Cosserat Rod Notation

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1 Notation Preliminaries

Reference Frames

There are two reference frames that need be be defined and that is is important to keep straight:

The global/laboratory/Eulerian frame: $\bar{\mathbf{x}} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ (1)

The local/material/Lagrangian frame: $\mathbf{x} = x_1\mathbf{d}_1 + x_2\mathbf{d}_2 + x_3\mathbf{d}_3$ (2)

Most quantities are expressed in the local frame, however, some, such as external forces or linear velocity, are often expressed with respect to their global reference frame. You can convert between the two by defining a rotation matrix $\mathbf{Q} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$, which allows us to to convert between the reference frames via $\mathbf{x} = \mathbf{Q}\bar{\mathbf{x}}$ and $\bar{\mathbf{x}} = \mathbf{Q}^T\mathbf{x}$. The local frame describes the orientation of the rod where $\mathbf{d}_3$ points along the centerline tangent ($\partial_s\bar{\mathbf{r}} = \bar{\mathbf{r}}_s = e\bar{\mathbf{t}}$) when there is no shear while $\mathbf{d}_1$ and $\mathbf{d}_2$ span the normal-binormal plane.

Other notation basics

Throughout this document scalers will be denoted as without bold (ρ), vectors will be lower-case bold ($\mathbf{x}$) and tensors will be uppercase bold ($\mathbf{Q}$). It will also be important to be able to distringuish between the original, undeformed reference state and the current, deformed state. The reference state will be denoted with an overhat ($\hat{s}$) will the deformed state will not have a hat (s). Notation generally follows Zhang et al. [2019].

2 Continuum Rod Notation

Cosserat rods are described by a centerline $\mathbf{r}(s,t)$ and local reference frame $\mathbf{Q}(s,t) = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ which consists of a triad of orthonormal basis vectors. The dynamics of the rod are then described by the equations for conservation of linear and angular momentum throughout the rod.

When describing the rod, a number of different quantities need to be defined. In the list below, keep in mind that all vectors can be defined in regards to both the global reference frame ($\bar{\mathbf{x}}$) or the local reference frame ($\mathbf{x}$).

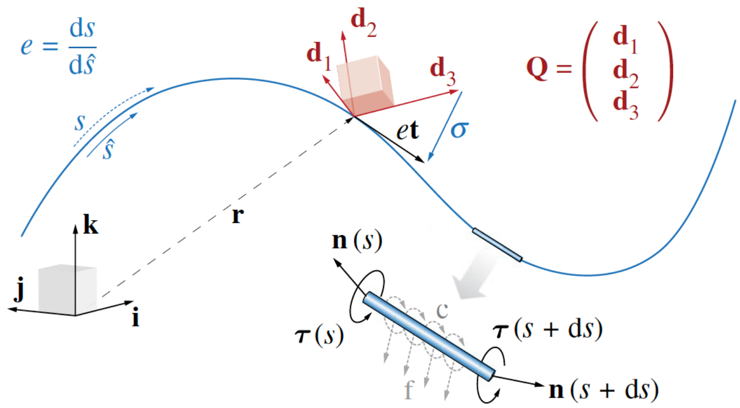

Figure 1: Schematic of Cosserat rod notation from Gazzola et al. [2018].

List of symbols:

s: arc-length – current configuration

$\hat{s}$: arc-length – reference/initial configuration

$\mathbf{d_1}$, $\mathbf{d_2}$, $\mathbf{d_3}$: local coordinate frame basis vectors

$\mathbf{Q} = \{\mathbf{d_1},\ \mathbf{d_2},\ \mathbf{d_3}\}$: Rotation matrix.

