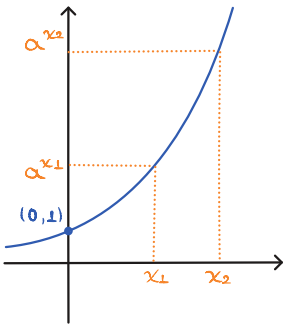
Εκθετικές Συναρτήσεις

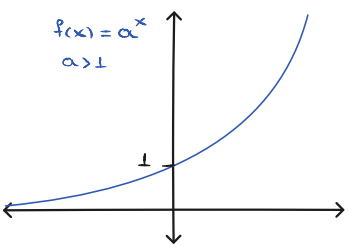


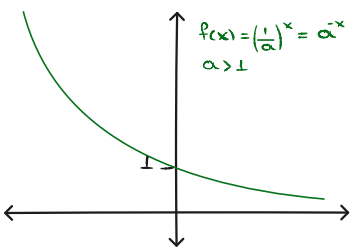
$$f(x) = a^x \quad , \quad a>0 \text{ και } a \neq 1$$

→ Εκθετική Συνάρτηση με βάση το a

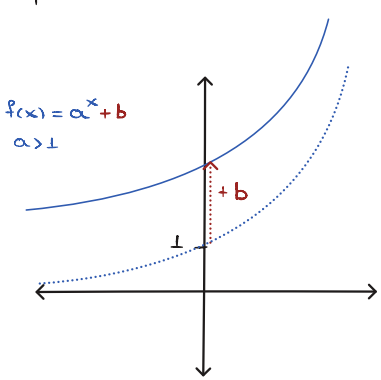
Σημείωση: Παρατηρούμε ότι για κάθε Εκθετική συνάρτηση $f(0) = a^0 = 1$

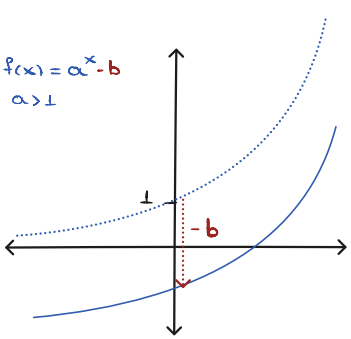
Γραφική $f(x)=a^x$





Συνάρτηση $f(x) = a^x \pm b$





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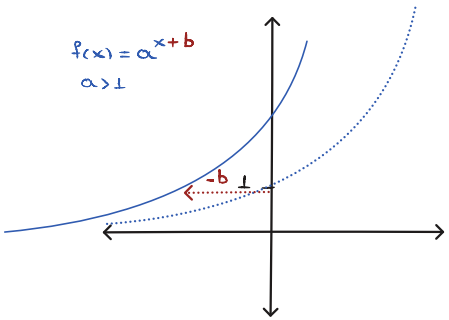
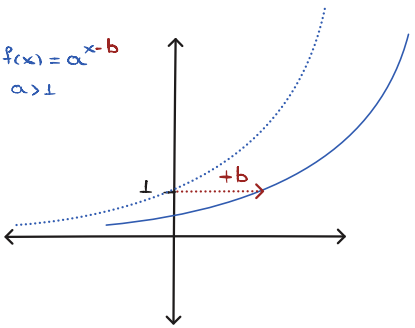


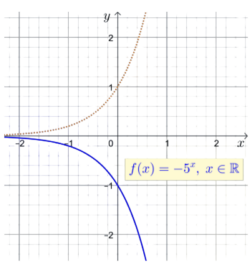


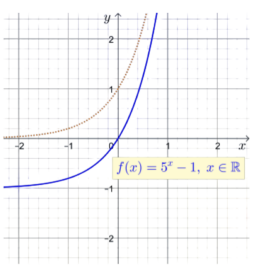
Άσκηση: Να κατασκευάσετε τις παρακάτω γραφικές

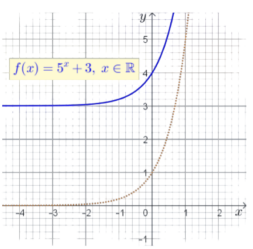
α. $f(x) = -5^x$

β. $f(x) = 5^x - 1$

γ. $f(x) = 5^x + 3$







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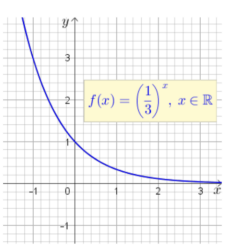
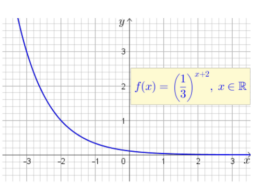
Άσκηση: Να κατασκευάσετε τις παρακάτω γραφικές

