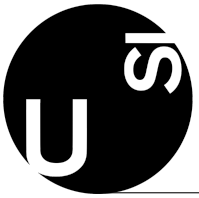


Bachelor Project

June 9, 2018

Barycentric Data Visualization for Triangle Meshes

Costanza Volpini

Abstract

The usual way to visualize data for a triangle mesh is to associate the data with the vertices and then use linear interpolation over the mesh triangles. While this is the obvious way to go for data given at the mesh vertices, it is less natural for data given at the edges or triangles, since it requires to first aggregate the data neighbouring each vertex, thus introducing an additional averaging step. In this project we want to explore alternative data visualization techniques, using the power of barycentric coordinates and GPU programming.

Advisor Prof.Dr. Kai Hormann
Assistant Jan Svoboda

Advisor's approval (Prof.Dr. Kai Hormann): Date:

Contents

1 Introduction

1.1 Barycentric coordinates

Barycentric coordinates, discovered by Möbius in 1827, represent one of the most progessive area of research in computer graphics and mathematics thanks to the numerous applications in image and geometry processing. [10] The position of any point in a triangle can be expressed using a linear combination of barycentric coordinates:

$$p = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3$$

where p_1, p_2 and p_3 are the vertices of a triangle and λ_1, λ_2 and λ_3 (the barycentric coordinates) are three scalars that respect the following barycentric coordinates properties.[7]

- partition of unity: $\sum_{i=1}^{3} \lambda_i(p) = 1$
- reproduction: $\sum_{i=1}^{3} \lambda_i(p) p_i = p$
- Lagrange-property: $\lambda_i(p_j) = \delta_{i,j}$
- linearity: $\lambda_i \in \prod_1$
- non-negativity: $\lambda_i(p) \geq 0$ for $p \in [p_1, p_2, p_3]$

A point is inside the triangle if and only if $0 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1$. If a barycentric coordinate is less than zero or greater than one, the point is outside the triangle. Barycentric coordinates allow the interpolation of values from a set of control points over the interior of a domain, using weighted combinations of values associated with the control points (Fig. 1). [10]

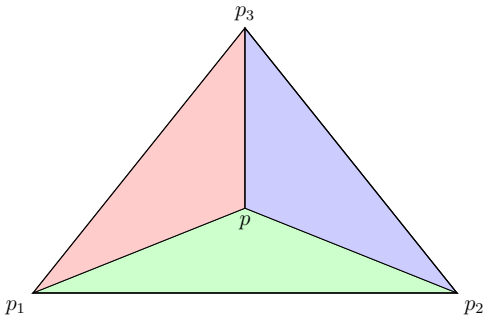


Figure 1: Let w_1 be the blue area, w_2 the red one and w_3 the green one. Normalizing each of them by the area of the triangle, we will get three values $(\lambda_1, \lambda_2, \lambda_3)$ that are the barycentric coordinates of p with respect to the triangle $[p_1, p_2, p_3]$.

1.2 Triangle meshes

A collection of triangles without any particular mathematical structure is called *triangle meshes*. To derive a global parameterization for an entire triangle mesh we can define a 2D position for each vertex. Let $\mathcal{M}$ be a triangle mesh that consists of a geometric and topological component represented by a graph structure with a set of vertices $\mathcal{V} = \{v_1, ..., v_V\}$ and a set of triangular faces connecting them $\mathcal{F} = \{f_1, ..., f_F\}$ with $f_i \in \mathcal{V} \times \mathcal{V} \times \mathcal{V}$. The connectivity of a triangle mesh can be expressed in termes of the edges of the respective graph $\mathcal{E} = \{e_1, ..., e_E\}$ where $e_i \in \mathcal{V} \times \mathcal{V}$. [4]

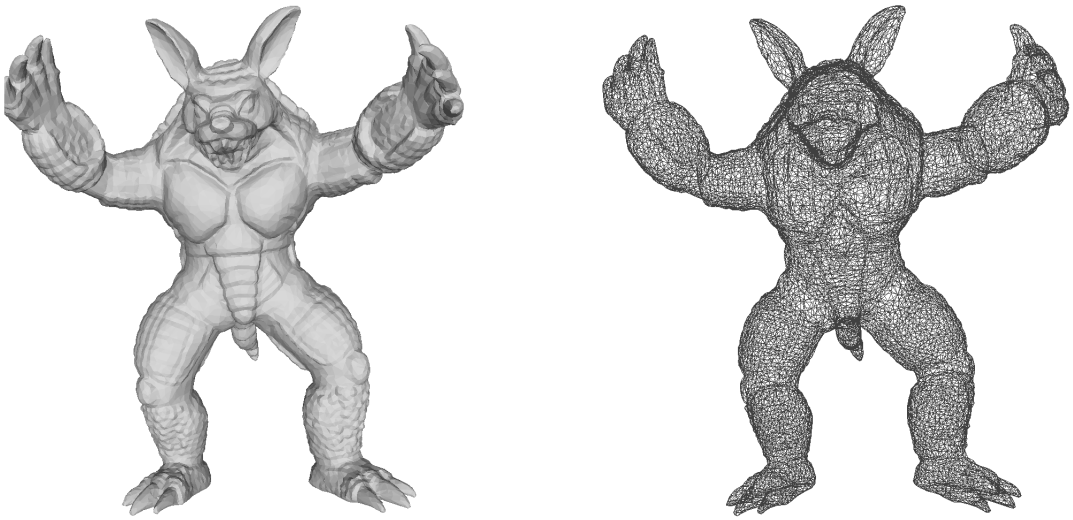


Figure 2: 3D triangle meshes.

1.3 Lighting - Phong lighting model

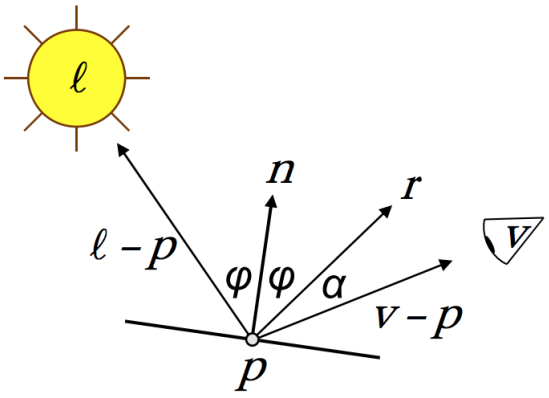


Figure 3: Lighting notation for Phong lighting model. [7]

Given a light source at position l with intensity I_l and a surface point at position p with normal n, we can define the angle between the incident light $(l-p)$ and the normal n as φ. Let r be our reflected light vector defined as $r = 2n \cdot < n, l-p > -(l-p)$ and α the angle between that vector and the view direction $(v-p)$.

