# 5 Integrals

In this chapter we will be looking at the third and final major topic that will be covered in this class, integrals. As with derivatives this chapter will be devoted almost exclusively to finding and computing integrals. Applications will be given in the following chapter. There are really two types of integrals that we'll be looking at in this chapter : Indefinite Integrals and Definite Integrals. The first half of this chapter is devoted to indefinite integrals and the last half is devoted to definite integrals.

As we investigate indefinite integrals we will see that as long as we understand basic differentiation we shouldn't have a lot of problems with basic indefinite integrals. The reason for this is that indefinite integration is basically "undoing" differentiation. In fact, indefinite integrals are sometimes called anti-derivatives to make this idea clear. Having said that however we will be using the phrase indefinite integral instead of anti-derivative as that is the more common phrase used.

We will also spend a fair amount of time learning the substitution rule for integrals. We will see that it is really just "undoing" the chain rule and so, again, if you understand the chain rule it will help when using the substitution rule. In addition, as we'll see as we go through the rest of the calculus course the substitution rule will come up time and again and so it is very important to make sure that we have that down so we don't have issues with it in later topics.

As we move over to investigating definite integrals we will quickly realize just how important it is to be able to do indefinite integrals. As we will see we will not be able to compute definite integrals unless we can fist compute indefinite integrals.

We will also take a look at an important interpretation of definite integrals. Namely, a definite integral can be interpreted as the net area between the graph of the function and the $x$-axis.

The following sections are the practice problems (without solutions) for this material.

If you are looking for the solutions to these problems you can go to the Practice Problems on the website and download a pdf with the solutions from there.

## 5.1 Indefinite Integrals

1. Evaluate each of the following indefinite integrals.

   **(a)** $\displaystyle\int 6x^5 - 18x^2 + 7\,dx$

   **(b)** $\displaystyle\int 6x^5\,dx - 18x^2 + 7$

2. Evaluate each of the following indefinite integrals.

   **(a)** $\displaystyle\int 40x^3 + 12x^2 - 9x + 14\,dx$

   **(b)** $\displaystyle\int 40x^3 + 12x^2 - 9x\,dx + 14$

   **(c)** $\displaystyle\int 40x^3 + 12x^2\,dx - 9x + 14$

For problems 3 - 5 evaluate the indefinite integral.

3. $\displaystyle\int 12t^7 - t^2 - t + 3\,dt$

4. $\displaystyle\int 10w^4 + 9w^3 + 7w\ dw$

5. $\displaystyle\int z^6 + 4z^4 - z^2\,dz$

6. Determine $f\left(x\right)$ given that $f'\left(x\right) = 6x^8 - 20x^4 + x^2 + 9$.

7. Determine $h\left(t\right)$ given that $h'\left(t\right) = t^4 - t^3 + t^2 + t - 1$.

# 5.2 Computing Indefinite Integrals

For problems 1 - 21 evaluate the given integral.

1. $\int 4x^6 - 2x^3 + 7x - 4\,dx$

2. $\int z^7 - 48z^{11} - 5z^{16}\,dz$

3. $\int 10t^{-3} + 12t^{-9} + 4t^3\,dt$

4. $\int w^{-2} + 10w^{-5} - 8\,dw$

5. $\int 12\,dy$

6. $\int \sqrt[3]{w} + 10\,\sqrt[5]{w^3}\,dw$

7. $\int \sqrt{x^7} - 7\,\sqrt[6]{x^5} + 17\,\sqrt[3]{x^{10}}\,dx$

8. $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}\,dx$

9. $\int \frac{7}{3y^6} + \frac{1}{y^{10}} - \frac{2}{\sqrt[3]{y^4}}\,dy$

10. $\int \left(t^2 - 1\right)\left(4 + 3t\right)\,dt$

11. $\int \sqrt{z}\left(z^2 - \frac{1}{4z}\right)\,dz$

12. $\int \frac{z^8 - 6z^5 + 4z^3 - 2}{z^4}\,dz$

13. $\int \frac{x^4 - \sqrt[3]{x}}{6\sqrt{x}}\,dx$

14. $\int \sin\left(x\right) + 10\csc^2\left(x\right)\,dx$

15. $\int 2\cos\left(w\right) - \sec\left(w\right)\tan\left(w\right)\,dw$

16. $\int 12 + \csc\left(\theta\right)\left[\sin\left(\theta\right) + \csc\left(\theta\right)\right]\,d\theta$

17. $\displaystyle\int 4\mathbf{e}^{z} + 15 - \frac{1}{6z}\,dz$

18. $\displaystyle\int t^{3} - \frac{\mathbf{e}^{-t} - 4}{\mathbf{e}^{-t}}\,dt$

19. $\displaystyle\int \frac{6}{w^{3}} - \frac{2}{w}\,dw$

20. $\displaystyle\int \frac{1}{1+x^{2}} + \frac{12}{\sqrt{1-x^{2}}}\,dx$

21. $\displaystyle\int 6\cos\left(z\right) + \frac{4}{\sqrt{1-z^{2}}}\,dz$

22. Determine $f\left(x\right)$ given that $f'\left(x\right) = 12x^{2} - 4x$ and $f\left(-3\right) = 17$.

23. Determine $g\left(z\right)$ given that $g'\left(z\right) = 3z^{3} + \frac{7}{2\sqrt{z}} - \mathbf{e}^{z}$ and $g\left(1\right) = 15 - \mathbf{e}$.

