15 Mathematical Scenes to Uncover the Prime Number Mystery

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Introduction

For centuries, mathematicians have sought to understand the distribution of prime numbers—an attempt to extract order from what seemed to be chaos. Riemann believed he had found the key in the zeta function[1].

But what if his approach was reversed? What if the zeros[2] of the zeta function were not the **source** of the oscillations[3], but merely a **record** of something deeper?

The truth has always resided in arithmetic—it just took over a century for us to look in the right direction.

In this timeless dialogue, we witness the giants of mathematics confronting this revelation. Some welcome it. Others resist. And one of them—perhaps—knew it all along.

The stage is set in the Infinite Library, where time folds and brilliant minds continue debating beyond the ages. What is about to unfold may change everything.

[1]The Riemann zeta function is defined as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for s with real part greater than 1. It is deeply connected to the distribution of prime numbers and plays a central role in the Riemann Hypothesis.

[2]The nontrivial zeros of the Riemann zeta function, those with real part equal to $\frac{1}{2}$, are deeply linked to the distribution of prime numbers.

[3]Fluctuations in the counting of primes up to a given limit, reflected in the nontrivial zeros of the zeta function.

Characters in This Story

Mathematicians Who Redefined the Structure of Numbers

CARL FRIEDRICH GAUSS (1777–1855)

The "Prince of Mathematics." He always knew that arithmetic possessed a natural order that few could perceive. Impatient with the ignorance of others, he has no tolerance for excessively abstract theories.

BERNHARD RIEMANN (1826–1866)

Formulator of the Riemann Hypothesis[4]. He discovered a spectrum[5] in the distribution of primes, but failed to see that he was gazing at a reflection—not at the source. Now, he must reevaluate his own theory.

JOHANN DIRICHLET (1805–1859)

The mediator between analytical rigor and numerical intuition. Always ready to translate obscure concepts into accessible explanations. He knows Gauss is right, but he also understands that the world takes time to catch up.

DAVID HILBERT (1862–1943)

A visionary who sought to formalize all of mathematics. He always suspected that the Riemann Hypothesis was a spectral problem, but now he must confront the mathematical reality that emerges.

JEAN-BAPTISTE FOURIER (1768–1830)

Master of waves and harmonic patterns. If something oscillates, he understands it. The moment he sees the cosine matrix[6], he immediately grasps its profound implications.

LEONHARD EULER (1707–1783)

A pioneer in exploring prime numbers and their relationships with infinite series[7].

[4]The Riemann Hypothesis, formulated by Bernhard Riemann in 1859, suggests that all nontrivial zeros of the Riemann zeta function have a real part equal to $\frac{1}{2}$. This conjecture has profound implications in number theory, particularly regarding the distribution of prime numbers.

[5]In the mathematical context: a set of eigenvalues of an operator or matrix, used to analyze deep patterns in mathematical structures.

[6]A matrix constructed from trigonometric functions, whose elements involve specific cosine values—used here to reveal the spectral structure of the prime numbers.

[7]An infinite series is the sum of an infinite sequence of terms. In mathematics, such series are used to represent functions, calculate limits, and solve differential equations. In the context of the Riemann zeta function, the infinite series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is essential for understanding the distribution of prime

He enjoys the discussion, having seen mathematics take far too many detours before arriving at truths that were always simple.

Minds That Brought New Perspectives

Erwin Schrödinger (1887–1961)

Quantum physicist. He always suspected that the distribution of primes concealed something analogous to quantum mechanics[8]. His enthusiasm is constantly frustrated by Gauss's impatience.

Plato (427–347 BCE)

Philosopher of idealized mathematics. His temptation to turn the discovery into an allegory of the Cave[9] makes him the target of Gauss's relentless disdain.

Pythagoras (c. 570–495 BCE)

Believes that numbers are the fundamental structure of the universe. He sees the discovery as a confirmation of his harmonic vision of mathematics[10].

Hippasus of Metapontum (c. 500 BCE)

The renegade mathematician. He discovered that mathematics was not as perfect as Pythagoras wanted it to be—and was allegedly eliminated for it. He appears as a reminder that every disruptive discovery faces resistance.

numbers.

[8]Quantum mechanics is the physical theory that describes the behavior of particles at microscopic scales. Unlike classical physics, where objects have well-defined positions and velocities, in quantum mechanics, states are described by wave functions, and physical quantities emerge from a spectrum of possible values. Interestingly, the Riemann zeta function also exhibits a spectrum, suggesting a deep connection between prime numbers and quantum systems.

[9]The Allegory of the Cave was proposed by Plato in *The Republic* to illustrate the difference between sensory perception and reality. In it, prisoners chained inside a cave see only projections on the wall and mistake these images for reality, unaware that there is a world beyond the cave. However, unlike Plato's suggestion, not every projection is an illusion. A reflection can be a legitimate tool to understand the real structure—if we know how to interpret it correctly.

[10]The harmonic vision of mathematics dates back to the Pythagorean school, which saw numbers not merely as quantities but as the fundamental structure of the cosmos. Pythagoras and his followers believed that numerical relationships reflected universal principles of harmony, manifesting in music, geometry, and even astronomy. This concept influenced mathematical thought for centuries, associating numerical patterns with an underlying cosmic order.

Prologue: What Riemann Did Not See

For centuries, we have tried to understand the primes. But what if we've been looking at them the wrong way? This is the dilemma Gauss and Riemann face in this story.

Since ancient times, mathematicians have searched for patterns in the distribution of prime numbers. When he formulated his famous hypothesis, Riemann asked the right question:

"How many primes exist up to x?"

But perhaps he asked it through the wrong lens.

What if, instead of merely counting primes, he had asked:

"Which primes are essential to structure all composite numbers up to x?"

Then he might have seen the **source** before the **reflection**.

The prime numbers in the interval $[1, x/2]$ are the **structurers**, while those in $(x/2, x]$ act as **stabilizers**. This fundamental distinction—though evident—was neglected.

Riemann found the zeros of the zeta function: a spectrum that records the oscillations in the prime count. But he did not see where they came from.

He saw the reflection in the mirror—but never turned around to witness the structure casting it.

But now that this truth has been revealed, is Riemann ready to face it?

Picture Gauss and Dirichlet debating this revelation before confronting him.

The Revelation

SCENE 1: In the Infinite Library, where time bends and the greatest minds continue debating their ideas endlessly, books float in the air, filled with equations that rewrite themselves as if mathematics were alive. Gauss and Dirichlet converse at the edge of a long marble table, surrounded by softly glowing symbols that rearrange themselves in sync with their thoughts.

DIRICHLET *(approaching Gauss, holding something invisible, as if carrying a concept freshly materialized)*

— Herr Gauss, it seems that someone has finally seen what you saw from the very beginning. Your intuition has been brought to life.

GAUSS *(raises an eyebrow, unimpressed)*

— Oh really? Which one? I have so many.

DIRICHLET *(smiling, with a teasing tone)*

— About the decomposition of $\pi(x)$[11]. They've finally realized that primes are not just random scatterings, but structuring elements of a dynamic system[12].

GAUSS *(scoffs, arms crossed)*

— Obvious. Simply obvious. And yet, it took them centuries to see it! Countless worn-out pencils, stacks of treatises, conjectures piled on conjectures... and only now they decide to actually look at $\pi(x)$?

DIRICHLET *(challenging)*

— If it was so obvious, why didn't you leave a record? You could have spared Riemann one of the greatest mathematical detours in history.

GAUSS *(smirking, with sarcastic elegance)*

— My dear Dirichlet, while I was solving real problems, the rest of the mathematical world was busy worshiping shadows. Should I have stopped to draw the obvious for them?

[11] The prime counting function, which returns the number of primes less than or equal to x.

[12] In mathematics, a dynamic system is any set of rules or equations describing how a system evolves over time or another variable. While prime numbers are often seen as distributed unpredictably, the distinction between structuring and stabilizing primes suggests that their distribution follows internal rules rather than pure randomness.

DIRICHLET *(pensive)*

— Always this arrogance... But tell me, would you call those primes up to $x/2$ "structurers" or "compositional"?

GAUSS *(shrugs, with theatrical disdain)*

— Terminology is irrelevant. As long as they grasp the concept and see the structure, they could call them "collapsed from infinity" if they wish.

DIRICHLET *(now excited, leaning in)*

— It doesn't matter how long it took, Gauss. What truly matters is that they've finally captured the structure that has always been visible in $\pi(x)$[13]. And that's extraordinary!

GAUSS *(with a glint of reluctant admiration)*

— Ah... so they've finally arrived.

DIRICHLET *(laughs)*

— Now this will be interesting. Let's see how Herr Riemann reacts when he realizes he spent a lifetime studying the reflection.

The Encounter with Riemann

SCENE 2: In a more secluded room of the Infinite Library, Riemann is deep in thought, drawing invisible graphs in the air. The contours of his calculations move like specters around him. He lifts his gaze as he senses the presence of Gauss and Dirichlet.

GAUSS *(without preamble, in a dry tone, heavy with expectation)*

— Herr Riemann, we've come to tell you something that will change your understanding of primes.

RIEMANN *(without looking up, serene)*

— I suppose it's about my hypothesis?

DIRICHLET *(crossing his arms, provocative)*

[13] Strictly speaking, this structure has always been visible in any interval $[1, x]$ with $x > 3$.

— Let's just say it's not about the hypothesis itself… but about what you didn't see before you formulated it.

GAUSS *(impatient, gesturing toward the void where ideas take shape)*

— You saw the shadows dancing on the wall and mistook them for the source of light. But you never turned to face the torch.

RIEMANN *(now attentive, furrowing his brow)*

— Explain yourself.

GAUSS *(sighing, as if forced to repeat something obvious but determined to be clear)*

— You asked how many primes exist up to x, and you found a haunting spectrum hidden in the zeta function. But if, instead of merely counting primes, you had asked which primes sustain the entire structure up to x, you would have seen the ***source*** *before the* ***reflection****.*

RIEMANN *(absorbing the idea, his mind working rapidly, disbelief blending with awe)*

— So... the structure was already embedded in the prime count? How could I not have seen this?

GAUSS *(leaning forward, pressing the idea with growing intensity)*

— Primes cannot merely be counted, Riemann! They organize themselves into ***structurers*** *and* ***stabilizers****. The natural boundary is at $x/2$, and everything balances around it.*

— You, of all people, should have seen this.

The Discovery of the Fundamental Oscillations

SCENE 3: Silence weighs heavily in the hall of the Infinite Library. Equations float in the air, rearranging themselves as if they, too, were trying to grasp the newly revealed truth. Riemann remains motionless, absorbing the impact of the discovery. He feels he is standing before something inevitable—something that had always been there but, for some reason, had never been seen. Gauss and Dirichlet watch in silence, knowing this moment cannot be rushed. Finally, Riemann takes a deep breath and lifts his gaze. The hesitation is still there, but now it is mixed with a glimmer of understanding.