$\mathbf{r}(s,t)$: centerline of rod

$\mathbf{t}(s,t)$: local tangent vector of rod

$e = \frac{ds}{d\hat{s}}$: local stretching or compression ratio

$\boldsymbol{\sigma} = \sigma_1\mathbf{d_1} + \sigma_2\mathbf{d_2} + \sigma_3\mathbf{d_3} = e\mathbf{t} - \mathbf{d_3}$: shear displacement vector

$\left.\begin{array}{l}\mathbf{n}(s)\text{: internal forces}\\ \boldsymbol{\tau}(s)\text{: internal torques}\end{array}\right\}$ Internal Forces

$\left.\begin{array}{l}\mathbf{f}(s)\text{: external forces}\\ \mathbf{c}(s)\text{: external force couple}\end{array}\right\}$ External Forces

ρ: density

A: cross-sectional area

$\mathbf{S}$: shearing stiffness matrix

$\mathbf{B}$: bending stiffness matrix

$\mathbf{I}$: second area moment of inertia

$\mathbf{v}(s,t)$: Linear velocity

$\boldsymbol{\omega}(s,t)$: Angular velocity

$\boldsymbol{\kappa}$: curvature

$\mathbf{r}_s = \frac{\partial \mathbf{r}}{\partial s}$

Linear Elasticity: To solve the momentum balance equations, we need to define a bending stiffness ($\mathbf{B}$) and shearing stiffness ($\mathbf{S}$). We assume the rod is perfectly elastic with a linear stress-strain response. For an elastic beam, the stiffness matrices are diagonal 3x3 matrices:

$$\mathbf{B} = \begin{bmatrix} E\ I_1 & & \\ & E\ I_2 & \\ & & G\ I_3 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} \alpha_c G\ A & & \\ & \alpha_c G\ A & \\ & & E\ A \end{bmatrix}$$

Here E is the elastic Young's modulus, G is the shear modulus, I_i is the second area moment of inertia, A is the cross sectional area and the constant α_c is $4/3$ (for circular cross sections). Additionally, because of the linear elastic assumption, we can define constitutive laws for both the load-strain relations as well as the torque-curvature relations. These are:

Load-strain: $\mathbf{n} = \mathbf{S}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathbf{o}})$

Torque-curvature: $\boldsymbol{\tau} = \mathbf{B}(\boldsymbol{\kappa} - \boldsymbol{\kappa}^{\mathbf{o}})$

$\boldsymbol{\sigma}^{\mathbf{o}}$ and $\boldsymbol{\kappa}^{\mathbf{o}}$ are reference curvatures which allow the rod to have a stress-free configuration in shapes other than a straight line.

Governing Equations

With all of the relevant quantities now defined, the governing equations for a Cosserat Rod are:

$$\text{Linear Velocity: } \partial_t \mathbf{r} = \mathbf{v} \tag{3}$$

$$\text{Angular Velocity: } \partial_t \mathbf{d}_k = (\mathbf{Q}^T \boldsymbol{\omega}) \times \mathbf{d}_k \qquad k = 1, 2, 3 \tag{4}$$

$$\text{Linear Momentum: } \rho A \cdot \partial_t^2 \bar{\mathbf{r}} = \overbrace{\partial_s \left(\frac{\mathbf{Q}^T \mathbf{S} \boldsymbol{\sigma}}{e} \right)}^{\text{internal shear/stretch force}} + \overbrace{e\, \bar{\mathbf{f}}}^{\substack{\text{external}\\ \text{force}}} \tag{5}$$

$$\begin{aligned} \text{Angular Momentum: } \frac{\rho \mathbf{I}}{e} \cdot \partial_t \boldsymbol{\omega} = {} & \overbrace{\partial_s \left(\frac{\mathbf{B} \boldsymbol{\kappa}}{e^3} \right) + \frac{\boldsymbol{\kappa} \times \mathbf{B} \boldsymbol{\kappa}}{e^3}}^{\text{internal bend/twist couple}} + \overbrace{\left(\mathbf{Q} \frac{\bar{\mathbf{r}}_s}{e} \times \mathbf{S} \boldsymbol{\sigma} \right)}^{\substack{\text{internal shear/}\\ \text{stretch couple}}} + \overbrace{\left(\rho \mathbf{I} \cdot \frac{\boldsymbol{\omega}}{e} \right) \times \boldsymbol{\omega}}^{\text{Lagrangian transport}} \\ & + \underbrace{\frac{\rho \mathbf{I} \boldsymbol{\omega}}{e^2} \cdot \partial_t e}_{\text{unsteady dilation}} + \underbrace{e\, \mathbf{c}}_{\substack{\text{external}\\ \text{couple}}} \end{aligned} \tag{6}$$

$$+ \text{ Initial and Boundary Conditions} \tag{7}$$

It is important to note that as written, the linear velocity and momentum equations are written in the global reference frame while the angular velocity and momentum equations are written in the local reference frame.

