabla_\alpha P^\mu = 0} \qquad \text{or} \qquad P^\alpha \left(\partial_\alpha P^\mu + \Gamma^\mu_{\alpha\beta} P^\beta\right) = 0 \,. \tag{A.2.49}$$

This form of the geodesic equation is particularly convenient since it also applies to massless particles.

A.2.2 Null Geodesics

For massless particles, we can't parameterize the geodesic in terms of proper time since $d\tau = 0$ for null separated points. Instead, it is conventional to define

$$\frac{dX^\mu}{d\tau} \equiv P^\mu \,, \tag{A.2.50}$$

where $P^\mu = (E, p^i)$ is the four-momentum of the particle. With this definition, the geodesic equation (A.2.49) holds even for massless particles.

Redshift.—Let us apply the geodesic equation to massless particles, e.g. photons, in an expanding universe described by the FRW metric (A.1.23). To study the evolution of the energy of the particles, we consider the $\mu = 0$ component of (A.2.49). Since E is independent of $\boldsymbol{x}$ (due to the homogeneity of space), we have

$$E\frac{dE}{dt} = -\Gamma^0_{ij} p^i p^j = -\dot{a}a\delta_{ij} p^i p^j \,, \tag{A.2.51}$$

where we have used that only the spatial components of $\Gamma^0_{\alpha\beta}$ are non-zero, namely $\Gamma^0_{ij} = \dot{a}a\delta_{ij}$. For a massless particle, we have

$$0 = g_{\mu\nu} P^\mu P^\nu = E^2 - a^2 \delta_{ij} p^i p^j \,. \tag{A.2.52}$$

Equation (A.2.51) can therefore be written as

$$\frac{1}{E}\frac{dE}{dt} = -\frac{\dot{a}}{a} \,. \tag{A.2.53}$$

This equation has the following solution

$$E \propto \frac{1}{a} \,. \tag{A.2.54}$$

Hence, the energy of a massless particle decreases as the universe expands. This effect is responsible for the redshifting of light, $\lambda \propto a$.

A.3 Einstein Equation

So far, we have described how test particles move in arbitrary curved spacetimes. Next, we want to discuss how the curvature of spacetime is determined by the local matter distribution. We are in search of the following relationship:

$$\begin{pmatrix} \text{a measure of local} \\ \text{spacetime curvature} \end{pmatrix} = \begin{pmatrix} \text{a measure of local} \\ \text{stress-energy density} \end{pmatrix} . \tag{A.3.55}$$

We will start on the left-hand side.

A.3.1 Curvature

It is important to realise that the motion of a single test particle tells us nothing about spacetime curvature. In a frame falling freely with the particle, the test particle remains at rest. Its motion is therefore indistinguishable from that of a test particle in flat spacetime. One test particle is not enough to detect curvature; the motion of at least two particles is needed. In the presence of spacetime curvature the separation between the two particles will change in time.

Tidal Forces

Consider two particles with positions $\boldsymbol{x}(t)$ and $\boldsymbol{x}(t) + \boldsymbol{b}(t)$. Let us first analyse the evolution of these particles in Newtonian gravity. In the presence of a gravitational potential $\Phi(\boldsymbol{x})$, the separation between the particles satisfies

$$\frac{d^2 b^i}{dt^2} \approx -\delta^{ij} \left(\frac{\partial^2 \Phi}{\partial x^j \partial x^k} \right) b^k . \tag{A.3.56}$$

Equation (A.3.56) holds as long as the separation $\boldsymbol{b}$ is small relative to the scale of variation of the potential $\Phi(\boldsymbol{x})$.