The *Phong lighting model* is defined as the sum of the self-emitting intensity, ambient term, diffuse reflection and specular reflection.

$$I = I_e + \rho_a \cdot I_A + \sum_{j=1}^{n} (\rho_d \cdot cos\varphi_j + \rho_s \cdot cos^k_{\alpha_j}) \cdot I_j$$

where I_e is the self-emitting intensity, ρ_a, ρ_d, ρ_s are the reflection constants (surface properties), n is the number of lights sources with intensities I_j and k is the shininess. [7]

1.4 Linear interpolation

Linear interpolation is a method that will return equal spacing between the interpolated values. Given two numbers n_1 and n_2 (the start and final values of the interpolant), a linear interpolation can be carried out using a parameter t ($t \in [0,1]$). [3]

$$n = n_1 + t(n_2 - n_1)$$

The standard linear interpolated visualisation is made passing three attributes (colours) to each vertex of a triangle. OpenGL will interpolate these colors linearly thanks to the barycentric coordinates that will tell how much of each color is being mixed at any position. Given a triangle $[p_1, p_2, p_3]$, where the color blue is passed to vertex p_1, red to p_2, and green to p_3, let be w_1 the blue area, w_2 the red area and w_3 the green area (See Fig. 1). Let us define the value at p as a *barycentric interpolation*

$$(w_1 p_1 + w_2 p_2 + w_3 p_3)/W$$

where W is the area of the triangle $[p_1, p_2, p_3]$.

1.5 Flat Shading

Flat shading is a way to compute the colour at each pixel (at a corner or the barycentre) using the triangle normal. Given a triangle $[p_1, p_2, p_3]$, the lighting is computed using the normal n

$$\widehat{n} = (p_2 - p_1) \times (p_3 - p_1) \qquad n = \frac{\widehat{n}}{||\widehat{n}||}$$

at $p = (p_1 + p_2 + p_3)/3$. This colour is then used for all pixels. Flat shading gives objects with flat facets. [7]

1.6 Gouraud Shading

Gouraud Shading is a way to compute the colour at each pixel assigning a normal to each corner of a triangle and after having computed the color for each corner it linearly interpolates these colour values (see Sections 1.1 and 1.4). Given a triangle $[p_1, p_2, p_3]$ and the normal at each corner n_1, n_2, n_3. The lighting is computed at p_i using normal n_i this applied to each corner return the colour values c_1, c_2, c_3. These colors are then linearly interpolate $c = \mu_1 c_1 + \mu_2 c_2 + \mu_3 c_3$. Gouraud shading gives objects that appear more smooth. [7]

2 Discrete differential geometry

2.1 Normals

For each triangle $T = [p_1, p_2, p_3]$ of a triangle mesh, the normal is defined as

$$n(T) = \frac{(p_2 - p_1) \times (p_3 - p_1)}{||(p_2 - p_1) \times (p_3 - p_1)||}$$

The normal along each edge E is the halfway between the normals of two adjacent triangles T_1 and T_2

$$n(E) = \frac{n(T_1) + n(T_2)}{||n(T_1) + n(T_2)||}$$

The normal at vertex V is obtained averaging the normals of the n adjacent triangles

$$n(V) = \frac{\sum_{i=1}^{n} \gamma_i n(T_i)}{||\sum_{i=1}^{n} \gamma_i n(T_i)||}$$

Where γ_i can be a constant value, equal to the triangle area or equal to the angle θ_i of T_i at V. [6]

2.2 Local averaging regions

A mesh can be constructed either as the limit of a family of smooth surfaces or as a linear approximation of an arbitrary surface. To derive a spatial average of geometric properties we mix finite elements (a linear interpolation between three vertices of a triangle) and finite volumes (finite-volume region on a triangulated surface using Voronoi cells or Barycentric cells, Fig. 4). Restricting the average to the neighbouring triangles (*1-ring*) for each vertex, we can choose an associated surface patch over which the average will be computed. Let $\mathcal{A}_{Barycenter}$ be the area formed using barycenters and $\mathcal{A}_{Voronoi}$ the one formed using *Voronoi* cell. The general case is represented by a point that can be anywhere, let's denote this surface area $\mathcal{A}_M$.

Voronoi cell of each vertex is an appropriate local region that provide a stable error bounds. The *Voronoi* region for a point P of a non-obtuse triangle $[P, Q, R]$ is expressed

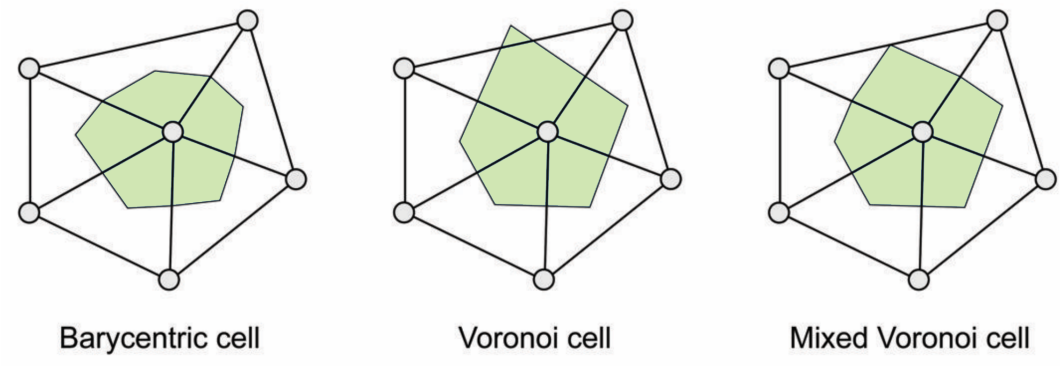


Figure 4: Local averaging regions used for computing discrete differential operators associated with the center vertex of the one-ring neighborhood. [4]

as $\frac{1}{8}(|PR|^2 cot\angle Q + |PQ|^2 cot\angle R)$. The sum of these areas for the whole *1-ring neighborhood* gives the non-obtuse *Voronoi* area for a vertex. The above expression for the *Voronoi* finite-volume area does not hold in case of obtuse angles. Let's define a new surface area for each vertex denoted $\mathcal{A}_{Mixed}$. Essentially the idea is to use the circumcenter point for each non-obtuse triangle and to use the midpoint of the edge opposite to the obtuse angle in case of an obtuse triangle. (See Pseudocode 7). [8]

2.3 Gaussian Curvature

The *Gaussian curvature* K is defined as the product of the principal curvatures (square of the geometric mean):

$$K = k_1 k_2$$

A basic interpretation would be to imagine the *Gaussian curvature* as a logical `AND` since it will check whether there is a curvature along both directions. The curvature of a surface is characterized by the principal curvatures. [2] Surfaces that have a zero gaussian curvature