24. Determine $h\left(t\right)$ given that $h''\left(t\right) = 24t^{2} - 48t + 2$, $h\left(1\right) = -9$ and $h\left(-2\right) = -4$.

# 5.3 Substitution Rule for Indefinite Integrals

For problems 1 - 16 evaluate the given integral.

1. $\displaystyle\int (8x-12)\left(4x^2-12x\right)^4 dx$

2. $\displaystyle\int 3t^{-4}\left(2+4t^{-3}\right)^{-7} dt$

3. $\displaystyle\int (3-4w)\left(4w^2-6w+7\right)^{10} dw$

4. $\displaystyle\int 5(z-4)\ \sqrt[3]{z^2-8z}\,dz$

5. $\displaystyle\int 90x^2 \sin\left(2+6x^3\right)\ dx$

6. $\displaystyle\int \sec(1-z)\tan(1-z)\ dz$

7. $\displaystyle\int \left(15t^{-2}-5t\right)\cos\left(6t^{-1}+t^2\right)\ dt$

8. $\displaystyle\int \left(7y-2y^3\right)\mathbf{e}^{y^4-7y^2}\,dy$

9. $\displaystyle\int \frac{4w+3}{4w^2+6w-1}\,dw$

10. $\displaystyle\int \left(\cos(3t)-t^2\right)\left(\sin(3t)-t^3\right)^5 dt$

11. $\displaystyle\int 4\left(\frac{1}{z}-\mathbf{e}^{-z}\right)\cos\left(\mathbf{e}^{-z}+\ln z\right)\ dz$

12. $\displaystyle\int \sec^2(v)\,\mathbf{e}^{1+\tan(v)}\,dv$

13. $\displaystyle\int 10\sin(2x)\cos(2x)\sqrt{\cos^2(2x)+5}\,dx$

14. $\displaystyle\int \frac{\csc(x)\cot(x)}{2-\csc(x)}\,dx$

15. $\displaystyle\int \frac{6}{7+y^2}\,dy$

16. $\displaystyle\int \frac{1}{\sqrt{4-9w^2}}\,dw$

17. Evaluate each of the following integrals.

**(a)** $\displaystyle\int \frac{3x}{1+9x^2}\,dx$

**(b)** $\displaystyle\int \frac{3x}{\left(1+9x^2\right)^4}\,dx$

**(c)** $\displaystyle\int \frac{3}{1+9x^2}\,dx$

# 5.4 More Substitution Rule

Evaluate each of the following integrals.

1. $\displaystyle\int 4\sqrt{5+9t}+12(5+9t)^7\,dt$

2. $\displaystyle\int 7x^3\cos\left(2+x^4\right)-8x^3\mathbf{e}^{2+x^4}\,dx$

3. $\displaystyle\int \frac{6\mathbf{e}^{7w}}{\left(1-8\mathbf{e}^{7w}\right)^3}+\frac{14\mathbf{e}^{7w}}{1-8\mathbf{e}^{7w}}\,dw$

4. $\displaystyle\int x^4-7x^5\cos\left(2x^6+3\right)\,dx$

5. $\displaystyle\int \mathbf{e}^z+\frac{4\sin(8z)}{1+9\cos(8z)}\,dz$

6. $\displaystyle\int 20\mathbf{e}^{2-8w}\sqrt{1+\mathbf{e}^{2-8w}}+7w^3-6\sqrt[3]{w}\,dw$

7. $\displaystyle\int (4+7t)^3-9t\sqrt[4]{5t^2+3}\,dt$

8. $\displaystyle\int \frac{6x-x^2}{x^3-9x^2+8}-\csc^2\left(\frac{3x}{2}\right)\,dx$

9. $\displaystyle\int 7(3y+2)\left(4y+3y^2\right)^3+\sin(3+8y)\,dy$

10. $\displaystyle\int \sec^2(2t)\left[9+7\tan(2t)-\tan^2(2t)\right]\,dt$

11. $\displaystyle\int \frac{8-w}{4w^2+9}\,dw$

12. $\displaystyle\int \frac{7x+2}{\sqrt{1-25x^2}}\,dx$

13. $\displaystyle\int z^7\left(8+3z^4\right)^8\,dz$

## 5.5 Area Problem

For problems 1 - 3 estimate the area of the region between the function and the x-axis on the given interval using $n = 6$ and using,

**(a)** the right end points of the subintervals for the height of the rectangles,

**(b)** the left end points of the subintervals for the height of the rectangles and,

**(c)** the midpoints of the subintervals for the height of the rectangles.

1. $f(x) = x^3 - 2x^2 + 4$ on $[1, 4]$
2. $g(x) = 4 - \sqrt{x^2 + 2}$ on $[-1, 3]$
3. $h(x) = -x\cos\left(\frac{x}{3}\right)$ on $[0, 3]$
4. Estimate the net area between $f(x) = 8x^2 - x^5 - 12$ and the x-axis on $[-2, 2]$ using $n = 8$ and the midpoints of the subintervals for the height of the rectangles. Without looking at a graph of the function on the interval does it appear that more of the area is above or below the x-axis?

## 5.6 Definition of the Definite Integral

For problems 1 & 2 use the definition of the definite integral to evaluate the integral. Use the right end point of each interval for $x_i^*$.