Derivation.—In an inertial frame, the equation of motion for the position $\boldsymbol{x}(t)$ of a particle moving in a gravitational potential $\Phi(\boldsymbol{x})$ is

$$\frac{d^2 x^i}{dt^2} = -\delta^{ij} \frac{\partial \Phi(x^k)}{\partial x^j} . \tag{A.3.57}$$

The equation of motion of the second particle is

$$\frac{d^2 (x^i + b^i)}{dt^2} = -\delta^{ij} \frac{\partial}{\partial x^j} \Phi(x^k + b^k) . \tag{A.3.58}$$

If the particles are close to each other, we can Taylor expand on the right-hand side,

$$\frac{\partial \Phi(x^k + b^k)}{\partial x^j} = \frac{\partial \Phi(x^k)}{\partial x^j} + \frac{\partial}{\partial x^k} \left(\frac{\partial \Phi(x^i)}{\partial x^j} \right) b^k + \cdots . \tag{A.3.59}$$

Subtracting (A.3.57) from (A.3.58), we get (A.3.56).

We notice the important role in (A.3.56) played by the *tidal (acceleration) tensor*

$$\frac{\partial^2 \Phi}{\partial x^i \partial x^j} . \tag{A.3.60}$$

It determines the forces that pull nearby particles apart or bring them closer together. The trace of the tidal tensor appears in the *Poisson equation*,

$$\nabla^2\Phi = \delta^{ij}\left(\frac{\partial^2\Phi}{\partial x^i \partial x^j}\right) = 4\pi G\rho\,. \tag{A.3.61}$$

We would like to generalise this equation to GR.

Geodesic Deviation

Let us first find the equivalent of (A.3.56) in GR. The algebra will be a bit more involved. Try not to get scared, the physics is the same as in the Newtonian example. The analog of the tidal tensor will give us a local measure of spacetime curvature.

We consider two particles separated by a four-vector B^μ. We wish to determine how B^μ evolves with respect to the proper time τ of an observer comoving with one of the particles. If the observer is freely falling, i.e. its four-velocity is $U^\mu = (1, 0, 0, 0)$, then the geodesic equation implies

$$\frac{d^2B^\mu}{d\tau^2} = -R^\mu{}_{0\alpha 0}\,B^\alpha\,, \tag{A.3.62}$$

where $R^\mu{}_{0\beta 0}$ are components of the *Riemann tensor*

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma\Gamma^\alpha_{\beta\delta} - \partial_\delta\Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\gamma\epsilon}\Gamma^\epsilon_{\beta\delta} - \Gamma^\alpha_{\delta\epsilon}\Gamma^\epsilon_{\beta\gamma}\,. \tag{A.3.63}$$

Derivation.—Let $X^\mu(\tau)$ and $X^\mu(\tau) + B^\mu(\tau)$ be two nearby geodesics. To find the evolution of the separation vector B^μ, it is necessary to compute its second derivative with respect to the observer's proper τ. This is a bit non-trivial. Instead of ordinary derivatives with respect to τ, we need to consider covariant derivatives along the geodesics. For example, the derivative of $B^\mu(X)$ evaluated on the observer's worldline $X^\mu(\tau)$ is

$$\begin{aligned}\frac{DB^\mu}{D\tau} \equiv U^\beta\nabla_\beta B^\mu &= \frac{dX^\beta}{d\tau}\nabla_\beta B^\mu \\ &= \frac{dB^\mu}{d\tau} + \Gamma^\mu_{\beta\gamma}U^\beta B^\gamma\,.\end{aligned} \tag{A.3.64}$$

Similarly, the second derivative along the geodesic is

$$\begin{aligned}\frac{D^2B^\mu}{D\tau^2} &\equiv U^\alpha\nabla_\alpha(U^\beta\nabla_\beta B^\mu) \\ &= \frac{d}{d\tau}\left(\frac{dB^\mu}{d\tau} + \Gamma^\mu_{\beta\gamma}U^\beta B^\gamma\right) + \Gamma^\mu_{\alpha\delta}U^\alpha\left(\frac{dB^\delta}{d\tau} + \Gamma^\delta_{\beta\gamma}U^\beta B^\gamma\right).\end{aligned} \tag{A.3.65}$$