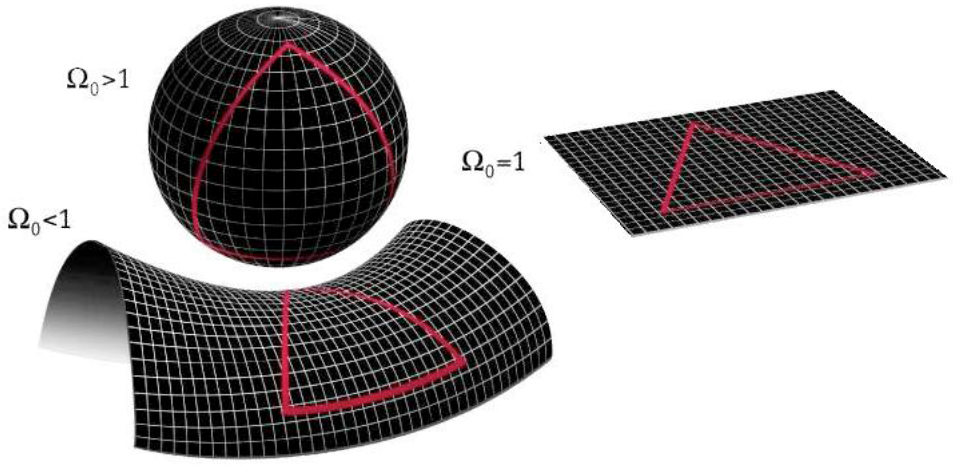


Figure 5: Positive curvature, negative curvature and zero curvature.

are called *developable surfaces* because they can be flattened out into the plane without any stretching. *Gaussian curvature* should be zero inside each mesh triangle and the same along edges since it can be flattened symmetrically into the plane by simply rotating one triangle about the common edge into the plane defined by the other. Consequently the *Gaussian curvature* is concentrated at the vertices of a triangle and defined as the *angle defect*

$$K(V) = 2\pi - \sum_{i=1}^{n} \theta_i$$

where θ_i are the angles of the triangle T_i adjacent to the vertex V at V. This should be seen as the integral of the Gaussian curvature over a certain region $S(V)$ around V, where these $S(V)$ form a partition of the surface of the entire mesh.

$$K(V) = \int_{S(V)} K dA$$

Negative curvature can be recognized by the fact that external directions curve in opposite directions, *zero curvature* has one external direction that has zero curvature, *positive curvature* has external directions that curve in the same direction (Fig. 5). The *Theorema Egregium*, discovered by C.F. Gauss in 1827, states that the *Gaussian curvature* is an intrinsic property of the surface that does not depend on the space, despite by the fact that it is define as the product of the principal curvatures (whose value depends on how the surface is immersed in the space). Technically, the *Gaussian curvature* is invariant under isometries. We can then notice that triangle angles add up to less than 180° in negative curvature, exactly 180° in zero curvature, and more than 180° in positive curvature. [6]

2.4 Mean Curvature

The *mean curvature* H is defined as the arithmetic mean of principal curvatures :

$$H = \frac{k_1 + k_2}{2}$$

A basic interpretation would be to imagine the *mean curvature* as a logical `OR` since it will check if there is a curvature along at least one direction.[2] The *mean curvature* inside each mesh triangle is zero, but it does not vanish at edges. The *mean curvature* associated with an edge is defined as $H(E) =\parallel E \parallel \theta_E/2$ where $\theta_E/2$ is the signed angle between the normals of adjacent triangles (see Fig. 6).

Let us think of an edge as a cylindrical patch $C(E)$ with a radius r that touches the planes defined by adjacent triangles. The *mean curvature* at any point of the cylindrical patch is defined as $1/(2r)$ and the area of $C(E)$ is $r||E||\theta_E$

$$H(E) = \int_{C(E)} H dA$$

. The *mean curvature* at a vertex V is defined as

$$H(V) = \frac{1}{2} \sum_{i=1}^{n} H(E_i)$$

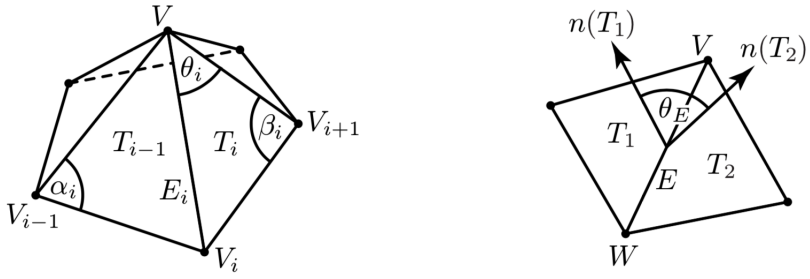


Figure 6: A vertex V with its neighbouring vertices V_i and adjacent triangles T_i. Angles opposite the edge E_i are denoted by α_i and β_i. The angle between the normals of adjacent triangles T_1 and T_2 with positive or negative sign is denoted as θ_E.[6]

Averaging the mean curvatures of its adjacent edges guarantee that *mean curvature* of an edge is divide uniformly to both end points. $H(E)$ and $H(V)$ should be seen as integral curvature values associated to regions $S(E)$ and $S(V)$. [6]

2.5 Mean Curvature Vector

Let be $H = Hn$ the surface normal vector scaled by the *mean curvature* to derive the discrete mean curvature vector associated to the mesh edge $E = [V, W]$

$$H(E) = \int_{C(E)} HdA = \frac{1}{2}(V - W) \times (n(T_1) - n(T_2))$$

The length of $H(E)$ gives the edge mean curvature $H(E) = ||H(E)|| = ||E||sin(\theta_E/2)$. The discrete mean curvature vector associated to V can be obtained averaging $H(E)$ over the edges adjacent to a vertex V

$$H(V) = \frac{1}{2}\sum_{i=1}^{n} H(E_i) = \frac{1}{4}\sum_{i=1}^{n}(\cot\alpha_i + \cot\beta_i)(V - V_i)$$

where α_i and β_i are angles opposite to E_i (see Fig. 6). [6]

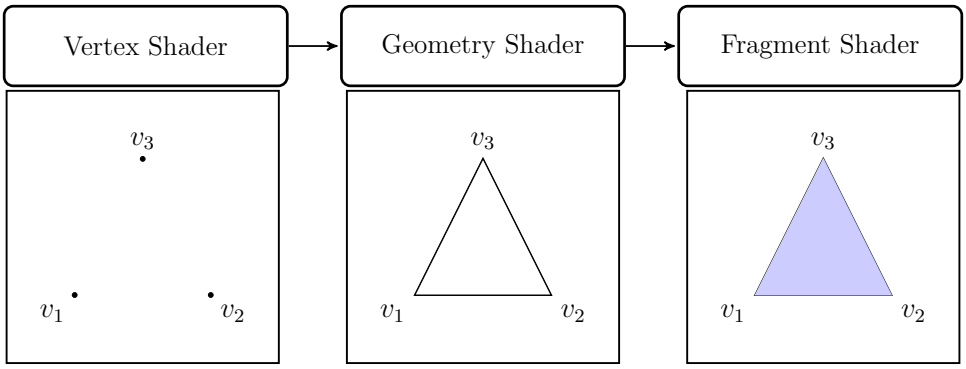


Figure 7: GPU pipeline [5]

3 GPU program

A program that runs on GPU is called *shader*. Shaders are principally used to modify the representation and the behaviour of 3D objects. They are also used to create lighting effects. Shaders can perform tasks efficiently thanks to the GPU. That guarantees faster results than CPU since GPU is designed to work in parallel.