1. $\displaystyle\int_1^4 2x + 3\,dx$

2. $\displaystyle\int_0^1 6x\left(x-1\right)\,dx$

3. Evaluate : $\displaystyle\int_4^4 \frac{\cos\left(\mathbf{e}^{3x}+x^2\right)}{x^4+1}\,dx$

For problems 4 & 5 determine the value of the given integral given that $\displaystyle\int_6^{11} f\left(x\right)\,dx = -7$ and $\displaystyle\int_6^{11} g\left(x\right)\,dx = 24$.

4. $\displaystyle\int_{11}^6 9f\left(x\right)\,dx$

5. $\displaystyle\int_6^{11} 6g\left(x\right) - 10f\left(x\right)\,dx$

6. Determine the value of $\displaystyle\int_2^9 f\left(x\right)\,dx$ given that $\displaystyle\int_5^2 f\left(x\right)\,dx = 3$ and $\displaystyle\int_5^9 f\left(x\right)\,dx = 8$.

7. Determine the value of $\displaystyle\int_{-4}^{20} f\left(x\right)\,dx$ given that $\displaystyle\int_{-4}^{0} f\left(x\right)\,dx = -2$, $\displaystyle\int_{31}^{0} f\left(x\right)\,dx = 19$ and $\displaystyle\int_{20}^{31} f\left(x\right)\,dx = -21$.

For problems 8 & 9 sketch the graph of the integrand and use the area interpretation of the definite integral to determine the value of the integral.

8. $\displaystyle\int_1^4 3x - 2\,dx$

9. $\displaystyle\int_0^5 -4x\,dx$

For problems 10 - 12 differentiate each of the following integrals with respect to x.

10. $\displaystyle\int_{4}^{x} 9\cos^2\left(t^2 - 6t + 1\right)\, dt$

11. $\displaystyle\int_{7}^{\sin(6x)} \sqrt{t^2 + 4}\, dt$

12. $\displaystyle\int_{3x^2}^{-1} \frac{\mathbf{e}^t - 1}{t}\, dt$

## 5.7 Computing Definite Integrals

1. Evaluate each of the following integrals.

 **(a)** $\displaystyle\int \cos\left(x\right) - \frac{3}{x^5}\,dx$

 **(b)** $\displaystyle\int_{-3}^{4} \cos\left(x\right) - \frac{3}{x^5}\,dx$

 **(c)** $\displaystyle\int_{1}^{4} \cos\left(x\right) - \frac{3}{x^5}\,dx$

Evaluate each of the following integrals, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

2. $\displaystyle\int_{1}^{6} 12x^3 - 9x^2 + 2\,dx$

3. $\displaystyle\int_{-2}^{1} 5z^2 - 7z + 3\,dz$

4. $\displaystyle\int_{3}^{0} 15w^4 - 13w^2 + w\,dw$

5. $\displaystyle\int_{1}^{4} \frac{8}{\sqrt{t}} - 12\sqrt{t^3}\,dt$

6. $\displaystyle\int_{1}^{2} \frac{1}{7z} + \frac{\sqrt[3]{z^2}}{4} - \frac{1}{2z^3}\,dz$

7. $\displaystyle\int_{-2}^{4} x^6 - x^4 + \frac{1}{x^2}\,dx$

8. $\displaystyle\int_{-4}^{-1} x^2\left(3 - 4x\right)\,dx$

9. $\displaystyle\int_{2}^{1} \frac{2y^3 - 6y^2}{y^2}\,dy$

10. $\displaystyle\int_{0}^{\frac{\pi}{2}} 7\sin\left(t\right) - 2\cos\left(t\right)\,dt$

11. $\displaystyle\int_{0}^{\pi} \sec\left(z\right)\tan\left(z\right) - 1\,dz$

12. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sec^2\left(w\right) - 8\csc\left(w\right)\cot\left(w\right)\,dw$

13. $\displaystyle\int_0^2 \mathbf{e}^x + \frac{1}{x^2+1}\,dx$

14. $\displaystyle\int_{-5}^{-2} 7\mathbf{e}^y + \frac{2}{y}\,dy$

15. $\displaystyle\int_0^4 f(t)\,dt$ where $f(t) = \begin{cases} 2t & t > 1 \\ 1-3t^2 & t \le 1 \end{cases}$

16. $\displaystyle\int_{-6}^{1} g(z)\,dz$ where $g(z) = \begin{cases} 2-z & z > -2 \\ 4\mathbf{e}^z & z \le -2 \end{cases}$

17. $\displaystyle\int_3^6 |2x-10|\,dx$

18. $\displaystyle\int_{-1}^{0} |4w+3|\,dw$

# 5.8 Substitution Rule for Definite Integrals

Evaluate each of the following integrals, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

1. $\displaystyle\int_0^1 3\left(4x+x^4\right)\left(10x^2+x^5-2\right)^6\,dx$

2. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{8\cos\left(2t\right)}{\sqrt{9-5\sin\left(2t\right)}}\,dt$

3. $\displaystyle\int_\pi^0 \sin\left(z\right)\cos^3\left(z\right)\,dz$

4. $\displaystyle\int_1^4 \sqrt{w}\,\mathbf{e}^{1-\sqrt{w^3}}\,dw$

5. $\displaystyle\int_{-4}^{-1} \sqrt[3]{5-2y}+\frac{7}{5-2y}\,dy$

6. $\displaystyle\int_{-1}^{2} x^3+\mathbf{e}^{\frac{1}{4}x}\,dx$

7. $\displaystyle\int_\pi^{\frac{3\pi}{2}} 6\sin\left(2w\right)-7\cos\left(w\right)dw$

8. $\displaystyle\int_1^5 \frac{2x^3+x}{x^4+x^2+1}-\frac{x}{x^2-4}\,dx$

9. $\displaystyle\int_{-2}^{0} t\sqrt{3+t^2}+\frac{3}{\left(6t-1\right)^2}\,dt$

10. $\displaystyle\int_{-2}^{1} \left(2-z\right)^3+\sin\left(\pi z\right)\left[3+2\cos\left(\pi z\right)\right]^3dz$

# 6 Applications of Integrals

The previous chapter dealt exclusively with the computation of definite and indefinite integrals as well as some discussion of their properties and interpretations. It is now time to start looking at some applications of integrals. Note as well that we should probably say applications of **definite** integrals as that is really what we'll be looking at in this section.