3 Discreet Rod Notation

We discretize the rod into a set of nodes connected by straight line segments. Just as a continuum Cosserat rod is associated with a centerline $\mathbf{r}(t)$ and local coordinate frame $\mathbf{Q}(t)$, here the discretized Cosserat rod is defined by a collection of centerline vertices $\mathbf{r}_i(t)$ that are connected together by line segments associated with reference frames $\mathbf{Q}_i(t)$.

Some Terminology: The rod is discretized into n nodes connected by $n+1$ straight segments.

- The terms segments, lines, elements, and edges all refer to the same thing.
- The terms nodes and vertices refer to the same thing.

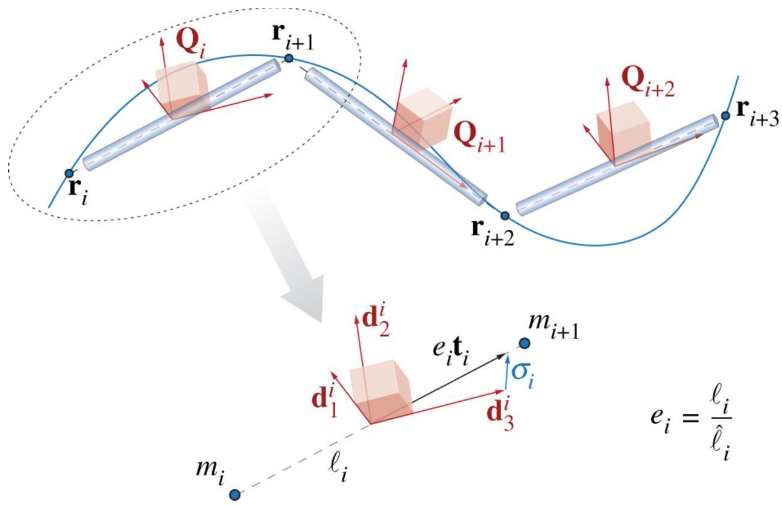

Figure 2: Schematic of discrete Cosserat rod notation from Gazzola et al. [2018].

List of symbols: Quantities are associated with either a segment or a node.

Quantities associated with each of the $n+1$ nodes ($i = [0, n]$):

- A position vector $\mathbf{r}_i$
- The linear velocity of the node $\mathbf{v}_i = \partial \mathbf{r}_i / \partial t$,
- The nodes pointwise mass m_i (related to ρ)
- Any external forces applied to the node $\bar{\mathbf{f}}_i$.

Quantities associated with each of the n segments ($j = [0, n-1]$):

- The local reference frame $\mathbf{Q}_j(t)$ (defines the orientation of each segment)
- An edge vector $\boldsymbol{\ell}_j = \mathbf{r}_{j+1} - \mathbf{r}_j$
- A current edge length $\ell_j = ||\boldsymbol{\ell}_j||$,
- A reference edge length $\hat{\ell}_j = ||\hat{\boldsymbol{\ell}}_j||$,
- A stretch ratio $e_j = \ell_j / \hat{\ell}_j$
- A unit tangent vector $\mathbf{t}_j = \boldsymbol{\ell}_j / \ell_j$.