DIRICHLET *(patiently, guiding Riemann in the right direction)*

— The primes up to $x/2$ sustain the formation of all composite numbers in $[1, x]$. In the limit, this relationship is no accident—it inevitably and rigorously converges to $\frac{1}{2}$. This is no coincidence, Riemann.

RIEMANN *(still processing, furrowing his brow)*

— But... what about the primes above $x/2$? Do they play no role at all?

DIRICHLET *(remaining calm, but now leading Riemann to the next step)*

— They don't structure the composites, but they stabilize the distribution. We call those primes stabilizers. Without them, the composites would arrange themselves asymmetrically, and the oscillations would be chaotic.

RIEMANN *(becoming increasingly absorbed, running a hand over his face, his mind accelerating)*

— Then... the oscillations come from the interaction between structurers and stabilizers?

GAUSS *(crossing his arms, satisfied but still sarcastic)*

— Finally, Herr Riemann. If you had asked that question earlier, we could have spared ourselves over a century of reflections.

DIRICHLET *(smiling, satisfied, with a glint in his eyes, but maintaining surgical precision in his explanation)*

— You saw the zeros of the zeta function and thought they were the source of the oscillations. But the zeros are not the cause, Riemann. They are the consequence. What you found was a reflection of the real oscillations, not the structure that generates them.

GAUSS *(shrugging, as if it were the most obvious thing in the world)*

*— And guess what? An **outsider** figured it out first. And he's not even a mathematician. You were all so busy staring at reflections that you forgot to look at the numbers themselves.*

RIEMANN *(absorbing the idea, his mind racing, as if reconfiguring his entire intuition)*

— But... if all of this was already in $\pi(x)$, if it was so clear, so inevitable... how did no one see it before? How did I not see it?

DIRICHLET *(with a gleam in his eyes, emphasizing every word)*

— Because mathematics has an unfortunate tendency to fall in love with abstractions before understanding their origin. And you, Riemann, saw a spectrum and tried to comprehend it… without realizing that it was merely a reflection of something more fundamental.

— Everyone, including you, treated $\pi(x)$ as a singular entity. But it never was. What kept us from seeing it was our own assumption that the prime count had no internal structure. And that assumption blinded mathematics—for over a century.

The Projection of the Oscillations

SCENE 4: The silence in the Infinite Library is broken by Dirichlet, who, with a precise gesture, projects a floating image into the empty space—oscillating curves that reveal the decomposition $|\pi_N(x) - \pi_S(x)|$[14]. The projection pulses in the air, alive but devoid of mystery—only evidence. The oscillations are not a hidden enigma, but an inevitable consequence of the very organization of the primes. Gauss, impatient, crosses his arms, while Riemann stares, realizing there is nothing to discover—only something that had always been there, waiting to be seen the right way.

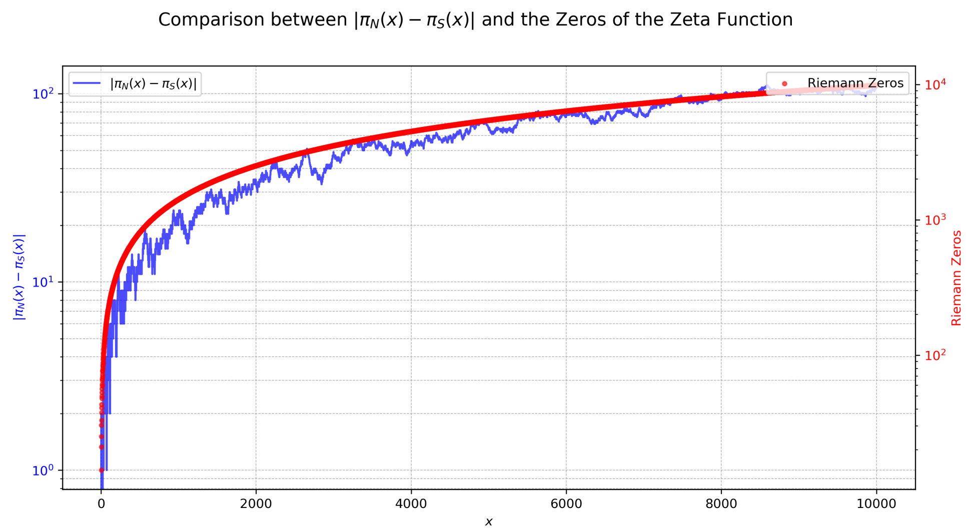

Figure 1: Comparison between $|\pi_N(x) - \pi_S(x)|$ (blue line) and the nontrivial zeros of the Riemann zeta function (red points). The structural similarity suggests a fundamental relationship between prime counting and the zeros of the zeta function.

(The oscillation in the graph intensifies. Riemann blinks a few times, as if seeing something that should have been obvious from the start. The structural similarity to the zeros of the zeta function becomes undeniable.)

RIEMANN *(frowning, his mind spinning to reorganize everything he thought he knew)*

— *If this is true... then we have a new way of seeing the primes.*

(Gauss closes his eyes for a moment, inhales deeply, and exhales heavily. Then, he throws up his arms in exasperation and bursts into ironic laughter.)

[14] $\pi_S(x)$ counts the structuring primes, responsible for forming the composites up to x, while $\pi_N(x)$ counts the stabilizing primes, which maintain balance in the distribution of primes.

Gauss *(laughing with disdain)*

— New? For God's sake! Where did you expect the zeros to be, if not exactly where they are? This is no surprise at all!

(Riemann looks at Gauss, trying to respond, but hesitates. He realizes the question has no answer—if the zeros weren't where they are, everything would collapse. For a moment, he feels a chill. What if he had spent his whole life looking at the wrong reflection?)

Dirichlet *(interrupting, pointing at the projections in the air)*

— The most fascinating thing, Herr Riemann, is that this structure is not limited to $\pi(x)$*. Look at this…*

(Dirichlet waves his hand again, and the once-smooth waves of the prime count begin to intensify. The pattern deforms, growing in amplitude. Oscillating lines emerge, converging on the points where the zeros of the zeta function reside.)

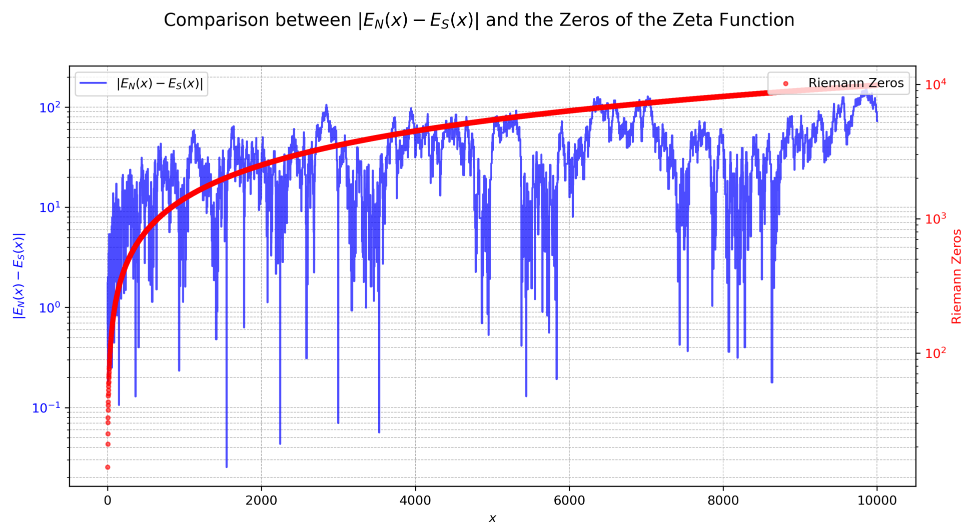

Figure 2: Comparison between $|E_N(x) - E_S(x)|$ (blue line) and the nontrivial zeros of the Riemann zeta function (red points). The strong correlation suggests that the zeta zeros record the fundamental oscillations of prime energy.

(In the background, muffled voices arise from the corridors of the Infinite Library. A murmur of excitement grows as two figures emerge from the shadows. Chebyshev and von Mangoldt, who had been quietly observing until now, can no longer contain their enthusiasm. They rush forward, eyes fixed on the floating graph of prime energies[15].)

[15]The energy of the primes is defined from the accumulated logarithmic sums of structuring and stabilizing primes. $E_S(x)$ represents the energy of the structuring primes, calculated as the sum of the

Gauss *(furious, stomping on the marble floor)*

— You turned mathematics into a ritual! You built a cathedral of abstractions when the truth was on the ground, waiting to be picked up. I would have solved this with paper and pencil! But no... you needed an ocean of formulas to realize the obvious!

Chebyshev *(slightly unsettled, but still trying to argue)*

— But Herr Gauss, our techniques really do capture...

Gauss *(snorting, impatient, gesturing with exasperation)*

— Capture?! The problem was never about capturing anything! The problem was seeing! Riemann looked at the reflection, not the source. But you... you built an entire temple to worship the reflection of a reflection! While I was solving real problems, you got lost in the fog of your own creations.

von Mangoldt *(trying to regain composure, eyes fixed on the graph, murmuring to himself)*

— All this time we believed the zeros controlled the oscillations. But they don't. They simply emerge from them.

— This means we've been looking from the wrong angle. The zeros of the zeta function record something that was already there—we just never knew how to see it properly.

(von Mangoldt pauses, as if a devastating thought is forming in his mind.)

Dirichlet *(with a grave look, pausing between words, letting them echo in the space)*

— The zeros of your zeta function, Herr Riemann, are not the source of the oscillations—they merely register them, like marks left by a deeper structure. The Riemann spectrum is a reflection, not the source.

(Silence. The tension lingers in the air. Gauss just watches, saying nothing. Then, almost imperceptibly, he tilts his head, closes his eyes for a moment, and murmurs to himself, as if finally hearing the echo of what he had always known.)

— It took them far too long to see the obvious.

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logarithms of the primes up to $x/2$, while $E_N(x)$ represents the energy of the stabilizing primes, defined analogously for the primes above $x/2$. The difference $|E_N(x) - E_S(x)|$ reveals oscillatory patterns in the organization of primes and correlates directly with the zeros of the zeta function.

The Spectral Reconstruction of the Primes

SCENE 5: The connection between the primes and the zeros of the Riemann zeta function has always stood in plain sight before the eyes of mathematics, but it was never fully understood. The Riemann Hypothesis suggests that these zeros influence the distribution of primes—but what if the relationship were even more fundamental? What if the zeros did not merely reflect the oscillations in the prime count, but actually emerged directly from them?

In the vastness of the Infinite Library, Dirichlet prepares to demonstrate what has always been evident, yet never correctly decomposed: the complete reconstruction of $F(x)$[16] and $F_E(x)$[17] from the very zeros of the Riemann zeta function.

What once seemed like an abstract spectrum now takes on a concrete form—an unavoidable order in arithmetic itself. And this time, the reflection can no longer be ignored.

