MSc. in HPC

Trinity College Dublin

Quantitative Finance (565b)

Exercises on Stochastic Calculus

(To be handed in by March 23^{rd}, 2015)

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February 13, 2015

1. Define a Brownian motion $W(t)\,, t > 0$.

   a) Given a Brownian motion $W(t)$ and a partition $\Pi = \{t_0\,, t_1\,, \ldots\,, t_n\}$ of $[0, T]$ $(0 = t_0 < t_1 < t_2 < \cdots < t_n = t)$, show that

   $$QV_W(T) = \lim_{||\Pi||\to 0} \sum_{i=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 = T\,,$$

   b) Derive the covariance of Brownian motion

   $$\mathrm{Cov}(W(t), W(s))\,,\ 0 \le s < t$$

   c) Show that Brownian motion is a martingale (again, for some $0 \le s < t$)

2. Let

   $$dX(t) = P(t)dt + 4Q(t)dW(t)\,,$$

   where $P(t)$ and $Q(t)$ are functions of time

   What is $dX(t)dX(t)$?

3. Itô's formula

   a) Let $f(t, W(t)) = W^2(t)$. Calculate $df(t, W(t))$
   b) Let $f(t, W(t)) = W^2(t) - t$. Calculate $df(t, W(t))$
   c) Let $f(t, W(t)) = \frac{1}{3}W^3(t) - W(t)t$. Calculate $df(t, W(t))$
   d) Let $f(t, W(t)) = e^{-rt}W(t)$, where r is a constant. Calculate $df(t, W(t))$
   e) $f(W(t)) = e^{2W(t)}$. Calculate $df(W(t))$
   f) $f(t, W(t)) = e^{(\sigma W(t) - \frac{1}{2}\sigma^2 t)}$, where σ is constant. Calculate $df(t, W(t))$

   Which of these are Martingales?

4. Using the function $f(t, W(t)) = tW(t)$ and Itô's formula, show that

   $$\int_0^t s dW(s) = tW(t) - \int_0^t W(s)ds\,.$$

5. Product Rule with Itô's formula

   Let $X(t, W(t))$ and $Y(t, W(t))$, so that $X \cdot Y$ is a function of t and $W(t)$, too.

   $$\begin{aligned} d[XY] &= \left(\frac{\partial(XY)}{\partial t} + \frac{1}{2}\frac{\partial^2(XY)}{\partial W^2}\right) dt + \frac{\partial(XY)}{\partial W}\, dW \\ &= X\, dY + Y\, dX + \frac{\partial X}{\partial W}\frac{\partial Y}{\partial W}\, dt \\ &= X\, dY + Y\, dX + dX\, dY \end{aligned}$$

Show that the last two lines are equivalent

6. Itô process. Let
$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)\,,$$
where μ and σ are constant. Let $f(t, S(t)) = e^{-rt}S(t)$. Calculate $df(t, S(t))$.

7. Solve the following stochastic differential equations for $S(T)$, where $S(0) = S_0$

   a) $dS(t) = \alpha dt + \sigma dW(t)$, where α and σ are constant
   b) $dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$, where α and σ are constant
   c) $dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t)$, where $\alpha(t)$ and $\sigma(t)$ are time dependent

8. For each of the solutions in the previous question, calculate $dS(t)$ from the solution and show that you retrieve the original SDE

9. Let $X(t) = \int_0^t e^{\beta s}dW(s)$. Let
$$f(t, X(t)) = e^{-\beta t}R_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}X(t)\,,$$
where $R_0, \alpha, \beta, \sigma$ are constants, and $t \geq 0$.

   a) Show that $f_t(t, X(t)) = \alpha - \beta f(t, X(t))$
   b) Show that
   $$df(t, X(t)) = (\alpha - \beta f(t, X(t)))dt + \sigma dW(t)\,.$$
   c) How is $X(t)$ distributed? What is its mean and variance?
   d) How is $f(t, X(t))$ distributed? What is its mean and variance?

10. Consider the general linear SDE
$$dS = (f(t) + \mu(t)S(t))\,dt + (g(t) + \sigma(t)S(t))\,dW(t).$$
This can be solved in the manner used for ordinary differential equations. Recall from ODEs the distinction between a homogeneous solution S_h and a solution S_p of the inhomogeneous equation. One can proceed by looking for a solution of the form
$$S_p = Z(t)S_h(t).$$
Using Itô's formula, show that
$$dZ = ((f(t) - g(t)\sigma(t))dt + g(t)dW(t))\,/S_h(t)$$
Hint: Use Ito product rule

Write down an expression for $S_p(t)$ for this SDE.