Stochastic price model simulation with CRR parameterization

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As part of ISYE6420 as an introduction to Bayesian Statistical Inference and applications, in this paper we define a state $\eta = \{t, \sigma, r, S_0\}$. For this state, we apply a stochastic dynamic following the CRR parameterization rules with a specific case where $S_0 = 1$, $\sigma = 0.3$, $r = 0.05$. With the model, we ran 1000 simulations where each simulation had $n = 10000$ (number of trials) within the state. As a core results we were able to calculate giving a target price ($K = 1.5$). That the probability $S_f > K$ is 0.084 and that for our case (large number of trials), we show that the binomial to normal distribution approximation is acceptable and that for $t = 1$ we obtain $\mu_{dist.} = 1.03$ and $\sigma_{dist.} = 0.32$. We also identified that $\sigma_{dist.}$ evolves with time getting larger behaving as a diffusion coefficient.

Key Words: Stochastic model, CRR parameterization, Bayesian probability.

I. INTRODUCTION

Option contracts are an ancient form of protection with historical roots traced back to Aristotelian anecdotes [1]. This study aims to present what are the key assumptions and definitions that are made within option valuation using statistical mechanics formalism and interpretation as the framework.

Option valuation is quite prestigious and a Nobel prize worth problem [2] where Black-Merton-Scholes model originally published on 1973 proposed a partial differential equation (PDE) that was a major scientific achievement and an elegant application for the Langevin equation and the nature of the Brownian motion from early 1900. As a starting point, we should fist understand that the Spot market and the Bonds market are equivalent. This is an initial base assumption where the Spot market is composed of financial instruments that have immediate delivery and where the Bonds market offers upfront delivery (B_0) with a promise to receive it back in a future time (t) the initial value plus interest (r) over time.

$$B = B_0 e^{rt} \tag{1}$$

We can write this equivalence principle from Spot to Bonds as a binomial step problem. Where we can define a probability (p) to move a step u in the upper direction and the complement probability $(1-p)$ to move a step d in the down direction. We can easily see this in action in the diagram in Fig. 1.

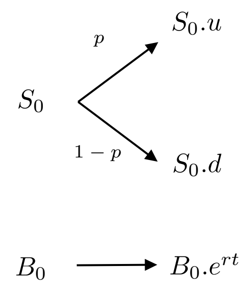

FIG. 1. equivalence diagram.

II. METHODS

A. CRR parameterization

$$B_0 e^{rt} = \langle S \rangle = S_0.u.p + S_0.d.(1-p) \tag{2}$$

Assuming that the initial investment is the same ($B_0 = S_0$) we can write the probability in terms of u, d and e^{rt}.

$$p = \frac{e^{rt} - d}{u - d} \tag{3}$$

Note that u and d are the size of the step while in the upwards and downwards direction respectively.

From this point, we will assume the CRR parameterization same as [3]:

$$u = e^{\sigma.\sqrt{t}} \tag{4}$$

$$d = e^{-\sigma.\sqrt{t}} \tag{5}$$

Where σ is the standard deviation of the price. With this assumption, we have that $u.d = 1$ and giving us a most welcome symmetry. For our model, we will assume that σ is constant and resulted from the prices $S \sim \mathcal{N}(\mu = \langle S \rangle, \sigma^2)$.

As a result, we can now use the binomial distribution e evaluate given a state $\eta = \{t, \sigma, r, S_0\}$ what will be the probability $P(S_f > K \mid \eta)$.

B. The Binomial Model

The binomial distribution accounts for giving n trials what would the probability of having k successes. This is particularly important in our application here as our

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problem with price simulation accounts exactly for two possible states (u, d) that we can interpret as success if going up and fail if going down.

The binomial distribution has the form:

$$P(x = k) = \frac{n!}{k!(n-k)!}.p^k.(1-p)^{n-k} \qquad (6)$$

This distribution is interesting as in the limit where n is large. The Binomial model $\mathcal{B}(n,p)$ can be approximated to a normal distribution $\mathcal{N}(n.p, n.p.(1-p))$.

Another approximation that can happen is that if the number of trials goes to infinity while the product $n.p$ remains fixed or if $p \to 0$. The Binomial distribution can be approximated to a poisson distribution with $\lambda = n.p$.