3.1 GPU pipeline

A program will allow us to control the rendering pipeline since by default there is no pipeline set in OpenGL. It takes a set of vertices as input, then the *vertex shader* transforms them (translation, rotation, projection...) and passes the transformed vertices to the *geometry shader*. This shader takes vertices to create primitive shapes and then it rasterizes them. These rasterized flat images are then passed as input to the *fragment shader* that adds the lighting, apply textures and color these images (Fig. 7).

3.2 Vertex Shader

The program that performs vertex operations is called *vertex shader*. It receives one vertex at a time and then it passes the output to a *fragment shader* or to a *geometry shader*, if any.

3.3 Fragment Shader

Fragment shader performs color computation for every visible pixel of the rasterized object. It works on a fragment at a time, but thanks to the power of GPU it can work in parallel for all vertices (*vertex shader*) and fragments (*fragment shader*).

3.4 Geometry Shader

Geometry shader is used for layered rendering. It takes as input a set of vertices (single primitive, example: triangle or a point) and it transforms them before sending to the next shader stage. In this way, we can obtain different primitives. Each time we call the function `EmitVertex()` the vector currently set to `gl_Position` is added to the primitive. All emitted vertices are combined for the primitive and output when we call the function `EndPrimitive()`. [1]

4 Vertex area based

This section shows alternative methods to extend the idea of flat shading from triangles to vertices. The idea of flat shading is to draw all the pixels of a triangle with the same colour. The extension of this approach is to split the surface of the triangle mesh likewise into regions around vertices and draw all pixels in these regions with the same colour (Fig. 9), thus visualizing data given at the vertices of the mesh in a piecewise constant, not necessarily continuous way, resembling the classical triangle flat shading. The aforementioned regions can easily be defined using barycentric coordinates and a simple GPU fragment program (Fig. 8) can be used for each pixel to find out to which region it belongs and which colour it should be painted with.

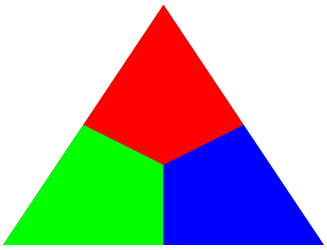


Figure 8: Max diagram

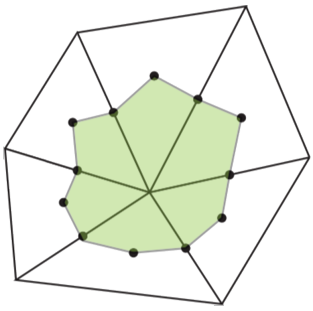


Figure 9: Region around a vertex

4.1 Max diagram - Vertex based area

Passing barycentric coordinates to the *fragment shader* will clearly demonstrate that we can get results different from the classic color interpolation.[9] There are various approaches to color interpolation focusing on the distance from vertices. For each point in a triangle, we can easily determine its closest vertex, which we use as a cue for coloring. Another approach, different from the above, can be defined as coloring vertex areas based on the maximum barycentric coordinate. The color is given by the region closest to a vertex (Fig. 8, Pseudocode 2).

4.2 Vertex Flat Shading

An extension of *flat shading* would be to have each vertex area to be in one constant color. This color can be taken using the normal at the vertex and the vertex position. The color will then be computed as in *Gouraud shading*. The idea is to compute the color per vertex but instead of linearly interpolating it in each triangle (as *Gouraud shading* does) we color regions around a vertex with that constant color (using the GPU fragment program: max diagram 5.1). To implement this approach, the barycentric coordinates, the vertex color, the normal at the vertex and the lighting calculations must be passed to the *fragment shader*. We want to avoid the automatic interpolation of colors. In order to return the resulting color using the *max diagram*, we have used a *Geometry shader* that has access to all three vertex colors in *fragment shader*. (Pseudocodes: 3, 4, 5)

Figure 10: Vertex flat shading.

4.3 Comparison between triangle flat shading, triangle Gouraud shading and vertex flat shading

The standard approaches: *triangle flat shading* and *triangle Gouraud shading* are then compared with the new technique *vertex flat shading*. In Fig. 11 we can see that the icosahedron where we have applied the vertex flat shading shader seems to be a good compromise since it preserves the original geometry and avoids creating triangle-like artifacts in the final result. Vertex regions look more realistic and less noisy than triangle regions. Moreover, Gouraud shading is prone to smoothing and loosing many details in some meshes.

4.4 Gaussian curvature

Another interesting alternative data visualization technique is to compute the *Gaussian curvature* per vertex. That can be done summing up, for each vertex, angles at this vertex with adjacent triangles and then subtracting this value from 2π. Having obtained this value, called *angle defect* (Fig. 13), we map it linearly to a color range. The resulting color will be the vertex flat shading visualisation of *Gaussian curvature* (See Section 2.2

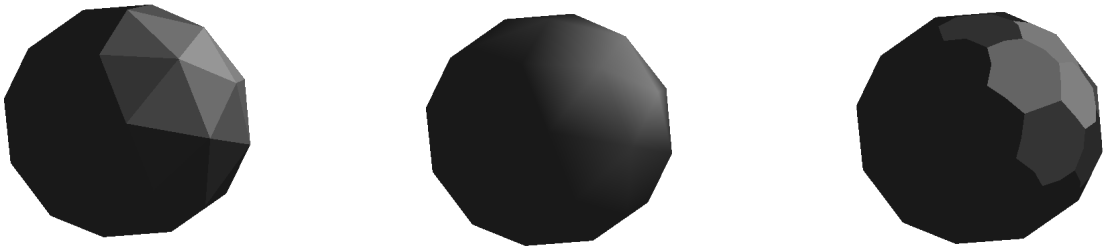


Figure 11: Comparison between: triangle flat shading, triangle Gouraud shading and vertex flat shading.

and 2.3).

$$K(V) = (2\pi - \sum_{j} \theta_j)/\mathcal{A}_{Mixed}$$

4.5 Constant Gaussian curvature per vertex

Constant Gaussian curvature per vertex returns a constant color around each vertex (Fig. 13, Pseudocode 6). Calculating the *Gaussian curvature* per vertex, this value is mapped into a color range to get the curvature color (see Fig. 14). This process is made separately for each vertex of the triangle and consequently, using the technique of max-diagram explained above (See section 5.1), the final resulting constant color is returned.