To manipulate the complex expression in (A.3.65), we use

$$\frac{d^2B^\mu}{d\tau^2} = -2\Gamma^\mu_{\alpha\beta}U^\alpha\frac{dB^\beta}{d\tau} - \partial_\gamma\Gamma^\mu_{\alpha\beta}B^\gamma U^\alpha U^\beta + \mathcal{O}(B^2)\,, \tag{A.3.66}$$

$$\frac{dU^\mu}{d\tau} = -\Gamma^\mu_{\alpha\beta}U^\alpha U^\beta\,, \tag{A.3.67}$$

$$\frac{d\Gamma^\mu_{\alpha\beta}}{d\tau} = U^\gamma\partial_\gamma\Gamma^\mu_{\alpha\beta}\,, \tag{A.3.68}$$

where (A.3.66) follows from subtracting the geodesic equations of the two particles and expanding the right-hand side to first order in B^μ. Notice that all time derivatives of B^μ cancel in (A.3.65). If you keep your head straight, you will then find

$$\frac{D^2 B^\mu}{D\tau^2} = -R^\mu{}_{\beta\gamma\delta}\, U^\beta U^\delta B^\gamma\,, \tag{A.3.69}$$

with $R^\alpha{}_{\beta\gamma\delta}$ as defined in (A.3.63). Equation (A.3.69) is called the *geodesic deviation equation.* For a freely falling observer, $U^\mu = (1,0,0,0)$, it reduces to (A.3.62).

We see that the Riemann tensor plays the role of the tidal tensor. In fact, for the metric (A.1.17), we have

$$R^i{}_{0j0} = \frac{\partial \Gamma^i_{00}}{\partial x^j} = \delta^{ik}\frac{\partial^2 \Phi}{\partial x^k \partial x^j}\,. \tag{A.3.70}$$

Hence, we recover the Newtonian limit (A.3.59). The "trace" of the Riemann tensor defines the *Ricci tensor*:

$$R_{\mu\nu} \equiv g^{\alpha\beta} R_{\alpha\mu\beta\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\nu\beta}\Gamma^\beta_{\mu\alpha}\,. \tag{A.3.71}$$

The relativistic generalisation of $\nabla^2\Phi = 0$ is the Einstein equation in vacuum,

$$\boxed{R_{\mu\nu} = 0}\,. \tag{A.3.72}$$

Next, we will show how this equation needs to be extended to take into account how the curvature of spacetime is sourced by the energy and momentum of matter.

A.3.2 Energy-Momentum

In relativity, the energy and momentum densities of a continuous distribution of matter are components of the energy-momentum tensor:

$$T_{\mu\nu} = \left(\begin{array}{c|c} T_{00} & T_{0i} \\ \hline T_{i0} & T_{ij} \end{array}\right) = \left(\begin{array}{c|c} \text{energy density} & \text{energy flux} \\ \hline \text{momentum density} & \text{stress tensor} \end{array}\right). \tag{A.3.73}$$

This is a natural object to appear on the right-hand side of the Einstein equation. An important property of the energy-momentum tensor is that it is locally conserved, in the sense that

$$\boxed{\nabla^\mu T_{\mu\nu} = 0}\,. \tag{A.3.74}$$

The conservation of the energy and momentum densities correspond to the $\nu = 0$ and $\nu = i$ components, respectively. To work with (A.3.74), we need to unpack the meaning of the covariant derivative, which we will do in the following insert.

Covariant derivative.—So far, we have only encountered the covariant derivative of a contravariant vector (i.e. a four-vector with upper index):

$$\boxed{\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\alpha} A^\alpha}\,. \tag{A.3.75}$$

In this insert, we will derive the action of the covariant derivative on general tensors:

