Online Appendix to "Accounting for Mexican Business Cycles"

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This appendix provides the detailed derivation for the small open-economy model. Equations are numbered as in the paper, whenever it is possible.

1 The economy

1.1 Households

Households maximize its present-valued expected lifetime utility,

$$\max_{c_t,l_t,d_t,k_{t+1}} E_t \sum_{t=0}^{\infty} \beta^t U(c_t, l_t), \tag{12}$$

s.t.

$$d_t = (1 + r_{t-1})d_{t-1} - q_t + c_t + x_t + \Lambda(k_{t+1}) + e_t m_t, \tag{8}$$

$$q_t = m_t^{\mu}(k_t^{\alpha} l_t^{1-\alpha})^{1-\mu}, \tag{9}$$

$$k_{t+1} = (1 - \delta)k_t + x_t - \Lambda(k_{t+1}), \tag{11}$$

taking $r_t = r + \psi(e^{d_t - d} - 1)$ and

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$$\ln e_t = \rho_e \ln e_{t-1} + \epsilon_t^e, \tag{10}$$

as given. We can write the Lagrangian as follows:

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) +$$
$$E_t \lambda_t \sum_{t=0}^{\infty} \beta^t [d_t - (1 + r_{t-1}) d_{t-1} + m_t^{\mu} (k_t^{\alpha} l_t^{1-\alpha})^{1-\mu} - c_t - (k_{t+1} - (1-\delta) k_t) - \frac{\eta}{2} (k_{t+1} - k_t)^2] - e_t m_t$$

The first order conditions with $U(c_t, l_t) \frac{[c_t - \omega^{-1} l_t^{\omega}]^{1-\gamma} - 1}{1-\gamma}$ are:

$$\frac{\partial \mathcal{L}}{\partial d_t} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1} (1 + r_t) \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Leftrightarrow \lambda_t = [c_t - \omega^{-1} l_t^{\omega}]^{-\gamma} \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial l_t} = 0 \Leftrightarrow -[c_t - \omega^{-1} l_t^{\omega}]^{-\gamma} l_t^{\omega-1} + \lambda_t (1-\alpha)(1-\mu) \frac{q_t}{l_t} = 0 \Leftrightarrow (1-\alpha)(1-\mu) q_t = l_t^{\omega} \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \Leftrightarrow \lambda_t (1 + \eta (k_{t+1} - k_t)) = \beta E_t \lambda_{t+1} (\eta (k_{t+2} - k_{t+1}) + 1 - \delta + (1-\mu) \alpha \frac{q_{t+1}}{k_{t+1}}) \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial m_t} = 0 \Leftrightarrow \frac{\mu q_t}{m_t} = e_t \tag{18}$$

The transversality condition is the following:

$$\lim_{j \to \infty} E_t \frac{d_{t+j}}{\prod_s^j (1 + r_s)} \leq 0 \tag{13}$$

To complete trade-interaction with the rest of the world,the last relation to be found is the trade-balance-to-GDP ratio (tby_t). Rearranging the terms of the debt equation yields:

$$\begin{aligned} q_t - c_t - x_t - \Lambda(k_{t+1}) - e_t m_t &= (1 + r_{t-1}) d_{t-1} - d_t \\ q_t - c_t - x_t - \Lambda(k_{t+1}) - e_t m_t &= tb_t \\ 1 - \frac{c_t}{y_t} - \frac{x_t}{y_t} - \frac{\Lambda(k_{t+1})}{y_t} &= tby_t \end{aligned} \tag{19}$$

2 Dynamic System

From the equations derived, the system of equations that provides the dynamics of the model (Variables: d_t, r_t, y_t, c_t, k_{t+1}, x_t, m_t, e_t, l_t, tby_t, λ_t) is given by:

$$d_t = (1 + r_{t-1})d_{t-1} - y_t + c_t + x_t + e_t m_t$$

$$y_t = m_t^{\mu}(k_t^{\alpha} l_t^{1-\alpha})^{1-\mu}$$

$$\ln e_t = \rho_e \ln e_{t-1} + \epsilon_t^e$$

$$k_{t+1} = (1 - \delta)k_t + x_t - \Lambda(k_{t+1})$$

$$\lambda_t = \beta E_t \lambda_{t+1}(1 + r_t)$$

$$\lambda_t = [c_t - \omega^{-1} l_t^{\omega}]^{-\gamma}$$

$$(1 - \alpha)(1 - \mu)y_t = l_t^{\omega}$$

$$\lambda_t(1 + \eta(k_{t+1} - k_t)) = \beta E_t \lambda_{t+1}(\eta(k_{t+2} - k_{t+1}) + 1 - \delta + (1 - \mu)\alpha \frac{y_{t+1}}{k_{t+1}})$$

$$\frac{\mu y_t}{m_t} = e_t$$

$$tby_t = 1 - \frac{c_t}{y_t} - \frac{x_t}{y_t} - \frac{g(k_{t+1})}{y_t} - e_t \frac{m_t}{y_t}$$

$$r_t = \psi(e^{d_t - d} - 1)$$

3 Steady State

The previous system of dynamic equations can be written at the steady-state (variables without the subscript t denote equilibrium values):