4.6 Gouraud Gaussian curvature

Gouraud Gaussian curvature returns an interpolated color per vertex. The idea is to calculate the *Gaussian curvature* as explained above (mapping the color into a color range to get the corresponding color per vertex) but instead of returning the constant color using a max-diagram approach, we just return the interpolation of values obtained for each triangle.

4.7 Evaluation and Comparison between constant Gaussian curvature per vertex and Gouraud Gaussian curvature

We compare the *constant Gaussian curvature per vertex* (Fig. 15) with the *Gouraud Gaussian curvature* (Fig. 16). In Fig. 15 each vertex area is colored applying the method *max diagram*. Instead, in Fig. 16 the color is obtained with a linear interpolation. Visualization of the principal curvatures of the model as colors from blue (highest values of curvature) to red (lower values of curvature) in Fig. 17 highlighs the geometry of meshes. These changes of curvature, positive (blue), flat (green) and negative regions (red), better emphasizes the 3-dimensionality of the model. *Gouraud Gaussian curvature* is smoother, which results in a loss of small details. This is particularly evident in armadillo's legs

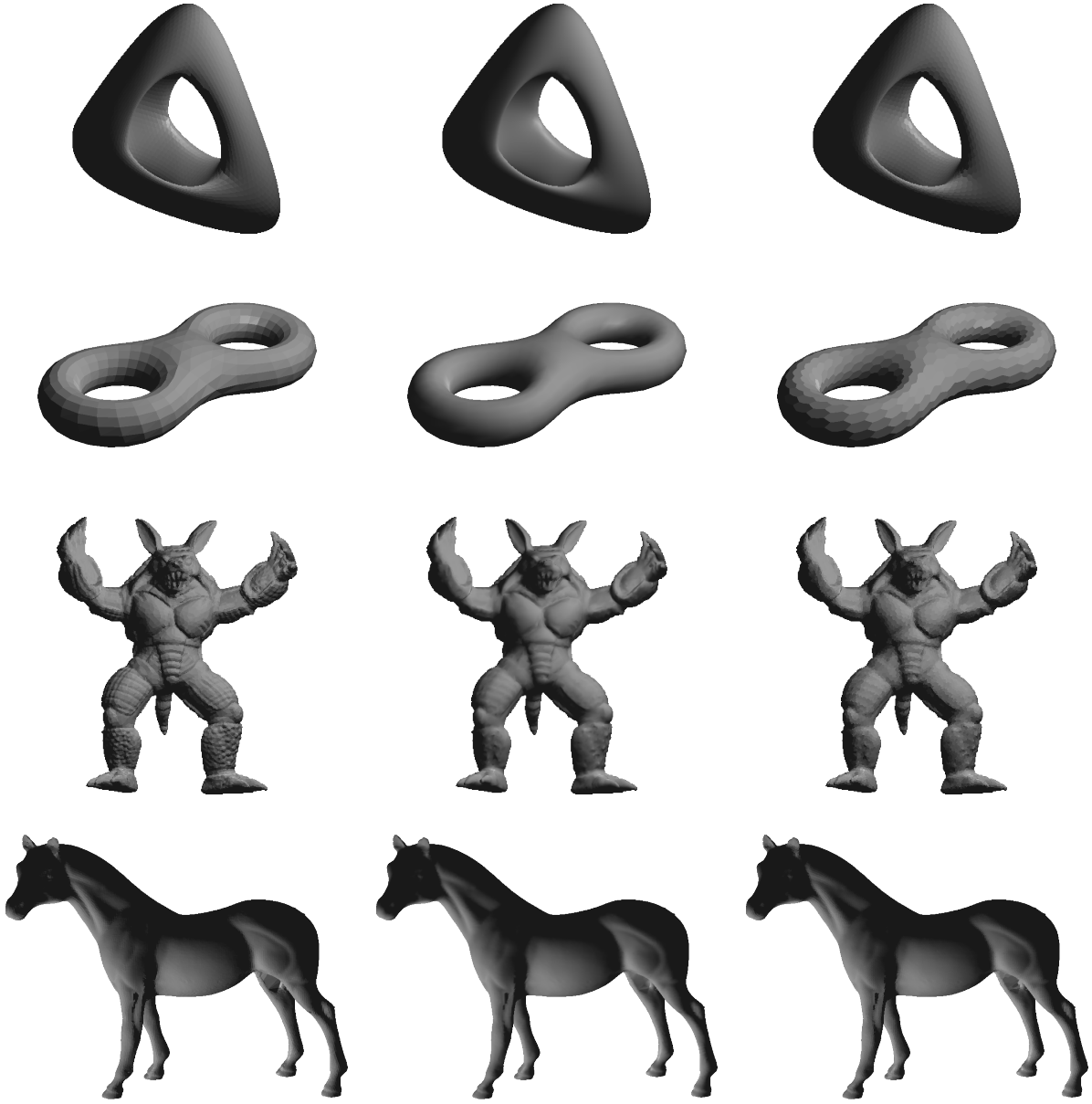


Figure 12: On the left: Triangle flat shading. On the center: Triangle Gouraud shading. On the right: Vertex flat shading.

mesh. Instead, *constant Gaussian curvature per vertex* generates sharper edges with piecewise-flat regions which slightly degrades the 3-dimensional perception of the model. On the other hand, *Constant Gaussian curvature per vertex* preserves the details of the given geometry, which can be particularly useful for data visualization purposes.

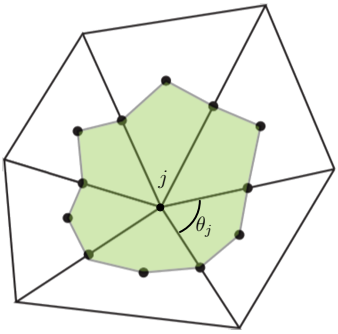


Figure 13: On the left: angle defect is denoted with θ_j.

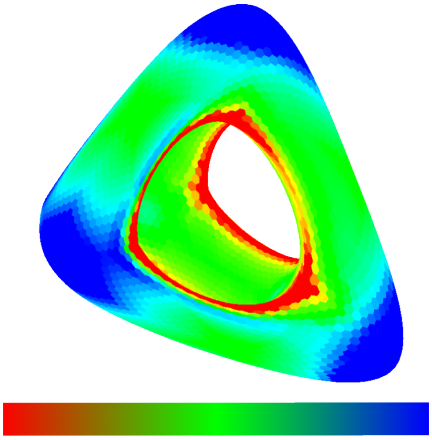


Figure 14: Color bar showing respective colors for negative, flat and positive curvatures. Negative curvatures are colored in red, flat curvatures in green and positive curvatures in blue.