- First, we note that there is no difference between the covariant derivative and the partial derivative if it acts on a scalar

$$\nabla_\mu f = \partial_\mu f \,. \tag{A.3.76}$$

- To determine how the covariant derivative acts on a covariant vector, B_ν, let us consider how it acts on the scalar $f \equiv B_\nu A^\nu$. Using (A.3.76), we can write this as

$$\begin{aligned}\nabla_\mu(B_\nu A^\nu) &= \partial_\mu(B_\nu A^\nu) \\ &= (\partial_\mu B_\nu)A^\nu + B_\nu(\partial_\mu A^\nu)\,.\end{aligned} \tag{A.3.77}$$

  Alternatively, we can also write

$$\begin{aligned}\nabla_\mu(B_\nu A^\nu) &= (\nabla_\mu B_\nu)A^\nu + B_\nu(\nabla_\mu A^\nu) \\ &= (\nabla_\mu B_\nu)A^\nu + B_\nu(\partial_\mu A^\nu + \Gamma^\nu_{\mu\alpha}A^\alpha)\,,\end{aligned} \tag{A.3.78}$$

  where we have used (A.3.75) in the second equality. Comparing (A.3.77) and (A.3.78), we get

$$(\nabla_\mu B_\nu)A^\nu + B_\nu\Gamma^\nu_{\mu\alpha}A^\alpha = (\partial_\mu B_\nu)A^\nu \,. \tag{A.3.79}$$

  Writing $B_\nu\Gamma^\nu_{\mu\alpha}A^\alpha$ as $B_\alpha\Gamma^\alpha_{\mu\nu}A^\nu$, this gives

$$(\nabla_\mu B_\nu)A^\nu = \left(\partial_\mu B_\nu - \Gamma^\alpha_{\mu\nu}B_\alpha\right)A^\nu \,. \tag{A.3.80}$$

  Since the vector A^ν is arbitrary, the factors multiplying it on each side must be equal, so we get

$$\boxed{\nabla_\mu B_\nu = \partial_\mu B_\nu - \Gamma^\alpha_{\mu\nu}B_\alpha}\,. \tag{A.3.81}$$

  Notice the change of the sign of the second term relative to (A.3.75) and the placement of the dummy index.

- The covariant derivative of the mixed tensor $T^\mu{}_\nu$, can be derived similarly by considering $f \equiv T^\mu{}_\nu A^\nu B_\mu$. This gives

$$\boxed{\nabla_\sigma T^\mu{}_\nu = \partial_\sigma T^\mu{}_\nu + \Gamma^\mu_{\sigma\alpha}T^\alpha{}_\nu - \Gamma^\alpha_{\sigma\nu}T^\mu{}_\alpha}\,. \tag{A.3.82}$$

  Writing out the covariant derivative in (A.3.74), we therefore find

$$\begin{aligned}0 = \nabla^\mu T_{\mu\nu} &= \nabla_\mu T^\mu{}_\nu \\ &= \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\alpha}T^\alpha{}_\nu - \Gamma^\alpha_{\mu\nu}T^\mu{}_\alpha \,.\end{aligned} \tag{A.3.83}$$

Perfect fluid.—In cosmology, the coarse-grained energy-momentum tensor takes the form of a perfect fluid:

$$T_{\mu\nu} = (\rho + P)\,U_\mu U_\nu - P\,g_{\mu\nu}\,, \tag{A.3.84}$$

where U^μ is the *four-velocity* of the fluid, and ρ and P are its *energy density* and the *pressure* as measured by a comoving observer (i.e. an observer with four-velocity U^μ). The energy-momentum tensor measured by a free-falling observer takes the following simple form

$$T_{\mu\nu} = \mathrm{diag}(\rho, +a^2P, +a^2P, +a^2P)\,, \tag{A.3.85}$$

$$T^\mu{}_\nu \equiv g^{\mu\alpha}T_{\alpha\nu} = \mathrm{diag}(\rho, -P, -P, -P)\,. \tag{A.3.86}$$

Using the FRW metric (A.1.23), the $\nu = 0$ component of (A.3.83) then leads to

$$\frac{d\rho}{dt} = -3\frac{\dot{a}}{a}(\rho + P)\,. \tag{A.3.87}$$

This equation determines how the energy densities of the different matter components that fill the universe evolve with time.