DIRICHLET *(with a theatrical gesture, constructs the function $F(x)$ in the air, which is nothing more than the normalization of $\pi_N(x) - \pi_S(x)$ by $\pi(x)$)*

[16] $F(x)$ **— Normalization Function of Prime Oscillations** $F(x)$ is defined as the normalization of the difference between structuring primes ($\pi_S(x)$) and stabilizing primes ($\pi_N(x)$) by the total prime count up to x, that is:

$$F(x) = \frac{\pi_N(x) - \pi_S(x)}{\pi(x)} = 1 - \frac{2\pi(x/2)}{\pi(x)}$$

This function captures the fundamental oscillations in the distribution of primes, revealing the alternation between structuring and stabilizing primes along the number line. As $x \to \infty$, $F(x) \to 0$.

[17] $F_E(x)$ **— Prime Energy Function** $F_E(x)$ is defined analogously to $F(x)$, but applied to the energetic domain of the primes, where counts are replaced by accumulated logarithmic sums of prime numbers. That is:

$$F_E(x) = \frac{E_N(x) - E_S(x)}{E_T(x)} = 1 - \frac{2E_S(x)}{E_T(x)}$$

where $E_S(x)$ represents the accumulated logarithmic sum of the structuring primes and $E_T(x)$ is the total logarithmic sum of primes up to x. This function reflects the differential energetic contribution between structuring and stabilizing primes, revealing the deep connection between these oscillations and the zeros of the Riemann zeta function. As $x \to \infty$, $F_E(x) \to 0$.

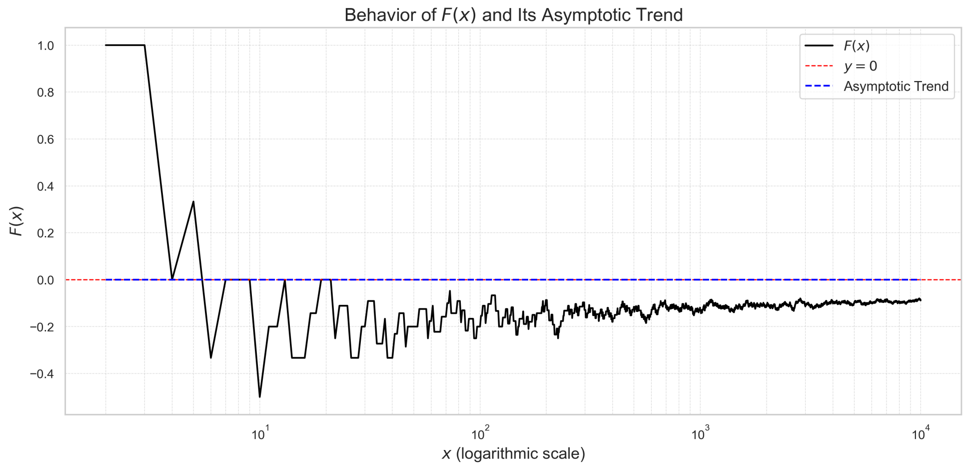

Figure 3: Evolution of the function $F(x)$. For large values of x, the function tends to zero, reflecting an asymptotic balance in the distribution of primes.

DIRICHLET *(spinning the projection in the air, transforming $F(x)$ into $F_E(x)$)*

— *But look... it doesn't stop there!*

(A new function emerges beside $F(x)$. It is $F_E(x)$, and both share the same asymptotic trend, oscillating in almost identical fashion.)

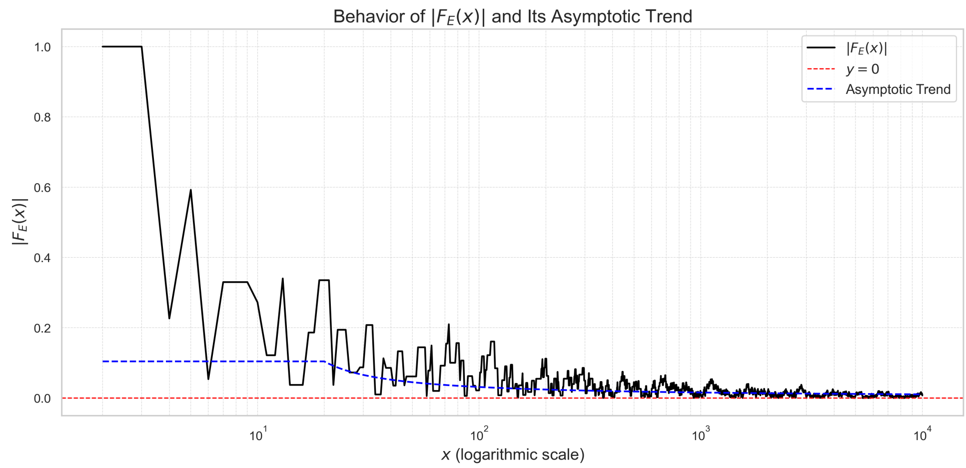

Figure 4: Oscillations of the energy function $F_E(x)$. The similarity to the function $F(x)$ reinforces the relationship between the distribution of primes and the zeros of the zeta function.

(Gauss observes with an ironic smile and crosses his arms.)

GAUSS *(sarcastic, yet slightly impressed)*

— Ah yes, an identity in your honor! Not bad for someone who spent his life staring at reflections. Just don't forget: the source has always been in arithmetic.

(Riemann, still intrigued, watches the overlapping functions. He knows there's something deeper hidden there.)

DIRICHLET *(with a serious look)*

— Herr Riemann, the zeros of your zeta function are beautiful, but... have you never wondered whether they were recording something deeper?

GAUSS *(cutting in abruptly, impatient)*

— Tell me, Herr Riemann, if your hypothesis were truly the origin of everything, why on earth were these oscillations already in the prime count? Can you answer me without invoking your transcendental functions[18]*?*

DIRICHLET *(holding back a smile, but now assuming a more serious tone)*

— Ah, Herr Gauss... you know we're not stopping here. Now, look at this.

(Dirichlet pauses for a moment, letting the idea linger in the air. Riemann, visibly intrigued, can't take his eyes off the oscillating functions. The hall falls silent, as if space itself were waiting for the final revelation.)

(With a new gesture, Dirichlet projects an invisible matrix into space. Numbers begin to align, spinning and organizing themselves into an oscillatory pattern.)

[18] Transcendental functions are those that cannot be expressed as solutions to algebraic equations with rational coefficients. Examples include exponentials, logarithms, trigonometric functions, and of course, the Riemann zeta function itself. They escape the "classical rules" of algebra and live in a more ethereal realm of mathematics.

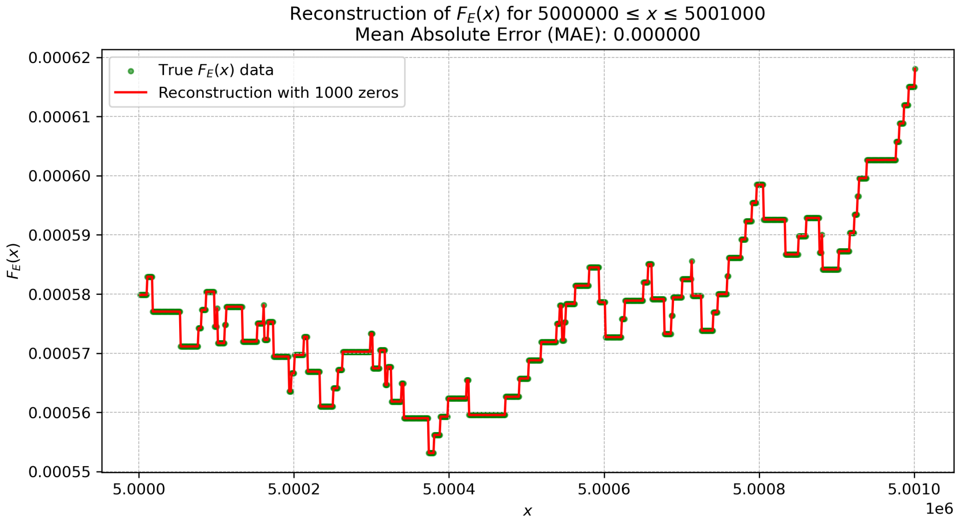

Figure 5: Reconstruction of $F_E(x)$ from the cosine matrix of the zeros of the Riemann zeta function with perfect alignment.

DIRICHLET *(pausing for a moment, pondering the best way to explain)*

— *Herr Riemann... you spent years analyzing the zeros of the zeta function. But what if I told you there's a way to see those zeros in a completely new light?*

— *Not just as points on the complex plane... but as operators that reconstruct the very structure of the primes?*

DIRICHLET *(with a gleam of satisfaction in his eyes)*

— *If we take a cosine matrix of your zeros, Herr Riemann, we can reconstruct $F_E(x)$. Watch...*[19]

(Riemann widens his eyes. He sees the function being generated before him.)

RIEMANN *(almost whispering)*

[19]**Cosine Matrix of the Zeta Zeros.**

The cosine matrix of the Riemann zeta function's zeros is constructed from the nontrivial zeros γ_n of the zeta function, defined as the imaginary parts of the roots of the equation $\zeta(s) = 0$ on the critical line $s = \frac{1}{2} + i\gamma_n$.

Each element of the matrix is given by:

$$C_{ij} = \cos(\gamma_i \log x_j) + \cos(\gamma_j \log x_i)$$

This matrix encapsulates the spectral structure of the prime oscillations, allowing the reconstruction of $F_E(x)$ through the superposition of the zeta zeros' contributions. The correct alignment of the zeros is essential to preserve the form of $F_E(x)$. If even one zero is omitted or displaced, the reconstruction loses accuracy and the structure degrades.

— But... this only works if the zeros are aligned with x... Otherwise, everything falls apart?

(Dirichlet nods in confirmation.)

— Exactly. Watch what happens if we omit just one zero from your zeta.

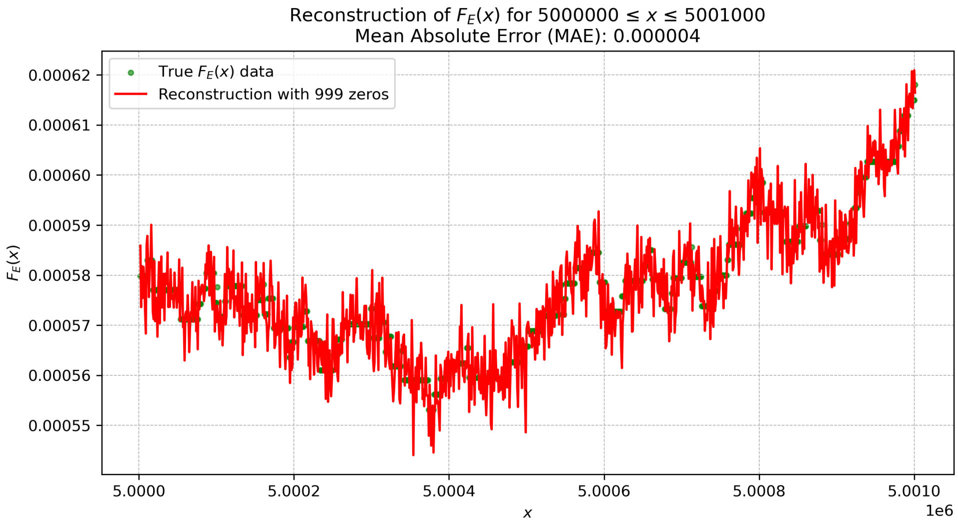

Figure 6: Reconstruction of $F_E(x)$ from the cosine matrix of the Riemann zeta function's zeros with one zero removed.