In addition, we should note that there are a lot of different applications of (definite) integrals out there. We will look at the ones that can easily be done with the knowledge we have at our disposal at this point. Once we have covered the next chapter, Integration Techniques, we will be able to take a look at a few more applications of integrals. At this point we would not be able to compute many of the integrals that arise in those later applications.

In this chapter we'll take a look at using integrals to compute the average value of a function and the work required to move an object over a given distance. In addition we will take a look at a couple of geometric applications of integrals. In particular we will use integrals to compute the area that is between two curves and note that this application should not be too surprising given one of the major interpretations of the definite integral. We will also see how to compute the volume of some solids. We will compute the volume of solids of revolution, *i.e.* a solid obtained by rotating a curve about a given axis. In addition, we will compute the volume of some slightly more general solids in which the cross sections can be easily described with nice 2D geometric formulas (*i.e.* rectangles, triangles, circles, *etc.*).

If you are looking for the solutions to these problems you can go to the Practice Problems on the website and download a pdf with the solutions from there.

# 6.1 Average Function Value

For problems 1 & 2 determine $f_{\text{avg}}$ for the function on the given interval.

1. $f(x) = 8x - 3 + 5\mathbf{e}^{2-x}$ on $[0, 2]$
2. $f(x) = \cos(2x) - \sin\left(\frac{x}{2}\right)$ on $\left[-\frac{\pi}{2}, \pi\right]$
3. Find $f_{\text{avg}}$ for $f(x) = 4x^2 - x + 5$ on $[-2, 3]$ and determine the value(s) of $c$ in $[-2, 3]$ for which $f(c) = f_{\text{avg}}$.

## 6.2 Area Between Curves

1. Determine the area below $f(x) = 3 + 2x - x^2$ and above the x-axis.

2. Determine the area to the left of $g(y) = 3 - y^2$ and to the right of $x = -1$.

For problems 3 - 11 determine the area of the region bounded by the given set of curves.

3. $y = x^2 + 2$, $y = \sin(x)$, $x = -1$ and $x = 2$

4. $y = \dfrac{8}{x}$, $y = 2x$ and $x = 4$

5. $x = 3 + y^2$, $x = 2 - y^2$, $y = 1$ and $y = -2$

6. $x = y^2 - y - 6$ and $x = 2y + 4$

7. $y = x\sqrt{x^2 + 1}$, $y = \mathbf{e}^{-\frac{1}{2}x}$, $x = -3$ and the y-axis.

8. $y = 4x + 3$, $y = 6 - x - 2x^2$, $x = -4$ and $x = 2$

9. $y = \dfrac{1}{x+2}$, $y = (x+2)^2$, $x = -\dfrac{3}{2}$, $x = 1$

10. $x = y^2 + 1$, $x = 5$, $y = -3$ and $y = 3$

11. $x = \mathbf{e}^{1+2y}$, $x = \mathbf{e}^{1-y}$, $y = -2$ and $y = 1$

## 6.3 Solids of Revolution / Method of Rings

For each of the following problems use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by the given curves about the given axis.

1. Rotate the region bounded by $y = \sqrt{x}$, $y = 3$ and the $y$-axis about the $y$-axis.
2. Rotate the region bounded by $y = 7 - x^2$, $x = -2$, $x = 2$ and the $x$-axis about the $x$-axis.
3. Rotate the region bounded by $x = y^2 - 6y + 10$ and $x = 5$ about the $y$-axis.
4. Rotate the region bounded by $y = 2x^2$ and $y = x^3$ about the $x$-axis.
5. Rotate the region bounded by $y = 6\mathbf{e}^{-2x}$ and $y = 6 + 4x - 2x^2$ between $x = 0$ and $x = 1$ about the line $y = -2$.
6. Rotate the region bounded by $y = 10 - 6x + x^2$, $y = -10 + 6x - x^2$, $x = 1$ and $x = 5$ about the line $y = 8$.
7. Rotate the region bounded by $x = y^2 - 4$ and $x = 6 - 3y$ about the line $x = 24$.
8. Rotate the region bounded by $y = 2x + 1$, $x = 4$ and $y = 3$ about the line $x = -4$.