For our analysis we will assume the case where n is large and we will approximate it to a normal distribution.

III. MODEL EVALUATION & RESULTS

Let's start our analysis by simulating 1000 cases with n = 10000. Following the CRR parameterization and

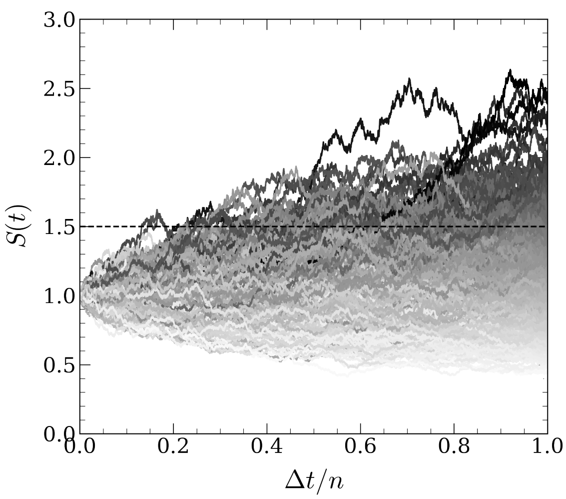

FIG. 2. CRR parameterization with initial conditions set to $S_0 = 1$, $\sigma = 0.3$, $r = 0.05$

The question that we will focus on paper is that, giving a target price ($K = 1.5$). What is the probability that will find $S_f > K$. Where S_f is the price at $t = 1$ following the stochastic dynamic for the η state. We can improve our visualization of the data by coloring the success cases and fail cases for the $S_f > K$ condition:

For our simulation with $n = 10000$ we fall in the case where a large number of trials exists and we can use the normal distribution approximation for our binomial problem.

We can finally calculate the probability $P(S_f > K \mid \eta)$ by counting the number of events after time t that have $S_f > K$. By doing this, we find that the probability for $K = 1.5$ is 0.084.

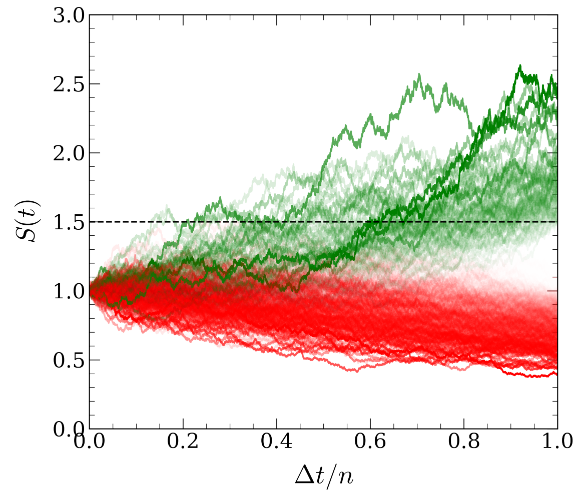

FIG. 3. Color coding with green when $S_f > K$ and using red for $S_f \leqslant K$. Colors have a strong line alpha when closer to the upper and lower limits.

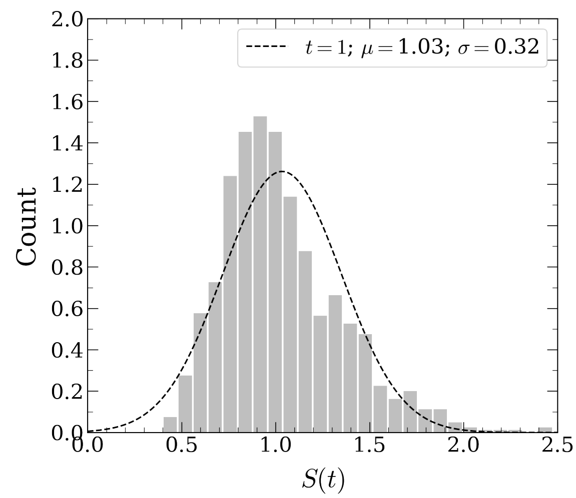

FIG. 4. Normal distribution aproximation fit.

We can compute the same for other values of K.

TABLE I. The probability measurement for when $S_f > K$.