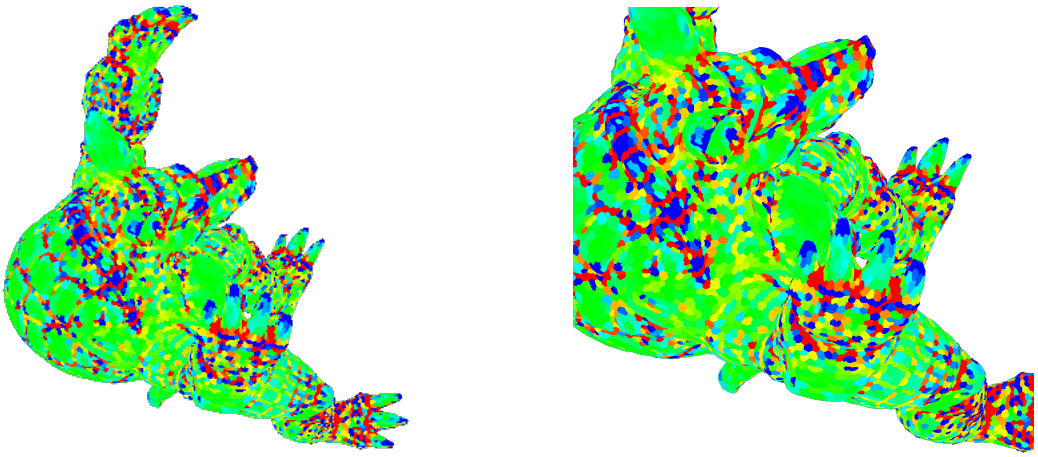
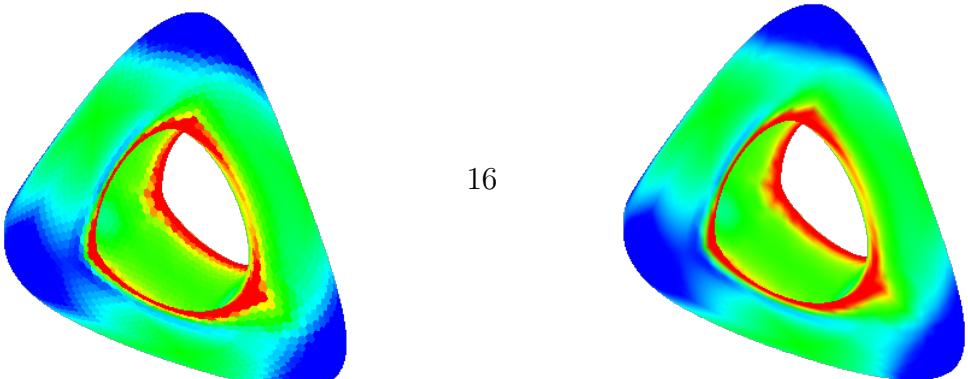


Figure 15: Constant Gaussian curvature per vertex



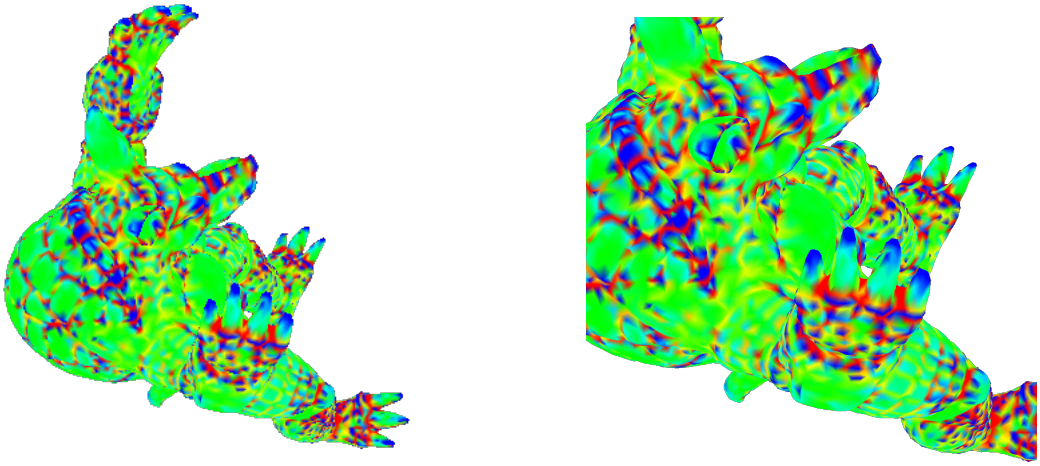


Figure 16: Gouraud Gaussian curvature

5 Edge area based

As the section before, we want now to show extensions where we split the surface of the triangle mesh likewise into regions around edges and we draw all pixels in these regions with the same colour (see Figure 19). This results in rhombus-shaped areas with constant color around each edge.

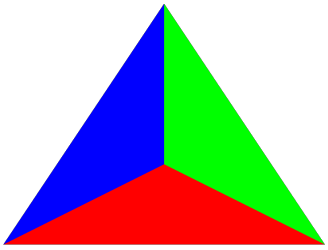


Figure 18: Min diagram

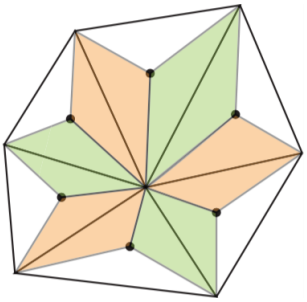


Figure 19: Region around an edge

5.1 Min diagram - Edge based area

For each point in a triangle, we can easily determine its closest edge, which we use as a cue for coloring. A different approach from interpolating, can be found coloring vertex areas based on the minimum barycentric coordinate. The color is given by the region farthest from a vertex (Fig. 18, Pseudocode insert Pseudocode).

5.2 Mean Curvature

Access to mesh edges requires to set up a list of edges over triangles. To avoid redundant datas we have decided to use the convention that an edge would be count only if it goes from a lower to higher vertex. An edge structure contains:

Listing 1: Edge structure

```
1     struct edge
2     {
3         float norm_edge;
4         int index_v1;
5         int index_v2;
6         Point3d n1;
7         Point3d n2;
8         float value_mean_curvature;
9         float cot_alpha;
10        float cot_beta;
11        float area_t1;
12        float area_t2;
13    };
```

indexes vertices, length of edge, normals of triangles, cotangents of opposite angles to the edge ($\cot\alpha,\ \cot\beta$), areas of triangles. (See edge struct 5.2).