# 6.4 Solids of Revolution / Method of Cylinders

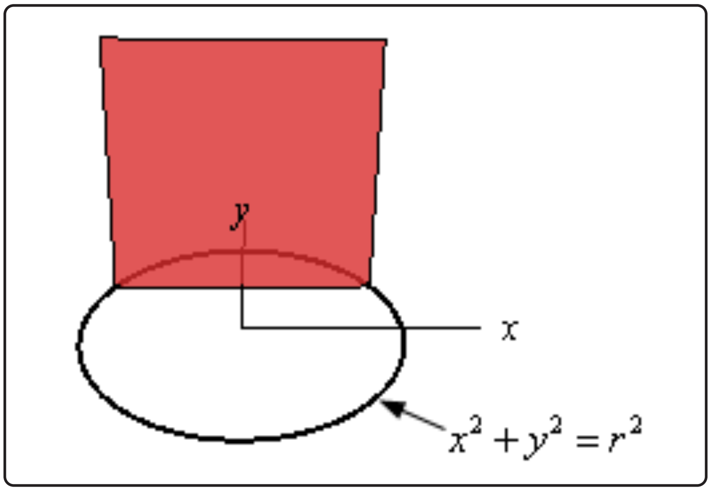
For each of the following problems use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by the given curves about the given axis.

1. Rotate the region bounded by $x = (y-2)^2$, the $x$-axis and the $y$-axis about the $x$-axis.
2. Rotate the region bounded by $y = \dfrac{1}{x}$, $x = \dfrac{1}{2}$, $x = 4$ and the $x$-axis about the $y$-axis.
3. Rotate the region bounded by $y = 4x$ and $y = x^3$ about the $y$-axis. For this problem assume that $x \ge 0$.
4. Rotate the region bounded by $y = 4x$ and $y = x^3$ about the $x$-axis. For this problem assume that $x \ge 0$.
5. Rotate the region bounded by $y = 2x+1$, $y = 3$ and $x = 4$ about the line $y = 10$.
6. Rotate the region bounded by $x = y^2 - 4$ and $x = 6 - 3y$ about the line $y = -8$.
7. Rotate the region bounded by $y = x^2 - 6x + 9$ and $y = -x^2 + 6x - 1$ about the line $x = 8$.
8. Rotate the region bounded by $y = \dfrac{\mathbf{e}^{\frac{1}{2}x}}{x+2}$, $y = 5 - \dfrac{1}{4}x$, $x = -1$ and $x = 6$ about the line $x = -2$.

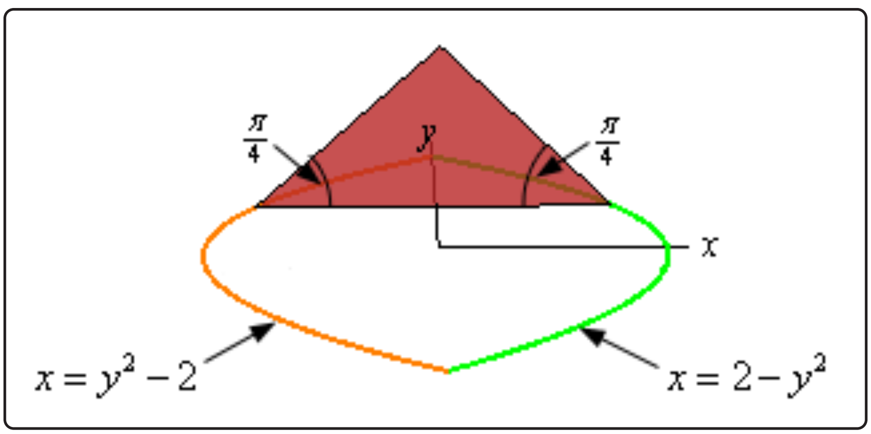
## 6.5 More Volume Problems

1. Find the volume of a pyramid of height $h$ whose base is an equilateral triangle of length $L$.

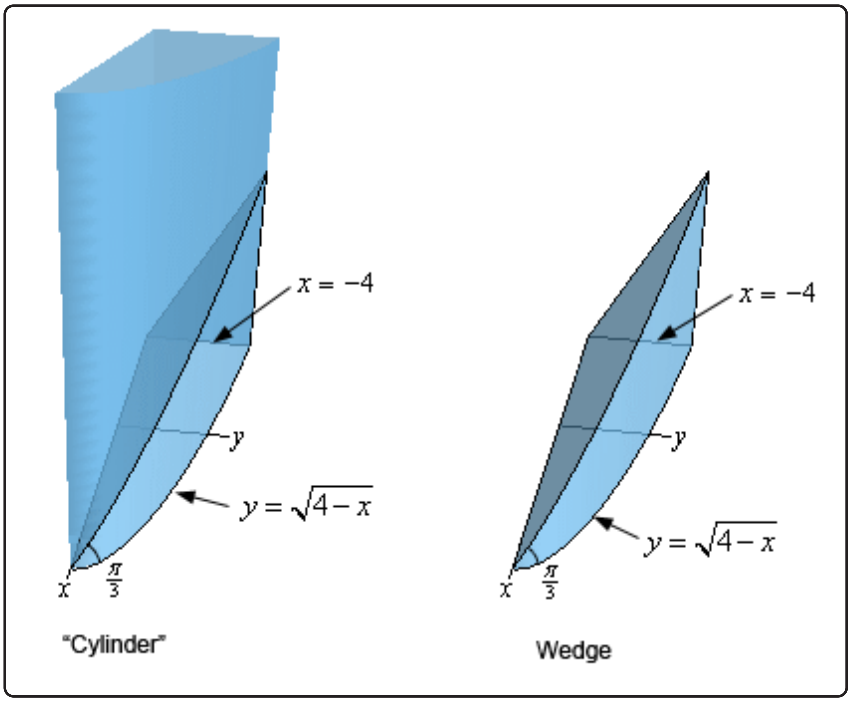
2. Find the volume of the solid whose base is a disk of radius $r$ and whose cross-sections are squares. See figure below to see a sketch of the cross-sections.