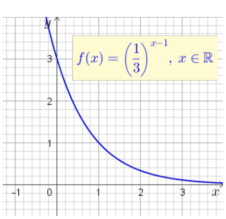
α. $f(x) = \left(\frac{1}{3}\right)^x$

β. $f(x) = \left(\frac{1}{3}\right)^{x+2}$

γ. $f(x) = \left(\frac{1}{3}\right)^{x-1}$







Λογαριθμική Συνάρτηση

$$\log_a \Theta = x \iff a^x = \Theta$$

→ Λογαριθμική του Θ με βάση το a

- $2^4 = 16 \dots\to \log_2 16 = 4$
- $5^3 = 125 \dots\to \log_5 125 = 3$

Ιδιότητες

- $\log_a 1 = 0$
- $\log_a a^x = x$
- $\log_a a = 1$
- $a^{\log_a x} = x$
- $\log_a(A \cdot B) = \log_a(A) + \log_a(B)$
- $\log_a\left(\frac{A}{B}\right) = \log_a(A) - \log_a(B)$
- $\log_a(A^\nu) = \nu \log_a(A)$

Ειδικός Συμβολισμός

Λογάριθμος με βάση το e: $\ln(\Theta) = x \iff e^x = \Theta$ $\quad \ln \equiv \log_e$

Λογάριθμος με βάση το 10: $\log(\Theta) = x \iff 10^x = \Theta$ $\quad \log \equiv \log_{10}$

Άσκηση: Να υπολογίσετε το x

α. $\log_x 64 = 2$ $\quad$ γ. $\log_x e^2 = 4$

β. $\log_{\sqrt{3}} 27 = x$ $\quad$ δ. $\log_4(\log_x 25) = 1$

α. $x^2 = 64 \Rightarrow x = \pm 8$

β. $\sqrt{3}^x = 27 \Rightarrow 3^{\frac{x}{2}} = 27 \Rightarrow 3^{\frac{x}{2}} = 3^3 \Rightarrow \frac{x}{2} = 3 \Rightarrow x = 6$

γ. $x^4 = e^2 \Rightarrow x = \sqrt{e}$

δ. $\log_x 25 = 4^1 \Rightarrow x^4 = 25 \Rightarrow x = \sqrt{5}$

Άσκηση: Να απλοποιήσετε τις παρακάτω σχέσεις:

α. $e^{2\cdot\ln(x)} + 10^{\log(7)} + \ln(e)$

β. $2\cdot\log(x+1) - \log(x)$

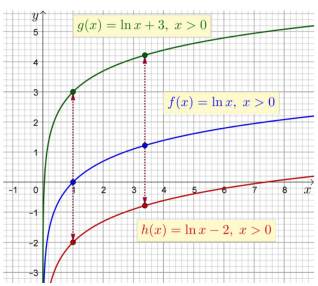
γ. $e^{2\ln(x)}$

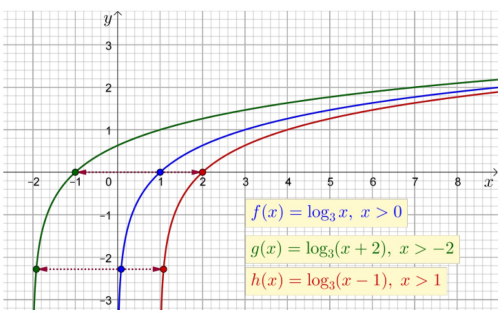
α. $e^{2\cdot\ln(x)} + 10^{\log(7)} + \ln(e) = e^{\ln(x^2)} + 7 + 1 = x^2 + 8$

β. $2\cdot\log(x+1) - \log(x) = \log(x+1)^2 - \log(x) = \log\left(\frac{(x+1)^2}{x}\right)$

γ. $e^{2\ln(x)} = e^{\ln(x^2)} = x^2$

Γραφική $f(x) = \log_a(x)$





Αλλαγή Βάσης Λογαρίθμου

$$\log_a x = \frac{\log_b x}{\log_b a}$$ ⟶ Αλλαγή βάσης από $a \to b$

Άσκηση: Να αποδείξετε ότι $\log_4 8 = \frac{3}{2}$

$$\log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{\log_2 2^3}{\log_2 2^2} = \frac{3}{2}$$

