Numerical Methods Homework 2 Report

Costanza Pennaforti

February 6, 2025

The goal is to solve

$$\partial_t^2 \phi = \Delta\phi$$

on the surface of a sphere.

To discretize ϕ in space, the first step is to expand it as a sum over modes l,m (m running from -l to l) in terms of spherical harmonics Y_{lm} like so

$$\phi(\theta, \varphi) = \sum_{l,m} c^{l,m}(t) * Y_{lm}(\theta, \varphi)$$

where $c^{l,m}(t)$ are time-dependent coefficients for each mode.

Thus, substituting this sum in the original equation and taking into account that $\Delta Y_{lm} = -l(l+1)Y_{lm}$, it is possible to obtain a second order differential equation for each coefficient:

$$\frac{\partial^2 c^{l,m}}{\partial t^2} = -l(l+1)c^{l,m}$$

.

To discretize ϕ in time, ODE methods can be used. However, standard ODE methods assume that there's only first order time derivatives, hence the need to define $\psi = \partial_t \phi$ such that the original equation gets "split" into two first order equations:

$$\psi = \partial_t \phi$$

and

$$\partial_t \psi = \Delta\phi$$

Substituting for the coefficients $c^{l,m}$ we get:

$$\partial_t c^{l,m} = \psi$$

and

$$\partial_t \psi = \Delta c^{l,m}$$

(thanks $https://en.wikipedia.org/wiki/Spherical_harmonics$ for the theory on this!!).

For part a) of the homework, the main goal was to find a way to numerically calculate the coefficients $c^{l,m}$ from the ϕ expansion. To do so we define a function to compute the coefficients $c^{l,m}$ given l, m and the initial function phi_ini, which will be defined in part b), as input. To do so we can use the following expression for the coefficients:

$$c^{l,m} = \int_0^{2\pi} \int_0^{\pi} \phi(\theta, \varphi) Y^*_{lm}(\theta, \varphi) \sin\theta d\theta d\varphi$$

The next step is to loop over all l and m to get an array of all coefficients up to some l_max. The latter is defined globally around the beginning of the code (unfortunately I encountered a few issues when this wasn't the case) The length of this array will be given by the total number of modes (so accounting for l going from 0 to l_max and m running from -l to l).

However, looking at part b), since we take the initial condition to be a Gaussian peaked at the north pole (i.e., around $\theta = 0$), we can say that the function is azimuthally symmetric and so doesn't really depend on φ. In other words $\phi(\theta, \varphi) \approx \phi(\theta)$

So if we look at the spherical harmonics formula

$$Y_{ml} = N * e^{im\varphi} P_{ml}(\cos\theta)$$

within the double integral for the

coefficients

$$c^{l,m} = \int_0^{2\pi} \int_0^{\pi} \phi(\theta, \varphi) N * e^{-im\varphi} P_{ml}(\cos\theta) \sin\theta d\theta d\varphi$$

we notice that the only φ dependent term is the exponential. The integral over φ of $e^{-im\varphi}$ is only non-zero when $m = 0$ so the total number of modes we need to account for is simply $l_max + 1$.

Thus, after this loop we get all coefficients.

In part b) we define the Gaussian mentioned above for ϕ as a function. The Gaussian is peaked around $\theta = 0$, North pole, with $\sigma = 0.2$ and is described by :

$$\phi(\theta, \varphi) = \frac{1}{\sqrt{2\pi} * 0.2} \exp\left(-1/2 \left(\frac{\theta}{0.2}\right)^2\right)$$

The references for the next parts are: $https://discourse.julialang.org/t/callback-function-example-in-differentialequations-jl/102371$ and $https://docs.sciml.ai/DiffEqDocs/sta$

For part c) and d) we'll use the $DifferentialEquations$ package to solve the system of equations.

To do so, we first define a single array U_0 which contains in the first half the initial $c^{l,m}$ coefficients for ϕ and in the second half the initial values for ψ which we can take to be 0 at t=0 (i.e., we assume the wave to be stationary initially).

After this we define a function, $wave_equation!$, containing the system of equations we want to solve. This function can then be passed into the ODE solver from the package alongside both the initial array U_0 and the time span we wish to evolve the wave for (from 0 to 10). U_0 is created by merging together the array containing the $t = 0$ coefficients $c^{l,m}$ calculated using one of the first few functions defined.

After this, we solve the system in the desired time span and define another function to plot the results at different time spans. l_max can be changed globally and the results for t=5 and different l_max are presented in the next page(all plots at different time stamps in the GitHub repository).

As L_max increases, the ripples appear to become more intense or more defined. This is because L indicated the number of modes included in the expansion of ϕ therefore a higher number of modes indicates a more "precise" representation of ϕ.