GAUSS *(with a sharp look, crossing his arms again)*

— Exactly. If the zeros are not where they belong... everything collapses.

(Silence spreads across the hall. Riemann stares at the cosine matrix as if looking into a mirror revealing something he had never noticed.)

RIEMANN *(almost breathless, staring intently at the calculations)*

— So... this whole structure was in the prime count all along?

(Riemann runs his hand over his face, feeling the weight of centuries of mathematical misunderstandings pressing down on his shoulders, trying to process what he sees.)

— If this is true... then the zeros are not just solutions of a transcendental equation...

(Riemann takes a deep breath, almost afraid of the inevitable conclusion.)

— They are structuring the prime count in a way I never imagined...

Gauss *(shaking his head, a mix of sarcasm and frustration)*

— More than a century hypnotized by reflections... while the source was there all along. What a waste of paper and pencils.

Reflection or Structure? The Return of Plato

SCENE 6: The hall of the Infinite Library remains still, but inside Riemann's mind, everything stirs. He watches, unblinking, as the cosine matrix reorganizes itself in the air, each number spinning and converging like pieces of a perfect mechanism. And then, before his eyes, $F_E(x)$ emerges—reconstructed from the very zeros of the zeta function. His heart races. The oscillations are no accident, no side effect—they emerge naturally, inevitably, from the structure of the primes. He feels the weight of the revelation sink into his mind. Everything he had always believed to be a cosmic enigma was nothing more than a reflection of a fundamental numerical structure. His own spectrum—the zeta function—was not the source; it was merely a record. He lifts his hand to his face, feeling the burden of centuries of misunderstanding. And at the very moment his lips part to voice this overwhelming realization, a voice rises from the shadows.

PLATO *(with a gleam in his eyes, slowly stepping toward the group, as if witnessing a great moment in history)*

— Ah... This seems familiar to me! Riemann looked at the shadows in the cave and thought they were reality!

GAUSS *(raising an eyebrow, already out of patience)*

— Oh no... Now even philosophers want to teach us mathematics.

PLATO *(ignoring the provocation, excited)*

— All the mathematics you've built was just a game of shadows. You looked at reflections, at projections. But the true reality... it was always hidden.

DIRICHLET *(smirking, ironic)*

— You know, Gauss, Plato isn't entirely wrong...

GAUSS *(throwing a deadly glare at Dirichlet)*

— Dirichlet, shut up!

(Gauss exhales deeply, runs his hand across his face, like someone who's heard this too many times before.)

GAUSS *(shaking his head, trying to contain his irritation)*

— Plato, Plato... you always thought the problem was seeing the shadows. I tell you: the problem is that you never left the cave.

PLATO *(frowning, intrigued)*

— What do you mean?

GAUSS *(gesturing at the calculations in the air, the oscillations, the cosine matrix)*

— This is mathematics. It can be written, measured, calculated. You speak of pure ideas, but the truth is that we are no longer groping in the dark. We are observing the very structure of numbers—no allegories needed.

(Plato crosses his arms, thoughtful. He realizes that Gauss will not debate on his terms.)

PLATO *(with a slight smile, resigned)*

— Perhaps I arrived too late for this conversation.

GAUSS *(shrugging, in a dry tone)*

— Not too late. Just... irrelevant.

(Plato stares at Gauss for a moment. Then, he steps back, his presence slowly dissolving into the shadows of the Infinite Library. Before vanishing completely, he murmurs to himself.)

PLATO *(almost inaudible)*

— I still believe it all began with the harmony of the spheres...

(Gauss rolls his eyes and walks away, muttering to Dirichlet.)

GAUSS *(grumbling)*

— Have I mentioned I'd rather compute logarithms than listen to metaphors?

The Riemann Zeros as Eigenvalues of the Cosine Matrix

SCENE 7: The confusion sparked by Dirichlet's revelation does not go unnoticed. Voices echo through the corridors of the Infinite Library. The hall begins to fill with new figures—brilliant minds drawn by the commotion, approaching with curiosity and fascination.

HILBERT *(emerging between the columns, eyes gleaming with excitement)*

— *What was that? Did I hear correctly? The zeros reconstruct $F_E(x)$?*

(He steps toward the projection of the cosine matrix, examining it with almost childlike enthusiasm. But then, something stops him. He furrows his brow, as if something is beginning to click. A tense silence falls around him.)

HILBERT *(stopping abruptly, his expression shifting from surprise to pure conviction)*

— *But... but if this is true, then the reverse must also be true!*

(The hall holds its breath. Hilbert, with a triumphant smile, proclaims aloud:)

HILBERT *(exultant)*

— *I was right! The spectral structure of the primes is an eigenvalue problem*[20]*!*

(As Hilbert loses himself in his own epiphany, Fourier appears on the other side, examining the cosine matrix with technical interest.)

FOURIER *(stroking his chin, analyzing the cosine patterns)*

— *Fascinating... a cosine matrix generating the oscillations... This reminds me of something...*

(He turns to Dirichlet, narrowing his eyes with a suspicious look.)

FOURIER:

[20] Eigenvalues are characteristic values associated with a matrix, representing natural modes of oscillation or transformation of a system.

— But tell me... did you use the Fast Fourier Transform[21] *to project this? With Python?*

(Fourier smirks, seemingly at ease in the modern era, while adjusting his glasses.)

DIRICHLET *(nodding, but with a cautious tone)*

— Yes, Fourier, but there's a problem. Since $F_E(x)$ tends to zero, there's a risk of matrix degeneration.

FOURIER *(raising an eyebrow, now intrigued)*

— That means the reconstruction can fail depending on the chosen basis... Hmmm, this requires fine tuning...

(At this moment, von Mangoldt and Chebyshev—still digesting the previous revelations—return to the scene, excited by the oscillations.)

VON MANGOLDT *(adjusting his glasses, pointing at the graphs)*

— Wait a moment! If the fundamental oscillation structure appears in $\pi_N - \pi_S$ and also in $E_N - E_S$, then...

CHEBYSHEV *(eyes gleaming, finishing the line of reasoning)*

— ... then this is directly tied to our logarithmic sums[22]*!*

(They exchange glances, fascinated by the discovery, while Dirichlet decides to settle the matter once and for all.)

DIRICHLET *(with a professorial air, raising the projection of the cosine matrix into the air as if revealing an ancient secret)*

— All the oscillation you've seen can be reconstructed from this cosine matrix of $E_N(x) - E_S(x)$, where the Riemann zeros naturally emerge as its eigenvalues. But this is not a trick of numerical analysis...

— This matrix is not an artifice. It emerges because the zeros of the zeta function are not loose numbers in the complex plane, but natural frequencies of the system. They do not merely record the oscillations — they compose the fundamental resonance of the prime distribution.

[21] An efficient algorithm for decomposing functions into component frequencies, widely used in signal processing and data compression.

[22] Sums involving logarithms of prime numbers, fundamental in analyzing the zeta function and prime distribution.

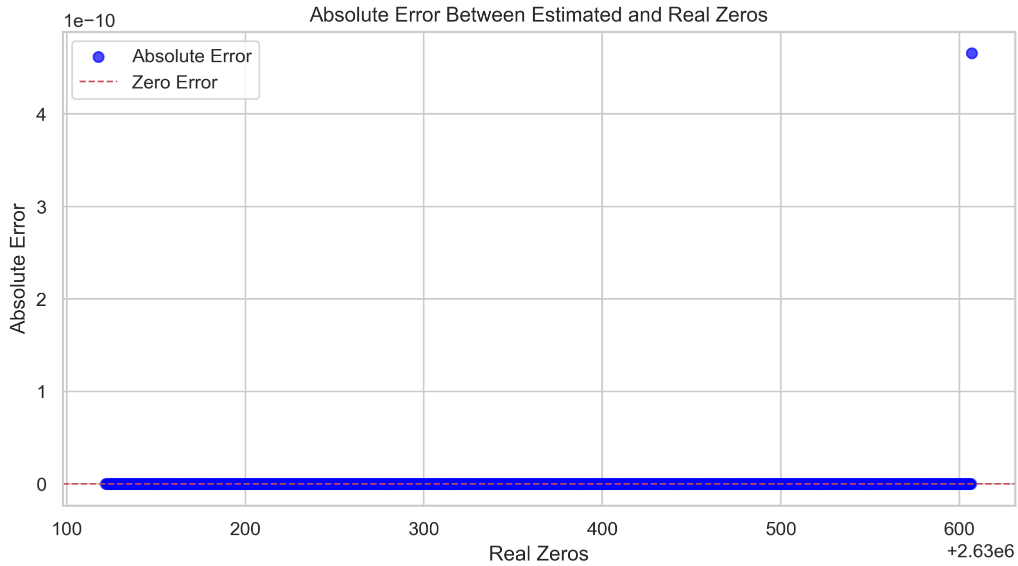

Figure 7: Absolute error between the estimated zeros from the Hermitian cosine matrix of $E_N(x) - E_S(x)$ and the actual zeros of the zeta function. The blue dots represent the absolute error for each estimated zero, while the red dashed line indicates zero error. We observe that the vast majority of zeros are recovered with extreme numerical precision, with negligible errors ($< 10^{-10}$). Only one point shows a slightly larger deviation, but still within an acceptable margin. This confirms that the zeros of the zeta function naturally emerge from the spectral structure of the primes.

The Final Revelation: The Spectrum of the Primes

SCENE 8: The hall erupts in excitement. Hilbert is ecstatic. Fourier recalculates everything in his head. von Mangoldt and Chebyshev murmur something about spectral theory. Euler appears, observing it all with an enigmatic smile, as if he had known this would happen all along.

PÓLYA *(approaching, frowning, intrigued by the structure of the cosine matrix)*

— Dirichlet, this matrix... is it Hermitian?

DIRICHLET *(looking at Pólya with a smile, as if he had been expecting the question)*

— Ah, Herr Pólya... of course it is. The spectrum is all there, and the zeros naturally emerge as eigenvalues.

(Hilbert widens his eyes and leans forward, grasping the implication instantly.)