5.3 Mean Curvature per edge

Mean Curvature per edge returns a constant color around each edge (See Fig. 21). It first calculate the mean curvature for each edge:

$$H(E) = ||E||(\theta_E/2)$$

where θ_E is the angle between the two normals of T_1 and T_2 (See Fig. 20).

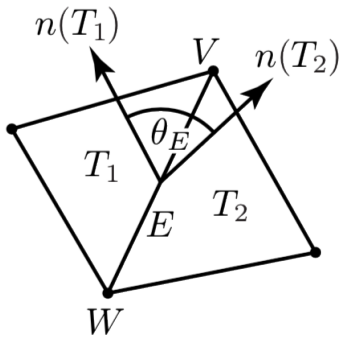


Figure 20: The dihedral angle θ_E at a mesh edge E is the angle between the normals of the adjacent triangles. [6]

The mean curvature for each vertex V of a mesh is defined as:

$$H(E) = \frac{1}{2\mathcal{A}_{Barycentre}} \sum_{i=1}^{n} ||E_i||(\theta_E/2)$$

This value is then normalized since it is an integral value. Every value is then mapped to positive or negative curvature, depending if the mesh at this edge is convex or concave, testing the $3D$ determinant of the 3-by-3 matrix $M = [e, n_1, n_2]$ with those three vectors as columns (e is the edge $[W, V]$, n_1 is the normal of T_1 and n_2 is the normal of T_2). If $det(M) > 0$ then the mesh is convex at $[W, V]$ and the mean curvature would be positive, else the mesh is concave and the mean curvature would be negative. This technique of edge flat shading represents an alternative to the classic triangle flat shading.

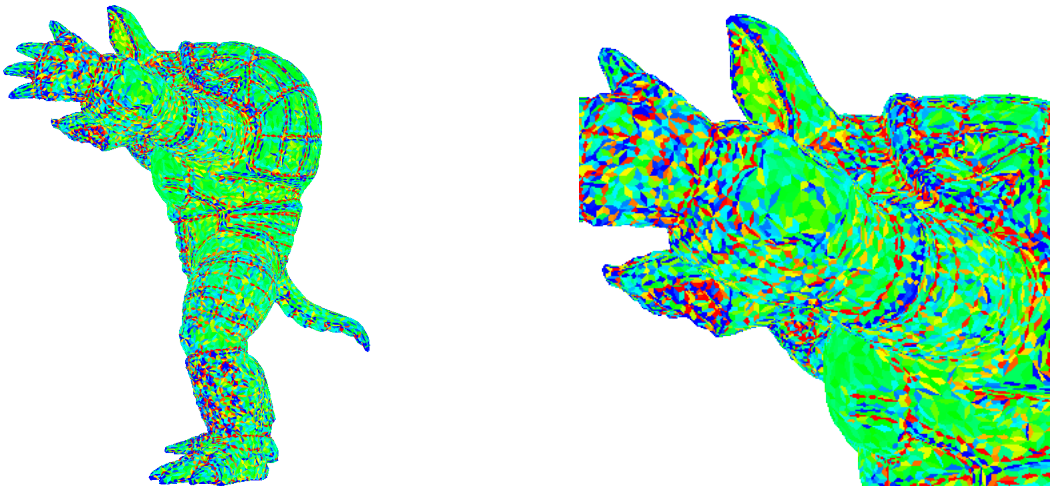


Figure 21: Mean curvature per edge

5.4 Mean Curvature per vertex

Mean Curvature per vertex returns an interpolated color around each vertex. The main idea is to calculate the mean curvature $H(V)$ for each vertex V. In a mesh, every edge has two opposite angles (let us denominate these with α and β), the mean curvature per vertex is defined as:

$$H(V) = \frac{1}{2\mathcal{A}_{Mixed}} \sum_{i=1}^{n} (\cot \alpha_i + \cot \beta_i)(V - V_i)$$

where V_i is one of the endpoints of the edge E_i (see Fig. 22).

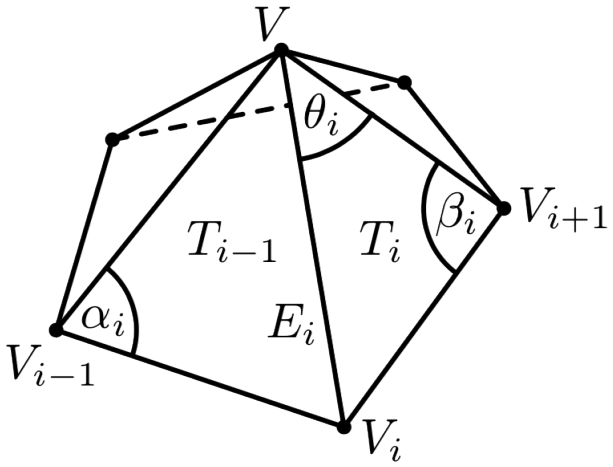


Figure 22: A vertex V of a triangle mesh with neighbouring vertices V_i and adjacent triangles T_i. The angle of T_i at V is denoted by θ_i and the angles opposite the edge E_i by α_i and β_i. [6]

These values are then interpolated using the automatic OpenGL interpolation resembling the classic Gouraud shading (see Fig. 23).

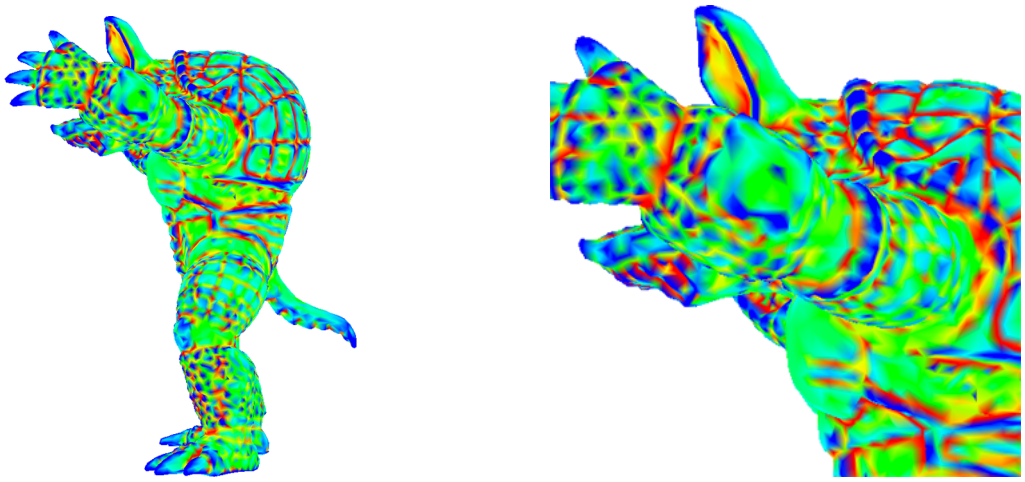


Figure 23: Mean curvature per vertex

5.5 Evaluation and Comparison between mean curvature per edge and mean curvature per vertex

// TODO: image of comparison

6 Conclusions

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

TODO: insert tables comparison with meshlab for gc and mc

7 References

[1] Learn Opengl `https://learnopengl.com`.