- A shear/axial strain vector $\boldsymbol{\sigma}_j = \mathbf{Q}_j(e_j\bar{\mathbf{t}}_j) - \mathbf{d}_{3,j}$,
- An angular velocity $\boldsymbol{\omega}_j$,
- The reference cross sectional area of the segment $\hat{A}_j$,
- The mass second moment of inertia $\hat{\mathbf{J}}_j$
- The bend/twist matrix of the segment $\hat{\mathbf{B}}_j$
- The shear/strain matrix of the segment $\hat{\mathbf{S}}_j$,
- All external couples applied to a segment $\mathbf{c}_j$.

There are are several quantities that are defined at each of the *internal* nodes.

Quantities associated with each of the $n-1$ internal nodes ($i = [1, n-1]$):

- The Voronoi region $\mathcal{D}_i$ at every interior node ($i = [2, n]$) $\mathcal{D}_i = (\ell_{i-1} + \ell_i)/2$
- The Voronoi region at rest $\hat{\mathcal{D}}_i$
- A dilation factor for the region $\mathcal{E}_i = \mathcal{D}_i/\hat{\mathcal{D}}_i$
- The curvature $\hat{\boldsymbol{\kappa}}_i$ integrated over the Voronoi region $\mathcal{D}$
- The bend/twist matrix $\hat{\mathcal{B}}_i$ integrated over the Voronoi region $\mathcal{D}$

$$\hat{\boldsymbol{\kappa}}_i = \frac{\log(\mathbf{Q}_i\mathbf{Q}_{i-1}^T)}{\hat{\mathcal{D}}_i} \quad \text{and} \quad \hat{\mathcal{B}}_i = \frac{\hat{\mathbf{B}}_i\ell_i + \hat{\mathbf{B}}_{i-1}\ell_{i-1}}{2\hat{\mathcal{D}}_i}, \qquad i = [1, n-1] \tag{8}$$

Governing Equations

With all of the relevant quantities now defined, the discrete form of the governing equations for a Cosserat Rod are:

$$\partial_t \bar{\mathbf{r}}_i = \bar{\mathbf{v}}_i \qquad i = [0, n] \tag{9}$$

$$\partial_t \mathbf{d}_{k,j} = (\mathbf{Q}_j^T\boldsymbol{\omega}_j) \times \mathbf{d}_{k,j} \qquad j = [0, n-1],\ k = 1, 2, 3 \tag{10}$$

$$m_i \cdot \partial_t \bar{\mathbf{v}}_i = \Delta^h\left(\frac{\mathbf{Q}_j^T\hat{\mathbf{S}}_j\boldsymbol{\sigma}_j}{e_j}\right) + \bar{\mathbf{f}}_i \qquad i = [0, n],\ j = [0, n-1] \tag{11}$$

$$\begin{aligned}\frac{\hat{\mathbf{J}}_j}{e_j} \cdot \partial_t\boldsymbol{\omega}_j =& \Delta^h\left(\frac{\hat{\mathcal{B}}_i\hat{\boldsymbol{\kappa}}_i}{\mathcal{E}_i^3}\right) + \mathcal{A}^h\left(\frac{\boldsymbol{\kappa}_i \times \hat{\mathcal{B}}_i\boldsymbol{\kappa}_i}{\mathcal{E}_i^3}\hat{\mathcal{D}}_i\right) + \left(\mathbf{Q}_j\ \bar{\mathbf{t}}_j \times \hat{\mathbf{S}}_j\boldsymbol{\sigma}_j\right)\hat{\ell}_j + \\ &\left(\hat{\mathbf{J}}_j \cdot \frac{\boldsymbol{\omega}_j}{e_j}\right) \times \boldsymbol{\omega}_j + \frac{\hat{\mathbf{J}}_j\boldsymbol{\omega}_j}{e_j^2} \cdot \partial_t e_j + \mathbf{c}_j \qquad j = [0, n-1],\ i = [1, n-1]\end{aligned} \tag{12}$$

Here we use two special operators, the discrete difference operator Δ^h and the averaging operator $\mathcal{A}^h$. An important property of these operators is that they take m points and return $m+1$ points, allowing consistency in the equations as written. As with the continuum equations, the linear velocity and momentum equations are written in the global reference frame while the angular velocity and momentum equations are written in the local reference frame.

References

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