3. Find the volume of the solid whose base is the region bounded by $x = 2 - y^2$ and $x = y^2 - 2$ and whose cross-sections are isosceles triangles with the base perpendicular to the $y$-axis and the angle between the base and the two sides of equal length is $\frac{\pi}{4}$. See figure below to see a sketch of the cross-sections.



4. Find the volume of a wedge cut out of a "cylinder" whose base is the region bounded by $y = \sqrt{4-x}$, $x = -4$ and the $x$-axis. The angle between the top and bottom of the wedge is $\frac{\pi}{3}$. See the figure below for a sketch of the "cylinder" and the wedge (the positive $x$-axis and positive $y$-axis are shown in the sketch - they are just in a different orientation).



## 6.6 Work

1. A force of $F(x) = x^2 - \cos(3x) + 2$, $x$ is in meters, acts on an object. What is the work required to move the object from $x = 3$ to $x = 7$?

2. A spring has a natural length of 18 inches and a force of 20 lbs is required to stretch and hold the spring to a length of 24 inches. What is the work required to stretch the spring from a length of 21 inches to a length of 26 inches?

3. A cable with mass ½ kg/meter is lifting a load of 150 kg that is initially at the bottom of a 50 meter shaft. How much work is required to lift the load ¼ of the way up the shaft?

4. A tank of water is 15 feet long and has a cross section in the shape of an equilateral triangle with sides 2 feet long (point of the triangle points directly down). The tank is filled with water to a depth of 9 inches. Determine the amount of work needed to pump all of the water to the top of the tank. Assume that the density of the water is 62 lb/ft$^3$.

# 7 Integration Techniques

By this point we've now looked at basic integration techniques. We've seen how to integrate most of the "basic" functions we're liable to run into : polynomials, roots, trig, exponential, logarithm and inverse trig functions to name a few. In addition, we've seen how to do basic $u$-substitutions allowing us to integrate some more complicated functions.

We've also taken a look at some basic applications of (definite) integrals. However, as was noted at the time, there are applications of (definite) integrals that will, on occasion, have integrals that need more than just a basic $u$-substitution. So, before we can take a look at those applications we'll need to first talk about some more involved integration techniques.

Before getting into the new techniques we first need to make it clear that in this chapter it is assumed at you are comfortable with basic integration, including $u$-substitutions. Many of the problems in this chapter will not have a lot, if any, discussion of the basic integration work under the assumption that you are comfortable enough with the basic work that discussion in simply not needed. In addition, we will usually, although not always, give the substitution that we're using for the $u$-substitution but we will generally not show the actual substitution work. Again, this is under the assumption that you are comfortable enough with basic $u$-substitutions that you can fill in the details if you need to.

The reason for skipping the discussion of the basic integration work and/or not showing the full substitution work is so we can concentrate our discussion on the particular method that we are covering in that particular section. This is not to "punish" you but simply to acknowledge that we only have so much time in which to discuss the material and just can't afford to spend a lot of time basically re-lecturing basic integration material. We realize that, for many of you, this is the start of your Calculus II course and so you may have had some time off and may well have some "rust" on your basic integration skills. This is a warning to start scraping that rust off. If you need do scrape some rust off you can check out the practice problems for some practice problems covering basic integration to refresh your memory on how basic integration works.

It is also very important for you to understand that most of the problems we'll be looking at in this chapter will involve $u$-substitutions in one way or another. In fact, many of the techniques in this chapter are really just substitutions. The only difference is that either they need a fair amount of work to get to the point where the substitutions can be used or they will involve substitutions used in ways that we've not seen to this point. So, again, if you have some rust on your $u$-substitution skills you'll need to get it scraped off so you can do the work in this chapter.

In addition, we will be doing indefinite integrals almost exclusively in most of the sections in this chapter. There are a few sections were we'll be doing some definite integrals but for the most part we'll keep the problems in most of the sections shorter by just doing indefinite integrals. It is assumed that if you were given a definite integral you could do the extra evaluation steps needed to finish the definite integral. Having said that, there are a few sections were definite integrals are done either because there are some subtleties that need to be dealt with for definite integrals or because the topic at hand, the last few sections in particular, involve only definite integrals.

So, with all that out of the way, here is a quick rundown of the new integration techniques we'll take a look at in this section.

Probably the most important technique, in this sense that it will be the most commonly seen technique out of this class, is integration by parts. This is the one new technique in this chapter that is not just $u$-substitutions done in new ways. Integration by Parts will involve $u$-substitutions at various steps the process on occasion but it will not be just a new way of doing a $u$-substitution.

As noted a lot of the techniques in this chapter are really just $u$-substitutions except they will need some manipulation of the integrand prior to actually doing the substitution. The techniques using this idea will include integrating some, but not all, products and quotients of trig functions, some integrands involving roots or quadratics that can't be done without manipulation of the integrand or "different" $u$-substitutions that we are used to. We'll also see how to use partial fractions to write some integrands involving rational expressions into a form that we can actually do the integral.

We'll also take a look at something called trig substitutions. This is probably the one technique that most find the most difficult, or at the least, the longest method. As we'll see a trig substitution is really a substitution but it is not a traditional $u$-substitution. However, having said that, if you understand how basic $u$-substitutions work it will help greatly when it comes to working with trig substitutions as the basic concepts are the same.