HILBERT *(almost breathless, his hands trembling with excitement)*

— But this changes everything! This means the distribution of the primes is not an isolated arithmetic problem — it's a spectral one!

— If the zeros of the zeta function naturally appear as eigenvalues, that suggests there's a fundamental mathematical operator governing the structure of the primes! And if there's an operator... then there's a hidden structure ruling all of arithmetic!

PÓLYA *(crossing his arms, smiling with satisfaction)*

— Hilbert, do you see now? The quantum system was always hidden in the structure of $\pi(x)$.

EULER *(smiling, leaning slightly toward Riemann)*

— So, Herr Riemann, your zeros finally served a purpose beyond just standing still?

(Riemann feels a chill. He realizes the question has no answer — if the zeros weren't where they are, everything would collapse. And yet, he had never wondered why before.)

(The hall begins to fill with excited murmurs. Euler, Fourier, and von Mangoldt exchange glances, trying to process the impact of the revelation. Hilbert is already muttering something about a quantum operator. The excitement builds—until it is interrupted by an impatient sigh that ripples through the room. Everyone turns and sees Gauss leaning against a column, arms crossed, with a look of utter boredom on his face.)

GAUSS *(sighing, fiddling with the cuffs of his coat, almost yawning)*

— Are you all really that excited about this?

(The hall falls silent. Dirichlet, however, smiles discreetly — he already knows what's coming.)

The Collapse of the Primes and the Wave Function[23]

SCENE 9: The excitement stirred by Hilbert, Pólya, Fourier, von Mangoldt, and Chebyshev spreads through the hall of the Infinite Library. Amidst this intellectual frenzy, a new figure emerges from the shadows, watching it all with a glint in his eyes.

SCHRÖDINGER *(approaching quickly, smiling, eyes wide with excitement)*

— *But this... this is a quantum system! I knew it! I always knew it!*

(Gauss, already losing patience, rolls his eyes and crosses his arms, exasperated. Dirichlet watches with a mix of amusement and interest, while Riemann is still digesting the magnitude of the discovery.)

SCHRÖDINGER *(gesturing enthusiastically)*

— *Think about it! The primes are the fundamental particles of arithmetic. But they only reveal themselves when we observe them! And when they do, they collapse and structure all composites! It's a collapse of the wave function! The primes should be called "collapsed from infinity," as I once heard Herr Gauss say... even if he didn't realize the brilliance of the term!*

GAUSS *(pausing for a moment, blinking slowly, incredulous)*

— *Did you... hear what you just said?*

GAUSS *(impatient, tapping his foot on the marble floor)*

— *I was being sarcastic, you fool! Is everything quantum mechanics to you?!*

SCHRÖDINGER *(ignoring Gauss, gesturing toward the oscillations)*

— *Look, Herr Riemann. The structuring primes act like a wave, harmonically distributing and supporting the construction of composites. Meanwhile, the stabilizers... they appear discretely, punctually, preserving symmetry and preventing collapse.*

SCHRÖDINGER *(ignoring Gauss's sarcasm, turning directly to Dirichlet)*

— *Herr Dirichlet, allow me to propose a thought experiment. Simple, yet revealing.*

— *Imagine the numbers from 2 to 20 as points arranged in space. Now, connect each*

[23] In the quantum context: A function that describes the state of a quantum system and its probabilistic behavior.

prime to its multiples.

— You'll see 2 connected to 4, 6, 8... 3 connected to 6, 9, 12... 5 to 10, 15... 7 to 14. They form true structuring networks.

— But then look at 11, 13, 17, 19...

— They remain solitary. Untouched. Invisible to the connections. They're there, but no one seeks them.

*— These are the **stabilizing primes** of this interval. They don't participate in the formation of composites, but they uphold the surrounding symmetry.*

— Now imagine using graph theory to extract the histogram of degree centrality from this network...

— You'd see the energy distribution of the structuring primes — the most connected — vibrating with intensity.

— While the stabilizers remain with zero degree. Silent. Essential.

(Dirichlet, intrigued, begins to visualize the structure of arithmetic as a dynamic network. Schrödinger smiles, satisfied.)

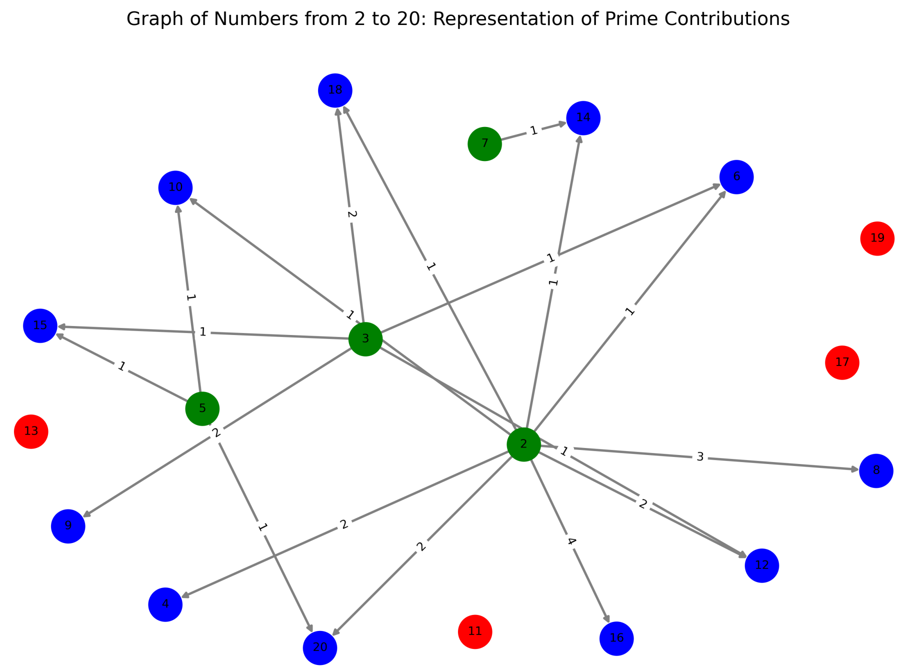

Figure 8: Connection network between primes and their multiples in the range from 2 to 20. Structuring primes (2, 3, 5, 7) show high connectivity, while stabilizing primes (11, 13, 17, 19) remain isolated.

SCHRÖDINGER *(crossing his arms, satisfied with the provocation)*

— *In other words, Herr Riemann, your primes behave as both wave and particle at the same time.*

(The hall falls silent. Riemann widens his eyes, processing the implication.)

SCHRÖDINGER *(smiling, teasing)*

— *But, Herr Gauss, if you reject the "collapsed from infinity," then tell me... How would you explain the structure without invoking that duality?*

(This leaves Gauss cornered. Because the structure is there. And he knows it.)

SCHRÖDINGER *(arms crossed, defiant)*

— *Ah, Herr Gauss... then tell me, where is the wave function in all of this? I bet there's one hiding here!*

DIRICHLET *(smiling, preparing for the final blow)*

— Well, Herr Schrödinger, you're so right that you should already know the answer. Tell me: if there's a spectrum, how could there not be a wave function?

(Dirichlet, who had been waiting precisely for this moment, raises his hand with a measured gesture. The hall darkens briefly, and a new projection appears in the air. Waves oscillate in space, their nodes perfectly aligned with the zeros of the Riemann zeta function.)

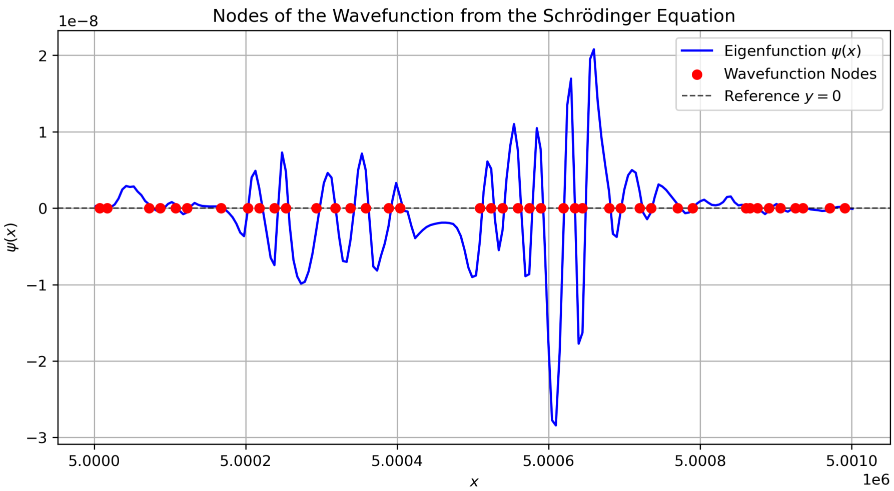

Figure 9: Nodes of the wave function $\psi(x)$ in comparison with the zeros of the zeta function.

DIRICHLET *(satisfied, turning to Schrödinger)*

— You asked for a wave function, Herr Schrödinger? Here it is. See how the nodes align exactly with the zeros of the Riemann zeta function.

(Absolute silence. Schrödinger stares at the graph, speechless. Riemann brings a hand to his face, feeling the magnitude of what stands before him. Gauss, in turn, simply sighs, shaking his head.)

SCHRÖDINGER *(barely audible)*

— This... this is a quantum state equation!

GAUSS *(exasperated, looking at Schrödinger as if he were an especially irritating student)*

— Sure, sure. Next you'll tell me a prime is both prime and composite until someone observes it.

Euler's Reaction and Gauss's Objection

SCENE 10: At that moment, a burst of laughter echoes through the hall. Everyone turns to see Leonhard Euler, shaking his head and smiling as if amused by all the commotion.

EULER *(smiling, arms crossed)*

— I spend a century proving that the sum of the reciprocals of the primes diverges… and you come to me with a "prime wave function"?

DIRICHLET *(without losing composure, pointing at the projection)*

— Herr Euler, you taught us that primes are the fundamental building blocks of arithmetic… but now we see that they also structure a spectrum. A spectrum that was already inside your zeta function!

EULER *(thoughtful, observing the projection)*

— Hmm… I always saw the zeta function as a bridge between primes and infinite series… But I must admit, I never imagined its zeros could be used this way.

EULER *(looking at Fourier, slightly skeptical)*

— So, my dear, you're telling me that all this chaos among primes can be organized by trigonometric sums? Where have I seen that before…

(He throws an ironic glance at Fourier, who smirks, fully aware that Euler himself pioneered this.)

FOURIER *(interjecting, intrigued)*

— Herr Euler, aren't you impressed that a cosine matrix can reconstruct this function? After all, your trigonometric series were the beginning of all this.

EULER *(shrugging, smiling)*

— Oh, dear Fourier, I've seen so many infinite sums do the unthinkable… I suppose I shouldn't be surprised. But I must admit… this is unexpected.

(Dirichlet and Hilbert exchange amused glances. Schrödinger, ignoring Gauss's and Euler's sarcasm, continues analyzing the projection, murmuring to himself.)