K (target price)	$P(S_f > K \mid \eta)$
0.55	0.974
1.00	0.477
1.05	0.401
1.20	0.254
1.50	0.084
1.81	0.023

Note from table I that the 95% confidence interval for K is between $[0.55, 1.81]$. Considering the specific state η with $\sigma = 0.3$, $r = 0.05$ and $S_0 = 1$.

We also identified that σ evolves with the time t. We can plot the data in table II.

Diffusion is the movements of molecules from high concentration to low concentration. This is a process that happens due to kinetic energy from these molecules and is associated with random motion. Most famously the

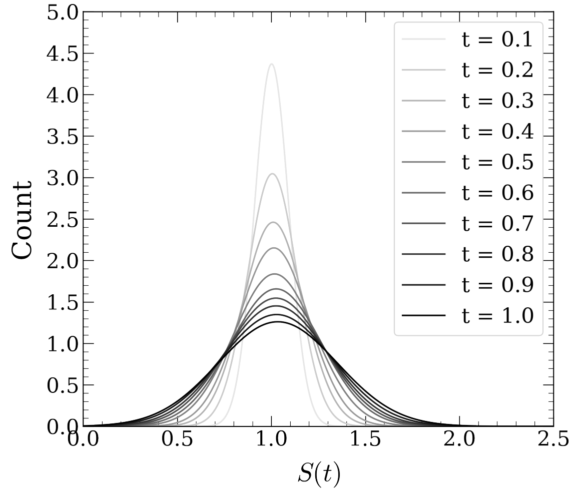

FIG. 5. time evolution for the Normal distribution fit.

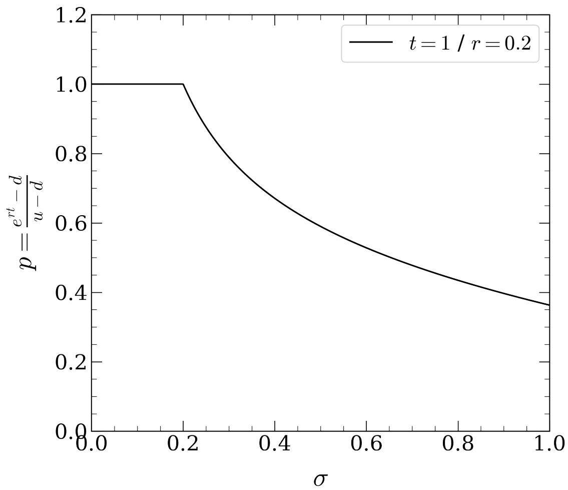

FIG. 6. phase transition characteristics for $r > \sigma$.

Brownian motion.

Its important to mention that the CRR parameterization has three main assumptions and one of which directly implements diffusion in the model.

- $\langle \Delta S^2 \rangle \sim \sigma^2.t$ (diffusion analogy)
- $e^{\beta t} \sim 1 + \beta t$ (small exponent: $\beta t << 1$)
- $\sqrt{t} >> t$ and $t^2 \to 0$ (dominating term)

TABLE II. Results for time, μ (mean) and σ standard deviation for the state η.

t (time)	μ (mean)	σ (std)
0.10	1.00	0.09
0.20	1.01	0.13
0.30	1.01	0.16
0.40	1.01	0.19
0.50	1.02	0.22
0.60	1.03	0.24
0.70	1.02	0.26
0.80	1.03	0.27
0.90	1.03	0.30
1.00	1.03	0.32

IV. CONCLUSION

As part of ISYE6420 as an introduction to Bayesian Statistical Inference and applications, we explored a stochastic simulation following the CRR parameterization considering a state $\eta = \{t, \sigma, r, S_0\}$, where $S_0 = 1$, $\sigma = 0.3$, $r = 0.05$.

When running 1000 simulations with $n = 10000$ (trials each) we were able to fit our data that originally started as a binomial model to a normal distribution and as a core result we were able to calculate giving any target price the probability of achieving that target. For $K = 1.5$ probability $P(S_f > K \mid \eta)$ is 0.084.

Unfortunately this work didn't explore in more details the time evolution of σ. This model clearly correlates to the Fokker-Planck differential equation behaving like a diffusion model. Also, CRR parameterization is not efficient for edge cases like.

For $r > \sigma$. The CRR parameterization fails as equation 3 would give us $p > 1$. A deeper investigation to this case is required as data suggests a phase transition model as we see in figure 6.

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