Next we'll be taking a look at a new kind of integral, Improper Integrals. This topic will address how to deal with definite integrals for which one or both of the limits of integration will be an infinity. In addition, we'll see how we can, on occasion, deal with discontinuities in the integrand (we'll focus on division by zero in the integrand).

We'll close our the section with a quick section on approximating the value of definite integrals.

We will leave this section with a warning. It is with this chapter that you will find that you can't just memorize your way through the class anymore. We will acknowledge that up to this point it is possible, for the most part, to just memorize your way through the class. You may not get the highest grades through just memorization as there are some topics that require a fair amount of understanding of the topic, but you can survive up to this point if your really good at memorization.

Integration by Parts is a really good example of this. While you will need to memorize/know the basic integration by parts formula simply memorizing that will not help you to actually use integration by parts on the problem. You will need to actually understand how integration by parts works and how to “assign” various portions of the integrand to the various portions of the integration parts formula.

Also while there are some basic formulas we can, and do on occasion, give for some of the methods there are also situations that just don’t fit into those formulas and so again you’ll really need to understand how to do those methods in order to work problems for which basic formulas just won’t work. Or, again, you can’t just memorize your way out of most the methods taught in this chapter. Memorization may allow you to get through the basic problems but will not help all that much with more complicated problems.

Finally, we also need to warn you about seeing “patterns” and just assuming that all the problems will fall into those patterns. Integration by Parts is, again, a good example of this. There are some “patterns” that seem to show up because a lot of the problems we do in that section do fall into the patterns. The problem is that there are also some problems for which the “patterns” simply don’t work and yet they still require integration by parts. If you get so locked into “patterns” you’ll find it all but impossible to do some problems because they simply don’t fall into those patterns.

This is not to say that recognizing that patterns in always a bad thing. Patterns do, on occasion, show up and they can be useful to understand/know as a possible solution method. However, you also need to always remember that there are problems that just don’t fit easily into the patterns. This is also a warning that will be valid in other chapters in a typical Calculus II course as well. Again, patterns aren’t bad per se, you just need to be careful to not always assume that every problem will fall into the patterns.

If you are looking for the solutions to these problems you can go to the Practice Problems on the website and download a pdf with the solutions from there.

## 7.1 Integration by Parts

Evaluate each of the following integrals.

1. $\displaystyle\int 4x\cos\left(2-3x\right)\,dx$

2. $\displaystyle\int_{6}^{0}\left(2+5x\right)\mathbf{e}^{\frac{1}{3}x}\,dx$

3. $\displaystyle\int\left(3t+t^2\right)\sin\left(2t\right)\,dt$

4. $\displaystyle\int 6\tan^{-1}\left(\frac{8}{w}\right)\,dw$

5. $\displaystyle\int\mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)\,dz$

6. $\displaystyle\int_{0}^{\pi}x^2\cos\left(4x\right)\,dx$

7. $\displaystyle\int t^7\sin\left(2t^4\right)\,dt$

8. $\displaystyle\int y^6\cos\left(3y\right)\,dy$

9. $\displaystyle\int\left(4x^3-9x^2+7x+3\right)\mathbf{e}^{-x}\,dx$

# 7.2 Integrals Involving Trig Functions

Evaluate each of the following integrals.

1. $\displaystyle\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)\,dx$

2. $\displaystyle\int \sin^8(3z)\cos^5(3z)\,dz$

3. $\displaystyle\int \cos^4(2t)\,dt$

4. $\displaystyle\int_{\pi}^{2\pi} \cos^3\left(\frac{1}{2}w\right)\sin^5\left(\frac{1}{2}w\right)\,dw$

5. $\displaystyle\int \sec^6(3y)\tan^2(3y)\,dy$

6. $\displaystyle\int \tan^3(6x)\sec^{10}(6x)\,dx$

7. $\displaystyle\int_{0}^{\frac{\pi}{4}} \tan^7(z)\sec^3(z)\,dz$

8. $\displaystyle\int \cos(3t)\sin(8t)\,dt$

9. $\displaystyle\int_{1}^{3} \sin(8x)\sin(x)\,dx$

10. $\displaystyle\int \cot(10z)\csc^4(10z)\,dz$

11. $\displaystyle\int \csc^6\left(\frac{1}{4}w\right)\cot^4\left(\frac{1}{4}w\right)\,dw$

12. $\displaystyle\int \frac{\sec^4(2t)}{\tan^9(2t)}\,dt$

13. $\displaystyle\int \frac{2+7\sin^3(z)}{\cos^2(z)}\,dz$

14. $\displaystyle\int \left[9\sin^5(3x)-2\cos^3(3x)\right]\csc^4(3x)\,dx$

# 7.3 Trig Substitutions

For problems 1 - 8 use a trig substitution to eliminate the root.

1. $\sqrt{4 - 9z^2}$
2. $\sqrt{13 + 25x^2}$
3. $\left(7t^2 - 3\right)^{\frac{5}{2}}$
4. $\sqrt{(w + 3)^2 - 100}$
5. $\sqrt{4(9t - 5)^2 + 1}$
6. $\sqrt{1 - 4z - 2z^2}$
7. $\left(x^2 - 8x + 21\right)^{\frac{3}{2}}$
8. $\sqrt{\mathbf{e}^{8x} - 9}$

For problems 9 - 16 use a trig substitution to evaluate the given integral.