Άσκηση: Να αποδείξετε ότι $9^{\log_3 5} + e^{1-\ln(2)} - 2^{\log_4 36} = 19 + \frac{e}{2}$

$$9^{\log_3 5} + e^{1-\ln(2)} - 2^{\log_4 36} = 3^{2\log_3 5} + e^{\ln(e)-\ln(2)} - 2^{\frac{\log_2 36}{\log_2 4}}$$

$$= 3^{\log_3 5^2} + e^{\ln\left(\frac{e}{2}\right)} - 2^{\frac{1}{2}\log_2 36} = 5^2 + \frac{e}{2} - 2^{\log_2 \sqrt{36}}$$

$$= 25 + \frac{e}{2} - \sqrt{36} = 19 + \frac{e}{2}$$

Λογαριθμικές Εξισώσεις

$$\log_a f(x_1) = \log_a f(x_2) \quad \longleftrightarrow \quad f(x_1) = f(x_2)$$

Λογαριθμικές Ανισώσεις

$$\log_a f(x_1) < \log_a f(x_2) \quad \longleftrightarrow \quad f(x_1) < f(x_2) \qquad a > 1$$

$$\log_a f(x_1) < \log_a f(x_2) \quad \longleftrightarrow \quad f(x_1) > f(x_2) \qquad 0 < a < 1$$

Άσκηση: Να λύσετε την ανισότητα $\log(x) > \ln(x)$

$$\frac{\ln(x)}{\log(x)} \geq \ln(x)$$

$$\ln(x) \geq \ln(x) \cdot \log(x)$$

$$\ln(x) \cdot (1 - \log(x)) \geq 0 \qquad , \quad (1 - \log(x)) < 0$$

$$\ln(x) \leq 0$$

$$e^{\ln(x)} \leq e^0$$

$$x \leq 1$$

Άσκηση: Να λύσετε την εξίσωση $3^x = 4$

$$3^x = 4 \Rightarrow \ln(3^x) = \ln(4) \Rightarrow x \cdot \ln(3) = \ln(4) \Rightarrow x = \frac{\ln(4)}{\ln(3)} = \log_3 4$$

Άσκηση: Να λύσετε την εξίσωση, $\ln(4e^{3x} + 3e^x) = -x$

$$\ln(4e^{3x} + 3e^x) = -x \Rightarrow \ln(4e^{3x} + 3e^x) = \ln(e^{-x})$$
$$4 \cdot e^{3x} + 3 \cdot e^x = e^{-x}$$
$$4 \cdot e^{4x} + 3 \cdot e^{2x} = 1$$

Θέτω $w = e^{2x}$, $\quad 4 \cdot w^2 + 3w - 1 = 0$

$$w_{1,2} = \frac{-\beta \pm \sqrt{\Delta}}{2a} = \frac{-3 \pm 5}{8}$$

$w_1 = -1$ (χ)

$w_2 = \frac{1}{4} \longrightarrow e^{2x} = \frac{1}{4}$

$$2x = \ln\left(\frac{1}{4}\right) \Rightarrow 2x = \ln 4^{-1} \Rightarrow 2x = -\ln(4) \Rightarrow x = -\frac{1}{2}\ln(4) = -\ln(\sqrt{4}) = -\ln(2)$$

Επομένως, $x = -\ln(2)$

1. Solve the following equation.

$$\log_4\left(x^2-2x\right)=\log_4\left(5x-12\right)$$

$$\begin{aligned} x^2-2x&=5x-12\\ x^2-7x+12&=0\\ (x-3)(x-4)&=0 \quad \rightarrow \quad x=3,\ \ x=4 \end{aligned}$$

2. Solve the following equation.

$$\log(6x)-\log(4-x)=\log(3)$$

$$\log\left(\frac{6x}{4-x}\right)=\log(3) \qquad \begin{aligned} \frac{6x}{4-x}&=3\\ 6x&=3(4-x)=12-3x\\ 9x&=12 \quad \rightarrow \quad x=\frac{12}{9}=\frac{4}{3} \end{aligned}$$

4. Solve the following equation.

$$\log_3\left(25-x^2\right)=2$$

$$\begin{aligned} 25-x^2&=9\\ 16&=x^2 \quad \rightarrow \quad x=\pm\sqrt{16}=\pm 4 \end{aligned}$$

5. Solve the following equation.

$$\log_2\left(x+1\right)-\log_2\left(2-x\right)=3$$

$$\log_2\left(\frac{x+1}{2-x}\right)=3 \qquad \frac{x+1}{2-x}=2^3=8 \qquad \begin{aligned} \frac{x+1}{2-x}&=8\\ x+1&=8(2-x)=16-8x\\ 9x&=15 \quad \rightarrow \quad x=\frac{15}{9}=\frac{5}{3} \end{aligned}$$