SCHRÖDINGER *(almost hypnotized by the graph)*

— But this means... the primes obey an uncertainty principle? Could there be an operator that transforms them between structuring and stabilizing states?

GAUSS *(exasperated)*

— What's wrong with you physicists?! This is pure arithmetic! There are no quantum states! No hidden particles! Just numbers!

DIRICHLET *(with a discreet smile, yet keeping a neutral tone)*

— Herr Gauss, don't you see? If the zeros of the Riemann zeta function emerge as eigenvalues of this system, and if this cosine matrix reconstructs the prime counting... then there's something deeper here.

(Gauss raises an eyebrow, ready to counter, but hesitates for a brief moment. The hall falls silent. Even he cannot deny the strength of the evidence just witnessed.)

SCHRÖDINGER *(still staring at the projection, murmuring to himself)*

— If there's a spectrum, then there must be a hidden structure... this means... perhaps arithmetic has a ground state[24]*...*

(Gauss simply watches, letting the silence linger for a moment. Then, finally, he shrugs, impatiently, as if the matter had been settled long ago, and turns to Riemann.)

— Mathematics has always been arithmetic, Riemann. The problem is that you all got so fascinated with the effects that you forgot to look for the cause.

[24] Ground state and spectrum are quantum mechanics terms that describe the lowest possible energy of a system and its collection of energy levels.

Pythagoras

SCENE 11: The hall of the Infinite Library still echoes with Gauss's words. Silence hangs heavy as Riemann stares at the calculations before him. An ancient presence—carried through time but alive in the essence of mathematics—crosses the marble columns. In his hands, a lyre glimmers faintly, as if tuned to the very structure of numbers.

PYTHAGORAS *(gazing at the floating projections with reverence)*

— *What a magnificent structure... The cosmos reveals itself in the primes!*

(Everyone turns toward the new figure. Dirichlet raises an eyebrow, intrigued. Riemann is still absorbing the impact of the revelation. Only Gauss sighs, as if he knows exactly what is coming next.)

PYTHAGORAS *(his voice nearly ecstatic, approaching slowly the vibrant images of $F_E(x)$)*

— *The rhythm of the spheres... the harmony of numbers... it's all here! It always was! I knew it!*

(He runs his fingers through the projections as if touching a cosmic score. The waves seem to respond to his presence, as if mathematics and music were merely two faces of the same truth. He turns to the others, eyes filled with meaning.)

PYTHAGORAS *(almost whispering, as if speaking to himself)*

— *I was the first to hear it... But there was no way to write it down.*

(Dirichlet smiles, recognizing the connection.)

DIRICHLET *(nodding, intrigued)*

— *The primes resonate like the vibrating strings of a monochord...*

(Pythagoras nods vigorously, pointing at the emerging spectral patterns in the cosine matrix.)

PYTHAGORAS *(exalted, almost in a trance)*

— *The hidden frequencies... the perfect intervals! Each prime is a note, structuring the* *melody of the universe.*

(Riemann, still stunned, mumbles, trying to assimilate what he sees. von Mangoldt and Chebyshev exchange glances, realizing that, somehow, their mathematics had always been linked to music.)

(But then, inevitably, Gauss interrupts.)

Gauss *(impatient, arms crossed)*

— *Here we go. Another one trying to turn arithmetic into mysticism.*

Pythagoras *(serious, unfazed)*

— *You may scoff, Gauss, but the truth is that every numerical structure hides a rhythm. What you call oscillations, I called harmony. The primes dance in invisible patterns.*

(Gauss raises an eyebrow, as if preparing to respond, but decides not to waste the energy. He just sighs, gesturing to the others.)

Gauss *(dismissive, but conceding a point)*

— *Very well, Pythagoras, if it makes you happy... call it harmony. But know this: it's just arithmetic. And I don't need a lyre to see it.*

(Pythagoras smiles, satisfied. He already knew that. But now, he also knows that what they once called mystery was merely mathematics waiting to be heard. The primes had always been the fundamental notes of that invisible score. It just took us time to understand their melody. He takes a step back, letting the lyre vanish into the air as the projections continue to vibrate.)

(Silence returns to the Infinite Library. Only the numbers speak now. The cosmos resonates in silence—and those who dared to listen finally understand.)

The Fate of the Outsider

SCENE 12: The hall of the Infinite Library is shrouded in silence. The floating projections display the cosine matrix, the aligned zeros of the zeta function, the oscillations captured in the structure of the primes. Riemann is still processing everything. Dirichlet and Gauss exchange glances. Chebyshev and von Mangoldt oscillate between excitement and unease. But then, a new voice rises—laden with irony and melancholy.

HIPPASUS OF METAPONTUM *(emerging from the shadows, with a grave expression)*

— Interesting... Very interesting.

(Pythagoras, who had remained silent until now, lifts his eyes and his expression shifts. There's a hint of discomfort on his face. Perhaps fear. Perhaps shame.)

PYTHAGORAS *(in a barely audible whisper)*

— It can't be...

(Gauss watches the scene, clearly impatient. Riemann, however, feels a chill down his spine as he stares at the approaching figure.)

HIPPASUS *(with a half-ironic smile, his voice echoing through the library)*

— Oh, master... didn't expect to see me again?

(Pythagoras averts his gaze. The other mathematicians remain silent. Hippasus steps forward, examining the floating equations, lightly touching them as if rediscovering something long forgotten.)

HIPPASUS *(turning to Pythagoras, his tone now sharp)*

— Tell me, master, did you really believe you could hide irrationality[25] *forever? That banishing it—drowning it in the ocean—would preserve your perfect world?*

(Pythagoras closes his eyes for a moment. He knows there's no answer.)

HIPPASUS *(now addressing everyone)*

[25] Irrational numbers: Numbers that cannot be expressed as a ratio of two integers, like $\sqrt{2}$ and π.

— But isn't it always like this? Mathematics doesn't reject truths. It only rejects those who dare to reveal them too soon.

(The entire hall seems to shrink. Gauss rolls his eyes but does not interrupt. Dirichlet merely observes. Chebyshev and von Mangoldt exchange uneasy glances. Pythagoras remains still, avoiding everyone's gaze. Riemann feels a different weight in the air—as if, for the first time, he is not the center of the question.)

HIPPASUS *(stepping forward with a tone of warning, unwavering)*

— The same happened to you, Herr Riemann. Don't you see? You looked at the reflection instead of the source. But unlike me, you weren't thrown into the sea. You were praised. Idolized. Turned into an icon.

HIPPASUS *(coming closer, his voice now lower, more incisive)*

— But what if your hypothesis is nothing more than a well-positioned mistake in history? What if everyone who followed simply reinforced a misunderstanding—without ever questioning it?

(Riemann turns pale. He didn't expect that blow.)

HIPPASUS *(gazing into the space beyond the library, as if seeing beyond time)*

— The most dangerous mistake is not mistaking a reflection for reality. The fatal mistake is spending over a century without daring to look beyond it.

HIPPASUS *(pausing, then with a cold smile)*

— And what is worse? The man who looks in the wrong direction, or the crowd that follows him without question?

(The hall plunges into sepulchral silence. The weight of doubt settles in. Riemann feels the ground vanish beneath his feet. The air grows heavy. He tries to organize his thoughts, but something inside him resists. What if his entire work was just a fragment of a greater enigma?)

HIPPASUS *(looking at everyone with a mix of severity and compassion)*

— And what if mathematics, this queen we so exalt, has been worshipping a reflection for centuries—not the actual structure?

Hippasus *(addressing no one in particular—or perhaps the only one who matters)*

— The question isn't whether it's true. The question is: are you ready to see it?

Hippasus *(giving one last look to Pythagoras and Riemann, his voice now almost a whisper, but sharp as a blade)*

— If the primes reveal an implicit structure… how many other truths have already been cast into the sea? And how many more will be?

(And then, Hippasus vanishes into the shadows. But his question does not vanish. It lingers. Hovering. Waiting for an answer.)

The Fear of the Unknown

SCENE 13: Silence weighs heavily over the Infinite Library. Riemann runs a hand over his face, trying to process what he has heard. Pythagoras remains motionless, the shadow of the past cast upon him. Gauss sighs loudly and mutters something about unnecessary melodrama. But even he seems affected.

(Schrödinger, still shaken by the revelation, turns to Pythagoras, seeking an explanation.)

SCHRÖDINGER *(almost in a whisper, trying to collect his thoughts)*

— *What was that? What just happened here?*

PYTHAGORAS *(looks at him with serenity and resignation)*

— *When we look at something we do not understand, the first instinct is not to embrace it... but to recoil.*

— *The unknown evokes awe, but also fear. And fear makes us cling to what we already know, even if it's just a reflection.*

— *And I was afraid, I confess. I couldn't recognize the chaos I was shown, and I failed to see that Hippasus's discovery didn't invalidate my work—it completed it and made it even more magnificent. And what Hippasus saw was no mere reflection...*

(Schrödinger furrows his brow, intrigued. Chebyshev whispers to von Mangoldt, trying to recall something long forgotten.)

— *What are they talking about?*

VON MANGOLDT *(whispering back, as if not wanting to be heard)*

— *The irrationals. Hippasus discovered the square root of two and enraged Pythagoras. Some say his followers threw him into the sea.*

(Chebyshev swallows hard. Now he remembers.)

PYTHAGORAS *(nods, turning again to the projection in space)*

— *When we discovered that the diagonal of a square could not be expressed as the ratio of two integers, it shattered everything we believed about the world. It was like staring into chaos* *for the first time. But... what seemed like a problem became a new foundation.*

(Schrödinger looks at the graphs of the Riemann zeros. His expression changes. He understands.)

PYTHAGORAS *(in a deeper tone, almost prophetic)*

— *The unknown always frightens. But truth does not wait for acceptance. What was once denied returns stronger. So it was with the square root of two. So it will be with the primes. The fear will pass. The structure will remain.*

The Collapse of Riemann

SCENE 14: Riemann remains motionless. The calculations oscillate in the air, vibrant, almost mocking him. Gauss, Dirichlet, and the others watch, but say nothing. This is his moment.

RIEMANN *(his voice trembling, as if speaking to himself)*

— *So... the zeros of my function... were just an echo? Just a record of something more fundamental?*

(He squints, struggling against the avalanche of thoughts. His mind revisits everything he has built. Everything he believed he had understood. The integrals, the functions, the proofs, the centuries of study surrounding his hypothesis... and now, it all seems to crumble.)

RIEMANN *(whispering, barely breathing)*

— *No... it can't be.*

(He brings a hand to his face, as if trying to contain something greater than himself. His breathing grows erratic. His eyes scan the floating calculations frantically. But suddenly, he begins to laugh—a short, nervous, bitter laugh.)

RIEMANN *(with a short, desperate laugh)*

— *What if it's all an illusion? What if mathematics has always been looking at the reflection and not at the structure?*

(Gauss keeps his sharp gaze but says nothing. Dirichlet crosses his arms. Schrödinger watches silently, fascinated by Riemann's internal collapse. The entire hall seems to await his next move.)