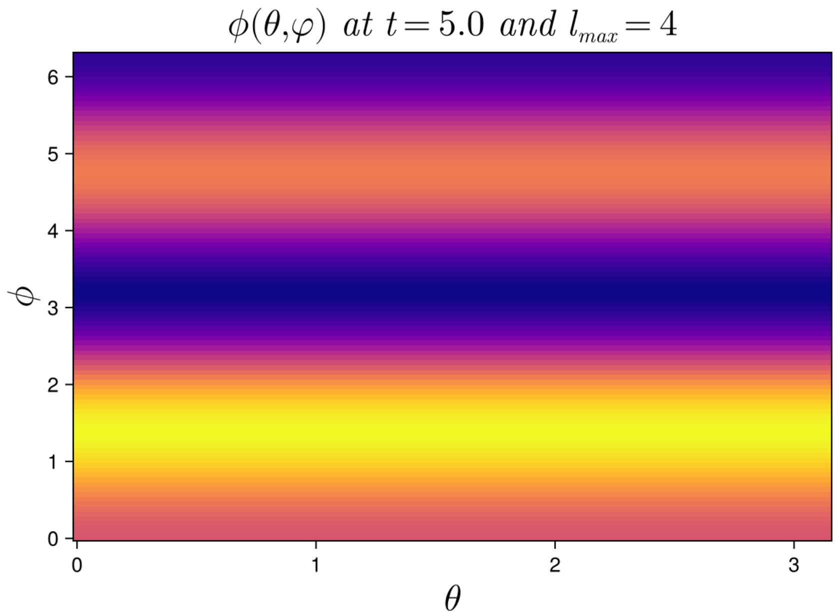

Figure 1: Time stamp $t = 5$ with $L_{\max} = 4$

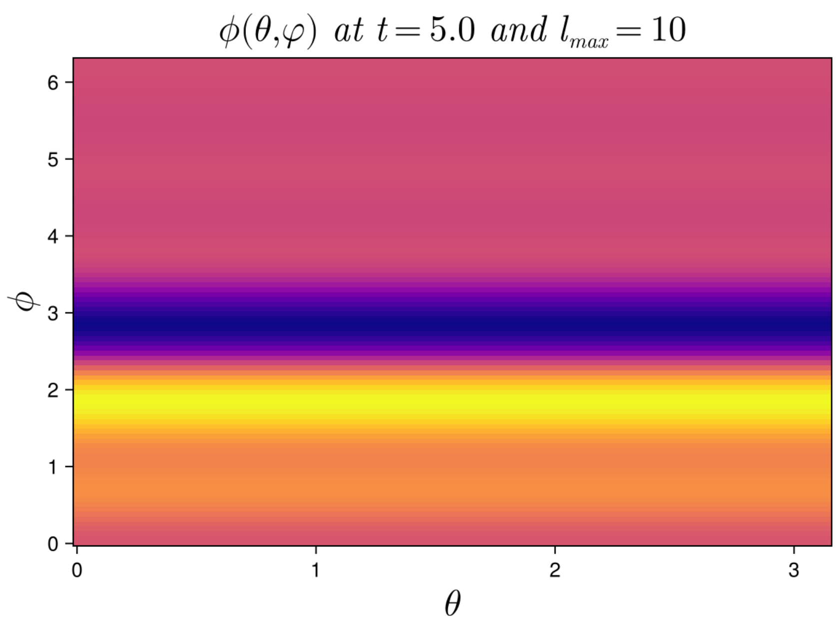

Figure 2: Time stamp $t = 5.0$ with $L_{\max} = 10$

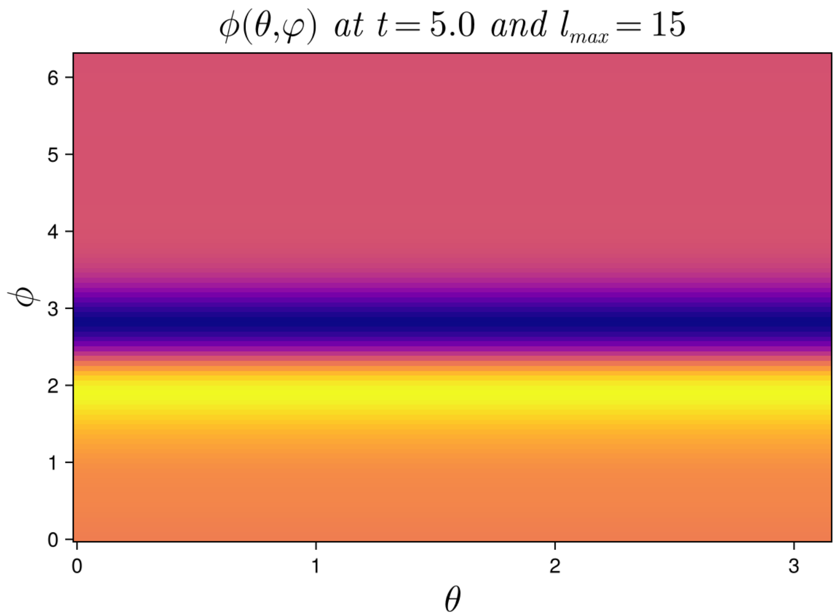

Figure 3: Time stamp $t = 5$ with $L_{\max} = 15$