$$q = rd + c + x - m$$

$$q_t = m^{\mu}(k^{\alpha} l^{1-\alpha})^{1-\mu}$$

$$e = 1$$

$$x = \delta k$$

$$r = \frac{1}{\beta} - 1$$

$$\lambda = [c - \omega^{-1} l^{\omega}]^{-\gamma}$$

$$(1-\alpha)(1-\mu)q = l^{\omega}$$

$$1 = \beta[(1-\mu)\alpha\frac{q}{k} + 1 - \delta]$$

$$\frac{\mu q}{m} = e$$

$$tby = 1 - \frac{c}{y} - \frac{x}{y} - \frac{m}{y}$$

$$r = r + \psi(\exp(0) - 1) = r$$

We calibrate the value of d. First, replace steady state values for k, l and m into the gross production function (already considering $e = 1$):

$$
\begin{aligned}
q &= m^{\mu}\left(k^{\alpha} l^{1-\alpha}\right)^{1-\mu} \\
q &= (\mu q)^{\mu}\left(\left[\left(\frac{1}{\beta}+\delta-1\right)^{-1}(1-\mu)\alpha q\right]^{\alpha}\left[\left((1-\alpha)(1-\mu)q\right)^{\frac{1}{\omega}}\right]^{1-\alpha}\right)^{1-\mu} \\
q &= q^{\mu}\mu^{\mu}\left[\left(\frac{1}{\beta}+\delta-1\right)^{-1}(1-\mu)\alpha\right]^{\alpha(1-\mu)} q^{\alpha(1-\mu)}\left[(1-\alpha)(1-\mu)\right]^{\frac{(1-\alpha)(1-\mu)}{\omega}} q^{\frac{(1-\alpha)(1-\mu)}{\omega}} \\
q &= q^{\mu+\alpha(1-\mu)+\frac{(1-\alpha)(1-\mu)}{\omega}}\mu^{\mu}\left[\left(\frac{1}{\beta}+\delta-1\right)^{-1}(1-\mu)\alpha\right]^{\alpha(1-\mu)}\left[(1-\alpha)(1-\mu)\right]^{\frac{(1-\alpha)(1-\mu)}{\omega}} \\
q^{1-\left(\mu+\alpha(1-\mu)+\frac{(1-\alpha)(1-\mu)}{\omega}\right)} &= \mu^{\mu}\left[\left(\frac{1}{\beta}+\delta-1\right)^{-1}(1-\mu)\alpha\right]^{\alpha(1-\mu)}\left[(1-\alpha)(1-\mu)\right]^{\frac{(1-\alpha)(1-\mu)}{\omega}} \\
q &= \left\{\mu^{\mu}\left[\left(\frac{1}{\beta}+\delta-1\right)^{-1}(1-\mu)\alpha\right]^{\alpha(1-\mu)}\left[(1-\alpha)(1-\mu)\right]^{\frac{(1-\alpha)(1-\mu)}{\omega}}\right\}^{\overline{1-\left(\mu+\alpha(1-\mu)+\frac{(1-\alpha)(1-\mu)}{\omega}\right)}}.
\end{aligned}
$$

Then, we obtain the steady-state values for hours of work, the stock of capital and intermediates goods imports from the first order conditions:

$$l = \left((1-\alpha)(1-\mu)q\right)^{\frac{1}{\omega}},$$

$$k = \left(\frac{1}{\beta}-1+\delta\right)^{-1}(1-\mu)\alpha q,$$

and

$$m = \mu q.$$

Investment in equilibrium is equal to

$$x = \delta k.$$

The Euler equation gives us

$$r = \frac{1}{\beta} - 1.$$

From the definition of value-added output, we know that

$$y = q - em,$$

and from the resource constraint we have

$$c = y - rd - x.$$

Finally, we have that

$$\lambda = [c - \omega^{-1} l^{\omega}]^{-\gamma}$$

and

$$tby = 1 - \frac{c}{y} - \frac{x}{y}$$

4 Log-linearization

Log-linearizing the equations that provide the dynamics of the model around the deterministic steady-state giver us the following system of log-linear equations (for any variable z, $\hat{z}_t = \ln z_t - \ln z$):

$$d\hat{d}_t = (1+r)d\hat{d}_{t-1} + rd\hat{r}_{t-1} - q\hat{q}_t + c\hat{c}_t + x\hat{x}_t + m\hat{e}_t + \hat{m}_t$$

$$\hat{q}_t = \mu\hat{m}_t + (1-\mu)\alpha\hat{k}_t + (1-\mu)(1-\alpha)\hat{l}_t$$

$$\hat{e}_t = \rho_e\hat{e}_{t-1} + \epsilon_t^e$$

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \delta\hat{x}_t$$

$$\hat{\lambda}_t = E_t\hat{\lambda}_{t+1} + \beta r\hat{r}_t$$

$$-\frac{1}{\gamma}\lambda^{-\frac{1}{\gamma}}\hat{\lambda}_t = c\hat{c}_t - l^{\omega}\hat{l}_t$$

$$(1-\alpha)(1-\mu)q\hat{q}_t = \omega l^{\omega}\hat{l}_t$$

$$\lambda\hat{\lambda}_t = \beta E_t[\lambda\hat{\lambda}_{t+1}(1-\delta+(1-\mu)\alpha\frac{q}{k}) + k\lambda\eta(\hat{k}_{t+2}-\hat{k}_{t+1}) + \frac{q}{k}(1-\mu)\alpha(\hat{q}_{t+1}-\hat{k}_{t+1})]$$

$$\mu\frac{q}{m}(\hat{q}_t - \hat{m}_t) = \hat{e}_t$$

$$ytbyt\hat{by}_t = (c+x)\hat{y}_t - c\hat{c}_t - x\hat{x}_t$$

$$r\hat{r}_t = \psi d\hat{d}_t$$

$$y\hat{y}_t = q\hat{q}_t - em(\hat{e}+\hat{q}) \tag{1}$$