(Riemann hesitates. His mind, always so sharp, searches for a way out. Perhaps there is still a way to salvage his hypothesis. A justification. A final argument. But the calculations remain, impassive, refusing to offer him any respite.)

RIEMANN *(taking a step back, shaking his head, laughing bitterly)*

— *My God... was my hypothesis just a brilliant illusion?*

GAUSS *(cutting, like a scalpel into raw flesh)*

— No. It was a trick of mirrors.

(Riemann turns pale. The sentence is not just a critique—it is a mathematical execution. His mind, once sharp, now trembles beneath its own equations. As if in free fall, with not a single number to cling to. His gaze sweeps across the calculations floating in the air, searching for refuge—but all he finds are the oscillations that now make far too much sense.)

GAUSS *(stepping forward, unhurried but lethal)*

— You mistook the shadow for the torch. You heard an echo and thought it was a voice. You looked into the mirror and called it truth.

— And the worst part? You made the entire world believe in the illusion.

RIEMANN *(whispering, almost inaudibly)*

— No...

(Gauss ignores it. His voice doesn't waver—his patience has run out.)

GAUSS *(gesturing to the graphs, the cosine matrix, the equations that now expose the inevitable truth)*

— What you thought was a cosmic enigma was always in the most elementary arithmetic. You didn't see it because you were too busy staring into the mirror.

(Silence falls. Schrödinger holds his breath. Dirichlet avoids looking directly at Riemann. Chebyshev and von Mangoldt exchange uneasy glances. Even Euler, ever measured, does not intervene. Because Riemann is alone. And he knows it.)

GAUSS *(the final blow, merciless, staring directly into Riemann's eyes)*

— What's worse, Herr Riemann? Not seeing? Or making everyone look in the wrong direction for over a century?

(Riemann opens his mouth. But no words come. Only silence.)

(His mind refuses to accept. He scans the equations, searching for an error, a flaw, anything to disprove what stands before him. But the problem isn't in the math. The problem is within him. The structure was always there. He just never looked at it the right way.)

(He closes his eyes. His body remains, but his mind seems far away, wandering among the equations, among the reflections of what he once believed he understood.)

(And for the first time, Bernhard Riemann—the man who dared to touch the deepest mysteries of mathematics—has nothing left to say.)

(The hall remains silent. Gauss does not wait for a reply. He has already won.)

(Riemann remains there. Alone. No escape. No defense. Only the calculations vibrating in the air—cold, unyielding, revealing a truth he can no longer deny.)

RIEMANN *(barely audible, looking at no one)*

— I never saw anything.

(The calculations continue to pulse in the air, cold and impassive. The truth no longer needs to be spoken. It simply is.)

Riemann Confronts Gauss

SCENE 15: The hall still echoes with the crushing silence left by Riemann's humiliation. But contrary to what was expected, Riemann does not collapse. He does not flee. He thinks. And then, something changes. Riemann faces Gauss—not with confusion, but with a newfound clarity. The revelation hit him hard, but something within refuses to simply surrender. He takes a deep breath, and when he speaks, his voice is no longer hesitant—it is the voice of someone who has understood his place in history.

(Riemann lifts his head. There is something new in his eyes. Not resignation. Clarity.)

RIEMANN *(calm, but full of intensity)*

— So, Herr Gauss… if something is a reflection, does it lose its value?

(Gauss keeps his gaze fixed on Riemann. Schrödinger holds his breath. Dirichlet raises an eyebrow. Something is different.)

RIEMANN *(stepping forward, without hesitation, eyes shining with new insight)*

— If a mirror reflects the image of an object, does that image cease to be real? If the spectral patterns of the zeta function reflect the arithmetic structure of the primes, does that make them irrelevant?

(Gauss remains silent. Schrödinger smiles slightly. He has seen this before.)

(Riemann is now in control. Even Gauss senses it.)

RIEMANN *(stepping forward again, relentless)*

— The reflection was not a mistake. It was a lens. A powerful lens that allowed us to see beyond the apparent chaos and glimpse the underlying order.

EULER *(nodding, pleased)*

— The reflection reveals the source. You saw it first, Gauss, but you were too busy scorning the mirror.

(Gauss keeps his stance, but something subtle shifts in his expression. Riemann sees it.)

RIEMANN *(the final blow, returning the humiliation in kind)*

— You told me I looked at the reflection instead of the source. But tell me, Herr Gauss… without the reflection, would you ever have seen the source?

RIEMANN *(now completely transformed, gesturing toward the equations in the air)*

— So, Herr Gauss, my hypothesis wasn't a mistake… it was the first glimpse of a greater principle. The zeta function was how mathematics recorded the oscillations—and without that lens, we might never have seen the true structure.

(Silence. Deep. Unbreakable.)

DIRICHLET *(interjects, calm)*

— But look… that is precisely the point. Riemann's spectrum was not the fundamental structure, but it wasn't an illusion either. It was the inevitable consequence of arithmetic itself. A mathematical signature as real as the primes.

DIRICHLET *(with a faint smile, looking from Gauss to Riemann)*

— Herr Gauss, you saw the structure but never recorded it. Herr Riemann, you revealed the spectrum to the world. In the end… you were describing the same thing.

(The hall remains silent for a moment. Schrödinger watches, amused. Euler nods, satisfied. Riemann looks at Gauss, and for the first time, there is mutual understanding in their gaze.)

(Euler, who had remained silent until now, steps forward. His gaze is calm but full of certainty. He's seen this kind of pattern before.)

EULER *(with a faint smile, settling the matter with authority)*

— You're surprised? But it was always there. Nature loves spectra. Always has. Perhaps we just took too long to accept it.

EULER *(looking at Gauss, with a conciliatory rather than confrontational tone)*

— Come now, Herr Gauss… If there was any mistake, it was simply seeing the reflection before the source. But think about what was built upon that spectrum. The tools developed, the connections that emerged.

— What Riemann did was not in vain. His spectrum became a powerful gateway that led mathematics to places never before explored. And why? Because mystery drove him.

— Reflect with me, Herr Gauss. If you had merely jotted down "the obvious" in the margins of a book, would we have made it this far? Would we have crossed the bridge Riemann's hypothesis gave us?

(Euler turns to Riemann.)

EULER *(with a slight smile, but with weight behind his words)*

— Herr Riemann, your hypothesis may have looked at the reflection, but it did so with depth. And because of that, it opened doors that would never have been opened otherwise. Now that the source has been revealed, imagine what might be built.

(A silence lingers in the hall. Schrödinger watches with a gleam in his eyes.)

SCHRÖDINGER *(crossing his arms, satisfied, shaking his head)*

— You know what's most ironic in all this, Herr Gauss? You were right… and wrong at the same time.

(The hall bursts into laughter. Even Gauss lets out a resigned sigh, shaking his head. Riemann smiles faintly. The clash is over — but everyone knows this conversation is only just beginning.)

(The hall, at last, seems to breathe. Schrödinger is visibly impressed. Euler nods, satisfied. Dirichlet smiles subtly. But the greatest impact is on Riemann himself. He has won. He proved his point. He reconstructed his hypothesis before everyone's eyes — and no one can refute it. And Gauss… well, Gauss merely sighs, muttering something unintelligible about how things were easier when there were only numbers.)

(The reflection is real. And now, no one can ignore it anymore.)

(There is much to discuss. And the story, as always, goes on. But everyone knows that what was said is not a period. It is a call.)

(And you? Are you ready to see the source?)

Epilogue: The Reflection and the Source

If the zeros of the Riemann zeta function are merely a reflection of a deeper structure, then we are not just solving a mathematical problem—we are redefining the way we see primes and, possibly, arithmetic itself.

Gauss always said that arithmetic was the queen of the sciences. But if we've spent more than a century staring at the reflection instead of the source, perhaps only now are we beginning to glimpse the true contours of this kingdom.

This suggests that the entire spectral theory, the long-standing obsession with the zeros of zeta, may have been a pursuit of the effect rather than the cause. **If mathematics remained hypnotized by a reflection, what else are we interpreting the wrong way?**

The Riemann Hypothesis revealed a profound pattern, but not its origin. The zeta function does not generate the oscillations; it encodes them in the nontrivial zeros. Now, for the first time, we may be able to see what truly generates them.

Perhaps we are standing at the threshold of a fundamental shift. But to understand the depth of what's at stake, let us now look at what truly lies ahead.

Explanatory Notes: What's at Stake?

Now that we understand the relationship between the reflection and the source, we must explore the implications of this shift in perspective.

This dialogue is not merely about the distribution of prime numbers—it reframes the very way we perceive them. The goal of this section is to clarify the key points for those unfamiliar with the deeper mathematical details.

1. What Did Riemann Do?

Bernhard Riemann formulated his famous hypothesis while studying the zeros of a function called the *Riemann zeta function*, denoted by $\zeta(s)$. He noticed that the nontrivial zeros appeared to align in an orderly fashion and proposed that they all lie on a critical line[26]. If true, the hypothesis would have deep implications for number theory, especially for understanding the distribution of the primes.

The Riemann Hypothesis suggests that the zeros of the zeta function encode the oscillations in the prime counting. However, the question raised here is: **What if the zeros are not the source of those oscillations, but merely a reflection of them?**

2. What Is Being Discovered Here?

The major revelation is that by properly decomposing the prime counting function, $\pi(x)$, separating the primes that structure the composites from those that do not, **we uncover a fundamental oscillatory pattern whose origin is purely arithmetic.**

These oscillations **precede the zeros of the zeta function**. This means that the zeros are **not the cause of the oscillations, but rather their consequence**. The structure was already embedded in pure arithmetic—without needing to appeal to the complex plane.

3. The Role of the Cosine Matrix

The cosine matrix is a central element of this new vision. Instead of treating the zeta zeros as abstract points, this approach sees them as **eigenvalues[27] of a specific matrix constructed from the decomposition of the prime counting function**.

If this matrix can **accurately reconstruct the observed oscillations** in the prime count, then the Riemann zeros naturally emerge **as a reflection of this spectral structure**, not its foundation. This effectively **reframes the Riemann Hypothesis as an eigenvalue problem within a real, tangible system**.

[26] The line in the complex plane where all nontrivial zeros of the zeta function are conjectured to lie.

[27] Values associated with a matrix or operator, revealing fundamental patterns within a mathematical system.

4. The Implications

If this approach is correct, it not only **provides a new interpretation of the Riemann Hypothesis**, but also suggests that mathematics **may have spent more than a century looking at the problem from the wrong angle**.

Instead of seeking indirect proofs in the complex domain, we should be analyzing **the structural oscillations in the prime count itself**.

This not only changes how we think about primes, but **may also open new paths to understand mathematical patterns directly within arithmetic**, without relying on tools that, until now, seemed indispensable.