A.3.3 Einstein Equation

Einstein's first guess for the field equation of GR was

$$R_{\mu\nu} \stackrel{?}{=} \kappa T_{\mu\nu}\,, \tag{A.3.88}$$

where κ is a constant. However, this doesn't work because, in general, we can have $\nabla^\mu R_{\mu\nu} \neq 0$, which wouldn't be consistent with $\nabla^\mu T_{\mu\nu} = 0$. When we add matter, the Ricci tensor $R_{\mu\nu}$ isn't the only measure of curvature. An alternative curvature tensor, made from $R_{\mu\nu}$ and $g_{\mu\nu}$, is

$$G_{\mu\nu} \equiv R_{\mu\nu} + \lambda g_{\mu\nu} R\,, \tag{A.3.89}$$

where $R = R^\mu{}_\mu = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, and λ is a constant that still needs to be determined. To use $G_{\mu\nu}$ in a field equation of the form $G_{\mu\nu} = \kappa T_{\mu\nu}$, we require $\nabla^\mu G_{\mu\nu} = 0$. This Bianchi identity is satisfied iff $\lambda = -\frac{1}{2}$. The combination in (A.3.89) is then called the *Einstein tensor*. The constant of proportionality between $G_{\mu\nu}$ and $T_{\mu\nu}$ is fixed by matching to the Newtonian limit (A.3.61). This gives $\kappa = 8\pi G$ and hence the final form of the Einstein equation is

$$\boxed{G_{\mu\nu} = 8\pi G\, T_{\mu\nu}} \qquad \text{or} \qquad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\, T_{\mu\nu}\,. \tag{A.3.90}$$

Note that if $T_{\mu\nu} = 0$, then $0 = g^{\mu\nu}G_{\mu\nu} = -R$, and the Einstein equation reduces to the vacuum form (A.3.72).

Newtonian limit.—The exotic appearance of the Einstein equation should not obscure the fact that it is a natural extension of Newtonian gravity. To illustrate this, consider the metric (A.1.17). At linear order in Φ, the Ricci tensor is

$$R_{00} = \nabla^2\Phi\,, \quad R_{0i} = 0\,, \quad R_{ij} = \nabla^2\Phi\,\delta_{ij}\,. \tag{A.3.91}$$

and the Ricci scalar is

$$R = -2\nabla^2\Phi\,. \tag{A.3.92}$$

The 00-component of the Einstein equation

$$\begin{aligned} G_{00} = R_{00} - \frac{1}{2}g_{00}R &= 2\nabla^2\Phi \\ &= 8\pi G\, T_{00} = 8\pi G\,\rho\,, \end{aligned} \tag{A.3.93}$$

therefore equals the *Poisson equation*

$$\nabla^2\Phi = 4\pi G\,\rho\,. \tag{A.3.94}$$

This confirms that the Einstein equation includes Newtonian gravity in the appropriate limit.

Friedmann equation.—The Ricci tensor associated with the metric (A.1.23) is

$$R_{00} = -3\,\frac{\ddot{a}}{a}\,, \quad R_{0i} = 0\,, \quad R_{ij} = (2\dot{a}^2 + \ddot{a}a)\,\delta_{ij}\,. \tag{A.3.95}$$

and the Ricci scalar is

$$R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]. \tag{A.3.96}$$

The 00-component of the Einstein equation therefore is

$$G_{00} = R_{00} - \frac{1}{2}g_{00}R \; = \; 3\left(\frac{\dot{a}}{a}\right)^2 \tag{A.3.97}$$

$$= \; 8\pi G\,T_{00} = 8\pi G\,\rho\,. \tag{A.3.98}$$

Defining the *Hubble parameter*, $H \equiv \partial_t \ln a$, this becomes the *Friedmann equation*

$$H^2 = \frac{8\pi G}{3}\,\rho\,. \tag{A.3.99}$$

This is the key evolution equation in cosmology.