9. $\displaystyle\int \frac{\sqrt{x^2 + 16}}{x^4}\,dx$
10. $\displaystyle\int \sqrt{1 - 7w^2}\,dw$
11. $\displaystyle\int t^3\left(3t^2 - 4\right)^{\frac{5}{2}}\,dt$
12. $\displaystyle\int_{-7}^{-5} \frac{2}{y^4\sqrt{y^2 - 25}}\,dy$
13. $\displaystyle\int_{1}^{4} 2z^5\sqrt{2 + 9z^2}\,dz$
14. $\displaystyle\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\,dx$
15. $\displaystyle\int \frac{(z + 3)^5}{\left(40 - 6z - z^2\right)^{\frac{3}{2}}}\,dz$
16. $\displaystyle\int \cos(x)\sqrt{9 + 25\sin^2(x)}\,dx$

## 7.4 Partial Fractions

Evaluate each of the following integrals.

1. $\displaystyle\int \frac{4}{x^2+5x-14}\,dx$

2. $\displaystyle\int \frac{8-3t}{10t^2+13t-3}\,dt$

3. $\displaystyle\int_{-1}^{0} \frac{w^2+7w}{(w+2)(w-1)(w-4)}\,dw$

4. $\displaystyle\int \frac{8}{3x^3+7x^2+4x}\,dx$

5. $\displaystyle\int_{2}^{4} \frac{3z^2+1}{(z+1)(z-5)^2}\,dz$

6. $\displaystyle\int \frac{4x-11}{x^3-9x^2}\,dx$

7. $\displaystyle\int \frac{z^2+2z+3}{(z-6)(z^2+4)}\,dz$

8. $\displaystyle\int \frac{8+t+6t^2-12t^3}{(3t^2+4)(t^2+7)}\,dt$

9. $\displaystyle\int \frac{6x^2-3x}{(x-2)(x+4)}\,dx$

10. $\displaystyle\int \frac{2+w^4}{w^3+9w}\,dw$

## 7.5 Integrals Involving Roots

Evaluate each of the following integrals.

1. $\displaystyle\int \frac{7}{2+\sqrt{x-4}}\,dx$

2. $\displaystyle\int \frac{1}{w+2\sqrt{1-w}+2}\,dw$

3. $\displaystyle\int \frac{t-2}{t-3\sqrt{2t-4}+2}\,dt$

## 7.6 Integrals Involving Quadratics

Evaluate each of the following integrals.

1. $\displaystyle\int \frac{7}{w^2+3w+3}\,dw$

2. $\displaystyle\int \frac{10x}{4x^2-8x+9}\,dx$

3. $\displaystyle\int \frac{2t+9}{\left(t^2-14t+46\right)^{\frac{5}{2}}}\,dt$

4. $\displaystyle\int \frac{3z}{\left(1-4z-2z^2\right)^2}\,dz$

## 7.7 Integration Strategy

Problems have not yet been written for this section.

I was finding it very difficult to come up with a good mix of new problems and decided my time was better spent writing problems for later sections rather than trying to come up with a sufficient number of problems for what is essentially a review section. I intend to come back at a later date when I have more time to devote to this section and add problems then.

# 7.8 Improper Integrals

Determine if each of the following integrals converge or diverge. If the integral converges determine its value.

1. $\displaystyle\int_0^{\infty} (1+2x)\,\mathbf{e}^{-x}\,dx$

2. $\displaystyle\int_{-\infty}^{0} (1+2x)\,\mathbf{e}^{-x}\,dx$

3. $\displaystyle\int_{-5}^{1} \frac{1}{10+2z}\,dz$

4. $\displaystyle\int_{1}^{2} \frac{4w}{\sqrt[3]{w^2-4}}\,dw$

5. $\displaystyle\int_{-\infty}^{1} \sqrt{6-y}\,dy$

6. $\displaystyle\int_{2}^{\infty} \frac{9}{(1-3z)^4}\,dz$

7. $\displaystyle\int_{0}^{4} \frac{x}{x^2-9}\,dx$

8. $\displaystyle\int_{-\infty}^{\infty} \frac{6w^3}{(w^4+1)^2}\,dw$

9. $\displaystyle\int_{1}^{4} \frac{1}{x^2+x-6}\,dx$

10. $\displaystyle\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx$

## 7.9 Comparison Test for Improper Integrals

Use the Comparison Test to determine if the following integrals converge or diverge.

1. $\displaystyle\int_1^\infty \frac{1}{x^3+1}\,dx$
2. $\displaystyle\int_3^\infty \frac{z^2}{z^3-1}\,dz$
3. $\displaystyle\int_4^\infty \frac{\mathbf{e}^{-y}}{y}\,dy$
4. $\displaystyle\int_1^\infty \frac{z-1}{z^4+2z^2}\,dz$
5. $\displaystyle\int_6^\infty \frac{w^2+1}{w^3\left(\cos^2(w)+1\right)}\,dw$

## 7.10 Approximating Definite Integrals

For each of the following integrals use the given value of n to approximate the value of the definite integral using

**(a)** the Midpoint Rule,

**(b)** the Trapezoid Rule, and

**(c)** Simpson's Rule.

Use at least 6 decimal places of accuracy for your work.

1. $\displaystyle\int_{1}^{7} \frac{1}{x^3+1}\,dx$ using $n = 6$

2. $\displaystyle\int_{-1}^{2} \sqrt{\mathbf{e}^{-x^2}+1}\,dx$ using $n = 6$

3. $\displaystyle\int_{0}^{4} \cos\left(1+\sqrt{x}\right)\,dx$ using $n = 8$