[2] Discrete differential geometry. `http://brickisland.net/cs177/?p=144`.

[3] Maths for computer graphics. `https://nccastaff.bmth.ac.uk/hncharif/MathsCGs/Interpolation.pdf`.

[4] M. Botsch, L. Kobbelt, M. Pauly, P. Alliez, and B. Lèvy. *Polygon Mesh Processing*. A K Peters, Ltd., 2010.

[5] Harsha. The world of shaders. goHarsha `https://goharsha.com/lwjgl-tutorial-series/world-of-shaders/`.

[6] K. Hormann. *Encyclopedia of Applied and Computational Mathematics*, chapter Geometry Processing, pages 593–606. Springer, Berlin, Heidelberg, 2015.

[7] K. Hormann. Slide of course: Computer graphics. iCorsi3.

[8] M. Meyer, M. Desbrun, P. Schrder, and A.H. Barr. *Discrete Differential-Geometry Operators for Triangulated 2-Manifolds*, pages 35–57. Springer Berlin Heidelberg, Berlin, Heidelberg, 2003.

[9] A. Patel. Red blob games. `https://www.redblobgames.com/x/1730-terrain-shader-experiments`, July 2017.

[10] J. Zhang, B. Deng, Z. Liu, G. Patanè, S. Bouaziz, K. Hormann, and L. Liu. Local barycentric coordinates, 2014.

A Pseudocodes

A.1 Chapter 4

TODO: CHECK AND UPDATE CODES

Listing 2: Max diagram - Vertex based area (Section: 5.1)

```
if (Coords.x > Coords.y && Coords.x > Coords.z) {
    vec3 blue = vec3(0.0f, 0.0f, 1.0f);
    FragColor = vec4(blue, 1.0f);
} else if(Coords.y > Coords.x && Coords.y > Coords.z) {
    vec3 green = vec3(0.0f, 1.0f, 0.0f);
    FragColor = vec4(green, 1.0f);
} else {
    vec3 red = vec3(1.0f, 0.0f, 0.0f);
    FragColor = vec4(red, 1.0f);
}
```

Listing 3: Vertex Shader for flat shading extension using lighting (Section: 4.2)

```
#version 330 core
layout (location = 0) in vec3 aPos;
layout (location = 1) in vec3 aNormal;
layout (location = 2) in vec3 aColor;

struct Light {
    // ...
};

out vec4 vertex_color;

uniform mat4 model;
uniform mat4 view;
uniform mat4 projection;

void main() {
    vec3 world_position = vec3(model * vec4(aPos, 1.0));
    vec3 world_normal = mat3(transpose(inverse(model))) *
       aNormal;

    // color obtained with lighting calculations
    vertex_color = get_result_color_lighting(...);

    gl_Position = projection * view * model * vec4(aPos, 1.0)
       ;
```

```
24      }
```

Listing 4: Geometry Shader for flat shading extension (Section: 4.2)

```
1  #version 330 core
2  layout (triangles) in;
3  layout (triangle_strip, max_vertices = 3) out;
4
5  in vec4 vertex_color[3];
6  out vec3 coords;
7  out vec4 wedge_color[3];
8
9  void main() {
10     wedge_color[0] = vertex_color[0];
11     wedge_color[1] = vertex_color[1];
12     wedge_color[2] = vertex_color[2];
13
14     coords = vec3(1.0, 0.0, 0.0);
15     gl_Position = gl_in[0].gl_Position;
16     EmitVertex();
17
18     coords = vec3(0.0, 1.0, 0.0);
19     gl_Position = gl_in[1].gl_Position;
20     EmitVertex();
21
22     coords = vec3(0.0, 0.0, 1.0);
23     gl_Position = gl_in[2].gl_Position;
24     EmitVertex();
25
26     EndPrimitive();
27 }
```

Listing 5: Fragment Shader for flat shading extension (Section: 4.2)

```
1  #version 330 core
2  in vec3 coords;
3  in vec4 wedge_color[3];
4  out vec4 fragColor;
5
6  void main() {
7      // max diagram
8      if (coords[0] > coords[1]) {
9          if (coords[0] > coords[2]) {
10             fragColor = wedge_color[0];
11         } else {
```

```
12              fragColor = wedge_color[2];
13          }
14      } else {
15          if (coords[1] > coords[2]) {
16              fragColor = wedge_color[1];
17          } else {
18              fragColor = wedge_color[2];
19          }
20      }
21  }
```

Listing 6: Vertex Shader for flat shading extension using gaussian curvature (Section: 4.4)

```
1  #version 330 core
2  layout (location = 0) in vec3 aPos;
3  layout (location = 2) in vec3 gaussian_curvature;
4
5  out vec4 vertex_color;
6
7  uniform mat4 model;
8  uniform mat4 view;
9  uniform mat4 projection;
10
11 uniform float min_gc;
12 uniform float max_gc;
13 uniform float mean_negative_gc;
14 uniform float mean_positive_gc;
15
16
17 vec3 interpolation(vec3 v0, vec3 v1, float t) {
18     return (1 - t) * v0 + t * v1;
19 }
20
21 vec4 get_result_color_gc() {
22     float val = gaussian_curvature[0];
23     vec3 red = vec3(1.0, 0.0, 0.0);
24     vec3 green = vec3(0.0, 1.0, 0.0);
25     vec3 blue = vec3(0.0, 0.0, 1.0);
26
27     //negative numbers until 0 -> map from red to green
28     if (val < 0) {
29         return vec4(interpolation(red, green, val/min_gc)/(
               mean_negative_gc/5), 1.0);
30     } else {
```

```
31          //map from green to blue, from 0 to positive
32          return vec4(interpolation(green, blue, val/max_gc)/(
                mean_positive_gc/5), 1.0);
33      }
34  }
35
36  void main() {
37      vec3 pos = vec3(model * vec4(aPos, 1.0));
38
39      vertex_color = get_result_color_gc();
40
41      gl_Position = projection * view * model * vec4(aPos, 1.0)
            ;
42  }
```

Listing 7: Region $\mathcal{A}_{Mixed}$ on an arbitrary mesh. [8] (Section: 2.2)

```
1       A_Mixed = 0
2       For each triangle T from the 1-ring neighborhood of x
3       If T is non-obtuse (Voronoi safe)
4           A_Mixed += Voronoi region of x in T
5       Else
6           If x is obtuse
7               A_Mixed += area(T)/2
8           Else
9               A_Mixed += area(T)/4
```