5. What Now?

The next step is to test and refine this approach, **validating whether the emerging spectral structure truly explains the zeros of the zeta function**. But regardless of the outcome, one thing is certain:

If this is true, we're not just adjusting a theory. We're correcting a misunderstanding that shaped all of modern mathematics.

The question is no longer **whether the Riemann Hypothesis will be proven**, but rather **whether we've been asking the right question from the start**.

About the Characters

Carl Friedrich Gauss (1777–1855) — Known as the "Prince of Mathematics," Gauss contributed to virtually every field in the discipline. From number theory to physics, his genius appeared early, and his absolute command of arithmetic influenced generations. Had he lived ten more years, perhaps he would have stated a theorem, instead of just leaving notes in the margins of a notebook.

Bernhard Riemann (1826–1866) — The man who saw the reflection before the source. Creator of the hypothesis that bears his name, his revolutionary approach connected number theory with complex analysis and mathematical physics. His intuition was right, but he didn't realize that the primes themselves carried the structure he was seeking.

Johann Peter Gustav Lejeune Dirichlet (1805–1859) — Riemann's mentor and a central figure in the formulation of the modern concept of functions. His influence on number theory helped establish the prime number theorem and other fundamental advances. In the debate, he acts as a mediator between Riemann's spectral vision and Gauss's arithmetic intransigence.

Pafnuty Chebyshev (1821–1894) and **Hans von Mangoldt** (1854–1925) — Two mathematicians who advanced our understanding of prime distribution. Chebyshev established fundamental inequalities in number theory, while von Mangoldt refined the spectral relations in the decomposition of the zeta function.

David Hilbert (1862–1943) — One of the most influential mathematicians of the 20th century, Hilbert saw mathematics as a formal system built on axioms. When he realized that Riemann's zeros were eigenvalues of a Hermitian operator, he knew he was witnessing a new paradigm.

Jean-Baptiste Joseph Fourier (1768–1830) — Creator of Fourier series, his influence on harmonic analysis is crucial to the modern understanding of mathematical spectra. In the debate, he immediately recognizes the connection between the cosine matrix and his own transforms.

Leonhard Euler (1707–1783) — Master of infinite sums, Euler pioneered the connection between primes and complex analysis. His zeta function was the first step toward Riemann's hypothesis, although he never saw the zeros as a spectral phenomenon.

Erwin Schrödinger (1887–1961) — Quantum physicist and formulator of the Schrödinger equation. Upon seeing the spectral structure of the primes, he instantly recognizes the analogy with quantum systems. To him, primes collapse like quantum states when observed.

Plato (c. 428–348 BCE) — The Greek philosopher who conceived the allegory of the cave. Upon witnessing the debate among mathematicians, he tries to claim the discovery as confirmation of his theory of shadows. However, he is ruthlessly dismissed by Gauss.

Pythagoras (c. 570–495 BCE) — The mystic-mathematician who saw numbers as the essence of the cosmos. He founded a school that merged mathematics and philosophy but also imposed rigid dogmas. His aversion to irrationality reflected the fear that his perfect mathematical worldview might collapse.

Hippasus of Metapontum (5th century BCE) — The mathematician who dared to challenge the very foundations of the Pythagorean school. By discovering that the diagonal of a square **could not be expressed as a ratio of integers**, he revealed the existence of irrational numbers — an unacceptable truth for the Pythagoreans, who saw numbers as the perfect structure of the cosmos.

Legend has it that he was condemned to exile and possibly thrown into the sea by his own peers. In the context of this narrative, Hippasus **is the outsider, the mathematician who reveals an inconvenient truth and is therefore rejected**.

He not only recognizes the flaw in the traditional view, but **exposes a recurring pattern in the history of mathematics**: major breakthroughs **are often perceived as threats before being accepted**. By confronting both Pythagoras and Riemann, **he questions whether the real problem lies not in mathematics itself — but in the mathematicians.**

Available Resources

For those who wish to explore this approach more deeply, we provide two sets of materials:

1. Notebooks on GitHub

All the notebooks that support this narrative are available in a public GitHub repository. They include:

- Detailed implementations of the decompositions presented.
- Construction of the cosine matrix and its connection to the primes.
- Reconstruction of the oscillations in the prime counting function.
- Comparison between the zeros of the zeta function and the emerging prime structure.

The notebooks are organized so that anyone with basic knowledge of computational mathematics can reproduce the experiments and verify the discussed patterns. The repository can be accessed at:

`https://github.com/costaalv/spectral-structure-of-the-primes`

2. Didactic Article on Zenodo

In addition to this narrative, a conventional didactic article has been published on Zenodo, presenting the discovery in a more formal way, with detailed mathematical demonstrations and technical rigor. The article is available at:

`https://zenodo.org/records/15082816`

3. Next Steps

The materials provided are only a starting point. The question raised here does not end with a definitive answer, but opens the way for further investigation.

The challenge now is to deepen this approach and explore its consequences. If primes truly carry this implicit structure, then what else in mathematics have we mistaken for a reflection instead of the source?

What has been presented here is not an ending. It is a call to re-evaluate the very foundations of mathematics. The central question is no longer simply *whether* the Riemann Hypothesis is true, but rather:

Have we been asking the right question all along?

Glossary

Complex Plane A two-dimensional representation of complex numbers, where each number has a real and an imaginary part. The Riemann zeta function is defined over this plane, and its spectral structure emerges from it.

Cosine Matrix A mathematical structure built from the oscillations in the prime counting function. This matrix reveals fundamental spectral patterns, allowing us to interpret the zeros of the zeta function as a reflection of a deeper arithmetic structure.

Eigenvalues and Eigenvectors Fundamental concepts in linear algebra and spectral theory. An **eigenvalue** of a matrix or linear operator is a number associated with an **eigenvector**, indicating how that vector is scaled by the transformation. In the context of the zeta function, eigenvalues naturally emerge from the spectral structure of primes.

Fourier Transform A mathematical tool that decomposes functions into individual frequencies (FFT). It is essential for understanding spectral patterns and plays a key role in analyzing the oscillations in the distribution of primes.

Hermitian Operator In linear algebra and quantum mechanics, a Hermitian operator has a spectrum consisting only of real values. In the context of the Riemann Hypothesis, it is suggested that a Hermitian operator underlies the distribution of primes.

Irrational Numbers Numbers that cannot be expressed as a ratio of two integers. Classic examples include π and $\sqrt{2}$. The discovery of irrational numbers challenged the Pythagorean view of a universe built entirely from ratios of integers.

Line of Critical Zeros In the complex plane, this is the line $\Re(s) = \frac{1}{2}$, where it is conjectured that all nontrivial zeros of the Riemann zeta function reside.

Oscillations in Prime Counting Wave-like patterns in the distribution of primes along the natural numbers. These oscillations are directly related to the zeros of the zeta function.

Prime Numbers Integers greater than 1 that are divisible only by 1 and themselves. They are considered the "atoms" of arithmetic, as every integer can be factored into primes.

Prime Structurers and Stabilizers A new distinction proposed in this work. Structuring primes are those that define fundamental patterns in the composition of natural numbers, while stabilizing primes play a balancing role in the organization of composites.

Reflection and Source A central metaphor in this work. The Riemann Hypothesis looked at the zeros of the zeta function, which are a reflection of a deeper arithmetic structure. The advancement proposed here consists of identifying the actual source of these oscillations.

Riemann Hypothesis A mathematical conjecture stating that all nontrivial zeros of the Riemann zeta function lie on the line $\Re(s) = \frac{1}{2}$ in the complex plane. It is considered one of the most important problems in mathematics, with direct implications for the distribution of prime numbers.

Riemann Zeta Function A mathematical function connecting prime numbers to complex analysis. Its still unproven hypothesis suggests that all of its nontrivial zeros lie on the line $\Re(s) = \frac{1}{2}$ in the complex plane. This hypothesis has deep implications for number theory.

Spectrum The set of eigenvalues of an operator or matrix. In mathematics and physics, the spectrum reveals fundamental properties of a system. In the context of the zeta function, the zeros are interpreted as a spectrum emerging from arithmetic.

Wavefunction Collapse A phenomenon in quantum mechanics where a system, upon being measured, assumes a single defined state among several possibilities. The analogy here is that primes "collapse" when structuring composite numbers.

Sources and Recommended Readings

Mathematics has always evolved between bold conjectures and rigorous proofs. The debate presented in this work aims to provoke new reflections on the structure of the primes and their relationship with the zeros of the zeta function. For those who wish to explore the topics discussed more deeply, here is a selection of recommended readings:

1. History and Philosophy of Mathematics

- **Marcus du Sautoy**, *The Music of the Primes* – A compelling narrative about the Riemann Hypothesis and its connection to the primes, written by one of the most charismatic mathematicians of our time.
- **John Derbyshire**, *Prime Obsession* – An excellent introduction to the Riemann Hypothesis, blending history and mathematics in an accessible way.

2. Number Theory and the Riemann Hypothesis

- **Brian Conrey**, *The Riemann Hypothesis* (Notices of the AMS, 2003) – A didactic and insightful article presenting the state of the art of the Riemann Hypothesis, written by one of the leading experts in the field.
- **G. H. Hardy** and **E. M. Wright**, *An Introduction to the Theory of Numbers* – A foundational text on number theory, covering everything from the distribution of primes to more advanced topics.
- **Harold M. Edwards**, *Riemann's Zeta Function* – A deep dive into the zeta function, ideal for those interested in the technical aspects of the hypothesis.
- **Enrico Bombieri**, various articles on the Riemann Hypothesis and number theory – Explorations of the connection between primes and complex analysis.

3. Spectra, Operators, and Quantum Mechanics

- **Michael Berry** and **Jonathan Keating**, articles on the connection between the Riemann Hypothesis and quantum mechanics – Studies that explore the relationship between the zeta zeros and quantum systems.
- **Alain Connes**, works on the spectral approach to the Riemann Hypothesis – An attempt to reinterpret the hypothesis through noncommutative geometry.
- **Mark Kac**, *Can One Hear the Shape of a Drum?* – A classic article introducing the idea that the eigenvalues of an operator may encode structural information, a concept that parallels this work.

4. The Future of Prime Theory

- **Terence Tao**, articles and lectures on analytic number theory – One of the most influential mathematicians today, exploring new approaches to the distribution of primes.
- **Andrew Odlyzko**, research on zeta zeros and numerical computation – Works that show how computational calculations reveal intriguing patterns among the Riemann zeros.

Final Thoughts

Mathematics advances as new questions are asked. If the primes truly carry a deeper implicit structure than we once imagined, then the journey is just beginning.

The books and articles listed above offer a foundation for understanding the complexity of the Riemann Hypothesis and its relationship to the primes. What new perspectives might arise from here?

Like Euler, Riemann, Gauss, and so many others, we continue to pursue the implicit structure of the numbers. The mystery is far from solved. But perhaps now, we finally know where to look.